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On the arithmetic product of combinatorial species

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Abstract

We introduce two new binary operations on combinatorial species; the arithmetic product and the modified arithmetic product. The arithmetic product gives combinatorial meaning to the product of Dirichlet series and to the Lambert series in the context of species. It allows us to introduce the notion of multiplicative species, a lifting to the combinatorial level of the classical notion of multiplicative arithmetic function. Interesting combinatorial constructions are introduced; cloned assemblies of structures, hyper-cloned trees, enriched rectangles, etc. Recent research of Cameron, Gewurz and Merola, about the product action in the context of oligomorphic groups, motivated the introduction of the modified arithmetic product. By using the modified arithmetic product we obtain new enumerative results. We also generalize and simplify some results of Canfield, and Pittel, related to the enumerations of tuples of partitions with the restrictions met.

Keywords: Combinatorial species; Arithmetic product; Euler product formula; Graph product; Partition meet

1. Introduction

Informally, a *combinatorial species* F (see [4,15]) is a class of labelled combinatorial structures that is closed by change of labels. Being more formal, F is a rule assigning to each finite set U, a finite set $F[U]$. The elements of $F[U]$ are called F-structures on the set U. The rule F not only acts on finite sets but also on bijections between finite sets. To each bijection $\sigma : U \longrightarrow V$, the rule F associates a bijection $F[\sigma] : F[U] \longrightarrow F[V]$ that is called the *transport of F-structures along* σ. In other words, F is an endofunctor of the category $\mathbb{B}$ of finite sets and bijections. A natural transformation $\kappa : F \to G$ between F and G as endofunctors of $\mathbb{B}$ is called an isomorphism of species, and the species F and G are called isomorphic if there exists such a natural transformation (see [4, p. 21]). In this paper the two isomorphic species are considered equal.

When $F[\sigma]f = f'$ we will say that f and f' are isomorphic F-structures and that σ is an isomorphism from f to f'.

For two species of the structures F and G, other species can be constructed through combinatorial operations; addition $F + G$, product $F \cdot G$, cartesian product $F \times G$, substitution $F \circ G$ (also denoted $F(G)$) and derivative F'

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(see [4] for details). To each species F are associated three main series expansions. The *exponential generating series*,

$$F(x) = \sum_{n\geq 0} |F[n]| \frac{x^n}{n!}, \tag{1}$$

where $|F[n]|$ is the number of F-structures on the set $[n] = \{1, 2, \ldots, n\}$. The *isomorphism type generating series*,

$$\widetilde{F}(x) = \sum_{n\geq 0} |F[n]/\sim| x^n, \tag{2}$$

where $F[n]/\sim$ denotes the set of isomorphism types of F-structures on $[n]$. The *cycle index series*,

$$Z_F(x_1, x_2, \ldots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in S_n} \text{fix}\, F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} \cdots. \tag{3}$$

Here S_n denotes the symmetric group on n symbols, $\text{fix}\, F[\sigma] := |\text{Fix} F[\sigma]|$, where $\text{Fix} F[\sigma]$ is the set of F-structures on $[n]$ fixed by the permutation $F[\sigma]$, and σ_k is the number of cycles of length k of σ. An alternative form of (3) is

$$Z_F(x_1, x_2, \ldots) = \sum_{n\geq 0} \sum_{\alpha\vdash n} \text{fix}\, F[\alpha] \frac{x_1^{\sigma_1} x_2^{\sigma_2} \cdots}{\text{aut}(\alpha)}, \tag{4}$$

where $\text{fix}\, F[\alpha]$ is the number of F-structures on $[n]$ fixed by $F[\sigma]$, σ being any permutation of cycle type $\alpha = (\sigma_1, \sigma_2, \ldots)$ and $\text{aut}(\alpha) = 1^{\sigma_1}\sigma_1! 2^{\sigma_2}\sigma_2! \cdots$.

Let F be a species of the structures and n a non-negative integer. Unless otherwise explicitly stated, we will denote by F_n the species F concentrated in the cardinality n,

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n, \\ \varnothing, & \text{if } |U| \neq n, \end{cases} \tag{5}$$

where U is a finite set. We will also denote by F_+ the species of F-structures over non-empty sets,

$$F_+[U] = \begin{cases} F[U], & \text{if } |U| \geq 1, \\ \varnothing, & \text{if } |U| = 0. \end{cases} \tag{6}$$

Let us denote by X the singleton (or singular) species,

$$X[U] = \begin{cases} U, & \text{if } |U| = 1, \\ \varnothing, & \text{if } |U| \neq 1. \end{cases} \tag{7}$$

Consider the operation of addition and product of two species:

$$(F + G)[U] := F[U] + G[U], \tag{8}$$

$$(F \cdot G)[U] := \sum_{U_1+U_2=U} F[U_1] \times G[U_2], \tag{9}$$

where the sum means disjoint union of sets. X^n, the product of the singleton species by itself n times, is the species of linear orders on length n;

$$X^n[U] = \begin{cases} \{l | l : [n] \to U, l \text{ a bijection}\}, & \text{if } |U| = n, \\ \varnothing, & \text{if } |U| \neq n. \end{cases} \tag{10}$$

A subgroup G of S_n acts on the elements of $X^n[U]$ as follows; $g \cdot l = l \circ g^{-1}$. We will denote by $\frac{X^n}{G}$ the species concentrated in n that assigns to a n-set U, the set of orbits of the linear orders on U under the action of G. A *molecular species* is one that is indecomposable under addition, or equivalently, one where all its structures are isomorphic. When M is molecular, there exists a non-negative integer n and a subgroup G of S_n such that $M = \frac{X^n}{G}$. Yeh [24,25] established the relationship between operations on actions of finite permutation groups and operations on species (product, substitution, cartesian product and derivative), via the decomposition of a species as a sum of molecular species. For example, if M and N are molecular, there are permutation groups $H \leq S_m$ and $K \leq S_n$, such that $M = \frac{X^m}{H}$, and $N = \frac{X^n}{K}$. We have that

$$\frac{X^m}{H} \cdot \frac{X^n}{K} = \frac{X^{m+n}}{H \times K}, \tag{11}$$

where the direct product $H \times K$ acts naturally over the disjoint union $[m] + [n] \equiv [m + n]$, in what is called the intransitive action. There is another natural action $H \times K : [m] \times [n]$, the product action, without an operation on the species counterpart. Some enumerative problems have been solved by Harary [12] and by Harrison and High [14] using the cycle index polynomial of the product action.

We can define the arithmetic product of two molecular species by the formula

$$\frac{X^m}{H} \boxdot \frac{X^n}{K} = \frac{X^{mn}}{H \times K}, \tag{12}$$

where the action of $H \times K$ over $[mn] \equiv [m] \times [n]$ is the product action. Then we can extend this product by linearity. But, in order to have a set-theoretical definition for the arithmetic product like formula (9) for the ordinary product, we need a notion of decomposition of a set into factors. In other words, a set-theoretical analogous of the factoring of a positive integer as a product of two positive integers. In this way we arrived at the concept of *rectangle* on a finite set. This concept was previously introduced in another context with the name of *cartesian decomposition* [2], and is a particular kind of what is called in [19] a *small transversal of a partition*.

Recall that the substitution of a species G, satisfying $G[\varnothing] = \varnothing$, into an arbitrary species F, is defined by the formula

$$F(G)[U] = \sum_{\pi \in \Pi[U]} F[\pi] \times \prod_{B \in \pi} G[B], \tag{13}$$

where the sum (disjoint union) runs over the set of set-partitions on U, $\Pi[U]$. An assembly a of G-structures on a finite set U is a family $\{g_B\}_{B \in \pi}$, where π is a partition of U and $g_B \in G[B]$ for each block B of π. An element of the set defined in (13) is a pair (f, a) where a is an assembly of G-structures on U and $f \in F[\pi]$, π being the partition subjacent in the assembly a. These pairs are usually called *F-assemblies of G-structures*.

The most interesting combinatorial construction associated to the arithmetic product is the *assembly of cloned structures*. Informally, an assembly of cloned G-structures is an assembly of G-structures in the above sense, where all structures in the assembly are isomorphic replicas of the same structure. Moreover, information about 'homologous vertices' or 'genetic similarity' between each pair in the assembly is also provided. The structures of $F \boxdot G$ have some resemblance with the structures of the substitution $F(G)$. An element of $F \boxdot G$ can be represented as a cloned assembly of G-structures together with an external F-structure (an F-assembly of cloned G-structures). Because of the symmetry $F \boxdot G = G \boxdot F$ this structures can also be represented as G-assemblies of cloned F-structures. For example, if L_+ denotes the species of non-empty lists, the structures of $F \boxdot L_+$ could be thought of either as F-assemblies of cloned lists, or as lists of cloned F-structures.

There is a link between *oligomorphic groups* [5] and combinatorial species, implicit in the work of Cameron, and which we hope to have made explicit here. To each oligomorphic group G we can associate a combinatorial species F_G. There is a correspondence between operations on oligomorphic groups and operations on species that is very similar to that established by Yeh between finite permutation groups and molecular species. For example, the intransitive product action of two oligomorphic groups translates to the ordinary product of the respective species and the wreath product to the operation of substitution. Recently, Cameron, Gewurz, and Merola have studied the product action of oligomorphic groups (see [6,10,11]). This have motivated us to introduce the *modified arithmetic product* in order to have the appropriated correspondence between operations. We have made use of this operation to obtain many new enumerative results. Using a simple manipulation of generating series (the *shift trick*) we simplified and generalized some results of Canfield [7], and Pittel [20].

2. The arithmetic product

Definition 1. For a finite set U, we say that an ordered pair (π, τ) of partitions of U is a *partial rectangle* on U when $\pi \wedge \tau = \hat{0}$. If moreover π and τ are independent partitions (every block of π meets every block of τ) we call it a

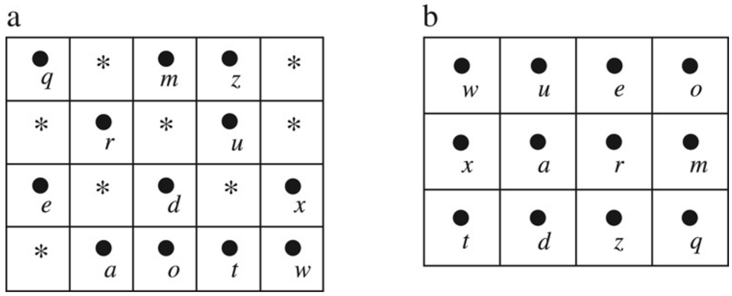

Fig. 1. (a) A partial rectangle, (b) a rectangle on the same set.

rectangle. More generally, a partial rectangle of dimension k, or a k-partial rectangle is a tuple $(\pi_1, \pi_2, \ldots, \pi_k)$ of partitions such that $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_k = \hat{0}$. It is called a k-rectangle if,

$$|B_1 \cap \cdots \cap B_k| = 1, \quad \text{for all } B_1 \in \pi_1, \ldots, B_k \in \pi_k. \tag{14}$$

This definition of the rectangle is equivalent to the “cartesian decomposition” of Baddeley, Praeger and Schneider [2]. If (π, τ) is a rectangle on U, we can arrange the elements of U in a matrix whose rows are the blocks of π and whose columns are the blocks of τ. Two matrices represent the same rectangle if we can obtain one from the other by interchanges of rows or columns. The same can be say about the partial rectangles except for the fact that some of the entries of the matrix could be empty. Fig. 1 shows an example of partial rectangle and rectangle on a set with 12 elements (the symbol $*$ means an empty intersection).

If (π, τ) is a rectangle on U obviously $|U| = |\pi||\tau|$. The *height* (ht) of a rectangle (π, τ) is defined to be $|\pi|$. Naturally $\mathrm{ht}(\pi, \tau)$ divides $|U|$. For $|U| = n$, we denote the number of rectangles of height d with the symbol $\left\{ {n \atop d} \right\}$. It is not difficult to see that

$$\left\{ {n \atop d} \right\} = \frac{n!}{d!\,(n/d)!}. \tag{15}$$

Consider $\mathcal{R}$ the species of rectangles, that is, for U a finite set,

$$\mathcal{R}[U] = \{(\pi, \tau) \mid (\pi, \tau) \text{ is a rectangle on } U\}. \tag{16}$$

If $n \geq 1$, we have

$$|\mathcal{R}[n]| = \sum_{d|n} \left\{ {n \atop d} \right\}. \tag{17}$$

In an analogous way, for the species $\mathcal{R}^{(k)}$, of k-rectangles, we have

$$|\mathcal{R}^{(k)}[n]| = \sum_{d_1 d_2 \cdots d_k = n} \left\{ {n \atop d_1, d_2, \ldots, d_k} \right\}, \tag{18}$$

where

$$\left\{ {n \atop d_1, d_2, \ldots, d_k} \right\} = \frac{n!}{d_1! d_2! \cdots d_k!}. \tag{19}$$

Definition 2 (*Arithmetic Product of Species*). Let M and N be species of structures such that $M[\varnothing] = N[\varnothing] = \varnothing$. The arithmetic product of M and N, is defined as follows

$$(M \boxdot N)[U] := \sum_{(\pi,\tau) \in \mathcal{R}[U]} M[\pi] \times N[\tau], \tag{20}$$

where as usual the sum means disjoint union and U is a finite set. In other words, the elements of $(M \boxdot N)[U]$ are tuples of the form (π, τ, m, n), where (π, τ) is a rectangle on U, $m \in M[\pi]$ and $n \in N[\tau]$. Recall that given a bijection

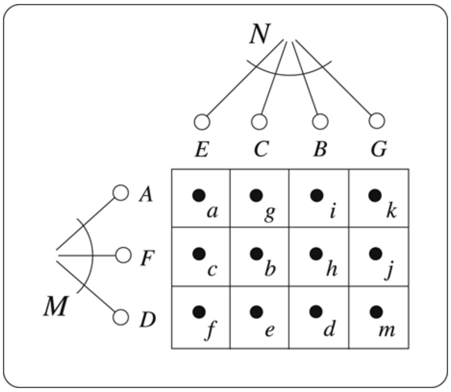

Fig. 2. Graphical representation of the arithmetic product.

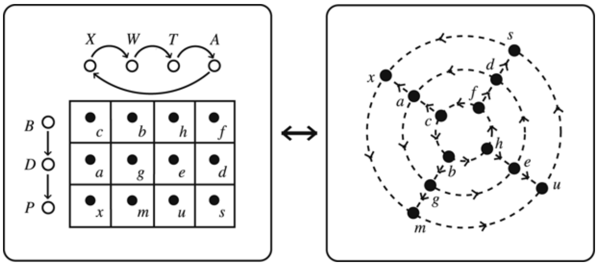

Fig. 3. An $(\mathcal{C} \boxdot L_+)$-structure on a 12-set.

$\sigma : U \longrightarrow V$ and a partition π of U, σ transports π to the partition $\pi' = \sigma(\pi) = \{\sigma(A) \mid A \in \pi\}$ of V. Therefore, σ induces a bijection $\sigma^{\pi} : \pi \longrightarrow \pi'$, which sends A to $\sigma(A)$, for every $A \in \pi$. Similarly for the partition τ.

The transport along a bijection $\sigma : U \longrightarrow V$ is carried out by $M \boxdot N$ as follows:

$$(M \boxdot N)[\sigma]((\pi, \tau, m, n)) = \left(\pi', \tau', M[\sigma^{\pi}](m), N[\sigma^{\tau}](n)\right). \tag{21}$$

In Fig. 2 an $(M \boxdot N)$-structure on a set with 12 elements is represented. The capital letters (except M and N) are the labels for the blocks of the two partitions constituting the rectangle.

Example 3. It is easy to see that the species $\mathcal{R}$ of rectangles satisfies the combinatorial equation $\mathcal{R} = E_+ \boxdot E_+$, where E_+ is the species of non-empty sets.

Using Eq. (15) it is easy to prove the following proposition.

Proposition 4. *Let M and N be the species of structures such that $M[\varnothing] = N[\varnothing] = \varnothing$. Then the exponential generating series of the species $M \boxdot N$ is*

$$(M \boxdot N)(x) = \sum_{n \geq 1} \sum_{d|n} \left\{ n \atop d \right\} |M[d]||N[n/d]| \frac{x^n}{n!}. \quad \Box \tag{22}$$

Example 5 (*Regular Octopuses [4, p. 56]*). Consider the species $\mathcal{C}$ of oriented cycles and L_+ of non-empty linear orders. Fig. 3 represents a $(\mathcal{C} \boxdot L_+)$-structure on a set with 12 elements. Since $|\mathcal{C}[n]| = (n-1)!$ and $|L_+[n]| = n!$, we obtain

$$\begin{aligned} |(\mathcal{C} \boxdot L_+)[n]| &= \sum_{d|n} \left\{ n \atop d \right\} |\mathcal{C}[d]||L_+[n/d]| \\ &= \sigma(n)(n-1)!, \end{aligned}$$

where $\sigma(n)$ is the sum of the positive divisors of n. Its exponential generating series is then

$$\begin{aligned}(\mathcal{C} \boxdot L_+)(x) &= \sum_{n\geq 1} \sigma(n)(n-1)!\frac{x^n}{n!} \\ &= \sum_{n\geq 1} \sigma(n)\frac{x^n}{n}.\end{aligned}$$

Example 6 (*Ordered Lists of Equal Size*). Since $|L_+[n]| = n!$, the number of $(L_+ \boxdot L_+)$-structures on a set of n elements is

$$\begin{aligned}|(L_+ \boxdot L_+)[n]| &= \sum_{d|n} \left\{ {n \atop d} \right\} |L_+[d]||L_+[n/d]| \\ &= d(n)n!,\end{aligned}$$

where $d(n)$ is the number of positive divisors of n. Then, we obtain the generating series

$$\begin{aligned}(L_+ \boxdot L_+)(x) &= \sum_{n\geq 1} n!d(n)\frac{x^n}{n!} \\ &= \sum_{n\geq 1} d(n)x^n.\end{aligned}$$

Example 7. Let $\mathcal{S}_+$ be the species of non-empty permutations. It is clear that

$$(\mathcal{S}_+ \boxdot \mathcal{S}_+)(x) = (L_+ \boxdot L_+)(x) = \sum_{n\geq 1} d(n)x^n.$$

The structures of $(\mathcal{S}_+ \boxdot \mathcal{S}_+)[U]$ are rectangles enriched with permutations on each side. Formally, they are tuples of the form $(\pi, \tau, \sigma_1, \sigma_2)$, where $(\pi, \tau) \in \mathcal{R}[U]$, $\sigma_1 \in \mathcal{S}_+[\pi]$, and $\sigma_2 \in \mathcal{S}_+[\tau]$. By the definition of rectangle, for each element $b \in U$ there exists a unique pair of sets $(A_b, B_b) \in \pi \times \tau$ such that $b \in A_b \cap B_b$. The pair (σ_1, σ_2) induces the permutation $\sigma_1 \boxtimes \sigma_2 \in \mathcal{S}_+[U]$, which sends the element $b \in A_b \cap B_b$ to the unique element in $\sigma_1(A_b) \cap \sigma_2(B_b)$ (see Fig. 4). Let $\widetilde{\mathcal{R}}$ be the species defined as follows

$$\widetilde{\mathcal{R}}[U] = \{(\pi, \tau, \sigma) \mid (\pi, \tau) \in \mathrm{Fix}\mathcal{R}[\sigma], \sigma \in \mathcal{S}[U]\}. \tag{23}$$

The function

$$\begin{array}{llll}\boxtimes_U : & (\mathcal{S}_+ \boxdot \mathcal{S}_+)[U] & \longrightarrow & \widetilde{\mathcal{R}}[U] \\ & (\pi, \tau, \sigma_1, \sigma_2) & \longmapsto & (\pi, \tau, \sigma_1 \boxtimes \sigma_2)\end{array} \tag{24}$$

is a natural bijection with the inverse $(\pi, \tau, \sigma) \mapsto (\pi, \tau, \sigma^\pi, \sigma^\tau)$, where σ^π and σ^τ denote the permutation induced on π by σ and τ respectively. The family $\{\boxtimes_U\}_{U\in\mathbb{B}}$, defines a species isomorphism $\boxtimes : \mathcal{S}_+ \boxdot \mathcal{S}_+ \longrightarrow \widetilde{\mathcal{R}}$.

Proposition 8. *Let M, N and R be species of structures such that $M[\varnothing] = N[\varnothing] = R[\varnothing] = \varnothing$, and X the singular species. The product $\boxdot$ has the following properties:*

$$M \boxdot N = N \boxdot M, \tag{25}$$

$$M \boxdot (N \boxdot R) = (M \boxdot N) \boxdot R, \tag{26}$$

$$M \boxdot (N + R) = M \boxdot N + M \boxdot R, \tag{27}$$

$$M \boxdot X = X \boxdot M = M, \tag{28}$$

$$(M \boxdot N)^\bullet = M^\bullet \boxdot N^\bullet, \tag{29}$$

$$M \boxdot X^n = M(X^n), \tag{30}$$

$$M \boxdot L_+ = \sum_{n\geq 1} M(X^n). \quad \square \tag{31}$$

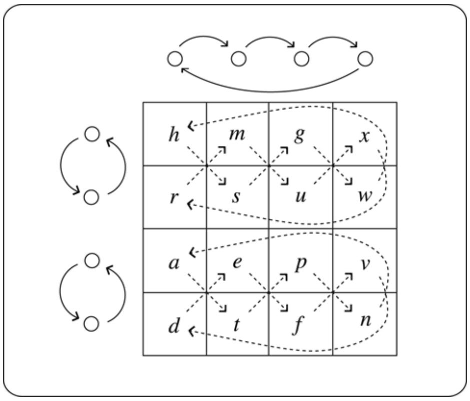

Fig. 4. A structure of $\mathcal{S}_+ \boxdot \mathcal{S}_+$ and induced permutation.

The properties above are not difficult to prove. In particular, the reader may verify that both sides of Eq. (26) evaluated at a set U, are naturally equivalent to the set

$$\sum_{(\pi_1,\pi_2,\pi_3)\in\mathcal{R}^{(3)}[U]} M[\pi_1] \times N[\pi_2] \times R[\pi_3]. \tag{32}$$

In general, for a family $\{M_i\}_{i=1}^k$ of species with $M_i[\varnothing] = \varnothing$, we have

$$(\boxdot_{i=1}^k M_i)[U] = \sum_{(\pi_1,\pi_2,\ldots,\pi_k)\in\mathcal{R}^{(k)}[U]} \prod_{i=1}^{k} M_i[\pi_i]. \tag{33}$$

Also observe that by Eq. (31), the structures of $M \boxdot L_+$ may be thought of as M-assemblies of lists of equal size.

2.1. *The arithmetic product and generating series*

Definition 9. For any two monomials x^n and x^m we define the arithmetic product $x^n \boxdot x^m := x^{nm}$. Extend this product by linearity to exponential formal power series with zero constant term.

We easily obtain that

$$\left(\sum_{n\geq 1} a_n \frac{x^n}{n!}\right) \boxdot \left(\sum_{n\geq 1} b_n \frac{x^n}{n!}\right) = \sum_{n\geq 1} c_n \frac{x^n}{n!}, \tag{34}$$

where

$$c_n = \sum_{d|n} \left\{ {n \atop d} \right\} a_d b_{n/d}. \tag{35}$$

The exponential formal power series with the arithmetic product form a ring with identity x. The substitution $x^n \leftarrow \frac{1}{n^s}$, makes it isomorphic to the ring of *modified formal Dirichlet series*,

$$\sum_{n\geq 1} \frac{a_n}{n!n^s}.$$

This motivates the following definition.

Definition 10. Let M be a species of structures satisfying the condition $M[\varnothing] = \varnothing$. Then the *modified Dirichlet generating series* of M is

$$\mathcal{D}_M(s) = \sum_{n\geq 1} \frac{|M[n]|}{n!n^s}. \tag{36}$$

Thus, for the species E_+, L_+, $\mathcal{C}$ and S_+, we have:

$$\mathcal{D}_{E_+}(s) = \sum_{n\geq 1} \frac{1}{n!n^s}, \tag{37}$$

$$\mathcal{D}_{L_+}(s) = \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s), \tag{38}$$

$$\mathcal{D}_{\mathcal{C}}(s) = \sum_{n\geq 1} \frac{1}{n^{s+1}} = \zeta(s+1), \tag{39}$$

$$\mathcal{D}_{S_+}(s) = \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s). \tag{40}$$

We have not found previous references of the modified Dirichlet series as tools in enumerative combinatorics or in analysis.

From Proposition 4 we obtain the following proposition.

Proposition 11. *For species of structures M and N with the condition $M[\varnothing] = \varnothing = N[\varnothing]$, we have:*

$$(M \boxdot N)(x) = M(x) \boxdot N(x), \tag{41}$$

and

$$\mathcal{D}_{M\boxdot N}(s) = \mathcal{D}_M(s) \cdot \mathcal{D}_N(s). \quad \square \tag{42}$$

For a formal power series $R(x)$ we have

$$x^n \boxdot R(x) = R(x^n). \tag{43}$$

Thus we have the generating series identity

$$(M \boxdot N)(x) = \sum_{n\geq 1} \frac{|M[n]|}{n!} N(x^n). \tag{44}$$

In particular

$$x^n \boxdot \frac{x}{1-x} = \frac{x^n}{1-x^n}, \tag{45}$$

and we obtain that the generating series of $M \boxdot L_+$ is the *Lambert series* (see for example [8, pp. 161–162])

$$(M \boxdot L_+)(x) = \sum_{n\geq 1} \frac{|M[n]|}{n!} \frac{x^n}{1-x^n}. \tag{46}$$

By (31) we also have

$$(M \boxdot L_+)(x) = \sum_{n\geq 1} M(x^n). \tag{47}$$

Using the previous two equations we get

$$(\mathcal{C} \boxdot L_+)(x) = \sum_{n\geq 1} \frac{x^n}{n(1-x^n)} = \sum_{n\geq 1} \ln\left(\frac{1}{1-x^n}\right) = \sum_{n\geq 1} \frac{\sigma(n)}{n} x^n \tag{48}$$

(see [8] and [23] for more properties of Lambert series). This identity translates to Dirichlet generating series as

$$\mathcal{D}_{\mathcal{C}\boxdot L_+}(s) = \zeta(s+1)\zeta(s) = \sum_{n\geq 1} \frac{\sigma(n)}{n} n^{-s}. \tag{49}$$

By Eq. (44) we also obtain:

$$(\mathcal{C} \boxdot M)(x) = \ln \prod_{n\geq 1} \left(\frac{1}{1-x^n}\right)^{\frac{|M[n]|}{n!}}, \tag{50}$$

$$E(\mathcal{C} \boxdot M)(x) = \prod_{n\geq 1} \left(\frac{1}{1-x^n}\right)^{\frac{|M[n]|}{n!}}. \tag{51}$$

Let M be a species of structures. To describe the compatibility of the product $\boxdot$ with the transformation $M \to Z_M$ it is necessary to define a product $\boxdot$ for two cycle index series (see [12]). First we have the following lemma.

Lemma 12. *Let $(\pi, \tau, \sigma_1, \sigma_2)$ be an element of $(\mathcal{S}_+ \boxdot \mathcal{S}_+)[U]$ and $\sigma = \sigma_1 \boxtimes \sigma_2 \in \mathcal{S}[U]$. If the cycle type of σ, σ_1, and σ_2, are respectively α, β and γ, then we have*

$$\alpha_k = \sum_{[i,l]=k} (i,l)\beta_i\gamma_l, \quad k = 1, 2, \ldots, [d, n/d], \tag{52}$$

where $d = \mathrm{ht}(\pi, \tau)$, $[i, l]$ denotes the least common multiple of i and l, and (i, l) the greatest common divisor.

Proof. Analogous to the proof of proposition 7(b) in ([4] p. 74). □

We will say that $\alpha = \beta \boxtimes \gamma$ when they satisfy the Eq. (52). Now define the operation $\boxdot$ on monomials by

$$\left(\prod_{i=1}^{n} x_i^{\beta_i}\right) \boxdot \left(\prod_{l=1}^{m} x_l^{\gamma_l}\right) := \prod_{i=1}^{n}\prod_{l=1}^{m} x_{[i,l]}^{\beta_i\gamma_l(i,l)} = \prod_{k=1}^{nm} x_k^{\alpha_k}, \tag{53}$$

where $\alpha_k = \sum_{[i,l]=k}(i,l)\beta_i\gamma_l = (\beta \boxtimes \gamma)_k$. Equivalently

$$\mathbf{x}^{\beta} \boxdot \mathbf{x}^{\gamma} := \mathbf{x}^{\beta\boxtimes\gamma}. \tag{54}$$

Finally extend linearly this operation to polynomials and formal power series. Note that $x_1^i \boxdot x_1^j = x_1^{ij}$ as in Definition 9.

Definition 13. For α, β, and γ as above, define the coefficient

$$\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} = \begin{cases} \dfrac{\mathrm{aut}(\alpha)}{\mathrm{aut}(\beta)\mathrm{aut}(\gamma)}, & \text{if } \beta \boxtimes \gamma = \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

Lemma 14. *Let σ be a permutation on a finite set U with cycle type α. For β and γ as above, let $S^{\sigma}_{\beta,\gamma}$ be the set of tuples $(\pi, \tau, \sigma_1, \sigma_2) \in (\mathcal{S}_+ \boxdot \mathcal{S}_+)[U]$, such that $\sigma_1 \boxtimes \sigma_2 = \sigma$, and the cycle types of σ_1 and σ_2 are respectively β and γ. Then*

$$|S^{\sigma}_{\beta,\gamma}| = \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}. \tag{56}$$

Proof. Denote by $\mathrm{Aut}(\sigma)$ the group of permutations commuting with σ. $\mathrm{Aut}(\sigma)$ acts transitively on $S^{\sigma}_{\beta,\gamma}$ in the following way

$$\eta \cdot (\pi, \tau, \sigma_1, \sigma_2) := (\eta(\pi), \eta(\tau), \eta^{\pi}\sigma_1(\eta^{\pi})^{-1}, \eta^{\tau}\sigma_2(\eta^{\tau})^{-1}) \; (\eta \in \mathrm{Aut}(\sigma)). \tag{57}$$

The order of the group fixing each element of $S^{\sigma}_{\beta,\gamma}$ is clearly $\mathrm{aut}(\beta)\mathrm{aut}(\gamma)$. □

Proposition 15. *Let M and N be two species of structures. Then, the cycle index series and the type generating series associated to the species $M \boxdot N$ satisfy the identities:*

$$Z_{M\boxdot N}(x_1, x_2, \ldots) = Z_M(x_1, x_2, \ldots) \boxdot Z_N(x_1, x_2, \ldots), \tag{58}$$

$$\widetilde{M \boxdot N}(x) = \widetilde{M}(x) \boxdot \widetilde{N}(x). \tag{59}$$

Proof. Using standard computations with formal power series it is not difficult to deduce the second identity from the first. Following Eq. (54) we obtain

$$Z_M(\mathbf{x}) \boxdot Z_N(\mathbf{x}) = \sum_{\alpha} \left(\sum_{\beta \boxtimes \gamma = \alpha} \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} \text{fix}\, M[\beta] \text{fix}\, N[\gamma] \right) \frac{\mathbf{x}^{\alpha}}{\text{aut}(\alpha)}. \tag{60}$$

Then, all we have to prove is that

$$\text{fix}(M \boxdot N)[\alpha] = |\text{Fix}(M \boxdot N)[\sigma]| = \sum_{\beta \boxtimes \gamma = \alpha} \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} \text{fix}\, M[\beta] \text{fix}\, N[\gamma], \tag{61}$$

where σ is any permutation on a finite set U, with cycle type α. Since

$$\text{Fix}(M \boxdot N)[\sigma] = \left\{ (\pi, \tau, m, n) \mid (\pi, \tau) \in \text{Fix}\mathcal{R}[\sigma], m \in \text{Fix}M[\sigma^{\pi}], n \in \text{Fix}N[\sigma^{\tau}] \right\}, \tag{62}$$

we have

$$\text{fix}(M \boxdot N)[\alpha] = \sum_{(\pi,\tau) \in \text{Fix}\mathcal{R}[\sigma]} |\text{Fix}M[\sigma^{\pi}]||\text{Fix}N[\sigma^{\tau}]| \tag{63}$$

$$= \sum_{\substack{(\pi,\tau,\sigma_1,\sigma_2) \in (\mathcal{S}_+ \boxdot \mathcal{S}_+)[U] \\ \sigma_1 \boxtimes \sigma_2 = \sigma}} |\text{Fix}M[\sigma_1]||\text{Fix}N[\sigma_2]|. \tag{64}$$

The last identity is obtained from bijection (24) in Example 7. Classifying the permutations σ_1 and σ_2 according to their cycle type, we get

$$\text{fix}(M \boxdot N)[\alpha] = \sum_{\beta,\gamma} \sum_{(\pi,\tau,\sigma_1,\sigma_2) \in S^{\sigma}_{\beta,\gamma}} \text{fix}\, M[\beta] \text{fix}\, N[\gamma]. \tag{65}$$

By Lemma 14 we obtain the result. □

2.1.1. The cyclotomic identity

There are various bijective proofs of the cyclotomic identity

$$\frac{1}{1 - \theta x} = \prod_{n \geq 1} \left(\frac{1}{1 - x^n} \right)^{\lambda_n(\theta)},$$

where $\lambda_n(\theta) := \frac{1}{n} \sum_{d|n} \mu(d)\theta^{n/d}$, μ being the classical Möbius function. See for example: Metropolis–Rota [18], Taylor [22], Bergeron [3], Labelle–Leroux [17] and Domocos–Schmitt [9]. We propose here a very simple one, as an application of the combinatorics of the arithmetic product.

Let $\mathcal{C}^{(\theta)}$ be the species of θ-colored cycles, or necklaces, following the terminology of Metropolis and Rota (see [18]). The elements of $\mathcal{C}^{(\theta)}[U]$ are pairs of the form (σ, f), where $\sigma \in \mathcal{C}[U]$ and $f : U \to \mathrm{A}$ is an arbitrary function assigning colors (letters) in a totally ordered set A (alphabet) with $|\mathrm{A}| = \theta$, to the labelled beads of the cycle σ. Denote by $\mathcal{S}^{(\theta)}$ the species of assemblies of necklaces. It is clear that

$$\mathcal{C}^{(\theta)}(x) = \ln \left(\frac{1}{1 - \theta x} \right), \tag{66}$$

$$\mathcal{S}^{(\theta)}(x) = \frac{1}{1 - \theta x}. \tag{67}$$

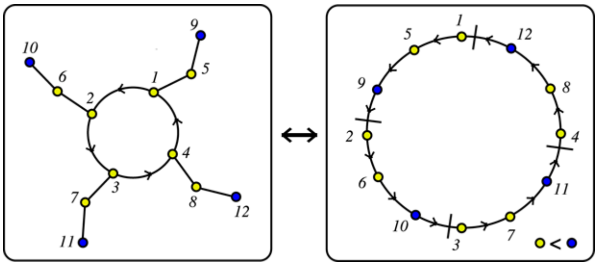

Fig. 5. The isomorphism $\mathcal{C}^{(\theta)} = \mathcal{C} \boxdot \overline{\mathcal{C}^{(\theta)}}$.

Let (σ, f) be a necklace in $\mathcal{C}^{(\theta)}[U]$, where $|U| = n$. The integer

$$d = \min\{1 \le k \le n \mid f \circ \sigma^k = f\}$$

is called the *period* of (σ, f). When $d = n$, the necklace is called *aperiodic*. The *flat part* $\overline{\mathcal{C}^{(\theta)}}$ of $\mathcal{C}^{(\theta)}$ (see [16]) is the species of aperiodic necklaces.

Let (σ, f) be an aperiodic necklace. To each of the n possible presentations of the cycle σ as an ordered tuple $\sigma = (a_1, a_2, \ldots, a_n)$ corresponds a different word $f(a_1)f(a_2)\cdots f(a_n)$ in the alphabet A. The lowest of them in the lexicographic order is called a *Lyndon* word. The ordering $\sigma = (a_1, a_2, \ldots, a_n)$ such that the corresponding word w is Lyndon will be called the *standard presentation* of σ. Thus, the necklace (σ, f) can be identified with the pair (w, l), where w is a Lyndon word, and l a linear order on U. It is the standard presentation of the cycle σ. The number of Lyndon words on A is well known to be $\lambda_n(\theta)$. Then, the decomposition of $\overline{\mathcal{C}^{(\theta)}}$ as a sum of molecular species is

$$\overline{\mathcal{C}^{(\theta)}} = \sum_{n\ge 1} \lambda_n(\theta) X^n. \tag{68}$$

Proposition 16. *We have the identities:*

$$\mathcal{C}^{(\theta)} = \mathcal{C} \boxdot \overline{\mathcal{C}^{(\theta)}}, \tag{69}$$

$$\mathcal{S}^{(\theta)} = E(\mathcal{C} \boxdot \overline{\mathcal{C}^{(\theta)}}). \tag{70}$$

Proof. Eq. (70) is immediate from (69).

To prove (69) observe that the structures of $\mathcal{C} \boxdot \overline{\mathcal{C}^{(\theta)}}$ are regular octopuses where each tentacle (linear order) is decorated with the same Lyndon word. Join the decorated tentacles following the external cycle of the octopus to obtain a necklace whose period is the common length of the tentacles. Conversely, given a necklace of period d, there is a unique way of cutting it into pieces of length d such that the word on each piece is Lyndon (see Fig. 5). It is easy to see how to get an element of $\mathcal{C} \boxdot \overline{\mathcal{C}^{(\theta)}}$ out of this sliced necklace. See for example [21, pp. 4–5], where a similar bijection is used to count ordinary octopuses. □

Eq. (70) can be interpreted as the cyclotomic identity lifted at a combinatorial level. By Eqs. (50) and (51), we get:

$$\mathcal{C}^{(\theta)}(x) = \ln \prod_{n\ge 1} \left(\frac{1}{1-x^n}\right)^{\lambda_n(\theta)}, \tag{71}$$

$$\mathcal{S}^{(\theta)}(x) = \frac{1}{1-\alpha x} = \prod_{n\ge 1} \left(\frac{1}{1-x^n}\right)^{\lambda_n(\theta)}. \tag{72}$$

3. Assemblies of cloned structures

In this section we will see that an $(M \boxdot N)$-structure can be interpreted as an "M-assembly of cloned N-structures". The intuition behind this is the following: an element of $(M \boxdot N)[U]$ consist of a rectangle (π, τ) on U enriched with

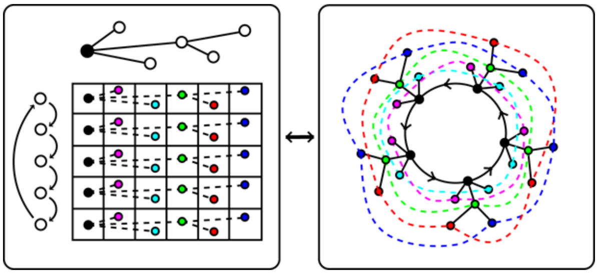

Fig. 6. A $\mathcal{C}$-assembly of cloned rooted trees.

an M-structure m on one side (π) and an N-structure n on the other side (τ). Because τ has the same number of elements as any block of π, we could lay an isomorphic copy (clone) of n on each block B of π. Those copies of n together with the "external structure" $m \in M[\pi]$ form an M-assembly of cloned N-structures.

To make this definition precise we need some formalism. Two elements of U belonging to the same block of τ will be called *homologous*. For example let $\mathcal{A}$ be the species of rooted trees. In Fig. 6 we represent in two ways a structure of $\mathcal{C} \boxdot \mathcal{A}$ as a $\mathcal{C}$-assembly of cloned rooted trees. In both of them homologous elements are represented with the same color (pattern). Roots of cloned trees in the right-hand side are connected like the original cycle on π in the left-hand side, the rest of homologous elements are connected with closed segmented curves.

We now express conveniently the relation among homologous elements. Let $B \in \pi$ and $b \in B$. It is clear that there is only one block $C \in \tau$ such that $\{b\} = B \cap C$. For $\left(B, B'\right) \in \pi \times \pi$, we define the bijection

$$\begin{aligned} \Phi^{\tau}_{B,B'} :\ & B \longrightarrow B' \\ & b \longmapsto b', \end{aligned}$$

where b' is the unique element of $B' \cap C$. In other words, $\Phi^{\tau}_{B,B'}$ sends each element of B to its homologous in B'. It is easy to verify that:

(i) $\Phi^{\tau}_{B,B} = \mathrm{Id}_B$ and
(ii) $\Phi^{\tau}_{B',B''} \circ \Phi^{\tau}_{B,B'} = \Phi^{\tau}_{B,B''}$, for all $B, B', B'' \in \pi$.

Definition 17. Let U be a finite set and M, N two species of structures such that $M[\varnothing] = N[\varnothing] = \varnothing$. An M-assembly of cloned N-structures is a triple $(\{n_B\}_{B\in\pi}, \tau, m)$, where:

(i) $(\pi, \tau) \in \mathcal{R}[U]$,
(ii) $\{n_B\}_{B\in\pi}$ is an assembly of N-structures ($n_B \in N[B]$, for each $B \in \pi$), along with the condition

$$N[\Phi_{B,B'}]n_B = n_{B'}, \tag{73}$$

for every pair $(B, B') \in \pi \times \pi$,
(iii) $m \in M[\pi]$.

Proposition 18. *Let be M and N be two species of structures. Then the species $M \boxdot N$ and the species of M-assemblies of cloned N-structures are isomorphic.*

Proof. Let U be a finite set and assume that $(\pi, \tau, m, n) \in (M \boxdot N)[U]$. For each $B \in \pi$, let $\Psi_{\tau,B} : \tau \longrightarrow B$ be the bijection that sends each block $C \in \tau$ to the unique element b in $C \cap B$. For $(B, B') \in \pi \times \pi$, we have

$$\Phi^{\tau}_{B,B'} = \Psi_{\tau,B'} \circ \Psi^{-1}_{\tau,B}. \tag{74}$$

Let Υ_U be the function

$$\Upsilon_U : (M \boxdot N)[U] \longrightarrow M\text{-assemblies of cloned } N\text{-structures on } U, \tag{75}$$

that sends (π, τ, m, n) to $(\{n_B\}_{B\in\pi}, \tau, m)$, where $n_B = N[\Psi_{\tau,B}](n)$, for each $B \in \pi$. From Eq. (74) condition (73) is satisfied.

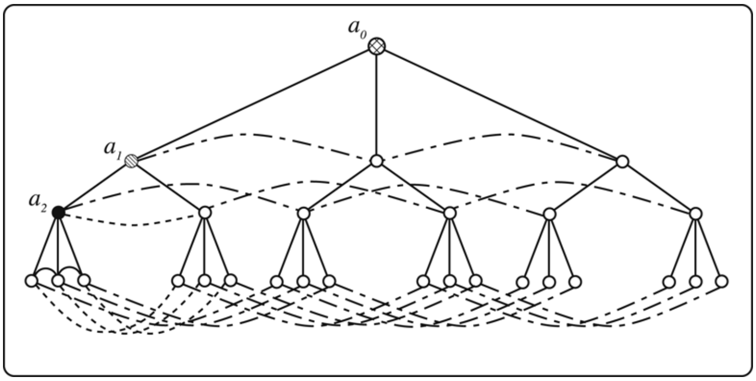

Fig. 7. Element of $\mathcal{H}_E$. Homologous elements according to $\tau_{a_0}, \tau_{a_1}, \tau_{a_2}$ are linked with different kinds of lines.

Now let $(\{n_B\}_{B\in\pi}, \tau, m)$ be an M-assembly of cloned N-structures on U. From condition (73) and Eq. (74), the N-structure $n \in N[\tau]$, where $n := N[\Psi^{-1}_{\tau,B}](n_B)$, remains the same independently of the block $B \in \pi$ that we choose. It is easy to check that Υ_U has as inverse the function that sends $(\{n_B\}_{B\in\pi}, \tau, m)$ to (π, τ, m, n).

The family of bijections $\{\Upsilon_U\}_{U\in\mathbb{B}}$ is the desired isomorphism. □

3.1. Hyper-cloned rooted trees

The species of R-enriched rooted trees could be defined by the implicit combinatorial equation

$$\mathcal{A}_R = X \cdot R(\mathcal{A}_R) \tag{76}$$

(see [4, p. 165]). When $|R[\varnothing]| = 1$ Eq. (76) becomes

$$\mathcal{A}_R = X + X \cdot R_+(\mathcal{A}_R). \tag{77}$$

By changing the operation of substitution of species by the arithmetic product in Eq. (77) we obtain the combinatorial implicit equation for a new kind of structures, the *R-enriched hyper-cloned rooted trees* (R-enriched HRT's)

$$\mathcal{H}_R = X + X \cdot (R_+ \boxdot \mathcal{H}_R). \tag{78}$$

This equation leads to the following recursive definition: an $\mathcal{H}_R$-structure on a set U is either a singleton vertex (when $|U| = 1$), or is obtained by choosing a vertex in $a_0 \in U$(the root) and attaching to it an R_+-assembly of cloned $\mathcal{H}_R$-structures on $U \setminus \{a_0\}$. To give an explicit description of this kind of structures we need some previous notation.

Let t_U be an R-enriched rooted tree on U. We denote by U^+ the set of elements of U that are not leaves of t_U. For $a \in U^+$, we denote by D_a the set of descendants of a in t_U, by π_a the partition of D_a induced by the forest of R-enriched trees that are attached to a, and by $\{t^a_B\}_{B\in\pi_a}$ such a forest. A vertex $a \in U^+$ is said to be of level one if all the elements of D_a are leaves of t_U. A branch of t_U is a path $a_0, a_1, \ldots, a_k$ on t_U, from the root a_0, to a vertex a_k of level one (see Fig. 7).

Definition 19. An *R-enriched HRT* on a finite set U is an R-enriched tree t_U together with a family of partitions, $\{\tau_a\}_{a\in U^+}$, $\tau_a \in \mathrm{Par}[U_a]$, satisfying the conditions:

(i) For every $a \in U^+$, (π_a, τ_a) is a rectangle on U,
(ii) For every pair $B, B' \in \pi_a$,
  (a) $\Phi^{\tau_a}_{B,B'} : t^a_B \longrightarrow t^a_{B'}$ is an isomorphism of R-enriched trees,
  (b) If $\Phi^{\tau_a}_{B,B'}(a') = a''$ with $a' \in B \cap U^+$, then $\Phi^{\tau_a}_{B,B'}(\tau_{a'}) = \tau_{a''}$.

Condition (ii)(b) reflects the fact that each of the bijections $\Phi^{\tau_a}_{B,B'}$, $a \in U^+$, have to preserve the partitions τ'_a, $a' \in B \cap U^+$, as parts of the structure of the R-enriched HRT's attached to a.

By condition (ii)(b), the family $\{\tau_a\}_{a\in U^+}$ is completely determined by the partitions $\{\tau_{a_0}, \tau_{a_1}, \ldots, \tau_{a_k}\}$ on any branch $a_0, a_1, \ldots, a_k \subseteq U^+$ of t_U.

Table 1
The first ten coefficients $|\mathcal{H}_E[n]|$

n	1	2	3	4	5	6	7	8	9	10
$\|\mathcal{H}_E[n]\|$	1	1	2	3	10	11	192	193	3554	10 080

From Eq. (78) we get the recursion:

$$|\mathcal{H}_R[1]| = 1, \tag{79}$$

$$|\mathcal{H}_R[n+1]| = \sum_{d|n} \left\{ {n \atop d} \right\} |R[d]||\mathcal{H}_R[n/d]|, \quad n \geq 1. \tag{80}$$

In particular, $\mathcal{H}_L$ is the species of rooted achiral trees (see [13]). The first ten coefficients of the sequence $|\mathcal{H}_E[n]|$ are shown in Table 1.

4. Multiplicative species

We have a notion of multiplicative species, the "categorified" analogous of a multiplicative arithmetic function in analytic number theory (see [1]).

Definition 20. Let M be a species of structures satisfying the condition $M[\varnothing] = \varnothing$. We say that M is *multiplicative* if,

$$M_{rs} = M_r \boxdot M_s, \tag{81}$$

whenever $(r, s) = 1$.

For example, since $L_n = \frac{X^n}{1}$, and $\mathcal{C}_n = \frac{X^n}{C_n}$, using the isomorphisms of groups $\{1\} \simeq \{1\} \times \{1\}$, $C_{rs} \simeq C_r \times C_s$ when $(r, s) = 1$, we obtain that the species L_+ and $\mathcal{C}$ are multiplicative. The following three propositions can be proved straightforwardly.

Proposition 21. *Let M and N be two multiplicative species of structures. Then the species $M^\bullet$ and $M \boxdot N$ are multiplicative.* □

Combining the above examples and the above proposition, we find that the species of regular octopuses $\mathcal{C} \boxdot L_+$ is multiplicative.

Proposition 22. *Let M be a species of structures. Then M is multiplicative, if and only if, $M_1 = X$, and, for $n \geq 2$,*

$$M_n = M_{p_1^{\alpha_1}} \boxdot M_{p_2^{\alpha_2}} \boxdot \cdots \boxdot M_{p_k^{\alpha_k}}, \tag{82}$$

where $n = \prod_{i=1}^k p_i^{\alpha_i}$ is the canonical prime factorization of the integer n. □

Proposition 23 (*Euler Product Formula*). *Let M be a multiplicative species of structures. Then*

$$M = \boxdot_{p \in \mathbf{P}} \left(X + M_p + M_{p^2} + \cdots \right), \tag{83}$$

where $\mathbf{P}$ denotes the set of prime numbers. □

The following corollary follows immediately by taking generating series in the previous proposition.

Corollary 24. *Let M be a multiplicative species of structures. Then:*

$$M(x) = \boxdot_{p \in \mathbf{P}} \left(x + M_p(x) + M_{p^2}(x) + \ldots \right), \tag{84}$$

$$\mathcal{D}_M(s) = \prod_{p \in \mathbf{P}} \left(1 + \sum_{k \geq 1} \frac{|M[p^k]|}{p^k! p^{-ks}} \right), \tag{85}$$

$$Z_M(x_1, x_2, \ldots) = \boxdot_{p \in \mathbf{P}} \left(x_1 + Z_{M_p}(x_1, x_2, \ldots) + Z_{M_{p^2}}(x_1, x_2, \ldots) + \cdots \right). \quad \square \tag{86}$$

Example 25. For the multiplicative species $\mathcal{C}$ of cyclic permutations we have the identities:

$$\mathcal{C}(x) = \boxdot_{p\in\mathbf{P}}\left(x + \sum_{k\geq 1}\frac{x^{p^k}}{p^k}\right) = \boxdot_{p\in\mathbf{P}}\left(x - \frac{x^p}{p}\right)^{[-1]}, \tag{87}$$

$$\mathcal{D}_{\mathcal{C}}(s) = \zeta(s+1) = \prod_{p\in\mathbf{P}}\left(1 - p^{-(s+1)}\right)^{-1}, \tag{88}$$

$$Z_{\mathcal{C}}(x_1, x_2, \ldots) = \boxdot_{p\in\mathbf{P}}\left((1-p^{-1})\sum_{j\geq 0}\sum_{k\geq j}\frac{x_{p^j}^{p^{k-j}}}{p^{k-j}}\right), \tag{89}$$

where in Eq. (87), the exponent $[-1]$ means inverse with respect to the arithmetic product of formal power series.

5. Oligomorphic groups and species

Let A be an at most countable set, and U a finite set. Denote by A^U and by $(A)_U$ the set of functions and injective functions from U to A respectively. A permutation group G on the set A is called *oligomorphic* if G has only finitely many orbits on A^U for every finite set U.

For $h \in A^U$ denote by $\ker(h)$ the partition of U whose blocks are the non-empty pre-images of elements of A by h. Recall that each function $f \in A^U$ can be identified with a pair $(\pi, \hat{h})$, where $\pi = \ker(h)$ and $\hat{h} \in (A)_\pi$ is the injective function $\hat{h}(B) := h(b)$, for every $B \in \pi$, b being an arbitrary element of B.

Definition 26. Let G, A and U be as above. Define the species of structures F_G^* by $F_G^*[U] = A^U/G$, the (finite) set of orbits of A^U under the action of G. For a bijection $\sigma : U \longrightarrow V$, define the bijection

$$\begin{aligned} F_G^*[\sigma] : F_G^*[U] &\longrightarrow F_G^*[V] \\ \overline{f} &\longmapsto \overline{f\circ\sigma^{-1}}. \end{aligned} \tag{90}$$

This bijection is well defined since σ commutes with the action of G over A^U. In an analogous way we define the species F_G of G-orbits of injective functions.

Recall that for a finite set A and a species M, the species of M-enriched functions is denoted by M^A (see [15]). Observe that when A is finite and G is the identity subgroup of S_A, $F_G[U] = A_U$ is isomorphic to $(1+X)^A$ and $F_G^*[U] = A^U$ is isomorphic to E^A.

Remark 27. Cameron [5] has studied the following three counting problems: how many elements in (a) $F_G[n]$, (b) $\widetilde{F_G}[n]$, (c) $F_G^*[n]$? Equivalently, how many G-orbits in (a) n-tuples of distinct elements, (b) n-sets, (c) all n-tuples?

Proposition 28. *We have the following combinatorial identity*

$$F_G^* = F_G(E_+). \tag{91}$$

Proof. Let $h \in A^U$, since the action of G does not affect the kernel of h, the orbit $\overline{h}$ of h can be identified with the pair $(\pi, \overline{\hat{h}})$, where $\pi = \ker(h)$. Obviously $\overline{\hat{h}} \in F_G[\pi]$. This defines a natural bijection from $F_G^*[U]$ to $F_G(E_+)[U]$. □

5.1. The modified arithmetic product

Let G and H be two oligomorphic groups of permutations on the sets A and B respectively. In [6] Cameron et al. deal with the enumerative problem in the remark above for the product group $G \times H$ acting over $A \times B$. We now introduce the analogous problem in the more general context of species of structures.

Definition 29 (*Modified Arithmetic Product of Species*). Let M and N be two species of structures. Denote by $P_{\mathcal{R}}$ the species of partial rectangles. We define the modified arithmetic product of M and N by

$$(M \circledast N)[U] := \sum_{(\pi,\tau)\in P_{\mathcal{R}}[U]} M[\pi] \times N[\tau], \tag{92}$$

where the sum represents the disjoint union and U is a finite set. For a bijection $\sigma : U \longrightarrow V$, the transport $(M \circledast N)[\sigma]$ is as in Definition 2.

Some of the properties of the arithmetic product in Proposition 8 have their analogous in this context.

Proposition 30. *Let M, N and R be species of structures. The product $\circledast$ has the following properties:*

$$M \circledast N = N \circledast M, \tag{93}$$

$$M \circledast (N \circledast R) = (M \circledast N) \circledast R, \tag{94}$$

$$M \circledast (N + R) = M \circledast N + M \circledast R, \tag{95}$$

$$X \circledast M = M_+, \tag{96}$$

$$M \circledast (1 + X) = (X + 1) \circledast M = M, \tag{97}$$

$$M \circledast (1 + X)^{[n]} = M((1 + X)^{[n]}_+). \tag{98}$$

Proof. We will only prove the identity (98). An element of $(M \circledast (1 + X)^{[n]})[U]$ is of the form $(\pi, \tau, m, \hat{f})$, where (π, τ) is a partial rectangle on U, $m \in M[\pi]$, and $\hat{f} : \tau \to [n]$ is an injective function. Recall that the pair $(\tau, \hat{f})$ can be identify with a function $f : U \to [n]$ whose kernel is τ. Since $\pi \wedge \tau = \hat{0}$, the restriction f_B of f to each block B of π is injective. Conversely, if all the functions in a family $\{f_B\}_{B\in\pi}$ are injective, then $\pi \wedge \tau = \hat{0}$, τ being the kernel of $f := \cup_{B\in\pi} f_B$. Then, the correspondence

$$\begin{aligned} \Omega_U : (M \circledast (1 + X)^{[n]})[U] &\longrightarrow M((1 + X)^{[n]}_+)[U] \\ (\pi, \tau, m, \hat{f}) &\longmapsto (\{f_B\}_{B\in\pi}, m), \end{aligned}$$

is a natural bijection. □

Like in Eq. (33) the product of a family $\{M_i\}_{i=1}^k$ of species of structures is given by

$$(\circledast_{i=1}^k M_i)[U] = \sum_{(\pi_1,\pi_2,\ldots,\pi_k)\in P^{(k)}_{\mathcal{R}}[U]} \prod_{i=1}^k M_i[\pi_i], \tag{99}$$

where $P^{(k)}_{\mathcal{R}}$ is the species of k-partial rectangles.

We have the following theorem.

Theorem 31. *Let G and H be two oligomorphic groups acting on sets A and B respectively. Then*

$$F_{G\times H} = F_G \circledast F_H. \tag{100}$$

Proof. Let $h : U \longrightarrow A \times B$ be an injective function. Let $h_1 : U \longrightarrow A$ and $h_2 : U \longrightarrow B$ be its components, i.e. $h(u) = (h_1(u), h_2(u))$ for $u \in U$. Let $\pi = \ker(h_1)$ and $\tau = \ker(h_2)$. It is clear that $\ker(h) = \ker(h_1) \wedge \ker(h_2) = \pi \wedge \tau$, and since h is injective, $\pi \wedge \tau = \hat{0}$. Then h can be identify with the tuple $(\pi, \tau, \hat{h}_1, \hat{h}_2)$, and its orbit under the action of $G \times H$ with $(\pi, \tau, \bar{\hat{h}}_1, \bar{\hat{h}}_2)$, where $\bar{\hat{h}}_1 \in F_G[\pi]$ and $\bar{\hat{h}}_2 \in F_H[\tau]$. This defines a natural bijection between $F_{G\times H}[U]$ and $(F_G \circledast F_H)[U]$. □

Take $A = [m]$, $B = [n]$, and G, H being the identity subgroups of S_m and S_n respectively. We obtain the isomorphism

$$(1 + X)^{[m]} \circledast (1 + X)^{[n]} = (1 + X)^{[n]\times[m]}. \tag{101}$$

The exponential generating series of the modified arithmetic product of species of structures is not as straightforward to compute as in the arithmetic product case. However, the identity

$$F^{*}_{G\times H}(x) = F^{*}_{G}(x) \times F^{*}_{H}(x), \tag{102}$$

proved in [6], provides a device to compute this series,

$$(F_G \boxtimes F_H)(x) = F_{G\times H}(x). \tag{103}$$

It has motivated the following general combinatorial identity.

Theorem 32. *Let $\{M_i\}_{i=1}^{k}$ be a family of species of structures. Then we have*

$$(\boxtimes_{i=1}^{k} M_i)(E_+) = \times_{i=1}^{k} M_i(E_+), \tag{104}$$

where $\times$ is the operation of cartesian product of species,

$$(M \times N)[U] = M[U] \times N[U]. \tag{105}$$

Proof. It is enough to prove the identity for $k = 2$. Consider the species Par of set partitions. For a finite set U, let $\leq_U$ be the refinement order on Par$[U]$. For $\eta \in$ Par$[U]$, let C_η be the set of elements of (Par$[U]$, $\leq_U$) greater than or equal to η. For $\pi \in \mathrm{C}_\eta$ and $B \in \pi$, let $\hat{B} = \{C \in \eta \mid C \subseteq B\}$ and $\hat{\pi} = \{\hat{B} \mid B \in \pi\}$. Clearly, $\hat{\pi}$ is a partition of η, and it is easy to see that the correspondence

$$\begin{aligned} \mathrm{C}_\eta &\longrightarrow (\mathrm{Par}[\eta], \leq_\eta) \\ \pi &\longmapsto \hat{\pi} \end{aligned}$$

is an order isomorphism. Then the partition $\pi \wedge \tau = \eta$ is an element of C_η if and only if $(\hat{\pi}, \hat{\tau})$ is a partial rectangle on η. The right-hand side of (104), for $k = 2$, evaluated in a set U is equal to

$$(M_1(E_+) \times M_2(E_+))[U] = \left(\sum_{\pi\in \mathrm{Par}[U]} M_1[\pi]\right) \times \left(\sum_{\tau\in \mathrm{Par}[U]} M_2[\tau]\right) \tag{106}$$

$$= \sum_{(\pi,\tau)\in \mathrm{Par}[U]\times \mathrm{Par}[U]} M_1[\pi] \times M_2[\tau] \tag{107}$$

$$= \sum_{\eta\in \mathrm{Par}[U]} \sum_{\pi\wedge\tau=\eta} M_1[\pi] \times M_2[\tau]. \tag{108}$$

For any partition ϱ in C_η, let $f_{\varrho,\eta} : \varrho \longrightarrow \hat{\varrho}$ be the bijection sending each block B of ϱ to $\hat{B}$. The family of bijections

$$\alpha_U : \sum_{\eta\in \mathrm{Par}[U]} \sum_{\pi\wedge\tau=\eta} M_1[\pi] \times M_2[\tau] \longrightarrow \sum_{\eta\in \mathrm{Par}[U]} \sum_{(\hat{\pi},\hat{\tau})\in P_{\mathcal{R}}[\eta]} M_1[\hat{\pi}] \times M_2[\hat{\tau}] \tag{109}$$

$$\alpha_U := \sum_{\eta\in \mathrm{Par}[U]} \sum_{\pi\wedge\tau=\eta} M_1[f_{\pi,\eta}] \times M_2[f_{\tau,\eta}] \tag{110}$$

defines a natural transformation

$$\alpha : M_1(E_+) \times M_2(E_+) \longrightarrow (M_1 \boxtimes M_2)(E_+). \quad \square \tag{111}$$

Taking exponential generating series and cycle index series in identity (104), we obtain the following corollary.

Corollary 33. *Let $\{M_i\}_{i=1}^{k}$ be as above. Then the following generating function identities hold:*

$$(\boxtimes_{i=1}^{k} M_i)(e^x - 1) = \times_{i=1}^{k} M_i(e^x - 1), \tag{112}$$

$$Z_{\boxtimes_{i=1}^{k} M_i}(\mathbf{x}) * Z_{E_+}(\mathbf{x}) = \times_{i=1}^{k}(Z_{M_i}(\mathbf{x}) * Z_{E_+}(\mathbf{x})), \tag{113}$$

where $\times$ *means coefficient-wise product or Hadamard product as in* [4], $*$ *means plethystic substitution, and*

$$Z_{E_+}(\mathbf{x}) = \exp\left(\sum_{n\geq 1}\frac{x_n}{n}\right) - 1. \quad \square \tag{114}$$

Using Eq. (112) with $M_i = E$, for $i = 1, \ldots, k$, we recover the first identity of Theorem 1 in [7],

$$P_{\mathcal{R}}^{(k)}(e^x - 1) = (e^{e^x-1})^{\times k} = \sum_{n\geq 0}(B_n)^k\frac{x^n}{n!}, \tag{115}$$

where B_n is the nth Bell number, the number of partitions of the set $[n]$.

In order to have the identity $(M\circledast N)(x) = M(x)\circledast N(x)$ for two species of structures M and N, following (112) we make the following definition.

Definition 34. For two formal power series $F(x)$ and $G(x)$ define the product $\circledast$ by

$$F(x)\circledast G(x) = \big(F(e^x - 1)\times G(e^x - 1)\big)\circ(\ln(1+x)). \tag{116}$$

It is easy to see that this product is commutative and distributive with respect to the sum,

$$(F(x) + H(x))\circledast G(x) = F(x)\circledast G(x) + H(x)\circledast G(x). \tag{117}$$

5.2. *The shift trick*

Sometimes the Eq. (116) is too clumsy to make computations. We will provide a more efficient method. Previous to that we need the following lemma.

Lemma 35. *Let $F(x)$ be a formal power series. For m and n non-negative integers we have the identities:*

$$F(x)\circledast(1+x)^n = F((1+x)^n - 1), \tag{118}$$

$$(1+x)^m\circledast(1+x)^n = (1+x)^{mn}. \tag{119}$$

Proof. Eq. (118) follows from identity (98). Eq. (119) follows from (118) or by taking generating functions in (101). $\square$

From this lemma we recover the following result of Pittel [20].

Proposition 36. *Let k be a fixed positive integer. The exponential generating series $P_{\mathcal{R},k}(x)$, of the number of partial rectangles (π, τ), with $|\pi| = k$, is*

$$P_{\mathcal{R},k}(x) = \frac{1}{k!e}\sum_{l\geq 0}\frac{1}{l!}\left((x+1)^l - 1\right)^k. \tag{120}$$

Proof. The required species is $P_{\mathcal{R},k} := E_k\circledast E$. Its exponential generating series is

$$(E_k\circledast E)(x) = \frac{x^k}{k!}\circledast e^x = \frac{1}{k!e}x^k\circledast e^{(x+1)} = \frac{1}{k!e}\sum_{l\geq 0}\frac{1}{l!}x^k\circledast(x+1)^l. \tag{121}$$

Use Eq. (118) to finish the proof. $\square$

The algorithm to compute the product $F(x)\circledast G(x)$, of two generating series $F(x)$ and $G(x)$, runs as follows:

(1) Express $F(x) = F_1(x+1)$ and $G(x) = G_1(x+1)$ as power series of $(x+1)$,
(2) use the distributive property and Eq. (119) to compute

$$H(x+1) = F_1(x+1)\circledast G_1(x+1),$$

(3) express back $H(x+1)$ as a power series of x.

Now we obtain some new formulas about enumerating (0, 1)-matrices.

Theorem 37. *The number $M(m,n,r)$ of $m \times n$ (0, 1)-matrices with exactly r entries equal to* 1 *and no zero row or columns, is given by*

$$M(m,n,r) = \sum_{l \geq r} \sum_{d|l} (-1)^{n+m-(d+l/d)} \binom{m}{d} \binom{n}{l/d} \binom{l}{r}. \tag{122}$$

Proof. The structures of the species $X^n \circledast X^m$ are the linearly ordered $m \times n$ partial rectangles. A structure of $(X^n \circledast X^m)[r]$ can be thought of as a $m \times n$ matrix with entries $1, 2, \ldots, r$, without repetitions, zero elsewhere, and no zero row or columns. Then, $M(m,n,r)$ is the coefficient of x^r in the generating series

$$(X^m \circledast X^n)(x) = x^m \circledast x^n. \tag{123}$$

By shifting we get

$$x^m \circledast x^n = (x+1-1)^m \circledast (x+1-1)^n \tag{124}$$

$$= \sum_{j,k} \binom{m}{j}\binom{n}{k} (-1)^{m+n-(j+k)} (x+1)^j \circledast (x+1)^k \tag{125}$$

$$= \sum_{j,k} \binom{m}{j}\binom{n}{k} (-1)^{m+n-(j+k)} (x+1)^{jk} \tag{126}$$

$$= \sum_{j,k,r} \binom{m}{j}\binom{n}{k}\binom{jk}{r} (-1)^{m+n-(j+k)} x^r. \tag{127}$$

Making the change $l = jk$, we obtain the result. □

Corollary 38. *The number $|P_{\mathcal{R},m,n}[r]|$ of $m \times n$ partial rectangles on r elements, is given by*

$$|P_{\mathcal{R},m,n}[r]| = \sum_{l \geq r} \sum_{d|l} \left\{ {l \atop d} \right\} \frac{(-1)^{m+n-(d+l/d)}}{(m-d)!(n-l/d)!(l-r)!}. \tag{128}$$

Proof. $|P_{\mathcal{R},m,n}[r]|$ is equal to $|(E_m \circledast E_n)[r]|$, which is the coefficient of $\frac{x^r}{r!}$ in the generating series

$$(E_m \circledast E_n)(x) = \frac{x^m}{m!} \circledast \frac{x^n}{n!}. \tag{129}$$

Then,

$$|P_{\mathcal{R},m,n}[r]| = r! \frac{M(m,n,r)}{m!n!} = \sum_{l \geq r} \sum_{d|l} \left\{ {l \atop d} \right\} \frac{(-1)^{m+n-(d+l/d)}}{(m-d)!(n-l/d)!(l-r)!}. \quad \square$$

We now give a very short direct proof of the beautiful formula obtained by Pittel [20].

Theorem 39. *The number $|P_{\mathcal{R}}^{(k)}[n]|$, of k-tuples of partitions $(\pi_1, \pi_2, \ldots, \pi_k)$ on $[n]$ satisfying $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_k = \hat{0}$, is given by*

$$|P_{\mathcal{R}}^{(k)}[n]| = e^{-k} \sum_{i_1, i_2, \ldots, i_k \geq 1} \frac{(i_1 \cdots i_k)_n}{i_1! \cdots i_k!}, \tag{130}$$

where $(m)_n = m(m-1)\cdots(m-n+1)$. □

Proof. By the definition of $\circledast$-product,

$$P_{\mathcal{R}}^{(k)} = E^{\circledast k} = \underbrace{E \circledast \cdots \circledast E}_{k \text{ factors}}.$$

The exponential generating series of this species is $(e^x)^{\circledast k}$. Following the algorithm, we have $e^x = e^{-1}e^{(x+1)}$. Then

$$E^{\circledast k}(x) = \left(e^{-1}e^{(x+1)}\right)^{\circledast k} \tag{131}$$

$$= e^{-k}\sum_{i_1,i_2,\ldots,i_k\geq 0}\frac{(x+1)^{i_1}}{i_1!}\circledast\cdots\circledast\frac{(x+1)^{i_k}}{i_k!} \tag{132}$$

$$= e^{-k}\sum_{i_1,i_2,\ldots,i_k\geq 0}\frac{(x+1)^{i_1 i_2\cdots i_k}}{i_1!i_2!\cdots i_k!} \tag{133}$$

$$= \sum_{n\geq 0}\left(e^{-k}\sum_{i_1,i_2,\ldots,i_k\geq 0}\frac{(i_1\cdots i_k)_n}{i_1!i_2!\cdots i_k!}\right)\frac{x^n}{n!}. \quad \square \tag{134}$$

As a corollary, we obtain a remarkable identity.

Corollary 40. *For $n \geq 1$,*

$$|P_{\mathcal{R}}^{(k)}[n]| = e^{-k}\sum_{l\geq n}\frac{|\mathcal{R}^{(k)}[l]|}{(l-n)!}. \tag{135}$$

Proof. Making the change $l = i_1 i_2\cdots i_k$ in (130) we obtain

$$|P_{\mathcal{R}}^{(k)}[n]| = e^{-k}\sum_{l\geq n}\sum_{i_1 i_2\cdots i_k = l}\frac{(l)_n}{i_1!\cdots i_k!} \tag{136}$$

$$= e^{-k}\sum_{l\geq n}\frac{1}{(l-n)!}\sum_{i_1 i_2\cdots i_k = l}\frac{l!}{i_1!\cdots i_k!}. \tag{137}$$

To finish the proof we recall Eq. (18). $\square$

Theorem 39 is a particular case of the following general result, that can be proved without much extra effort.

Theorem 41. *Let $\{M_i\}_{i=1}^k$ be a family of species of structures whose exponential generating series, $M_i(x) = F_i(x+1)$, expressed as power series of $(x+1)$, are given,*

$$M_i(x) = F_i(x+1) = \sum_{n\geq 0} b_n^{(i)}\frac{(x+1)^n}{n!}, \quad i = 1,\ldots,k. \tag{138}$$

Then,

$$(\circledast_{i=1}^k M_i)(x) = \sum_{n\geq 0}\left(\sum_{i_1,i_2,\ldots,i_k\geq 0} b_{i_1}^{(1)}\cdots b_{i_k}^{(k)}\frac{(i_1\cdots i_k)_n}{i_1!i_2!\cdots i_k!}\right)\frac{x^n}{n!} \tag{139}$$

$$= a_0 + \sum_{n\geq 1}\left(\sum_{l\geq n}\frac{1}{(l-n)!}\left[\frac{x^n}{n!}\right](F_1\boxdot\cdots\boxdot F_k)(x)\right)\frac{x^n}{n!}, \tag{140}$$

where $a_0 = |(\circledast_{i=1}^k M_i)[0]| = \prod_{i=1}^k |M_i[0]|$. $\square$

Finally, as an application of Theorem 41, we obtain four new formulas for the enumeration of (0, 1)-matrices and partial rectangles.

Example 42. Given the expansions:

$$L(x) = \frac{1}{2\left(1-\frac{x+1}{2}\right)} = \sum_{n\geq 0}\frac{(x+1)^n}{2^{n+1}}, \tag{141}$$

$$\mathcal{C}(x) = \ln\left(\frac{1}{2\left(1 - \frac{x+1}{2}\right)}\right) = \ln(2^{-1}) + \sum_{n\geq 1}\frac{(x+1)^n}{n2^n}. \tag{142}$$

We obtain formulas for:

- The number of (0, 1)-matrices of any size, with n ones, and with no zero row or column,

$$\frac{|(L \circledast L)[n]|}{n!} = \frac{1}{4n!}\sum_{r,s\geq 0}\frac{(rs)_n}{2^{r+s}}. \tag{143}$$

- The number of matrices of any size up to column permutations, with n different elements, zero elsewhere and with no zero row or column,

$$|(L \circledast E)[n]| = \frac{1}{2e}\sum_{r,s\geq 0}\frac{(rs)_n}{2^r s!}. \tag{144}$$

- The number of linearly ordered k-partial rectangles on $[n]$,

$$|L^{\circledast k}[n]| = \frac{1}{2^k}\sum_{i_1\cdots i_k\geq 0}\frac{(i_1\cdots i_k)_n}{2^{i_1+\cdots+i_k}}. \tag{145}$$

- The number of cyclic k-partial rectangles on $[n]$,

$$|\mathcal{C}^{\circledast k}[n]| = \sum_{i_1\cdots i_k\geq 1}\frac{(i_1\cdots i_k - 1)_{n-1}}{2^{i_1+\cdots+i_k}}. \tag{146}$$

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