Stinespring's construction as an adjunction

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This talk is part of a series on recently published papers in the journal *Compositionality*. The present talk is about the paper:

Stinespring's construction as an adjunction
2019-12-20, Volume 1, Issue 2
Compositionality 1, 2 (2019)
arXiv:1807.02533

The work has its origins from the earlier paper

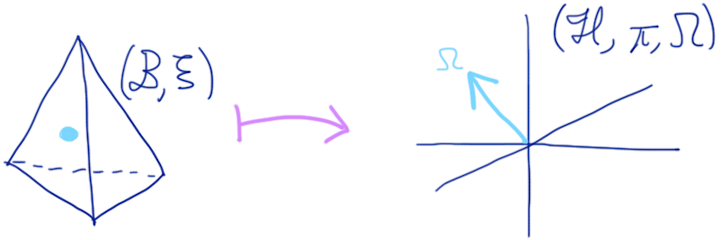
From Observables and States to Hilbert Space and Back: A 2-Categorical Adjunction
Applied Categorical Structures Volume 26, pages 1123–1157 (2018)
arXiv:1609.08975

and there is also overlap with

- arXiv:1708.00091 (stochastic Gelfand–Naimark theorem)
- arXiv:2009.07125 (functorial von Neumann entropy)

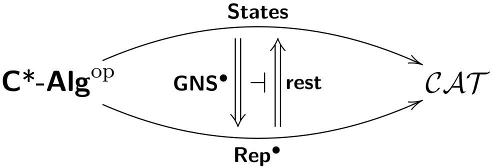
The main idea behind this talk

The general idea behind this talk came from an attempt to understand the sense in which the GNS and Stinespring constructions are functorial. These are constructions that take certain kinds of states and (non-deterministic) processes on (abstract) C^*-algebras and realize them as vectors and operations on Hilbert spaces that arise as representations of the given C^*-algebra.



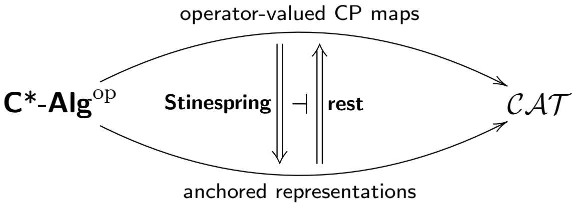
The main idea behind this talk

These realizations are *universal* and *functorial* in a certain sense. In the case of states, we have an adjunction:



The main idea behind this talk

And in the case of non-deterministic processes, we have an adjunction



Since these adjunctions look very similar, and since the first one is a bit easier to understand, I will mainly focus on the former. I will briefly describe what happens in the latter towards the end of this talk.

Pure states in quantum mechanics

Normally, a **(pure) state** in quantum mechanics is often first defined as a unit vector ψ (or more precisely a ray) in a complex Hilbert space $(\mathcal{H}, \langle\ \cdot\ ,\ \cdot\ \rangle)$.[1] It gives rise to a positive unital (linear) functional

$$\mathcal{B}(\mathcal{H}) \xrightarrow{\langle \psi,\ \cdot\ \psi \rangle} \mathbb{C}$$

sending a bounded operator $A \in \mathcal{B}(\mathcal{H})$ to the complex number $\langle \psi, A\psi \rangle$.[2] An **observable** is a self-adjoint element of $\mathcal{B}(\mathcal{H})$. The real number $\langle \psi, A\psi \rangle$ is interpreted as the **expectation value of the observable A in the state ψ**.

[1] If it makes you feel more comfortable thinking about linear algebra than analysis, you can assume $\mathcal{H} = \mathbb{C}^n$ for some natural number n.

[2] And if you're thinking $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{B}(\mathcal{H}) = \mathcal{M}_n(\mathbb{C})$, the algebra of $n \times n$ matrices.

Mixed states in quantum mechanics

From this point of view, it is natural to generalize the definition of a (pure) state to allow for any positive unital functional $\mathcal{B}(\mathcal{H}) \overset{\omega}{\rightsquigarrow} \mathbb{C}$. Such a functional is called a **(mixed) state** (or sometimes even a **family of expectation values**). An example of a (mixed) state is obtained if you have a collection of pure states $\{\psi_x \in \mathcal{H}\}_{x \in X}$, indexed by some finite set X, and a probability measure $\{\bullet\} \overset{p}{\rightsquigarrow} X$ on X. In this case, a state can be defined by

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\overset{\omega}{\rightsquigarrow} \mathbb{C} \\ A &\mapsto \sum_{x \in X} p_x \langle \psi_x, A\psi_x \rangle. \end{aligned}$$

This also explains the terminology "mixed" since this describes a mixture of pure states ψ_x weighted by a probability distribution.

Density matrices in quantum mechanics

In general, there is a nice theorem that says what all (sufficiently continuous) states look like.

Theorem 1 (A corollary of the Riesz representation theorem)

Let $\mathcal{B}(\mathcal{H}) \overset{\omega}{\rightsquigarrow} \mathbb{C}$ *be a positive unital (and strongly continuous) functional. Then there exists a non-negative operator* ρ *with* $\mathrm{tr}(\rho) = 1$ *such that* $\omega = \mathrm{tr}(\rho\,\cdot\,)$*. Conversely, given any such operator* ρ*, the functional* $\mathrm{tr}(\rho\,\cdot\,) : \mathcal{B}(\mathcal{H}) \rightsquigarrow \mathbb{C}$ *is positive unital (and strongly continuous).*

Such a ρ is called a **density matrix**. As in our previous example, if we have a set $\{\psi_x, p_x\}_{x \in X}$ of unit vectors with probabilities, then the density matrix corresponding to the state is $\rho = \sum_{x \in X} p_x |\psi\rangle\langle\psi|$, where $|\psi_x\rangle\langle\psi_x|$ denotes the projection onto the span of ψ_x.

States on C^*-algebras

We can generalize the previous situation even more, which will be helpful in giving more familiar examples. Let $\mathcal{A}$ be a C^*-algebra.[3] A **positive element** in $\mathcal{A}$ is an element of the form A^*A for some $A \in \mathcal{A}$. A **state** on $\mathcal{A}$ is a positive unital functional $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$. This means $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

[3] A C^*-**algebra** is a normed unital algebra with an involution $*$ satisfying certain conditions. That's basically all you need to know to follow this talk.

States on $\mathbb{C}^X$

An example is $\mathcal{A} = \mathbb{C}^X$, the C^*-algebra of complex-valued functions on a finite set X with the involution being complex conjugation. Given any state $\mathbb{C}^X \overset{\omega}{\rightsquigarrow} \mathbb{C}$, there exists a unique probability measure $\{\bullet\} \overset{p}{\rightsquigarrow} X$ such that

$$\omega(A) = \sum_{x \in X} p_x A_x =: \langle A \rangle_p .$$

Here, A is a function on X whose value at x is written as A_x. This gives the usual expectation value of a function (the **random variable** A) on X with respect to the probability measure p.

Finite-dimensional C^*-algebras

In general, all finite-dimensional C^*-algebras are of the form (up to $*$-isomorphism)

$$\mathcal{A} \cong \bigoplus_{x \in X} \mathcal{M}_{m_x}(\mathbb{C}),$$

where X is a finite set and $m_x \in \mathbb{N}$. Given a state ω on $\mathcal{A}$ there exist a unique probability measure $\{\bullet\} \overset{p}{\rightsquigarrow} X$ and a (not necessarily unique) family of density matrixes $\rho_x \in \mathcal{M}_{m_x}(\mathbb{C})$ such that

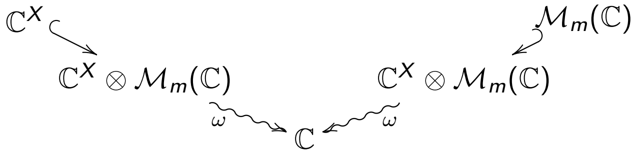
$$\omega\left(\bigoplus_{x \in X} A_x\right) = \sum_{x \in X} p_x \mathrm{tr}(\rho_x A_x).$$

States on $\mathbb{C}^X \otimes \mathcal{M}_m(\mathbb{C})$

As an example, let's think about a state ω on

$$\mathbb{C}^X \otimes \mathcal{M}_m(\mathbb{C}) \cong \bigoplus_{x \in X} \mathcal{M}_m(\mathbb{C}).$$

The **marginals**



define a probability distribution $\{\bullet\} \overset{p}{\rightsquigarrow} X$ and a density matrix ρ, respectively.

States on $\mathbb{C}^X \otimes \mathcal{M}_m(\mathbb{C})$

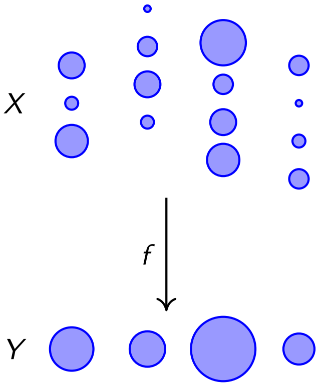
Meanwhile, restricting to one of the matrix factors in the direct sum gives a positive (not necessarily unital!) functional

$$\omega_x := \left(\mathcal{M}_m(\mathbb{C}) \xhookrightarrow{e_x} \mathbb{C}^X \otimes \mathcal{M}_m(\mathbb{C}) \overset{\omega}{\rightsquigarrow} \mathbb{C} \right)$$
$$A \longmapsto e_x \otimes A \longmapsto \omega(e_x \otimes A)$$

with $p_x = \omega_x(\mathbb{1}_m) \in [0,1]$. If $p_x \neq 0$, then $\frac{1}{p_x}\omega_x$ is a state on $\mathcal{M}_m(\mathbb{C})$. Hence, there exist a family of density matrices $\rho_x \in \mathcal{M}_m(\mathbb{C})$ such that $\omega_x = p_x \mathrm{tr}(\rho_x \,\cdot\,)$. Thus, $\{\rho_x\}_{x\in X}$ represents a probabilistic ensemble, where ρ_x is the state of the system, which occurs with probability p_x. The marginal $\mathrm{tr}(\rho\,\cdot\,)$ is the mixed state whose density matrix is $\rho = \sum_{x\in X} p_x \rho_x$.

Probability-preserving functions

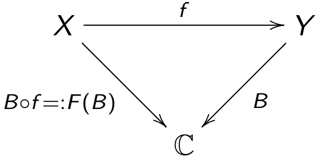
If (X, p) and (Y, q) are finite probability spaces, a probability-preserving function $(X, p) \xrightarrow{f} (Y, q)$, called a **deterministic process**, can be visualized in terms of combining water droplets:[4]



[4] I learned this powerful picture from Gromov.

State-preserving $*$-homomorphisms

There is a one-to-one correspondence between such deterministic processes $(X, p) \xrightarrow{f} (Y, q)$ and $*$-homomorphisms $\mathbb{C}^X \xleftarrow{F} \mathbb{C}^Y$ such that $\omega \circ F = \xi$, where ω and ξ are the states corresponding to p and q, respectively. This assignment sending f to F is defined via pullback:



Thus, we write $(\mathbb{C}^X, \omega) \xleftarrow{F} (\mathbb{C}^Y, \xi)$ and say that F is **state-preserving**.

The EPR state

A quantum example of a deterministic state-preserving process is the following. Let $\mathcal{M}_4(\mathbb{C}) \cong \mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C}) \overset{\omega}{\rightsquigarrow} \mathbb{C}$ be the state with corresponding (rank 1) density matrix

$$\rho = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\sigma := \frac{1}{2}\mathbb{1}_2$ be the density matrix corresponding to a state $\mathcal{M}_2(\mathbb{C}) \overset{\xi}{\rightsquigarrow} \mathbb{C}$. The inclusion $\mathcal{M}_2(\mathbb{C}) \hookrightarrow \mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C})$ to either of these factors is a state-preserving $*$-homomorphism $(\mathcal{M}_2(\mathbb{C}), \xi) \to (\mathcal{M}_4(\mathbb{C}), \omega)$. This example is special because it illustrates that a deterministic process can take a pure state (in this case ω) to a mixed state (ξ). This cannot happen in the classical setting (think about water droplets!).

State spaces and deterministic evolution

Associated to every C^*-algebra $\mathcal{A}$, we get a convex set of states

$$\mathcal{S}(\mathcal{A}) := \left\{ \mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C} \ : \ \omega \text{ positive unital} \right\}.$$

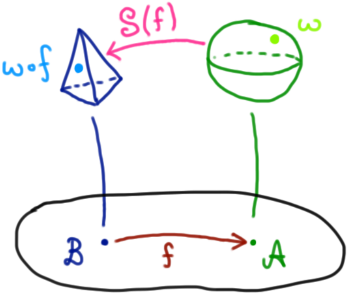
Furthermore, to each $*$-homomorphism $\mathcal{A} \xleftarrow{F} \mathcal{B}$, we get a pullback map

$$\mathcal{S}(\mathcal{A}) \xrightarrow{\mathcal{S}(F)} \mathcal{S}(\mathcal{B}).$$

As a fibration or an indexed category

Two viewpoints:

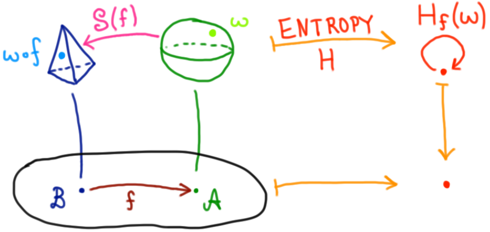
1. View state-spaces and pullbacks as a fibration over **C*-Alg** (C^*-algebras and $*$-homomorphisms).
2. Or view as a functor $\mathbf{C^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\mathcal{S}} \mathbf{Set}$ (sometimes called an indexed category).



I will go back and forth between these viewpoints.

A brief aside on fibrations and entropy

Viewing states as a fibration turns out to be useful in other contexts, too. One can characterize the von Neumann entropy as a certain functor



of fibrations (using finite-dimensional C^*-algebras) in such a way so that it agrees with the Baez–Fritz–Leinster functorial characterization of the Shannon entropy.[5]

[5]BTW, I'll be giving a talk on this in about two weeks on December 16th (details: http://www.sci.brooklyn.cuny.edu/~noson/CTseminar.html).

Pointed representations

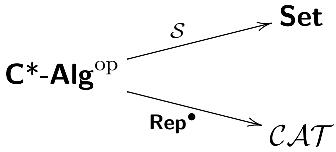
Here's another fibration over **C*-Alg** involving representations. It is described by a functor $\mathbf{C^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\mathbf{Rep}^\bullet} \mathcal{CAT}$ defined as follows. To each $\mathcal{A}$ we get a category $\mathbf{Rep}^\bullet(\mathcal{A})$ whose objects are triples $(\mathcal{H}, \pi, \Omega)$ with $\mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$ a representation and Ω a unit vector in $\mathcal{H}$. A morphism $(\mathcal{H}, \pi, \Omega) \xrightarrow{L} (\mathcal{K}, \rho, \Theta)$ is a vector-preserving isometric intertwiner of representations, i.e.

$$L\Omega = \Theta, \quad L^*L = \mathrm{id}_{\mathcal{H}}, \quad \text{and} \quad \begin{array}{ccc} \mathcal{B}(\mathcal{H}) & \xrightarrow{\pi(A)} & \mathcal{B}(\mathcal{H}) \\ {\scriptstyle L}\downarrow & // & \downarrow{\scriptstyle L} \\ \mathcal{B}(\mathcal{K}) & \xrightarrow[\rho(A)]{} & \mathcal{B}(\mathcal{K}) \end{array} \quad \forall A \in \mathcal{A}.$$

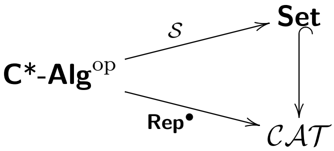
Now, to a $*$-homomorphism $\mathcal{A} \xleftarrow{F} \mathcal{B}$ of C^*-algebras, we get a functor $\mathbf{Rep}^\bullet(\mathcal{A}) \xrightarrow{F^*} \mathbf{Rep}^\bullet(\mathcal{B})$, since if π is a representation of $\mathcal{A}$, we can pull it back to a representation $\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$ of $\mathcal{B}$.

Pointed representations and states as functors

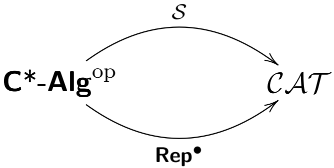
So we have two different indexed categories (or fibrations)



Are they related? Well, every set can be viewed as a discrete category, so we can view these functors as having the same codomain as well.

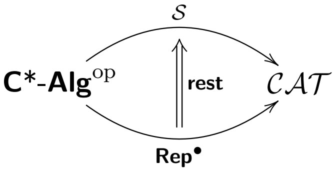


or just



From pointed representations to states

Let's first construct a *natural* transformation

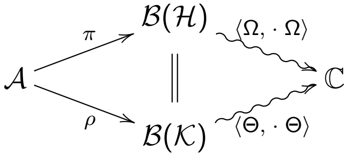


by associating to each C^*-algebra $\mathcal{A}$ a functor $\mathbf{Rep}^{\bullet}(\mathcal{A}) \xrightarrow{\mathbf{rest}_{\mathcal{A}}} \mathcal{S}(\mathcal{A})$. It takes a pointed representation $(\mathcal{H}, \pi, \Omega)$ to a state defined by the composite

$$\mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}) \xrightarrow{\langle \Omega, \,\cdot\, \Omega \rangle} \mathbb{C}.$$

From pointed representations to states

Since the category $\mathcal{S}(\mathcal{A})$ is discrete, the functor $\mathbf{Rep}^{\bullet}(\mathcal{A}) \xrightarrow{\mathbf{rest}_{\mathcal{A}}} \mathcal{S}(\mathcal{A})$ must send an isometric intertwiner $(\mathcal{H}, \pi, \Omega) \xrightarrow{L} (\mathcal{K}, \rho, \Theta)$ to the identity on states, i.e.

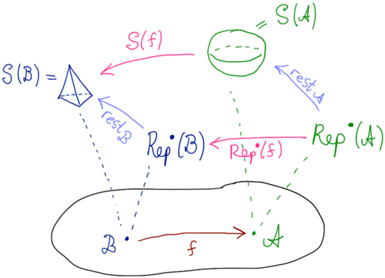


So is it true? Yes!

$$\begin{aligned} \langle \Theta, \rho(A)\Theta \rangle &= \langle L\Omega, \rho(A)L\Omega \rangle = \langle L\Omega, L\pi(A)\Omega \rangle \\ &= \langle \Omega, L^{*}L\pi(A)\Omega \rangle = \langle \Omega, \pi(A)\Omega \rangle \end{aligned}$$

From pointed representations to states

Naturality of **rest** says that to every $*$-homomorphism $\mathcal{A} \overset{F}{\longleftarrow} \mathcal{B}$,



$$
\begin{array}{ccc}
\mathcal{S}(\mathcal{A}) & \xrightarrow{\mathcal{S}(F)} & \mathcal{S}(\mathcal{B}) \\
\uparrow \scriptstyle{\mathbf{rest}_{\mathcal{A}}} & /\!\!/ & \uparrow \scriptstyle{\mathbf{rest}_{\mathcal{B}}} \\
\mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow[\mathbf{Rep}^{\bullet}(F)]{} & \mathbf{Rep}^{\bullet}(\mathcal{B})
\end{array}
$$

Note that this is an equality of functors on the nose! Does the natural transformation **rest** have an inverse or an adjoint in any sense? Yes! But we shouldn't expect things to be on the nose anymore because $\mathbf{Rep}^{\bullet}(\mathcal{A})$ and $\mathbf{Rep}^{\bullet}(\mathcal{B})$ have nontrivial morphisms.

The GNS construction

Although we've thrown away Hilbert space by looking at C^*-algebras and states instead of representations, there is a natural way to construct a Hilbert space from these data. Namely, given $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, first define the nullspace[6]

$$\mathcal{N}_\omega := \big\{ A \in \mathcal{A} \ : \ \omega(A^*A) = 0 \big\}.$$

This turns out to be a left ideal in $\mathcal{A}$. The sesquilinear form $A_1, A_2 \mapsto \omega(A_1^* A_2)$ descends to an inner product on $\mathcal{A}/\mathcal{N}_\omega$. Completing this normed vector space with respect to this inner product gives a Hilbert space

$$\mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}.$$

[6] This is the same nullspace that appears in the non-commutative definition of a.e. equivalence and agrees with the string-diagrammatic definition of Cho and Jacobs in synthetic probability theory (cf. earlier talks at this seminar by Fritz and Rischel) when instantiated in the quantum Markov category of C^*-algebras.

The GNS construction

The C^*-algebra $\mathcal{A}$ also acts on $\mathcal{H}_\omega$ by left-multiplication (on representatives). This defines a representation $\mathcal{A} \xrightarrow{\pi_\omega} \mathcal{B}(\mathcal{H}_\omega)$. Together with the vector $[1_\mathcal{A}]_\omega \in \mathcal{H}_\omega$ associated to the representative $1_\mathcal{A} \in \mathcal{A}$, these three data define a pointed representation $(\mathcal{H}_\omega, \pi_\omega, [1_\mathcal{A}]_\omega)$. Thus, we've specified a functor $\mathcal{S}(\mathcal{A}) \xrightarrow{\mathbf{GNS}^\bullet_\mathcal{A}} \mathbf{Rep}^\bullet(\mathcal{A})$ for each C^*-algebra $\mathcal{A}$. But given a $*$-homomorphism $\mathcal{A} \xleftarrow{F} \mathcal{B}$, does the diagram

$$\begin{array}{ccc} \mathcal{S}(\mathcal{A}) & \xrightarrow{\mathcal{S}(F)} & \mathcal{S}(\mathcal{B}) \\ {\scriptstyle \mathbf{GNS}^\bullet_\mathcal{A}} \downarrow & & \downarrow {\scriptstyle \mathbf{GNS}^\bullet_\mathcal{B}} \\ \mathbf{Rep}^\bullet(\mathcal{A}) & \xrightarrow[\mathbf{Rep}^\bullet(F)]{} & \mathbf{Rep}^\bullet(\mathcal{B}) \end{array}$$

commute?

The GNS construction

$$\begin{array}{ccc} \mathcal{S}(\mathcal{A}) & \xrightarrow{\mathcal{S}(F)} & \mathcal{S}(\mathcal{B}) \\ \mathbf{GNS}^{\bullet}_{\mathcal{A}} \Big\downarrow & & \Big\downarrow \mathbf{GNS}^{\bullet}_{\mathcal{B}} \\ \mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow[\mathbf{Rep}^{\bullet}(F)]{} & \mathbf{Rep}^{\bullet}(\mathcal{B}) \end{array}$$

$$\begin{array}{ccc} \omega & \xmapsto{\quad\mathcal{S}(F)\quad} & \omega \circ F \\ & & \Big\downarrow \mathbf{GNS}^{\bullet}_{\mathcal{B}} \\ \mathbf{GNS}^{\bullet}_{\mathcal{A}} \Big\downarrow & & (\mathcal{H}_{\omega\circ F}, \pi_{\omega\circ F}, [1_{\mathcal{B}}]_{\omega\circ F}) \\ & & \Big\downarrow \color{red}{\mathbf{GNS}^{\bullet}_{F,\omega}} \\ (\mathcal{H}_{\omega}, \pi_{\omega}, [1_{\mathcal{A}}]_{\omega}) & \xmapsto[\mathbf{Rep}^{\bullet}(F)]{} & (\mathcal{H}_{\omega}, \pi_{\omega} \circ F, [1_{\mathcal{A}}]_{\omega}) \end{array}$$

The GNS construction

So the two representations are not necessarily the same![7] Nevertheless, there is a canonical map $\mathbf{GNS}^{\bullet}_{F,\omega} : \mathcal{B}/\mathcal{N}_{\omega\circ F} \to \mathcal{A}/\mathcal{N}_{\omega}$ obtained from the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad F \quad} & \mathcal{A} \\ \big\downarrow & & \big\downarrow \\ \mathcal{B}/\mathcal{N}_{\omega\circ F} & \xrightarrow[\mathbf{GNS}^{\bullet}_{F,\omega}]{} & \mathcal{A}/\mathcal{N}_{\omega} \end{array}$$

and explicitly given by

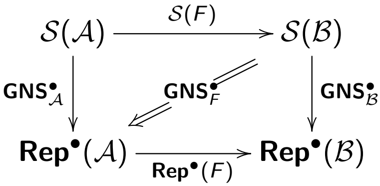
$$\mathbf{GNS}^{\bullet}_{F,\omega}([B]) := [F(B)] \qquad \forall\, [B] \in \mathcal{B}/\mathcal{N}_{\omega\circ F}.$$

This is then extended to the full Hilbert spaces, which is possible since this assignment is an isometry. Furthermore, it is a vector-preserving isometric intertwiner.

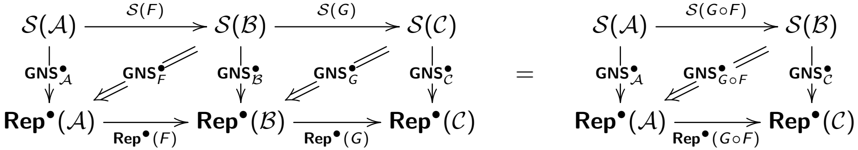
[7] For example, if F is the unique map $\mathcal{A} \leftarrow \mathbb{C}$, then $\mathbf{GNS}^{\bullet}_{\mathbb{C}}(\omega \circ F) \cong (\mathbb{C}, \mathrm{id}, 1)$.

The GNS oplax-natural transformation

This provides us with a natural transformation



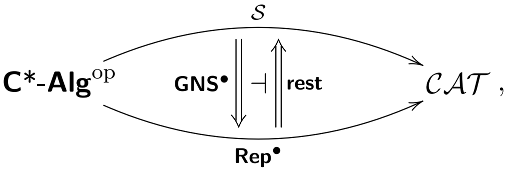
Oplax-naturality of **GNS**$^{\bullet}$ says (in particular)



for every pair of composable $*$-homomorphisms $\mathcal{A} \xleftarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$.

The GNS adjunction

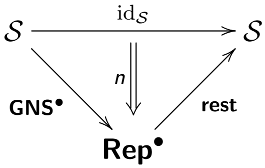
All this gives us an adjunction



but one must be careful about the meaning of the 2-morphisms $\Rightarrow$ drawn here. Only **rest** is a natural transformation between functors. However, $\mathbf{GNS}^{\bullet}$ is only an oplax-natural transformation! Therefore, this defines an adjunction in the 2-category $\mathbf{Fun}(\mathbf{C^*\text{-}Alg}^{\mathrm{op}}, \mathcal{CAT})$ whose objects are functors, 1-morphisms are oplax-natural transformations, and 2-morphisms are modifications.

The GNS adjunction

But such an adjunction should come equipped with modifications



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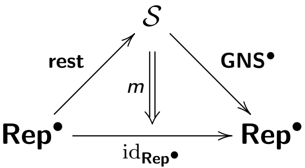
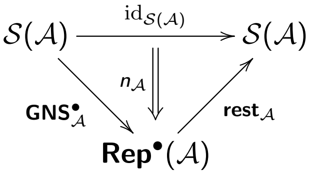


On a C^*-algebra $\mathcal{A}$, this should provide us with natural transformations (2-morphisms in $\mathcal{CAT}$)



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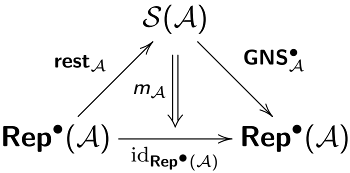
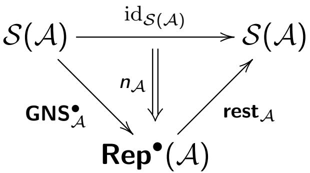
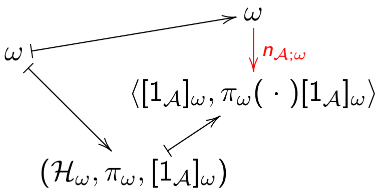


The modification n

Let's see what these natural transformations are.





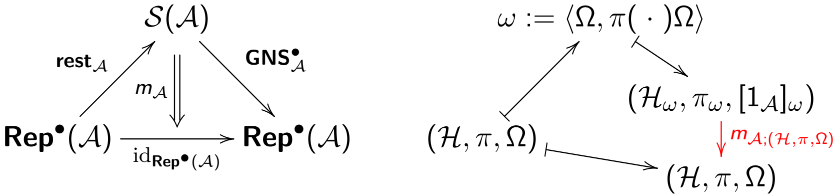
But by definition of the inner product on $\mathcal{H}_\omega$,

$$\langle [1_{\mathcal{A}}]_\omega, \pi_\omega(A)[1_{\mathcal{A}}]_\omega \rangle = \langle [1_{\mathcal{A}}]_\omega, [A]_\omega \rangle = \omega(1_{\mathcal{A}}^* A) = \omega(A).$$

Therefore, $n_{\mathcal{A}}$ is the identity modification!

The modification m

As for m, we have



There is a unique vector-preserving isometric intertwiner

$$(\mathcal{H}_{\omega}, \pi_{\omega}, [1_{\mathcal{A}}]_{\omega}) \xrightarrow{m_{\mathcal{A};(\mathcal{H},\pi,\Omega)}} (\mathcal{H}, \pi, \Omega)$$

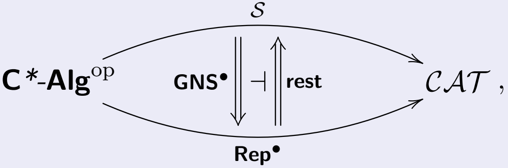
uniquely determined by the condition

$$m_{\mathcal{A};(\mathcal{H},\pi,\Omega)}\left([1_{\mathcal{A}}]_{\omega}\right) = \Omega.$$

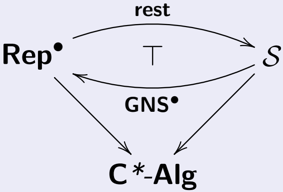
The GNS adjunction

Theorem 2 (P. 2018)

GNS$^\bullet$ *is the left adjoint of* **rest***:*



or equivalently



in terms of fibrations.

GNS provides the minimal realization of your state

- Given a pointed representation $(\mathcal{H}, \pi, \Omega)$ of $\mathcal{A}$, one obtains a state on $\mathcal{A}$ via pullback $\mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}) \overset{\langle\Omega, \,\cdot\, \Omega\rangle}{\rightsquigarrow} \mathbb{C}$.
- Given a state $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$, one obtains a pointed representation $(\mathcal{H}_\omega, \pi_\omega, [1_\mathcal{A}]_\omega)$.
- The pointed representation $(\mathcal{H}_\omega, \pi_\omega, [1_\mathcal{A}]_\omega)$ satisfies the universal property that it is the "smallest" realization of the state as a pure state on some Hilbert space which is acted upon by the algebra. Smallest here actually means smallest because for any other $(\mathcal{H}, \pi, \Omega)$ satisfying $\omega = \langle\Omega, \pi(\,\cdot\,)\Omega\rangle$, there exists a unique vector-preserving isometric intertwiner $(\mathcal{H}_\omega, \pi_\omega, [1_\mathcal{A}]_\omega) \to (\mathcal{H}, \pi, \Omega)$.
- The GNS adjunction in the 2-category $\mathbf{Fun}(\mathbf{C^*\text{-}Alg}^{\mathrm{op}}, \mathcal{CAT})$ also explains the "functoriality" of the GNS construction as coming from oplax-naturality.

Operator-valued CP maps

One can generalize the previous discussion further by the following schematic. First, instead of working with states $\mathcal{A} \overset{\omega}{\rightsquigarrow} \mathbb{C}$ on C^*-algebras, work with **operator-valued completely positive (OCP) maps** $\mathcal{A} \overset{\varphi}{\rightsquigarrow} \mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is some Hilbert space.[8] These are written as pairs $(\mathcal{K}, \varphi)$. Such a completely positive map represents the evolution of a system under some dynamics. Unlike in the case of states, there are non-trivial morphisms between such OCP maps. A **morphism of OCP maps** $T : (\mathcal{K}, \varphi) \to (\mathcal{L}, \psi)$ is a bounded linear map $T : \mathcal{K} \to \mathcal{L}$ such that

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\varphi(A)} & \mathcal{K} \\
{\scriptstyle T}\downarrow & // & \downarrow{\scriptstyle T} \\
\mathcal{L} & \xrightarrow[\psi(A)]{} & \mathcal{L}
\end{array}
\qquad \forall A \in \mathcal{A}.
$$

[8] A positive linear functional is obtained in the special case $\mathcal{K} = \mathbb{C}$.

Operator-valued CP maps

OCP maps and their morphisms for a given C^*-algebra $\mathcal{A}$ form a category $\mathbf{OCP}(\mathcal{A})$. In fact, for every $*$-homomorphism $\mathcal{A} \xleftarrow{F} \mathcal{B}$,

$$\begin{aligned} \mathbf{OCP}(\mathcal{A}) &\xrightarrow{\mathbf{OCP}_F} \mathbf{OCP}(\mathcal{B}) \\ (\mathcal{K}, \varphi) &\longmapsto (\mathcal{K}, \varphi \circ F) \\ \Big((\mathcal{K}, \varphi) \xrightarrow{T} (\mathcal{L}, \psi)\Big) &\longmapsto \Big((\mathcal{K}, \varphi \circ F) \xrightarrow{T} (\mathcal{L}, \psi \circ F)\Big) \end{aligned}$$

defines a functor. Combining these gives a functor

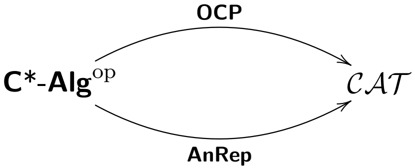
$$\mathbf{C^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\mathbf{OCP}} \mathcal{CAT}.$$

Anchored representations

Second, we need a category **AnRep**$(\mathcal{A})$ (replacing **Rep**$^{\bullet}(\mathcal{A})$, pointed representations). For this, we define an **anchored representation** of $\mathcal{A}$ to be a quadruple $(\mathcal{K}, \mathcal{H}, \pi, V)$ consisting of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and a bounded linear map $V : \mathcal{K} \to \mathcal{H}$. Morphisms are a bit technical, so I'll skip them. Once this is done, you get **AnRep**$(\mathcal{A})$. Much like in the case of OCP maps, one obtains a functor

$$\mathbf{C^*\text{-}Alg}^{\mathrm{op}} \xrightarrow{\mathbf{AnRep}} \mathcal{CAT}.$$

So again, we have two functors and we can ask how/if they're related.

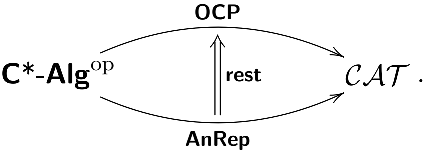


The restriction natural transformation

Given an anchored representation $(\mathcal{K}, \mathcal{H}, \mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}), \mathcal{K} \xrightarrow{V} \mathcal{H})$, one obtains an OCP map $\mathcal{A} \rightsquigarrow \mathcal{B}(\mathcal{K})$ via the composite

$$\mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}) \overset{\mathrm{Ad}_{V^*}}{\rightsquigarrow} \mathcal{B}(\mathcal{K}),$$

where $\mathrm{Ad}_{V^*}(A) := V^*AV$.[9] One can also extend this to morphisms of OCP maps and this defines a natural transformation



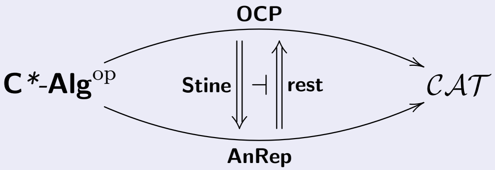
[9] Special case: If $\mathcal{K} = \mathbb{C}$, then $\mathbb{C} \xrightarrow{V} \mathcal{H}$ picks a vector $\Omega := V(1)$ and $\mathrm{Ad}_{V^*}(A) = V^*AV = \langle \Omega, A\Omega \rangle$.

The Stinespring adjunction

By using Stinespring's construction, a generalization of the GNS construction to OCP maps, one can construct a left adjoint of **rest**.

Theorem 3 (P. 2019)

There exists a left adjoint **Stine** : **OCP** $\Rightarrow$ **AnRep** *to the natural transformation* **rest** : **AnRep** $\Rightarrow$ **OCP**



in the 2-category **Fun**$(\mathbf{C^*\text{-}Alg}^{\mathrm{op}}, \mathcal{CAT})$.

Note: **Stine** is only an oplax-natural transformation.

Summary

- Representations and vectors produce states and vice versa. The two are related by an adjunction in a 2-category of functors.
- Representations and bounded linear maps produce certain completely positive maps and vice versa. The two are also related by an adjunction in *the same* 2-category of functors.
- The adjunction realizes the Stinespring construction associated to an OCP map as having a universal property describing it as a minimal anchored representation for that OCP map.
- The functoriality under $*$-homomorphisms is realized as oplax-naturality from this adjunction.
- Intricate details of von Neumann algebras and normality are not needed for this characterization.
- Question: Can one use Paschke's construction to generalize this to *all* completely positive maps between C^*-algebras?

Thank you!

Thanks for your attention and thanks to *Compositionality*!

For more details, including some of the physics related to this work, see

- arXiv:1807.02533/https://doi.org/10.32408/compositionality-1-2
- arXiv:1609.08975/https://doi.org/10.1007/s10485-018-9522-6
- arXiv:1708.00091
- arXiv:2009.07125

and the references therein.

Advertisement: A talk on the last paper (functoriality of the von Neumann entropy) will be given in Noson Yanofsky's category theory seminar http://www.sci.brooklyn.cuny.edu/∼noson/CTseminar.html on December 16th.