# Continuous pullbacks of categories

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Lemma 0.1. *Consider a pullback*

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "G"] \\ A \arrow[r, "F"'] & C \end{tikzcd}$$

*in* **CAT**. *Then:*

- *if $F$ and $G$ lie in* **ACC**, *and furthermore at least one of $F$ and $G$ is an isofibration, then the whole square lies in* **ACC**;
- *if $A$ and $B$ are complete and $G$ is a continuous fibration, then $P$ is complete.*

Proof. For the first point, **ACC** is known to be closed under bipullbacks in **CAT** [**?** , Theorem 5.1.6], and pullbacks along isofibrations are bipullbacks [**?** , Proposition 5.1.1] (citer Joyal-Street!).

For the second point, consider any functor $D\colon \mathbb{I} \to P$. We let $a_i = \pi_1(D(i))$ and $b_i = \pi_2(D(i))$ for all $i \in \mathbb{I}$, and choose limiting cones

$$\lambda_i \colon a \to a_i \qquad\qquad \mu_i \colon b \to b_i$$

for $\pi_1 D$ and $\pi_2 D$, respectively. By continuity of $G$, the cone $G\mu$ is limiting, so we get a unique morphism $\xi\colon Fa \to Gb$ making each square of the form below commute.

$$\begin{tikzcd} Fa \arrow[r, "F\lambda_i"] \arrow[d, "\xi"'] & Fa_i \arrow[d, equal] \\ Gb \arrow[r, "G\mu_i"'] & Gb_i \end{tikzcd}$$

Since $G$ is a fibration, we find a cartesian lifting $\xi' : b' \to b$ of $\xi$ along $b$, so in particular $Gb' = Fa$. We claim that the cone of all $(\lambda_i, (\mu_i \circ \xi'))$ as below

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$$\begin{tikzcd}
Fa \arrow[r, equal] \arrow[d, equal] & Fa \arrow[r, "F\lambda_i"] \arrow[d, "\xi"] & Fa_i \arrow[d, equal] \\
Gb' \arrow[r, "G\xi'"'] & Gb \arrow[r, "G\mu_i"'] & Gb_i
\end{tikzcd}$$

is limiting in $P$.

Indeed, consider any $(x, y) \in P$ and cone

$$\alpha_i : x \longrightarrow a_i \qquad\qquad \beta_i : y \longrightarrow b_i$$

to $D$. By universal property of $a$ and $b$, we obtain unique cone morphisms

$$m : x \longrightarrow a \qquad\qquad n : y \longrightarrow b.$$

Furthermore, by continuity of $G$, the cone $G\mu$ is limiting, so by uniqueness in its universal property, the left-hand square below commutes.

$$\begin{tikzcd}
Fx \arrow[r, "Fm"] \arrow[d, equal] \arrow[rr, bend left, "F\alpha_i"] & Fa \arrow[r, "F\lambda_i"] \arrow[d, "\xi"] & Fa_i \arrow[d, equal] \\
Gy \arrow[r, "Gn"'] \arrow[rr, bend right, "G\beta_i"'] & Gb \arrow[r, "G\mu_i"'] & Gb_i
\end{tikzcd}$$

Indeed, upon composing with $G\mu_i$, both morphisms give $G\beta_i$, by chasing the above diagram. We now use the fact that $G$ is a fibration to find a unique $m' : y \longrightarrow b'$ such that $Gm' = Fm$ and the top triangle commute.

$$\begin{tikzcd}
y \arrow[dr, dashed, "m'"'] \arrow[drr, bend left=15, "n"] & & \\
& b' \arrow[r, "\xi'"] & b
\end{tikzcd}$$

$$\begin{tikzcd}
Gy \arrow[dr, "Fm"'] \arrow[drr, bend left=15, "Gn"] & & \\
& Gb' \arrow[r, "\xi"] & Gb
\end{tikzcd}$$

We thus have the followin situation,

$$\begin{tikzcd}
Fx \arrow[r, "Fm"'] \arrow[d, equal] \arrow[rrr, bend left, "F\alpha_i"] & Fa \arrow[rr, "F\lambda_i"'] \arrow[d, equal] \arrow[dr, "\xi"] & & Fa_i \arrow[d, equal] \\
Gy \arrow[r, "Gm'"] \arrow[rr, bend right, "Gn"'] \arrow[rrr, bend right=50, "G\beta_i"'] & Gb' \arrow[r, "G\xi'"] & Gb \arrow[r, "G\mu_i"] & Gb_i
\end{tikzcd}$$

so that $(m, m')$ is a cone morphism $(\alpha, \beta) \to (\lambda, \mu \circ \xi')$. We further claim its uniqueness. Indeed, any other such cone morphism, say $(p, q)$, has $p = m$ by uniqueness in the universal property of $a$. Furthermore, $\xi' \circ q = n$ by uniqueness in the universal property of $b$. And finally, $q = m'$ by uniqueness in the universal property of $\xi'$. $\square$