Operads with restriction

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May 28, 2026

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1 Introduction

Let C be a small multicategory. A functor $X : C \to \mathbf{Set}$ associates to each c in X a set $X(c)$, and to each multimorphism $f : c_1, \ldots, c_n \to c$ an associated function $X(f) : X(c_1) \times \cdots \times X(c_n) \to X(c)$. C can be seen as an algebraic theory with many sorts and n-ary function symbols, and X a model of the algebraic theory. This perspective has been useful for the study of coherence theorems in algebraic structures. For the sake of this paper, we will understand a "coherence theorem" to mean a theorem that proves that one operad or multicategory, $\mathcal{O}$ - say, one presented by generators and relations - is equivalent to another operad or multicategory whose elements in each degree are expicitly enumerated. For example, the monoidal category operad, which is freely generated by $\otimes$, I, and the associator and unitor isomorphisms, is equivalent to the terminal operad in **Cat**, whose objects and morphisms in each degree can be explicitly enumerated (there is one object and one morphism in each degree.)

Consider the category **Graph** equipped with the free category monad **FreeCat**. In this setting, it is possible to construct three generalized multicategories - $\mathcal{O}_{\mathbf{Cat}}$, $\mathcal{O}_{\mathbf{Fun}}$, and $\mathcal{O}_{\mathbf{Nat}}$, whose algebras are, respectively,

- small categories
- pairs $(C, D, F : C \to D)$, where C, D are small categories and F is a functor between them

- tuples $(C, D, F : C \to D, G : C \to D, \tau : F \Rightarrow G)$

We might define $\mathcal{O}_{\mathbf{Cat}}$ to be the operad freely generated by the identity symbol in degree 1 and the composition symbol in degree 2, quotiented out by the associativity and left/right unitality axioms for morphism composition. Then it is a coherence theorem to prove that $\mathcal{O}_{\mathbf{Cat}}$ is isomorphic to the terminal operad, and the other two operads have their own associated coherence theorems.

The next most common categorical structure we encounter is the adjunction. Is there a multicategory over $(\mathbf{Graph}, \mathbf{FreeCat})$ whose algebras are precisely the adjunctions? It does not appear to be the case. The challenge is that the "interface" offered by an operad does not clearly offer a way to express the existence of a map $\mathbf{Hom}(x, Gy) \to \mathbf{Hom}(Fx, y)$.

On the other hand, there is a virtual equipment $\mathcal{O}_{\mathbf{Adj}}$ such that morphisms of equipments $\mathcal{O}_{\mathbf{Adj}} \to \mathbf{Set}$ do indeed correspond to adjunctions. In other words, there is a generalized multicategory $\mathcal{O}_{\mathbf{Adj}}$ over $(\mathbf{Graph}, \mathbf{FreeCat})$, whose models contain all small adjunctions as a subcategory; we are able to recover the adjunctions by observing that $\mathcal{O}_{\mathbf{Adj}}$ has extra structure (this concept of "restriction" in a virtual double category) and noticing that the models of $\mathcal{O}_{\mathbf{Adj}}$ which respect this additional structure are exactly the adjunctions.

This is the basic motivation for the paper. We introduce a concept of "generalized multicategory with restriction", which is a generalized multicategory with additional structure. We define an algebra for a generalized multicategory with restriction to be an algebra for the generalized multicategory, respecting the additional restriction structure. In this way we hope that we are able to broaden the class of algebraic structures which can be studied using the operadic point of view.

2 Main results

Let E, B be two categories with finite limits. We draw intuition from the case where E is **Graph** and **B** is **Set**. Let $p : E \to B$ be a Grothendieck fibration (for intuition, the forgetful functor), which we will assume to be cloven. Assume that p preserves finite limits. Let $T_0 : B \to B, T_1 : E \to E$ be Cartesian monads, with T_1 lying strictly above T_0. We are motivated by the case where T_1 is the free category monad and T_0 is the identity monad.

We will rely here on the notion of "generalized multicategory" due to Leinster, see *Higher categories, Higher operads.*

Let $A_1, X_1, g_1 : X_1 \to T_1 A_1, u_1 : X_1 \to A_1$ be a generalized multicategory in E relative to T_1. We will write $(\cdot)_1$ or $(- \cdot -)_1$ for the multiplication morphism, and e_1 for the unit morphism. Our basic definition establishes what it means for this generalized multicategory to be a generalized category "with restriction" relative to the fibration p. (We will see later that a generalized multicategory is always "with restriction" with respect to the unique fibration to the terminal category; so generalized multicategories "with restriction" generalize generalized multicategories.)

Because p preserves finite limits, it sends the generalized multicategory for (E, T_1) to a generalized multicategory for (B, T_0). We use a subscript 0 to denote the components of the projected operad.

Notationally, if $u : X \to Y$ is any morphism in the base category B and S is an object in E over Y, then the chosen Cartesian lift of u with codomain Y is denoted $\overline{u}$, the domain of $\overline{u}$ is called u^*Y, and if $f : Z \to Y$ is a morphism with $p(f) = u$, then the unique factoring of f through $\overline{u}$ is called $\hat{f}$.

Let $X_1 \otimes X_1$ denote the apex of the operad composed with itself once, i.e., $T(u_1) \times_{T_1(A_1)} g_1$. Let $\pi_r : X_1 \otimes X_1 \to X_1$ denote the projection onto the second component. Then in the diagram

$$\begin{array}{ccccc} X_1 \otimes X_1 & \xrightarrow{\pi_r} & X_1 & & \\ {\scriptstyle \hat{(\cdot)}}\downarrow & & {\scriptstyle \hat{u}_1}\downarrow & & \\ (\cdot)^*_0 X_1 & & u^*_0 A_1 & \xrightarrow{\overline{u}_0} & A_1 \end{array} \tag{1}$$

we can construct a filler for the bottom row $c_1 : (\cdot)^*_0 X_1 \to u^*_0 A_1$ which is uniquely characterized by the requirement that $p(c_1) = p(\pi_r)$ and that $\overline{u}_0 \circ c_1 = u_1 \circ \overline{(\cdot)}_0$, by appealing to the fact that $\overline{u}_0$ is Cartesian and $u_0 \circ p(\pi_r) = u_0 \circ (\cdot)_0 : X_0 \otimes X_0 \to X_0 \to A_0$.

Now we are able to explain what it means to be a "generalized multicategory with restriction." We require that in the commutative square we have just constructed, there are morphisms χ and σ,

$$\begin{array}{ccc} X_1 \otimes X_1 & \xrightarrow{\pi_r} & X_1 \\ {\scriptstyle \hat{(\cdot)}}\downarrow \uparrow{\scriptstyle \sigma} & & {\scriptstyle \hat{u}_1}\downarrow \uparrow{\scriptstyle \chi} \\ (\cdot)^*_0 X_1 & \xrightarrow[c_1]{} & u^*_0 A_1 \end{array} \tag{2}$$

such that

- $\hat{(\cdot)} \circ \sigma = 1$
- $\hat{u}_1 \circ \chi = 1$
- $\chi \circ c_1 = \pi_r \circ \sigma$
- σ is the equalizer of π_r and $\chi \circ c_1 \circ \hat{(\cdot)}$

Note that the third condition follows from the first and last.

Note that σ is uniquely determined by χ if it exists.

3 Example - virtual double categories

Let $(X_1, A_1, \dots)$ be a virtual double category; let V denote its underlying category of vertical arrows. $u^*_0 A_1$ is a graph whose nodes are morphisms f in V, and where an edge $f \to f'$ is a horizontal 1-cell $\operatorname{cod} f \to \operatorname{cod} f'$. $(\cdot)^*_0 X_1$ is a graph whose nodes are pairs of composable morphisms (f, g) in V, and whose edges $(f_1, g_1) \to (f_2, g_2)$ are two-cells with left and right boundary $g_1 \circ f_1$ and $g_2 \circ f_2$. c_1 is a graph homomorphism that sends the node (f, g) to g and the 2-cell α to the bottom horizontal 1-cell bounding α. $\hat{(\cdot)}$ is an operation which, for each pair of pairs of composable vertical morphisms $(f_0, g_1), (f_n, g_2)$, accepts as argument a single 2-cell $\alpha : g_1 \Rightarrow g_2$ whose source is a list of n edges and a list of 2-cells $(\beta_1, \ldots, \beta_n)$ where the left boundary of β_1 is f_0 and the right boundary of β_n is f_n; the composition operator $\hat{(\cdot)}$ composes the 2-cells, forgetting their factoring, while *remembering* the factoring of the underlying vertical 1-cells. The fact that χ is a right inverse to $\hat{u}_1$ means that χ associates to any "cup" $(f, g, h : \operatorname{cod} f \to \operatorname{cod} g)$ a 2-dimensional filler, although we don't specify here the number of morphisms in the arity. The fact that σ is right inverse to $\hat{(\cdot)}$ means that σ returns a factoring of the 2-cell it is given. The commutativity of the square when going from bottom left to top right indicates that the factoring supplied by σ has its second component determined by $\chi \circ c_1$. The equalizer condition implies that whenever (α, x) is a composable group of 2-cells (with α a list, and x a single 2-cell) such that x is the canonical filler associated to its boundary, α is uniquely determined by this requirement, as (α, x) is the canonical factorization of its own composite.

It is not clear that every virtual double category "with restrictions" in our sense, also has restrictions in the usual sense, because in the usual sense the restriction of a horizontal cell along two vertical cells is required to have an upper boundary which is a single morphism rather than a chain of morphisms. We can suggest two ways to rectify this. The first is to alter the definition we have given above so that instead of working with $X_1 \otimes X_1 = \{(\alpha, x) \mid \alpha \in T_1(X_1), x \in X_1, T_1(u_1)(\alpha) = g_1(x)\}$, we instead work with $\{(\alpha, x) \mid \alpha \in X_1, x \in X_1, T_1(u_1)(\eta(\alpha)) = g_1(x)\}$. The second is to impose the requirement that there is an algebra map $h : T(A_0) \to A_0$ such that $\eta_0 \circ h \circ g_0 = g_0$, so that it is possible to impose the assumption that $\chi : u_0^* A_1 \to X_1$ factors through $\eta : A_1 \to T_1 A_1$.

For now we will accept the inelegance that the choice of function χ is a structure rather than a property and may not be unique.

Theorem 1. *Let $\mathcal{C}$ be a virtual double category. The following are equivalent:*

- *$\mathcal{C}$ has (chosen) restrictions in the established sense of virtual double category theory.*
- *$\mathcal{C}$ has restrictions with respect to the forgetful fibration $p : \mathbf{Graph} \to \mathbf{Set}$, in our sense, i.e., there are morphisms χ, σ as defined above, subject to the additional constraint that there exists a morphism $h_1 : u_0^* A_1 \to A_1$ such that $g_1 \circ \chi = \eta_1 \circ h_1$. (Note that this implies that g_0 factors through η_0 by $h_0 = p(\eta_0)$.)*

Proof. It is well known that virtual double categories are equivalent to generalized multicategories in **Graph** with respect to the **FreeCat** monad. We regard $(\mathbf{Graph}, \mathbf{FreeCat})$ as being fibered over $(\mathbf{Set}, \mathbf{Id})$ by the usual forgetful functor. The forgetful functor p is a fibration that preserves finite limits; it sends the virtual double category to an ordinary category $V = (X_0, g_0 : X_0 \to T_0 A_0, u_0 : X_0 \to A_0)$, where A_0 is the set of objects of the virtual double category, X_0 the set of vertical morphisms, g_0 and u_0 the domain and codomain projections respectfully.

First, note that a splitting χ of the map $\hat{u}_1 : X_1 \to u_0^* A_1$ is exactly a map that associates to each pair of vertical morphisms $(f : x_1 \to y_1, g : x_2 \to y_2)$ and each horizontal 1-cell $s : y_1 \to y_2$ a 2-dimensional "filler" $\chi(f, g, s)$ with left and right boundary f, g respectively, and horizontal lower boundary s. The top boundary of $\chi(f, g, s)$ may be any chain of composable horizontal cells, but the constraint $g_1 \circ \chi = \eta_1 \circ h_1$ forces the vertical domain of $\chi(f, g, s)$ to be a single 1-cell.

We will now prove that the fillers given by χ are Cartesian iff there exists σ satisfying the axioms above. Observe that elements of $(\cdot)_0^* X_1$ are tuples $((f_1, g_1), (f_2, g_2), k, s)$, where (f_1, g_1) and (f_2, g_2) are pairs of composable vertical 1-cells, $k : \operatorname{cod} g_1 \to \operatorname{cod} g_2$ is a horizontal 1-cell, and s is a 2-cell with left and right boundary $g_1 \circ f_1$ and $g_2 \circ f_2$ respectively, and whose lower boundary is k. A section σ of $\hat{(\cdot)}$ is equivalent to a choice of factoring of the given 2-cell s into composable 2-cells $([a_1, \ldots, a_n], s')$, where s' has lower horizontal boundary k, the upper boundary of s' is exactly the list of lower boundaries of the a_i, the left and right boundary of s' are g_1, g_2 respectively, the left boundary of a_1 is f_1, and the right boundary of a_n is f_n. The constraint $\chi \circ c_1 = \pi_r \circ \sigma$ states that in the factoring $\sigma((f_1, g_1), (f_2, g_2), k, s) = ([a_1, \ldots, a_n]; s')$, the morphism s' is independent of s, f_1 and f_2; it is forced to be $\chi(g_1, g_2, k)$. Because of our earlier assumption that $g_1 \circ \chi$ factors through η_1, in the expression $[a_1, \ldots, a_n]$ we always have $n = 1$.

Thus, if χ, σ satisfy the first three bullets of 2, they give a filling 2-cell for every "cup", and a factorization function that assigns each 2-cell equipped with an appropriate factoring of its boundaries, to a factoring of the associated 2-cell through the designated filler.

The equalizer condition will then imply that this factorization is unique. Suppose that $\chi(f, g, k)$ is always Cartesian. A relatively straightforward reading of this as a universal property would be to say that for any object Z_1 and any $p_1 : Z_1 \to (\cdot)_0^* X_1$, there is a unique morphism $a : Z_1 \to T(X_1)$ such that $T(u_1) \circ a = g_1 \circ \chi \circ c_1 \circ p_1$ and $\hat{(\cdot)} \circ (a, \chi \circ c_1 \circ p_1) = p_1$. (Applying this to the case where p_1 is the identity morphism, we recover σ.) Then it is evidently equivalent to say that there is a unique $a' : Z_1 \to X_1 \otimes X_1$ (with $a = \pi_\ell \circ a'$) such that $\pi_r \circ a' = \chi \circ c_1 \circ p_1$ and $\hat{(\cdot)} \circ a' = p_1$.

So, if $a' : Z_1 \to X_1 \otimes X_1$, letting $p_1 = \hat{(\cdot)} \circ a'$, if and $\pi_r \circ a' = \chi \circ c_1 \circ p_1$, then a' is the unique lift of p_1 along $\hat{(\cdot)}$ with this property.

But σp_1 is also a lift of p_1 along $\hat{(\cdot)}$ with this property, so $a' = \sigma \circ \hat{(\cdot)} \circ a'$, which gives $\hat{(\cdot)} \circ a'$ as the desired factoring of a' through σ, which is unique as σ is a (split) monomorphism. The converse direction is similar.

□

In the case of virtual double categories we were interested in the special case of restriction where the domain of the filler was of shape A_1 rather than of the more general shape $T_1(X_1)$. It is interesting to note that in this case, the factoring of any 2-cell through the Cartesian morphism is also of shape X_1 rather than of the more general shape $T_1(X_1)$

Lemma 1. *Let $(X_1, A_1, u_1 : X_1 \to A_1, g_1 : X_1 \to T_1A_1)$ be a generalized multicategory "with restriction", i.e., having morphisms (χ, σ) as above.*

Suppose that χ factors through η as for virtual double categories with restriction, i.e., there is $h_1 : u_0^*A_1 \to A_1$ such that $\eta_1 \circ h_1 = g_1 \circ \chi$.

Then $\pi_\ell \circ \sigma : u_0^*A_1 \to X_1 \otimes X_1 \to T_1X_1$ is of the form $\eta_1 \circ h_2 : \hat{(\cdot)}_0^* \to X_1 \to T(X_1)$ for a unique morphism h_2.

Proof. Use that η is a Cartesian natural transformation. □

4 Algebras for generalized multicategories with restriction

Let $s_1 : E_1 \to A_1$ be an algebra for our generalized multicategory

We will discuss when s_1 respects the "restriction structure" χ. In the diagram below, the meaning of the maps $g_1, \hat{u}_1, \overline{u}_0$, and χ have already been established. Let $p : g_1 \times_{T_1A_1} T_1s_1 \to E_1$ be the action map of the operad; then p' is constructed by appeal to the universal property of the pullback, as $u_1 \circ \pi_{X_1} = s_1 \circ p$ by assumption.

Definition 1. s_1 is compatible with the restriction action of $\chi : u_0^*A_1 \to X_1$ when there exists a morphism χ' as in the diagram below, such that $\hat{p}_1 \circ \chi' = 1$ and $\pi_{X_1} \circ \chi' = \chi \circ \pi_{u_0^*A_1}$ is a pullback diagram.

$$\begin{array}{ccccccc} T_1E_1 & \longleftarrow & g_1 \times_{T_1A_1} T_1s_1 & \underset{\chi'}{\overset{\hat{p}_1}{\rightleftarrows}} & p_0^*E_1 & \xrightarrow{\overline{p}_0} & E_1 \\ \downarrow{\scriptstyle T_1s_1} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s_1} \\ T_1A_1 & \xleftarrow{g_1} & X_1 & \xleftarrow{\chi} & u_0^*A_1 & \xrightarrow[\overline{u}_0]{} & A_1 \end{array} \tag{3}$$

I will not be explaining further.

Theorem 2. *Let $\mathcal{C}$ be a virtual double category with restriction. Then algebras respecting the restriction structure of the multicategory are in one to one correspondence with covariant functors $\mathbf{C} \to \mathbf{Set}$ sending chosen Cartesian cells to Cartesian cells in $\mathbf{Set}$.*

Proof. Let $(X_1, A_1, g_1, u_1, (\cdot)_1, e_1)$ be the generalized multicategory associated to $\mathcal{C}$. It is already known that algebras for this multicategory correspond to covariant functors $\mathcal{C} \to \mathbf{Set}$ of virtual double categories. What is interesting here is to show that the algebras respecting the restriction structure correspond to functors sending chosen Cartesian 2-cells to Cartesian 2-cells.

Therefore, let $s_1 : E_1 \to B_1$ be an algebra which respects the restriction structure χ. Let $(f : x_1 \to y_1, g : x_2 \to y_2)$ be two vertical morphisms, and

$h : y_1 \to y_2$ a horizontal morphism. We have already argued that $\chi(f, g, h)$ is a Cartesian 2-cell; we will prove that the functor associated to s_1 sends $\chi(f, g, h)$ to a Cartesian 2-cell. Fix $w_1 \in E_1(x_1), w_2 \in E_1(x_2)$; let $a \in E_1(h)$ lying over $E_1(f)(w_1)$ and $E_1(g)(w_2)$ respectively. Then a can be viewed as an edge in $p_0^* E_1$ from (w_1, f) to (w_2, g), and $\chi'(a)$ is an edge from (w_1, f) to (w_2, g) in $g_1 \times_{T_1 A_1} T_1 s_1$, equivalently an edge in E_1 from w_1 to w_2 over $\operatorname{dom} \chi(f, g, h)$; the requirement that $\hat{p}_1(\chi'(a)) = a$ gives that $\chi'(a)$ is a lift of a along $E_1(\chi(f, g, h), w_1, w_2)$. To see that $\chi'(a)$ is unique, note that if $b : w_1 \to w_2$ is any edge in $E_1(\operatorname{dom} \chi(f, g, h))$, then because of the requirement that χ' is the pullback of χ, there is a (unique) edge b' from (w_1, f) to (w_2, g) in $p_0^* E_1$ over the edge $h : f \to g$ in $u_0^* A_1$, such that $\chi'(b') = b$; but then $\hat{p}_1(b) = \hat{p}_1(\chi'(b')) = b'$, so if $\hat{p}_1(b) = a$, $b = \chi(a)$.

Let us prove the converse direction. Suppose that we have a functor of virtual double categories $E_1 : \mathcal{C} \to \mathbf{Set}$ preserving chosen Cartesian cells. We will construct a map χ' satisfying the requirements above. This is equivalent to saying that for each pair of vertical morphisms $(f : x_1 \to y_1, g : x_2 \to y_2)$ in X_0 and $h : y_1 \to y_2$ in A_1, the associated map $E_1(\chi(f, g, g))$ is fiber-wise a bijection (i.e., a bijection over each $w_1 \in E_1(x_1), w_2 \in E_1(x_2)$. This is what it means for a 2-cell in $\mathbf{Set}$ to be Cartesian, so we are done. □