Logic in Color

A story and a language of category theory

(Forthcoming dissertation with John Baez)

I believe we can bring category theory into the public mind by presenting it as *logic*, and we can share the language *in color* — that is, in two dimensions.

Below is a proposed curriculum for students of all ages. The material is founded on a logical system developed in the author's dissertation [2] [4]; upon completion, it is meant to become a book and an education program.

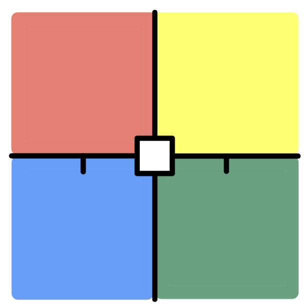
For ACT 2022, I hope to demonstrate a lesson designed for young students, exploring how we can reflect on our thinking with colors, strings, and beads.

Binary Logic

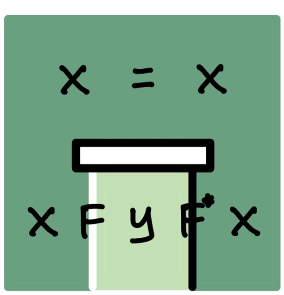
The story begins with the simplest kind of logic, the one we learn in school: binary logic is embodied in Rel, the universe of sets, functions, relations, and implications.

The types, terms, judgements, and inferences of a logic form a *fibrant double category*, understood as a language in the complementary forms of syntax and imagery [3]. Below is the general form of an inference $\mathtt{s}(\mathtt{r}) : \mathsf{R}(x, y) \vdash \mathsf{S}(a(x), b(y))$.

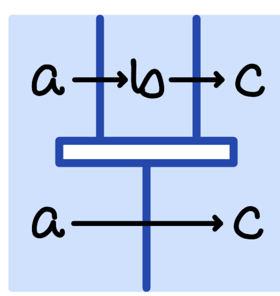
$$\frac{\begin{array}{c}\mathbb{X} : x \,.\, x\mathsf{R}y \,.\, y : \mathbb{Y}\\ \mathtt{r}\end{array}}{\begin{array}{c}\mathtt{s}\\ \mathbb{A} : a \,.\, a\mathsf{S}b \,.\, b : \mathbb{B}\end{array}}$$



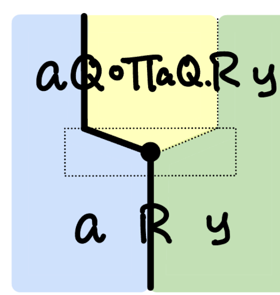
I believe the simplest way we think about the world is in terms of *collections* of things, *relations* of collections, and *implications* of relations. This is why I believe that Rel is the natural setting to begin mathematical education. Addition and multiplication are shadows of the sum and product of sets — why not teach the way we actually think?



adjunction



monad



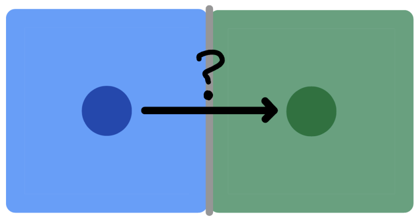
extension

The language is more than a fun new visual approach to a standard curriculum of logic. In two dimensions, one sees concepts in their general forms: co/functions are adjunctions, orders are monads, universals are extensions. With good story and examples, I believe that these ideas can be understood in public education.

Matrix Logic

Binary logic is the simplest form of logic: judgements are relations, matrices of 0 and 1. Yet in real life, judgements contain all kinds of content beyond truth values: besides the proposition "I can see the tree", a judgement could be one's actual *perspective* of the tree, e.g. as a map from your field of view to the color spectrum.

Beyond *whether* A is connected to B, we can think of the *set of connections* from A to B. This is the main way that type theory has generalized propositional logic. One can present matrices of sets as the natural expansion of relations, and guide lessons in proof or code using indexed sums and products as expanded quantification.

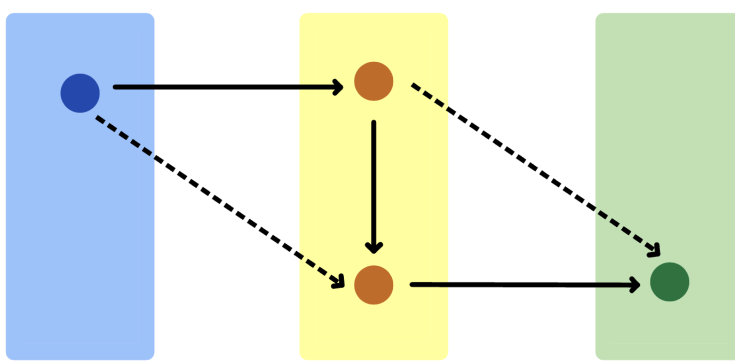


Yet the greatest aspect of matrix logic is the *exploration*. Students can explore logics of distance, resources, probability, knowledge... who knows how many. Imagine all of their surprising "what if's", all of their unique ideas, with such a myriad of possibilities.

Active Logic

The third part is *category theory*. The limitation of matrix logic is that types are only sets: in reality, types have inner content. In a judgement of distance, "I am 1 meter from a tree", the tree is not a formless collection of atoms — it is really a *space*.

Categories can be motivated as "microcosm logics"; reasoning can "flow through" types. In this sense, the construction from Mat(V) to VCat *completes* the logic of V. [1]



The main difference is that now, types *act* on judgements; categories act on profunctors. Hence the author contends that the language of category theory is the *coend calculus*.

Coend is the "bilinear existential" which composes judgements. End is the "natural universal" which reifies inferences.

I propose that this perspective is the key to understanding category theory as logic; and we can bring categorical thinking into education by presenting the subject as logic.

References

[1] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion, 2013. URL: https://arxiv.org/abs/1301.3191, doi:10.48550/ARXIV.1301.3191.

[2] Fosco Loregian. Coend calculus, 2015. URL: https://arxiv.org/abs/1501.02503, doi:10.48550/ARXIV.1501.02503.

[3] David Jaz Myers. String diagrams for double categories and equipments, 2016. URL: https://arxiv.org/abs/1612.02762, doi:10.48550/ARXIV.1612.02762.

[4] Michael Shulman. Enriched indexed categories. 2012. URL: https://arxiv.org/abs/1212.3914, doi:10.48550/ARXIV.1212.3914.