Globularly generated double categories

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MIT categories seminar

September 24, 2020

Plan for the talk

1. Preliminaries on bicategories and double categories
2. The bicategory of von Neumann algebras
3. Globularly generated double categories

Preliminaries on bicategories and double categories

Bicategories

A bicategory is a category enriched by categories. [Bénabou 67']. Has:

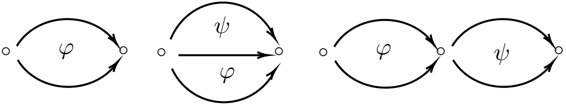
- Objects, (1-) morphisms, and morphisms between morphisms (2-morphisms)
- Horizontal composition of 1-morphisms. Horizontal 1 dim identities.
- Vertical composition of 2-morphisms. Vertical identities.
- Horizontal composition on 2-morphisms induced by horizontal 1-dim composition. Horizontal 2 dim identities.

Vertical composition is assumed to be strictly associative and unital. Horizontal composition is only assumed to be associative and unital up to compatible natural isomorphisms. Vertical and horizontal composition satisfy the exchange property. **Notation:** We denote by **bCat** the category of bicategories and pseudofunctors.

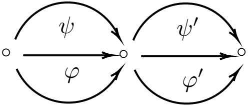
Pictorial representation

Let B be a bicategory. We represent objects, 1-morphisms, and horizontal 1 dim composition as:

Represent 2-morphisms, vertical composition, and horizontal composition:



Exchange relation:



Double categories

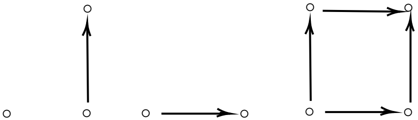
A double category is a category internal to categories. [Ehresmann 63'].
A double category C thus has:

1. Category of objects and category of morphisms C_0, C_1.
2. Source, target functors $s, t : C_1 \to C_0$.
3. (Horizontal) dentity functor $i : C_0 \to C_1$.
4. (Horizontal) composition functor $* : C_1 \times_{C_0}^{t,s} C_1 \to C_1$.

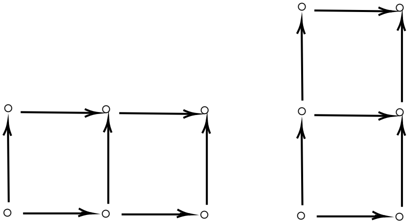
Satisfying functorial versions of usual conditions defining a category.
Think of: Categories with every set turned into a category and structure function into a functor. Non-strict version: Pseudo-double category.
Write **dCat** for the category of double categories and double functors.

Pictorial representation

C double category. Objects and morphisms of C_0 are referred to as the objects and vertical morphisms of C. Objects and morphisms of C_1 are referred to as the horizontal morphisms and the squares of C. Drawn as:

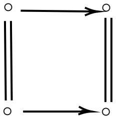


Vertical and horizontal composition are implemented by vertical and horizontal concatenation, i.e. as:



The horizontal bicategory

Let C be a double category. A square in C is globular if it is of the form:



Objects, horizontal morphisms and globular squares of C form a bicategory, denoted by HC and called the horizontal bicategory of C. The function $C \mapsto HC$ extends to a functor $H : \mathbf{dCat} \to \mathbf{bCat}$.

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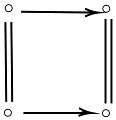

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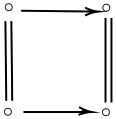

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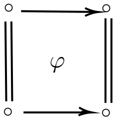
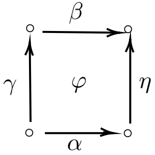


where φ is a 2-morphism in B. $\mathbb{H}B$ referred to as the trivial double category associated to B. The function $B \mapsto \mathbb{H}B$ extends to an embedding $\mathbb{H} : \mathbf{bCat} \to \mathbf{dCat}$. H and $\mathbb{H}$ are related via $\mathbb{H} \dashv H$.

Another right inverse to H

Let B be a 2-category. Write $\mathbf{Q}B$ for the double category whose squares are of the form:

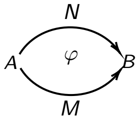


where φ is a 2-morphism, in B, from $\eta\alpha$ to $\beta\gamma$. We denote any such square by a quintet $(\varphi; \alpha, \gamma, \beta, \eta)$ and we call $\mathbf{Q}B$ the Ehresmann double category of quintets of B. Thus defined $\mathbf{Q}B$ satisfies the equation $H\mathbf{Q}B = B$. The double category $\mathbf{Q}B$ is edge-symmetric and admits a connection [Brown,Mosa 99']. The function $B \mapsto \mathbf{Q}B$ extends to an equivalence from **bCat** to the category $\mathbf{dCat}^!$ of edge-symmetric double categories with connection. When B is a proper bicategory $\mathbf{Q}B$ is not a double category but a Verity double category.

The main difference between $\mathbf{Q}$ and $\mathbb{H}$ is the category of objects of the corresponding double category. Both minimal filling of squares with globular data.

The bicategory of algebras

Write **Mod** for the bicategory whose 2-morphisms are of the form:

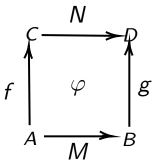


where A and B are unital complex algebras, ${}_AM_B$ and ${}_AN_B$ are bimodules and $\varphi : M \to N$ is a bimodule morphism. Horizontal identity and horizontal composition in **Mod** are defined by the functions $A \mapsto {}_A A_A$ and $(M_B, {}_B N) \mapsto M \otimes_B N$. The exchange identity in **Mod** follows from functoriality of $M \otimes_B N$ on M and N.

Observation: A, B algebras. A, B are isomorphic in **Mod** if and only if A and B are Morita equivalent.

The double category of algebras

Write [**Mod**] for the double category whose squares are of the form:



where A, B, C and D are algebras, ${}_A M_B$ and ${}_C N_D$ are bimodules, $f : A \to C$ and $g : B \to D$ are unital algebra morphisms, and $\psi : M \to N$ is a linear transformation such that the equation:

$$\varphi(a\xi b) = f(a)\psi(\xi)g(b)$$

holds i.e. the squares of [**Mod**] are equivariant bimodule morphisms.

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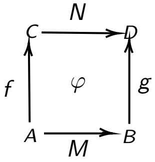

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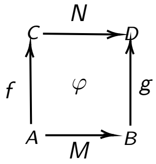

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Tensor product of algebras, vector spaces, and linear transformations morally provide **Mod** with the structure of a symmetric monoidal bicategory. **We should have:** Coherence invertible bimodules satisfying a bunch of very complicated equations presented by, e.g. [Kapranov, Voevodski 94'], [Baez, Neuchl 95'], [Crans 98'], [Schommer-Pries 11'], A bit excessive for our purposes. Coherence data for $\otimes$ of algebras is naturally defined in terms of unital morphisms, and satisfies MacLane equations strictly. Need a different language to express this.

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Tensor product on vertices, edges and squares of $[\mathbf{Mod}]$ provide $[\mathbf{Mod}]$ with the structure of a symmetric pseudomonoid in **dCat**, i.e. of a symmetric monoidal double category. Moreover, $[\mathbf{Mod}]$ is fibrant and thus the coherence isomorphisms of $[\mathbf{Mod}]$ descend to coherence isomorphisms of a symmetric monoidal structure on **Mod** with tensor porduct $H\otimes$. Shulman M. A., Constructing symmetric monoidal bicategories. arXiv:1004.0993. $[\mathbf{Mod}]$ is the correct framework to equip algebras with a 2 dim symmetric monoidal structure.

Mod-like bicategories

Observation: There are essentially two types of bicategories, exemplified by **Cat** and **Mod**. **Cat** has objects, 'functions' between objects as 1-morphisms, and morphisms between these 'functions' as 2-morphisms. **Mod** has objects, 'other objects' as 1-morphisms, and 'functions' between 1-dimensional 'objects' as 2-morphisms. There is a correct notion of morphism between objects in **Mod**, not directly included in **Mod**.

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Slogan: A **Mod**-like bicategory B should have a category of 'function/correct' morphisms B^*. It is expected that there should be a clear lift of B to a double category C, such that $C_0 = B^*$ and such that $HC = B$. A natural symmetric monoidal structure on B should better be expressed as a symmetric monoidal structure on C.
⋮

The bicategory of von Neumann algebras

von Neumann algebras

$\mathcal{H}$ Hilbert space. Write $\mathbf{B}\mathcal{H}$ for the set of bounded operators on $\mathcal{H}$, i.e. $||T\xi|| \leq M||\xi||$, $\forall \xi \in \mathcal{H}$, and some $M \in \mathbb{R}_+$. $\mathbf{B}\mathcal{H}$ is a unital *-algebra.

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Examples:

- $\mathbf{B}\mathcal{H}$ is a vN algebra for any Hilbert space $\mathcal{H}$. In particular $M_n(\mathbb{C})$.
- (X, μ) 'nice' measure space. $L^\infty(X, \mu)$ vN algebra on $L^2(X, \mu)$. All commutative von Neumann algebras are of this form.
- G group, $\mathcal{H}$ Hilbert space. $\lambda : G \to U\mathcal{H}$. G' is a vN algebra. In particular if G is discrete, G has left regular representation in $\ell^2(G)$. G' vN algebra. Group vN algebra of G. $L(G)$.

Factors

Let A von Neumann algebra on $\mathcal{H}$. $A \cap A'$ center of A. A is a factor if $A \cap A' = \mathbb{C}1_A$. Factors are simple vN algebras and also most noncommutative vN algebras.

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Examples:

- $\mathbf{B}\mathcal{H}$ factor. $M_n(\mathbb{C})$ factor. Closure of $M_2(\mathbb{C}) \subseteq M_4(\mathbb{C}) \subseteq \ldots$ factor.
- Let G be ICC, i.e. every non-trivial conjugacy class of G is infinite. $L(G)$ factor.

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A Subfactor is an inclusion of factors $A \subseteq B$. **Example:** If $H \subseteq G$ and both are ICC, then $L(H) \subseteq L(G)$ is a subfactor. The Jones index $[B : A]$ of a subfactor $A \subseteq B$ [Jones 83'] is a generalized quantized dimension, taking values in $\left\{4cos(\pi/n)^2 : n \in \mathbb{N}\right\} \cup [4, \infty]$ measuring how A fits into B. Subfactors express how observables interact when one region is contained in the other. Index measure this.

Morphisms and bimodules

Let A, B be vN algebras. $f : A \to B$ be a unital *-morphism. We say f is a normal morphism if f is 'continuous'. Write *vN* for the category of vN algebras and normal morphisms. Write *Fact* for the full subcategory of factors.

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A, B vN algebras, an A, B-Hilbert bimodule ${}_A\mathcal{H}_B$ is a Hilbert space $\mathcal{H}$ together with normal morphisms $A \to \mathbf{B}\mathcal{H}$ and $B^{op} \to \mathbf{B}\mathcal{H}$ such that. $A \subseteq B'$. Given bimodules ${}_A\mathcal{H}_B$ and ${}_A\mathcal{K}_B$ an intertwiner from ${}_A\mathcal{H}_B$ to ${}_A\mathcal{K}_B$ is a bounded operator $T : \mathcal{H} \to \mathcal{K}$ such that $T(a\xi b) = aT(\xi)b$ $\forall \xi \in \mathcal{H}$, $a \in A, b \in B$.

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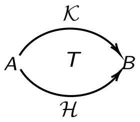

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We wish to organize the above pictures into a bicategory W^*. **Have:** Pictures, i.e. Objects, 1-morphisms, 2-morphisms and the usual composition of intertwiners as vertical 2-dim composition. **Need:** Horizontal identity and horizontal composition. Highly nontrivial/Very technical.

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With this structure W^* is a bicategory. We write W^*_{fact} for the sub-bicategory of W^* generated by factors. Landsman, N. P., Bicategories of operator algebras and Poisson manifolds, Fields Inst. Comm 30, 271–286 (2001)]. Obviously Mod-like bicategory.

Things to say about W^*

- [Bartels, Douglas, Hénriques 14'] prove a subfactor $A \subseteq B$ is such that $[B : A] < \infty$ if and only if ${}_AL^2(B)_B$ is dualizable in W^*_{fact} and in this case $[B : A]$ is the square root of the trace of ${}_AL^2(B)_B$.

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• Strong Morita equivalence is not the strictest notion of isomorphism between vN algebras (*-isomorphisms is)

Things to say about W^*

• [Bartels, Douglas, Hénriques 14'] prove a subfactor $A \subseteq B$ is such that $[B : A] < \infty$ if and only if ${}_A L^2(B)_B$ is dualizable in W^*_{fact} and in this case $[B : A]$ is the square root of the trace of ${}_A L^2(B)_B$.

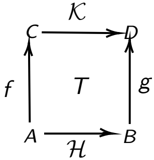
• [Landsman 01'] proves that two von Neumann algebras A, B are isomorphic in W^* if and only if A, B are strong Morita equivalent [Rieffel 74'], i.e. if and only if there exists a faithful ${}_A\mathcal{H}_B$ such that $A' = B^{op}$.

• Strong Morita equivalence is not the strictest notion of isomorphism between vN algebras (*-isomorphisms is)

• There are obvious tensor product operations on vN algebras/factors, bimodules and intertwiners morally making W^* into a symmetric monoidal bicategory. Coherence data is defined in terms of *-morphisms. **Mod**-like bicategory situation: Extend to a (fibrant) double category!

Lifting to a double category

We follow the construction of [**Mod**]. Consider squares of the form:



with A, B, C, D von Neumann algebras, ${}_A\mathcal{H}_B$ and ${}_C\mathcal{K}_D$ bimodules,
$f : A \to C$, $g : B \to D$ *-morphisms and $T : H \to K$ bounded s.t:

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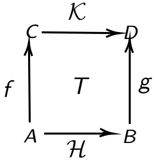

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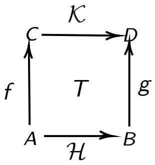

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BDH identity and composition

Let A, B factors. $f : A \to B$ *-morphism. Observe that $f(A) \subseteq B$ subfactor. f finite if $[f(A) : B] < \infty$. $Fact^{<\infty}$ category of factors and finite morphisms. $Mod_1^{<\infty}$ subcat of $[W^*]_1$ gen. by squares with factor vertices and finite vertical edges, i.e. finite equivariant bounded intertwiners.

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Theorem (Bartels, Douglas, Henriques '14)

There exist functors

$$L^2 : \mathit{Fact}^{<\infty} \to \mathit{Mod}_1^{<\infty}$$

and

$$\boxtimes_\bullet : \mathit{Mod}^{<\infty} \times_{\mathit{Fact}^{<\infty}} \mathit{Mod}^{<\infty} \to \mathit{Mod}^{<\infty}$$

such that $L^2(A)$ is the Haagerup standard form for every A and $\boxtimes_\bullet(\mathcal{H}_B, {}_B\mathcal{K})$ is $\mathcal{H} \boxtimes_B \mathcal{K}$ for every $(\mathcal{H}_B, {}_B\mathcal{K})$.

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Technique: Use of the theory of minimal conditional expectations for finite index subfactors [Kosaki 91'] in an essential way. No version of these techniques for infinite index avialable!

The double category BDH

With the above functors $(Fact^{<\infty}, Mod_1^{<\infty})$ is a double category. We denote this double category by BDH. BDH satisfies the equation $HBDH = W^*_{fact}$.

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• BDH directly recognizes strong Morita equivalence, finite index, isomorphisms of semisimple von Neumann algebras.

Questions

Question: Is there a double category of general von Neumann algebras (not-necessarily factors) and normal *-morphisms C such that $HC = W^*$ and such that BDH is a sub-double category of C? **Strategy:** Consider an easier question: Does there exist a double category of factors and not-necessarily finite index morphisms D such that $HD = W^*_{fact}$ and such that BDH is a sub-double category of D? If so, use direct integral methods.

Question: Peterson, Ishan and Ruth define von Neumann couplings between von Neumann algebras in the preprint Ishan I., Peterson J., Ruth L., Von Neumann equivalence and properly proximal groups. arXiv:1910.08682 as von Neumann algebras satisfying certain conditions. Can we define a tricategory of von Neumann algebras, von Neumann couplings, bimodules and bounded intertwiners? If so, provide this tricategory with a symmetric monoidal structure and study 3-dualizable objects. **Prospects:** Associate 3 dim local TFT's to von Neumann algebras. **Strategy:** Define a bicategory internal to **SMC**. This must pass through a double category of von Neumann algebras.

Globularly generated double categories

Finding a double category of factors: Strategy

The theory of von Neumann algebras does not give us direct tools to extend BDH to general morphisms. **Strategy:** Solve the problem categorically, i.e. understand any such extension in terms of its 'surrounding' categorical structure, i.e. in terms of other double categories of factors.

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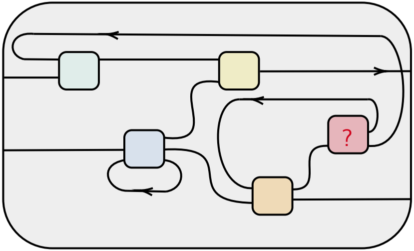

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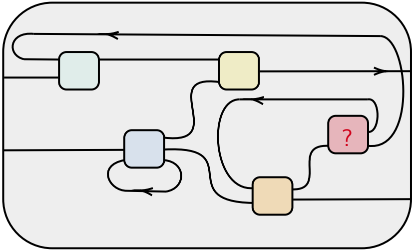
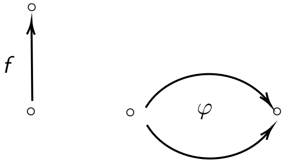


Question: Are there double categories of factors at all? i.e. is the above shaded square $\neq \emptyset$?

Decorated bicategories

A decorated bicategory is a pair (B^*, B) where B^* is a category and B is a bicategory such that the objects of B^* and B are the same. Represent a decorated bicategory as a bunch of diagrams of the form:



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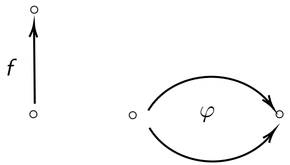

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Example: Let C be a double category. The pair (C_0, HC) is a decorated bicategory. Write H^*C and call it the decorated horizontalization of C.

Internalizations

Problem: Given a decorated bicategory (B^*, B). Find double categories C such that $H^*C = (B^*, B)$. We call any such C an internalization.

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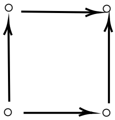
Problem of existence of internalizations: Is the decorated horizontalization construction generic? We think of the above problem as a problem of coherently 'filling' 'hollow' squares of the form:



which we form with the 1-dimensional data provided to us by (B^*, B) in such a way that the 1-dimensional and the globular data we started with is fixed. Problems of filling squares with globular data appear in Brown's proof of the 2-dimensional Seifert-van Kampen theorem [Brown, Higgins, Sivera 11']

The globularly generated piece construction

Let C be a duble category. Write γC for the minimal sub-double category of C containing all vertical morphisms and all globular squares of C.

Lemma (O 18')

Let C be a double category.

1. $H^*C = H^*\gamma C$.
2. *If D is a sub-double category of C satisfying the equation $H^*C = H^*D$ then γC is a sub-double category of D.*

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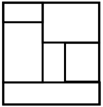
C is a solution to internalization for H^*C. 1 says that so is γC. 2 says that γC is the minimal solution on C. We call γC the globularly generated piece of C. **Question:** Can we understand these 'minimal' solutions outside of the context of C?

Globularly generated double categories

We say that a double category C is globularly generated if any of the following three equivalent conditions is satisfied:

1. $\gamma C = C$.
2. C is generated, as a double category, by its globular squares.
3. C contains no proper sub-double categories D such that $H^* C = H^* D$.

Intuitively C is globularly generated if every square in C admits a subdivision, say as:



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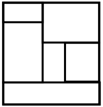

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1. $V_n \subseteq V_{n+1} \subseteq C_1$.
2. $\varinjlim V_n = C_1$.

i.e. the chain of V_n's is a filtration for C_1. Call the filtration $... \subseteq V_n \subseteq V_{n+1} \subseteq ...$ of C_1 the vertical filtration of C.

Vertical length

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Intuition: ℓC measures the complexity of mixed compositions of horizontal identity and globular squares in C, e.g. $\ell C = 1$ iff every square in C can be written as vertical composition of globular and horizontal identity squares. **Examples:** $\ell\mathbb{H}B = 1$, $\ell\mathbf{Q}B = 1$, $\gamma[\mathbf{Mod}] = 1$ and $\ell BDH = 1$. **Question:** Is ℓ trivial?

Constructing GG double categories

Let (B^*, B) be a decorated bicategory. We wish to associate to (B^*, B) a globularly generated double category defined only through the data of (B^*, B). **Idea:** Formally reconstruct a vertical filtration with the data of (B^*, B) and then turn that into a globularly generated double category.

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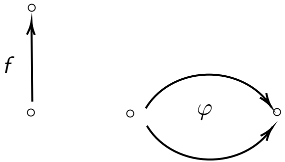

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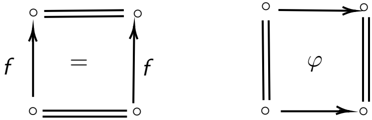
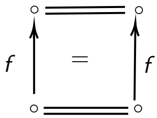
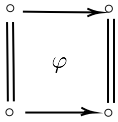


Constructing GG double categories

Stack the above diagrams vertically. Formally, write F_1 for the free category generated by diagrams of the form:





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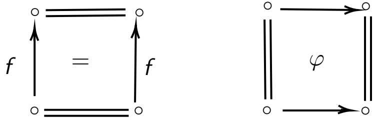
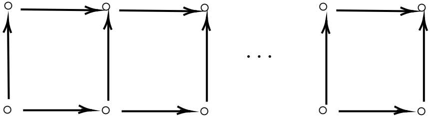


Write E_1 for the collection of formal words on compatible elements of F_1, i.e. E_1 is the collection of formal expressions of the form:



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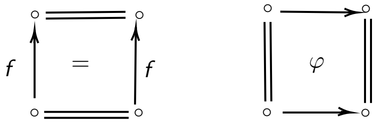

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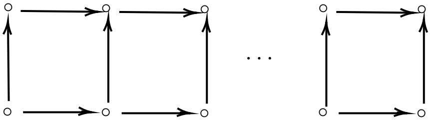

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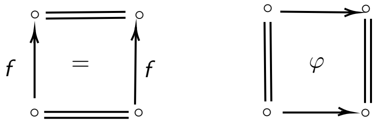

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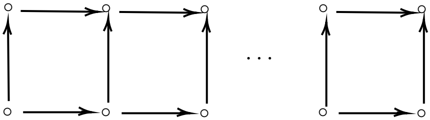

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Carefully choose an equivalence relation R on F_∞ containing both the exchange relation and the composition information of (B^*, B). Write $Q_{(B^*,B)}$ for F_∞/R.

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We call $Q_{(B^*,B)}$ the free globularly generated double category associated to (B^*, B). **Warning:** The equlity $H^*Q_{(B^*,B)} = (B^*, B)$ does not hold in general. **Example:** Let A be an abelian group. Let G be a group. $H^*Q_{(\Omega G,\Omega^2 A)} = (\Omega G, \Omega^2(G * A))$.

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Example: If B^* has no sections or retractions, i.e. (B^*, B) is reduced. In particular if B^* is free or ΩM for a reduced monoid M then (B^*, B) is saturated.

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Example: If B^* has no sections or retractions, i.e. (B^*, B) is reduced. In particular if B^* is free or ΩM for a reduced monoid M then (B^*, B) is saturated.

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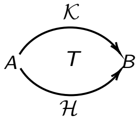
If (B^*, B) is not saturated we can always enlarge (B^*, B) canically in order to obtain a saturated decorated bicategory.

An internalization of factors

Write *Fact* for the category whose objects are factors and whose morphisms are possibly infinite *-monomorphisms. Recall that W^*_{fact} is the bicategory of diagrams of the form:



with corners being factors, $\mathcal{H}$ and $\mathcal{K}$ bimodules and φ a bounded intertwiner. Thus defined $(Fact, W^*_{fact})$ is a decorated bicategory.

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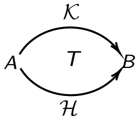

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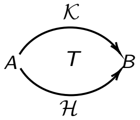

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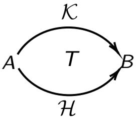

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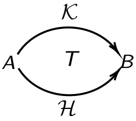

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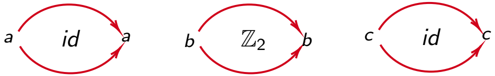
Length is not trivial

We also use the free globularly generated double category construction to prove that vertical length is non-trivial. Consider the bicategory B:



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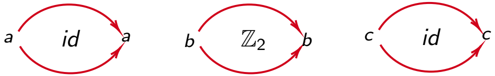

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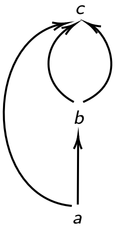

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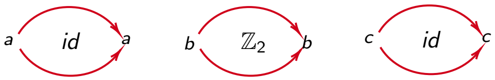

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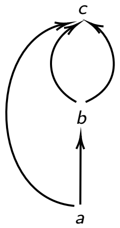

The free globularly generated double category $Q_{(B^*,B)}$ of (B^*, B) has a square of vertical length 2. Same method: Double categories of arbitrarily large and infinite length. Length is non-trivial.

Final remarks

• The free globularly generated double category construction is a free object in **gCat** with respect to H^*. We can thus describe every globularly generated double category as a canonical double quotient of the free globularly generated double category of its decorated horizontalization.

• We can use this to construct an extension of γBDH to arbitrary *-morphisms of factors. This provides a second non-double equivalent double category of factors. **Question:** How many of these can we build? **Partial answer:** We can build one of vertical length one for every special endofunctor monoidal fibration. There is evidence that this is somehow controlled by a cohomology theory. **Problem:** Build this cohomology.

References

1. J. Orendain. Internalizing decorated bicategories: The globularily generated condition. Theory and Applications of Categories, Vol. 34, 2019, No. 4, pp 80-108.
2. J. Orendain. Free globularily generated double categories. Theory and Applications of Categories, Vol. 34, 2019, No. 42, pp 1343-1385.
3. J. Orendain. Free Globularly Generated Double Categories II: The Canonical Double Projection. arXiv:1905.02888.

Thank you :)