Logic in color

a visual language of categorical thinking

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Thanks

Mom and Dad, Catherine, John, Mike, Sarah, and Nathanael.

Category theory is logic.

Category theory is known as a language of mathematics.

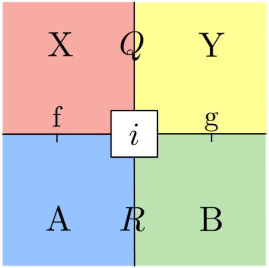
Recently, the Applied Category Theory community has begun to explore its potential as a language for many kinds of science.

I propose that category theory is the language of *logic*.

Logics

The language of category theory
— composition and identity, adjunction and representation —
is the language of a *fibrant double category*,
also known as proarrow equipment, or framed bicategory.

We understand this structure as a **logic**.

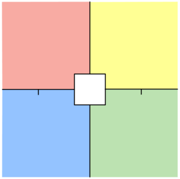


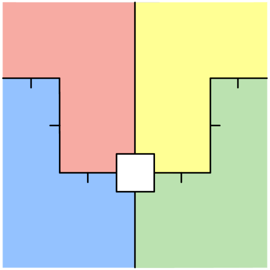
fib. dbl. cat.	dim.	**logic**
object	0	type
tight arrow	V	process
loose arrow	H	relation
square	2	inference

The discovery of logics

- Street, Wood et al. : formal CT $\sim$ proarrow equipment.
  - Bicategories: monads live in two dimensions
  - Equipments add *representation* $\mathbb{A}(f, 1)$ and $\mathbb{A}(1, f)$.
- Shulman: equipment $\sim$ fibrant double category.
  - Functions and relations each form a dimension.
  - Category of relations *varies* over the category of types.
- Myers: string diagrams for fibrant double categories.
  - Duality swaps focus from types to inferences.
  - Substitution and image are drawn simply as *bending*.

Logics





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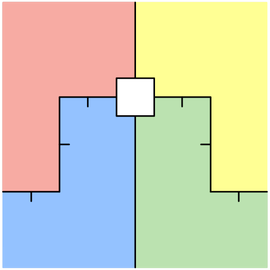
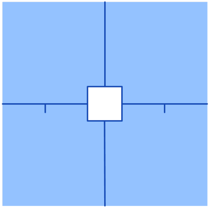


substitution

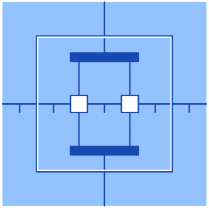
The metalogic of logics

Now, we construct the metalanguage of all logics.
The key is to see a logic as like a category.

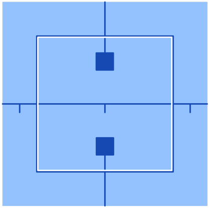
A category is a matrix with composition and identity;
a logic is a *matrix category* with composition and identity.



logic



composition



identity

Outline

Today, we develop the language of logics.

- A *span of categories* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$
$\sim$ a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$.
- The *weave double category* $\langle\mathbb{A}\rangle$ is the logic of morphisms and equations in $\mathbb{A}$; then $\mathbb{A} \leftarrow \langle\mathbb{A}\rangle \rightarrow \mathbb{A}$ can act on $\mathcal{R}$.
- A *matrix category* is a bimodule of weave double categories.

A *logic* is a matrix category $\underline{\mathbb{C}} \leftarrow \mathbb{C} \rightarrow \underline{\mathbb{C}}$ with comp. and identity.

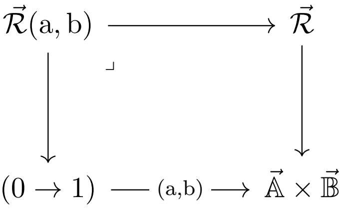
The construction

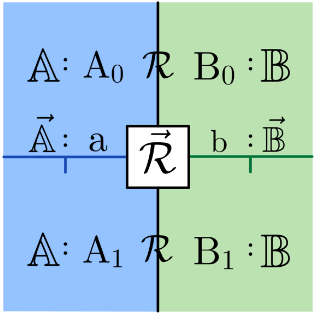
	$\mathrm{Mat}\mathbb{C}\mathrm{at}$	$\mathrm{H.PsMnd}(-)$	$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$	Logic
0	category	(H)-pseudomonad	bifibrant double category	logic
V	profunctor	(H)-vertical monad	vertical profunctor	meta process
H	matrix category	(H)-pseudobimodule	horizontal profunctor	meta relation
VH	matrix profunctor	(H)-vertical bimodule	double profunctor	meta inference
T	functor	ps. mnd. morphism	double functor	flow type
TV	transformation	v. mnd. morphism	vertical transformation	flow process
TH	matrix functor	ps. bim. morphism	horizontal transformation	flow relation
TVH	matrix transform.	v. bim. morphism	double transformation	flow inference

Spans of categories

The basic data of a logic is a span of categories:
relations and inferences, over pairs of types and processes.

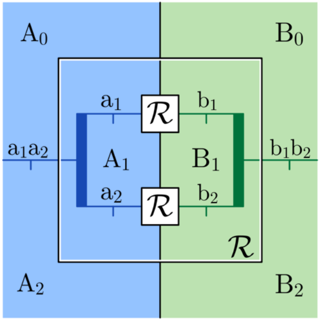
A span of categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B} \;\sim\;$ a matrix of categories:
a **displayed category** is a normal lax functor $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$.





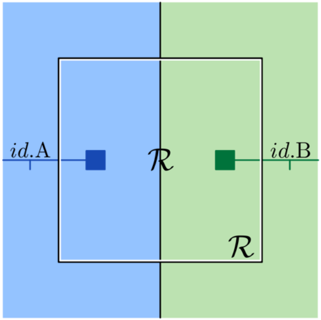
$\vec{\mathcal{R}}(a, b) : \mathcal{R}(A_0, B_0) \mid \mathcal{R}(A_1, B_1)$

Spans of categories



composition

$\vec{\mathcal{R}}(a_1, b_1) \circ \vec{\mathcal{R}}(a_2, b_2) \Rightarrow \vec{\mathcal{R}}(a_1a_2, b_1b_2)$

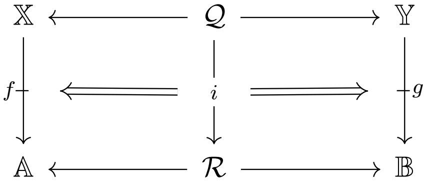


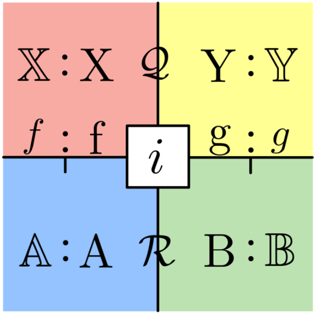
identity

$\mathcal{R}(A, B) \Rightarrow \vec{\mathcal{R}}(id.A, id.B)$

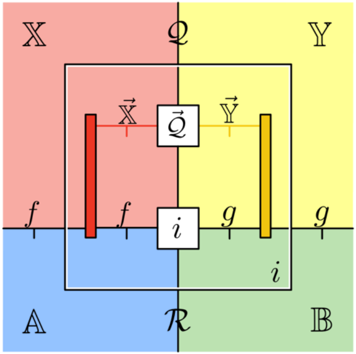
Spans of profunctors

This idea generalizes to spans of profunctors $f \leftarrow i \rightarrow g$.
A **displayed profunctor** is a map $i(f,g) : \mathrm{Prof}$ which forms a bimodule of lax functors $\mathcal{Q}(\mathbb{X},\mathbb{Y})$ and $\mathcal{R}(\mathbb{A},\mathbb{B})$.



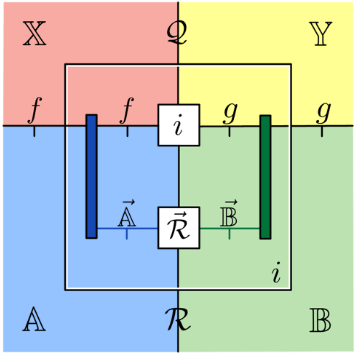


$i(\mathrm{f},\mathrm{g}) : \mathcal{Q}(\mathrm{X},\mathrm{Y}) \mid \mathcal{R}(\mathrm{A},\mathrm{B})$



precomposition

$\vec{\mathcal{Q}}(x, y) \circ i(f, g) \Rightarrow i(xf, yg)$



postcomposition

$i(f, g) \circ \vec{\mathcal{R}}(a, b) \Rightarrow i(fa, gb)$

Equivalence: spans are matrices

Inverse image is functorial, defining “displayed functors” and “displayed transformations”.

Theorem

The double category of span categories is equivalent to that of displayed categories.

	$\mathrm{Span}\mathbb{C}\mathrm{at}$	$\simeq$	$\mathrm{Dis}\mathbb{C}\mathrm{at}$
0	*span cat.* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$	$\sim$	*dis. cat.* $\mathcal{R}(\mathrm{A}, \mathrm{B}) : \mathrm{Cat}$
V	*span fun.* $[\![\mathcal{R}]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$	$\sim$	*dis. fun.* $[\![\mathcal{R}]\!] : \mathcal{R}_0(\mathrm{A}_0, \mathrm{B}_0) \rightarrow \mathcal{R}_1([\![\mathrm{A}_0]\!], [\![\mathrm{B}_0]\!])$
H	*span prof.* $f \leftarrow i \rightarrow g$	$\sim$	*dis. prof.* $i(f, g) : \mathrm{Prof}$
2	*span trans.* $[\![i]\!] : i_0 \rightarrow i_1$	$\sim$	*dis. trans.* $[\![i]\!] : i_0(\mathrm{f}_0, \mathrm{g}_0) \Rightarrow i_1([\![\mathrm{f}_0]\!], [\![\mathrm{g}_0]\!])$

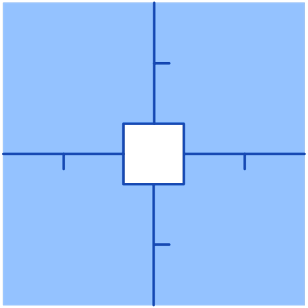
Arrow double categories

If $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is to be *relations* from $\mathbb{A}$ to $\mathbb{B}$, then relations should *vary* over processes in $\mathbb{A}$ and $\mathbb{B}$.

The **arrow double category** $\overrightarrow{\mathbb{A}}$ is that of commuting squares.

$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{\mathrm{a}}_0^1} & \mathrm{A}_0^1 \\ {\scriptstyle \mathrm{a}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_1^0 & \xrightarrow{\hat{\mathrm{a}}_1^1} & \mathrm{A}_1^1 \end{array}$$

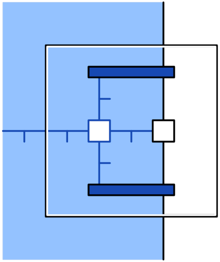
$$(\mathrm{a}_0, \mathrm{a}_1) : \overrightarrow{\mathbb{A}}(\hat{\mathrm{a}}_0^1 \rightarrow \hat{\mathrm{a}}_1^1)$$



Now $\mathbb{A} \leftarrow \overrightarrow{\mathbb{A}} \rightarrow \mathbb{A}$ and $\mathbb{B} \leftarrow \overrightarrow{\mathbb{B}} \rightarrow \mathbb{B}$ can *act* on $\mathcal{R}$.

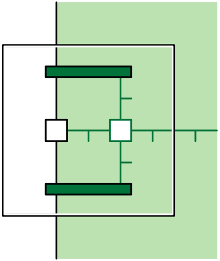
Fibered and opfibered categories

A **fibered category** over $\mathbb{A}$ is a left $\overrightarrow{\mathbb{A}}$-module.
An **opfibered category** over $\mathbb{B}$ is a right $\overrightarrow{\mathbb{B}}$-module.



substitution

$\odot : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1) \to \mathcal{R}(\mathrm{A}_0)$



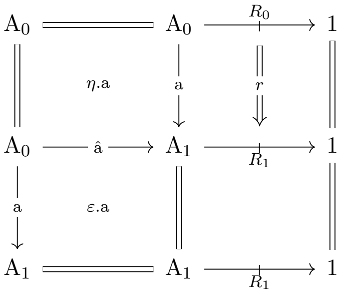
image

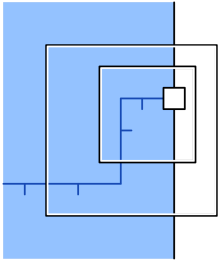
$\odot : \mathcal{R}(\mathrm{B}_0) \times \overrightarrow{\mathbb{B}}(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{B}_1)$

These are often denoted $\mathrm{a}^* R$ "pullback" and $\mathrm{b}_! R$ "pushforward".

Fibered and opfibered categories

In a fibered category $\mathcal{R}$ over $\mathbb{A}$, a morphism $r : R_0 \to R_1$ over $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1)$ is equivalent to $\eta.\mathrm{a} \circ r : R_0 \to \hat{\mathrm{a}} \odot R_1$ over $\mathrm{id}.\mathrm{A}_0$, by factoring through the **cartesian** morphism $\varepsilon.\mathrm{a} \circ \mathrm{id}.R_1$.



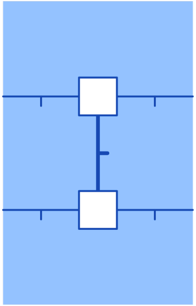


This gives a contravariant representation of morphisms over a.

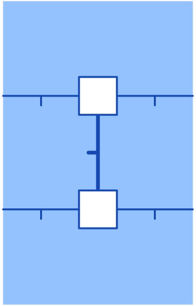
$$\vec{\mathcal{R}}(\mathrm{a})(R_0, R_1) \cong \mathcal{R}(R_0, \mathrm{a} \odot R_1)$$

Weave double category

Yet an arrow double category is not a *logic*.
There is a limitation to the equational reasoning of $\overrightarrow{\mathbb{A}}$.



$(\mathrm{a}_0, \mathrm{a}_1 \cdot \mathrm{a}_2) = (\mathrm{a}_0 \cdot \mathrm{a}_1, \mathrm{a}_2)$

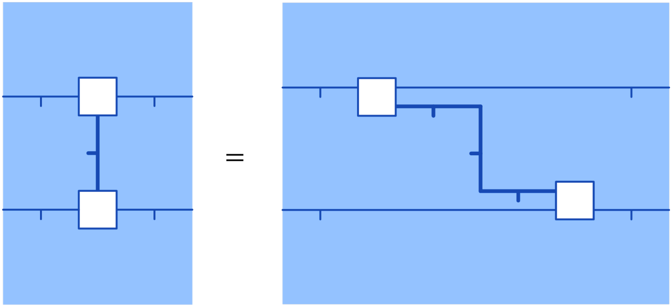


$(\mathrm{a}_0 \cdot \mathrm{a}_1, \mathrm{a}_2) = (\mathrm{a}_0, \mathrm{a}_1 \cdot \mathrm{a}_2)$

Composable pairs are only defined *up to associativity*.

Weave double category

The latter cannot be expressed in the arrow double category.



So, we define the *weave double category*:
the union of the arrow double category $\overrightarrow{\mathbb{A}}$ with its opposite $\overleftarrow{\mathbb{A}}$.

Weave double category

Let $\mathbb{A}$ be a category, with arrow double category $\overrightarrow{\mathbb{A}}$.
The **op-arrow double category** $\overleftarrow{\mathbb{A}}$ is the horizontal opposite.

$$\overleftarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \equiv \overrightarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$$

Denote an **arrow** $\hat{\mathrm{a}} : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1)$, and an **op-arrow** $\check{\mathrm{a}} : \overleftarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$.

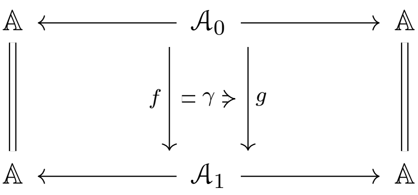
$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{\mathrm{a}}_1^0} & \mathrm{A}_1^0 \\ {\scriptstyle \mathrm{a}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_0^1 & \xrightarrow[\hat{\mathrm{a}}_1^1]{} & \mathrm{A}_1^1 \end{array} \qquad \begin{array}{ccc} \mathrm{A}_0^0 & \xleftarrow{\check{\mathrm{a}}_1^0} & \mathrm{A}_1^0 \\ {\scriptstyle \mathrm{a}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_0^1 & \xleftarrow[\check{\mathrm{a}}_1^1]{} & \mathrm{A}_1^1 \end{array}$$

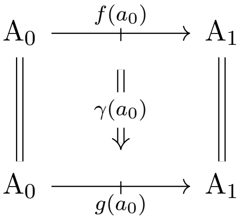
We use $\bar{\mathrm{a}}$ for objects of $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$.

Weave double category

Define $\mathrm{Dbl}_{\mathbb{A}}$ be the 2-category of double categories on $\mathbb{A}$, double functors over $\mathrm{id}.\mathbb{A}$, and identity-component transformations.

Given double categories $\mathcal{A}_0$ and $\mathcal{A}_1$ on $\mathbb{A}$, and double functors $f, g : \mathcal{A}_0 \to \mathcal{A}_1$ over $\mathrm{id}.\mathbb{A}$, an icon $\gamma : f \Rightarrow g$ gives for each $a_0 : \mathcal{A}_0$ a 2-morphism $\gamma(a_0) : f(a_0) \Rightarrow g(a_0)$, subject to naturality.





Weave double category

Let $\mathbb{A}$ be a category. The **weave double category** $\langle\mathbb{A}\rangle$ is the coproduct of the arrow and op-arrow double categories in $\mathrm{Dbl}_{\mathbb{A}}$.

$$\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$$

$\langle\mathbb{A}\rangle$ is generated by squares of $\overrightarrow{\mathbb{A}}$, opsquares of $\overleftarrow{\mathbb{A}}$, and isomorphisms of identity arrows and op-arrows.

$$\hat{\mathrm{id}}.A \cong \check{\mathrm{id}}.A$$

Theorem

$\langle\mathbb{A}\rangle$ is a logic.

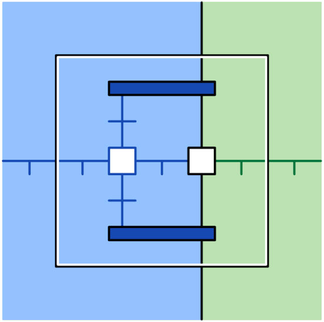
Theorem

$\langle\mathbb{A}\rangle$-modules are bifibered categories over $\mathbb{A}$.

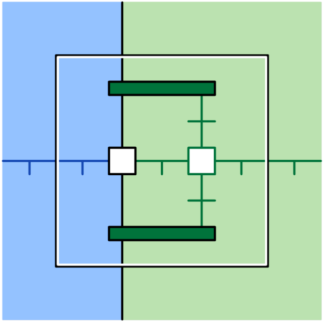
Matrix categories

Let $\mathbb{A}, \mathbb{B}$ be categories, with weave double categories $\langle\mathbb{A}\rangle, \langle\mathbb{B}\rangle$.

A **matrix category** or **two-sided bifibration** $\mathcal{R} : \mathbb{A} \| \mathbb{B}$ is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ which is a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$.



$\odot_{\mathbb{A}} : \langle\mathbb{A}\rangle(A_0, A_1) \times \mathcal{R}(A_1, B) \rightarrow \mathcal{R}(A_0, B)$

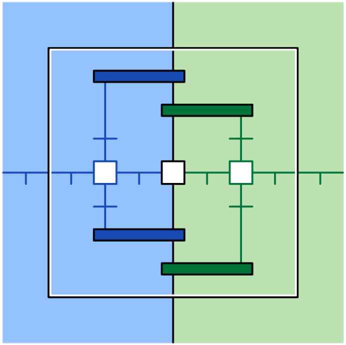


$\odot_{\mathbb{B}} : \mathcal{R}(A, B_0) \times \langle\mathbb{B}\rangle(B_0, B_1) \rightarrow \mathcal{R}(A, B_1)$

Matrix categories

The actions of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ on $\mathcal{R}$ are associative and unital up to coherent isomorphism.

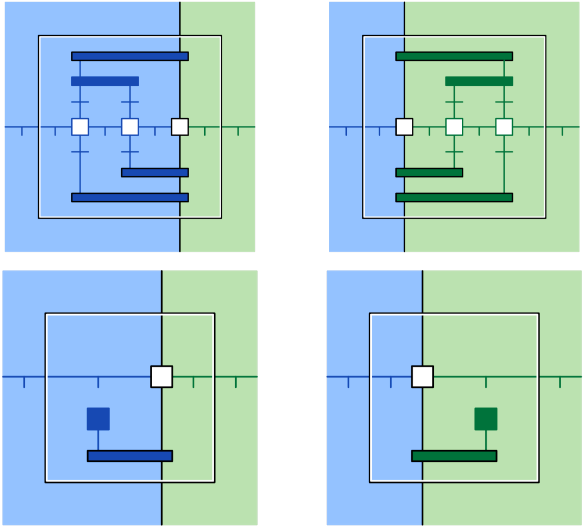
$$\begin{array}{ccc} \langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle & \xrightarrow{\langle\mathbb{A}\rangle * \odot_{\mathbb{B}}} & \langle\mathbb{A}\rangle * \mathcal{R} \\ \downarrow{\scriptstyle \odot_{\mathbb{A}} * \langle\mathbb{B}\rangle} & \Downarrow{\scriptstyle \alpha_{\mathcal{R}}} & \downarrow{\scriptstyle \odot_{\mathbb{A}}} \\ \mathcal{R} * \langle\mathbb{B}\rangle & \xrightarrow{\odot_{\mathbb{B}}} & \mathcal{R} \end{array}$$



center associator

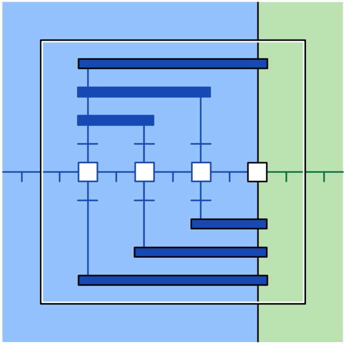
$$\alpha_{\mathcal{R}} : \ \bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) \cong (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$$

Matrix categories

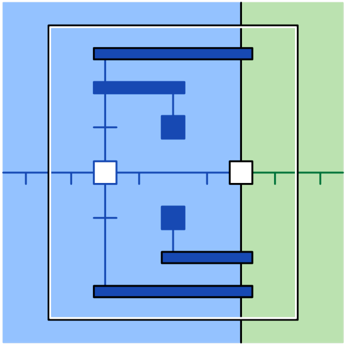


Matrix categories

The coherence means that reassociating a composite is well-defined, and reassociating a unit is well-defined.



$$(\langle \bar{a}_k \rangle \circ \langle \bar{a}_\ell \rangle \circ \langle \bar{a}_m \rangle) \odot R \Rrightarrow \langle \bar{a}_k \rangle \odot (\langle \bar{a}_\ell \rangle \odot (\langle \bar{a}_m \rangle \odot R))$$



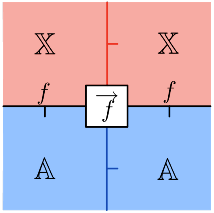
$$(\langle \bar{a}_k \rangle \circ \mathrm{id.A}) \odot R \Rrightarrow \langle \bar{a}_k \rangle \odot (\mathrm{id.A} \odot R)$$

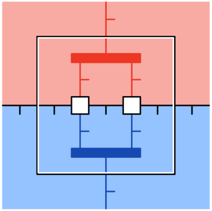
Matrix profunctors

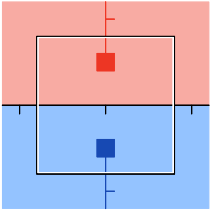
We now define relations of matrix categories.

Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor; then the **arrow profunctor** of arrow categories $\overrightarrow{f} : \overrightarrow{\mathbb{X}} \mid \overrightarrow{\mathbb{A}}$ consists of commutative squares; its projections form a span profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$.

$$\overrightarrow{f}(\hat{\mathrm{x}}, \hat{\mathrm{a}}) = \{(\mathrm{f}_0 : f(\mathrm{X}_0, \mathrm{A}_0), \mathrm{f}_1 : f(\mathrm{X}_1, \mathrm{A}_1)) \mid \mathrm{a} \cdot \mathrm{f}_0 = \mathrm{f}_1 \cdot \mathrm{x}\}$$







This forms a *vertical profunctor* of arrow double categories.

Matrix profunctors

Dually, the **op-arrow profunctor** of f is the profunctor of op-arrow categories $\overleftarrow{f} : \overleftarrow{\mathbb{X}} \mid \overleftarrow{\mathbb{A}}$.

$$\overleftarrow{f}(\check{\mathrm{x}}, \check{\mathrm{a}}) = \{\mathrm{f}_0 : f(\mathrm{X}_0, \mathrm{A}_0), \mathrm{f}_1 : f(\mathrm{X}_1, \mathrm{A}_1) \mid \mathrm{x} \cdot \mathrm{f}_0 = \mathrm{f}_1 \cdot \mathrm{a}\}$$

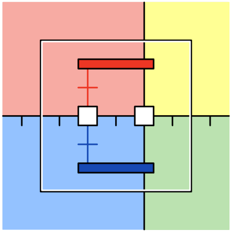
The **weave vertical profunctor** of weave double categories $\langle f \rangle : \langle \mathbb{X} \rangle \mid \langle \mathbb{A} \rangle$ is the coproduct of $\overrightarrow{f}$ and $\overleftarrow{f}$ in the category of vertical profunctors over f.

Just like the weave double category, this is generated from squares and opsquares in f, plus the actions of $\mathbb{X}$ and $\mathbb{A}$, subject to naturality with respect to the isomorphisms $\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$.

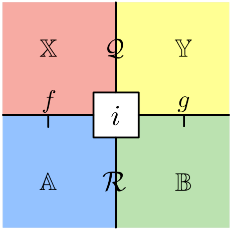
Matrix profunctors

Let $\mathcal{Q}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{R}(\mathbb{A}, \mathbb{B})$ be matrix categories.
Let $f : \mathbb{X} \,|\, \mathbb{A}$ and $g : \mathbb{Y} \,|\, \mathbb{B}$ be profunctors,
with weave profunctors $f \leftarrow \langle f \rangle \rightarrow f$ and $g \leftarrow \langle g \rangle \rightarrow g$.

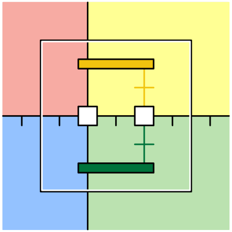
A **matrix profunctor** $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a span profunctor which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$, coherent with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$.



$\odot_f : \langle f \rangle * i \rightarrow i$



$i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A}, \mathbb{B})$

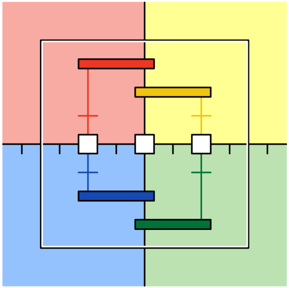


$\odot_g : i * \langle g \rangle \rightarrow i$

Matrix profunctors

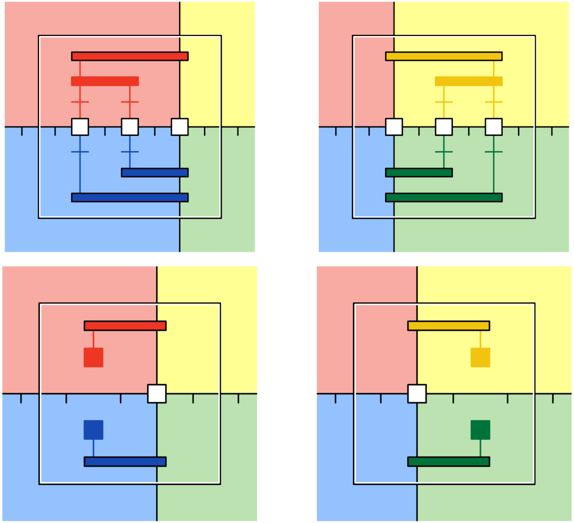
The matrix profunctor $i(f, g)$ is a relation of matrix categories $\mathcal{Q}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{R}(\mathbb{A}, \mathbb{B})$, so it coheres with associators and unitors.

$$\begin{array}{ccc}
\bar{\mathrm{x}} \odot (Q \odot \bar{\mathrm{y}}) & \xrightarrow{\alpha_{\mathcal{Q}}} & (\bar{\mathrm{x}} \odot Q) \odot \bar{\mathrm{y}} \\
\downarrow{\scriptstyle \mathrm{f}\odot(i\odot\mathrm{g})} & & \downarrow{\scriptstyle (\mathrm{f}\odot i)\odot\mathrm{g}} \\
\bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) & \xrightarrow[\alpha_{\mathcal{R}}]{} & (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}
\end{array}$$



associator coherence

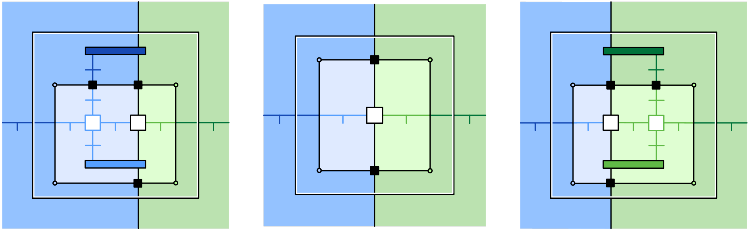
Matrix profunctors



Matrix functors and transformations

Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $\mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ and $\mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix categories.

A **matrix functor** $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ is a morphism of bimodules, preserving composition and identity up to coherent isos.



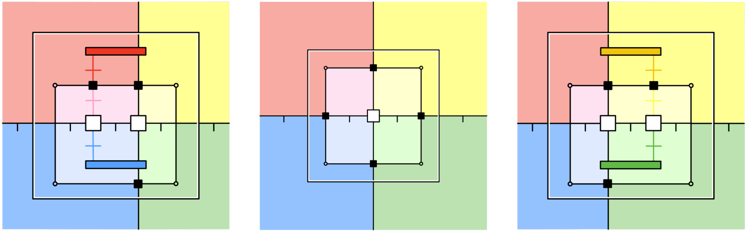
left join **right join**

$[\![\langle \bar{a}_k \rangle]\!] \odot_1 [\![R]\!] \cong [\![\langle \bar{a}_k \rangle \odot_0 R]\!]$ $[\![R]\!] \odot_1 [\![\langle \bar{b}_\ell \rangle]\!] \cong [\![R \odot_0 \langle \bar{b}_\ell \rangle]\!]$

Matrix functors and transformations

Let $[\![\mathcal{Q}]\!](\mathbb{X}, \mathbb{Y})$ and $[\![\mathcal{R}]\!](\mathbb{A}, \mathbb{B})$ be matrix functors, and let $i_0(f_0, g_0) : \mathcal{Q}_0 \mid \mathcal{R}_0$ and $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be matrix profunctors.

A **matrix transformation** $[\![i]\!] : i_0 \to i_1$ is a span transformation which coheres with the left and right joins of $[\![\mathcal{Q}]\!]$ and $[\![\mathcal{R}]\!]$.

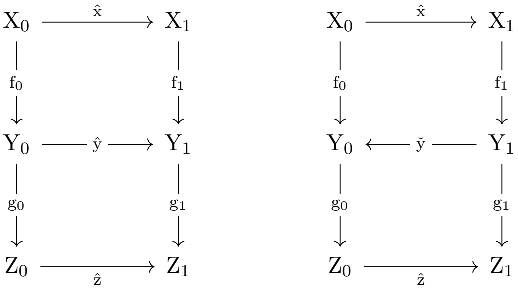


$[\![\mathrm{x}]\!] \odot [\![Q]\!] \rightrightarrows [\![\mathrm{a} \odot R]\!]$ $\qquad$ $[\![Q]\!] \odot [\![\mathrm{y}]\!] \rightrightarrows [\![R \odot \mathrm{b}]\!]$

Sequential composition

We now see how matrix categories and functors,
matrix profunctors and transformations form a *logic*.

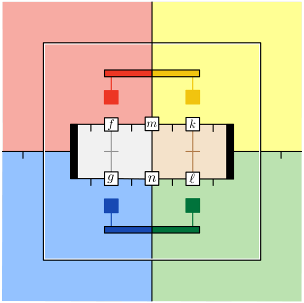
How do we compose matrix profunctors? By using *weaves*.

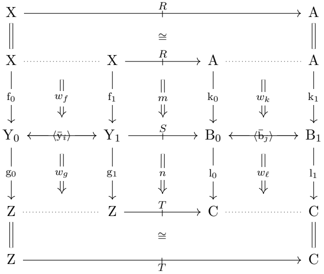


Both squares of $\langle f \circ g \rangle$ can be expressed in $\langle f \rangle \circ \langle g \rangle$ —
so an action by $\langle f \rangle$ and one by $\langle g \rangle$ defines an action by $\langle f \circ g \rangle$.

Sequential composition

So, we ensure the actions are *well-defined* on the *identities*, associativity zig-zags in $\langle f \circ g \rangle$ and $\langle k \circ \ell \rangle$: so to compose $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$, we quotient $m \circ n$ by their actions.





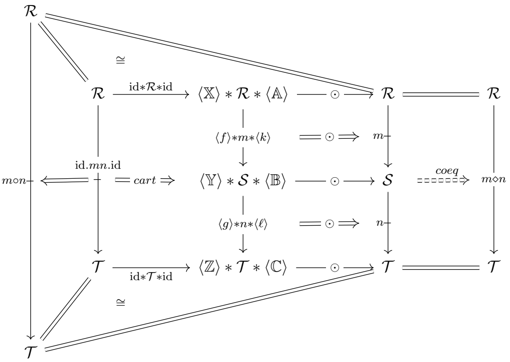
$$[S.(m,n)] \;\equiv\; [u_{\mathcal{R}} \cdot (\langle \bar{y}_i \rangle \odot S \odot \langle \bar{b}_j \rangle).(w_f \odot m \odot w_k,\; w_g \odot n \odot w_\ell) \cdot u_{\mathcal{T}}^{-1}]$$

Sequential composition

Let $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ be matrix profunctors. The **sequential composite**

$$(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$$

is the following coequalizer.



The logic of matrix categories

Theorem

Matrix categories form a logic.

Proof.

As sequential composition of matrix profunctors is defined by coequalizer, it is canonically functorial. The associator and unitors are inherited from $\mathrm{Span}\mathbb{C}\mathrm{at}$, because the coequalizer is orthogonal to span profunctor composition.

Hence $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a double category. Moreover it is a logic: substitution of matrix functors in matrix profunctors is exactly analogous to that of functors in profunctors, in $\mathbb{C}\mathrm{at}$. □

The logic of matrix categories

A **double fibration** is a category in the 2-category of fibrations.

$$\begin{array}{ccccc} \mathrm{MatCat} & \longleftarrow & \mathrm{MatProf} & \longrightarrow & \mathrm{MatCat} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cat} \times \mathrm{Cat} & \longleftarrow & \mathrm{Prof} \times \mathrm{Prof} & \longrightarrow & \mathrm{Cat} \times \mathrm{Cat} \end{array}$$

Theorem

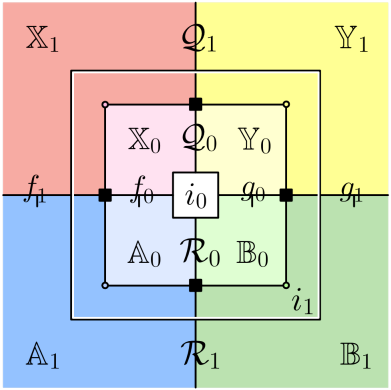
Matrix categories are fibered over pairs of categories.

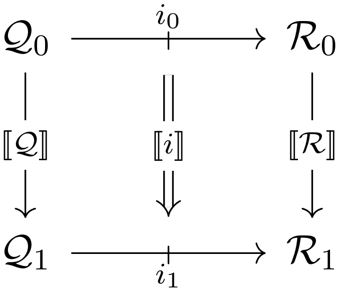
Proof.

Substitution of functors in matrix categories, and transformations in matrix profunctors, is defined by pullback. Sequential composition of matrix profunctors preserves substitution, by the coequalizer. □

The logic of matrix categories

This is the logic of matrix categories, over pairs of categories.





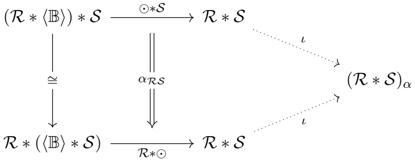
$\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$

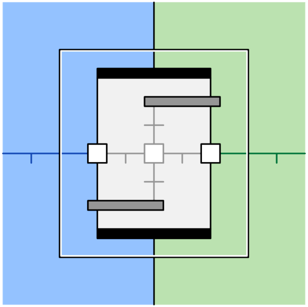
Now, we define *parallel composition* of matrix categories.

Parallel composition

Now, we define composition of matrix categories.

Let $\mathcal{R} : \mathbb{A} \,||\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,||\, \mathbb{C}$ be matrix categories.
The **parallel composite** $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,||\, \mathbb{C}$ is constructed as follows.
On $\mathbb{A} \leftarrow \mathcal{R} * \mathcal{S} \rightarrow \mathbb{C}$ we form the *iso-coinserter* of actions by $\langle \mathbb{B} \rangle$.

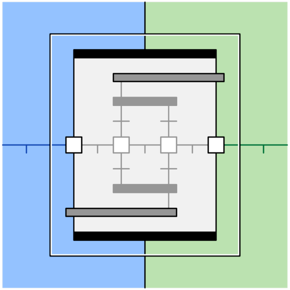




This adjoins an associator $\alpha_{\mathcal{R}\mathcal{S}} : \mathrm{B}_0.(R, \bar{\mathrm{b}} \odot S) \;\cong\; \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, S)$.

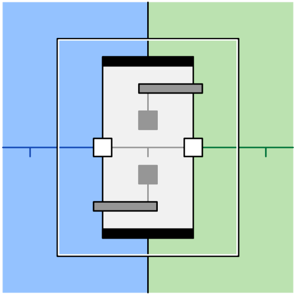
Parallel composition

On the associator, two equations are imposed by *coequifier*, for reassociating a composite and a unit.



associator coherence

$(R, \bar{\mathrm{b}}_1 \odot (\bar{\mathrm{b}}_2 \odot S))$
$\rightrightarrows ((R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2), S)$



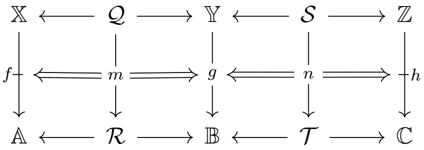
unitor coherence

$(R, \overline{\mathrm{id}}.\mathrm{B} \odot S)$
$\rightrightarrows (R \odot \overline{\mathrm{id}}.\mathrm{B}, S)$

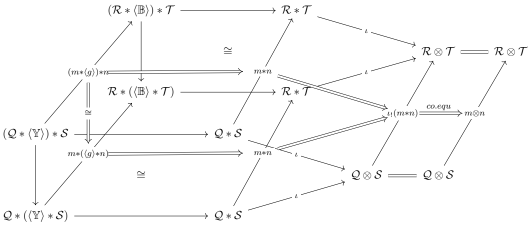
Hence $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,||\, \mathbb{C}$ is a *codescent object*.

Parallel composition

Let $m(f,g)$ and $n(g,h)$ be matrix profunctors.



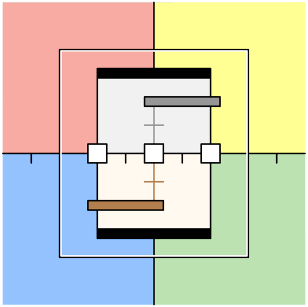
The **parallel composite** matrix profunctor $m \otimes n : \mathcal{Q} \otimes \mathcal{S} \,|\, \mathcal{R} \otimes \mathcal{T}$ is the following coequalizer.



Parallel composition

So the elements of $(m \otimes n)(f, h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X}, \mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A}, \mathbb{C})$ are composites of: morphisms $\mathrm{y}.(q, s)$, associators $\alpha_{\mathcal{QS}}$, elements $\mathrm{g}.(m, n)$, associators $\alpha_{\mathcal{RT}}$, and morphisms $\mathrm{b}.(r, t)$, such that for any $[\mathrm{g}_0, \mathrm{g}_1] : \langle g \rangle(\bar{\mathrm{y}}, \bar{\mathrm{b}})$ and $m : m(\mathrm{f}, \mathrm{g}_0)$, $n : n(\mathrm{g}_1, \mathrm{h})$ the following commutes.

$$\begin{array}{ccc}
\mathrm{Y}_0.(Q, \bar{\mathrm{y}} \odot S) & \xrightarrow{\alpha_{\mathcal{QS}}} & \mathrm{Y}_1.(Q \odot \bar{\mathrm{y}}, S) \\
\Big\downarrow {\scriptstyle \mathrm{g}_0.(m, [\mathrm{g}_0, \mathrm{g}_1] \odot n)} & & \Big\downarrow {\scriptstyle \mathrm{g}_1.(m \odot [\mathrm{g}_0, \mathrm{g}_1], n)} \\
\mathrm{B}_0.(R, \bar{\mathrm{b}} \odot T) & \xrightarrow[\alpha_{\mathcal{RT}}]{} & \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, T)
\end{array}$$

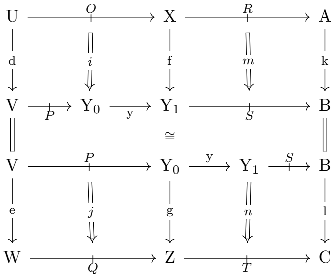


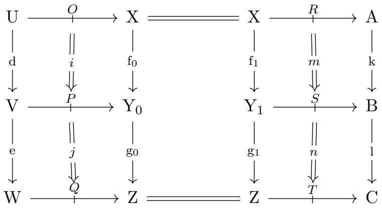
Parallel composition

Parallel composition does *not* preserve sequential composition.

$$(i \otimes m) \diamond (j \otimes n) \qquad \nleftrightarrow \qquad (i \diamond j) \otimes (m \diamond n)$$

Parallel composition *creates* an associator element, while sequential composition *equates* elements.





The metalogic of matrix categories

A **metalogic** is a logic $\mathbb{C}$ and a fibered logic $\mathbb{C} \leftarrow \mathbb{M} \to \mathbb{C}$ which forms an *intramonad* in $\mathrm{Span}(\mathrm{Span}\mathbb{C}\mathrm{at})$: analogous to an intermonad in an intercategory, but vertically 1-weak, horizontally 2-weak, and no interchange.

Theorem

Matrix categories form a metalogic.

$$\begin{array}{ccccc}
\mathrm{C} & \longleftarrow & \mathrm{MC} & \longrightarrow & \mathrm{C} \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{P} & \longleftarrow & \mathrm{MP} & \longrightarrow & \mathrm{P} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{C} & \longleftarrow & \mathrm{MC} & \longrightarrow & \mathrm{C}
\end{array}$$

This is a “bifibrant triple category” without interchange between the metalogical and logical parallel compositions.

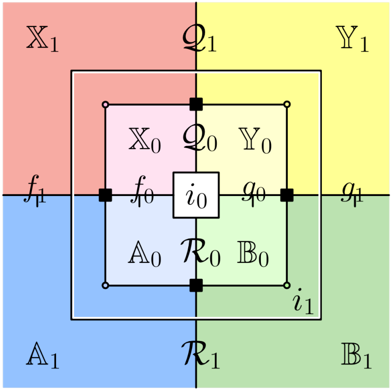
The metalogic of logics

A **logic** is a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

Theorem

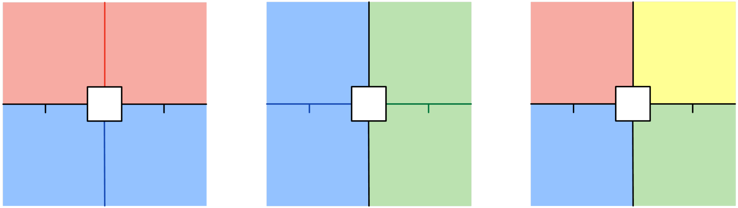
Logics form a metalogic.

$$\begin{array}{ccccc}
L & \longleftarrow & R & \longrightarrow & L \\
\uparrow & & \uparrow & & \uparrow \\
P & \longleftarrow & I & \longrightarrow & P \\
\downarrow & & \downarrow & & \downarrow \\
L & \longleftarrow & R & \longrightarrow & L
\end{array}$$



The metalogic of logics

There are two kinds of relations between logics.
a *vertical* profunctor consists of *processes* between logics, and
a *horizontal* profunctor consists of *relations* between logics.

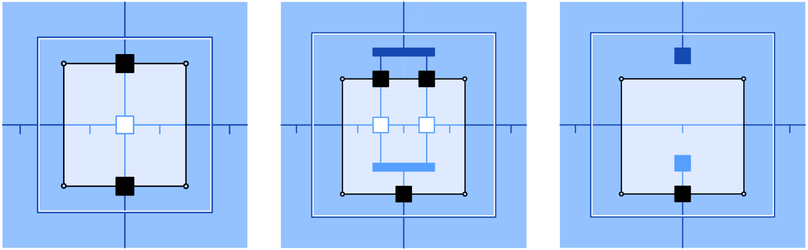


meta process (v-prof.) | meta relation (h-prof.) | meta inference (d-prof.)

Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.

The metalogic of logics

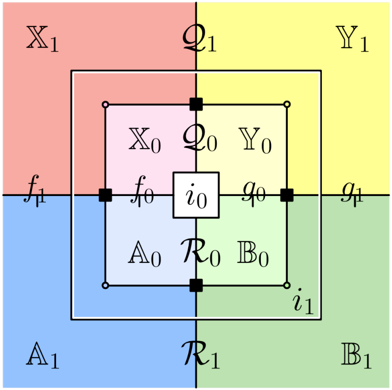
Logics have two kinds of relation, and one kind of function: a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to $\mathbb{A}_1$, preserving relation composition and unit up to coherent iso.



This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition and unit.

The metalogic of logics

All together, logics form a metalogic.



A cube is a double transformation, the fully general notion of what is known as a modification. We call it a *flow inference*.

Thank you.