Relevant Dialectica Categories

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MIT Category Theory

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Outline

1. Introduction
2. Why Relevance logic?
3. Dialectica Categories
4. Relevant Dialectica

Thanks to David and Brendan for the invitation!

Personal stories

Personal stories

I'm a logician, a proof-theorist, a computational linguist and a category theorist.

I have worked in the computer industry for the last 20 years

Personal stories

Introduction

Many people claim that *implication* in logic is counterintuitive.

The formulae below fail to be valid if we interpret $\rightarrow$ as our "natural" notion of logical implication:

$$(p \rightarrow q) \vee (q \rightarrow p)$$

$$(p \wedge \neg p) \rightarrow q$$

$$p \rightarrow (q \rightarrow q)$$

$$p \rightarrow (q \rightarrow p)$$

These laws (all classically valid) are unsettling: the antecedent seems irrelevant to the consequent in most.

Relevance Logic

Relevantists want to construct logics rejecting theses and arguments that commit 'fallacies of relevance'. (Mares on SEP 2012)
To 'stay on topic' need a *variable sharing principle*:
1. No formula $A \to B$ can be proven in relevance logic if A and B do not have at least one propositional variable in common.
2. No inference can be shown valid if the premises and conclusion do not share at least one propositional variable.

But the variable sharing principle is only a necessary condition a logic must satisfy. It is not sufficient: it does not give a criterion that eliminates all paradoxes and fallacies.
The whole system needs to satisfy some harmony principles too.

This Talk

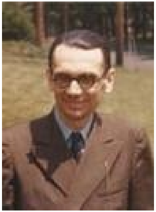
How to devise a relevant system you can live with

Use good proof theory $\Rightarrow$ Linear Logic, Girard.

Use (non-degenerated) categorical models

Add bits you think are necessary...

Gödel's Dialectica Interpretation



The interpretation is named after the Swiss journal Dialectica where it appeared in a special volume dedicated to Paul Bernays 70th birthday in 1958.

I was originally trying to provide an internal categorical model of the Dialectica Interpretation. The categories I came up with proved to be a model of Linear Logic...

Dialectica (from Wikipedia)

$A_D(x;y)$ quantifier-free formula defined inductively:

$$
\begin{array}{lcl}
(P)_D & \equiv & P \text{ (} P \text{ atomic)} \\
(A \wedge B)_D(x,v;y,w) & \equiv & A_D(x;y) \wedge B_D(v;w) \\
(A \vee B)_D(x,v,z;y,w) & \equiv & (z=0 \to A_D(x;y)) \wedge (z \neq 0 \to B_D(v;w)) \\
(A \to B)_D(f,g;x,w) & \equiv & A_D(x;fxw) \to B_D(gx;w) \\
(\exists z A)_D(x,z;y) & \equiv & A_D(x;y) \\
(\forall z A)_D(f;y,z) & \equiv & A_D(fz;y)
\end{array}
$$

Theorem (Dialectica Soundness, Gödel 1958)

Whenever a formula A is provable in Heyting arithmetic then there exists a sequence of closed terms t such that $A_D(t;y)$ is provable in system T. The sequence of terms t and the proof of $A_D(t;y)$ are constructed from the given proof of A in Heyting arithmetic.

Dialectica Categories

Gödel's Dialectica: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type T.

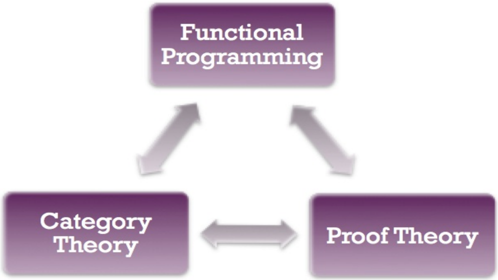
basic idea: translate every formula A of HA to $A^D = \exists u \forall x. A_D$, where A_D is quantifier-free.

Use: If HA proves A then system T proves $A_D(t, y)$ where y is string of variables for functionals of finite type, t a suitable sequence of terms not containing y

Goal: to be as constructive *as possible* while being able to interpret all of classical arithmetic (Troelstra)

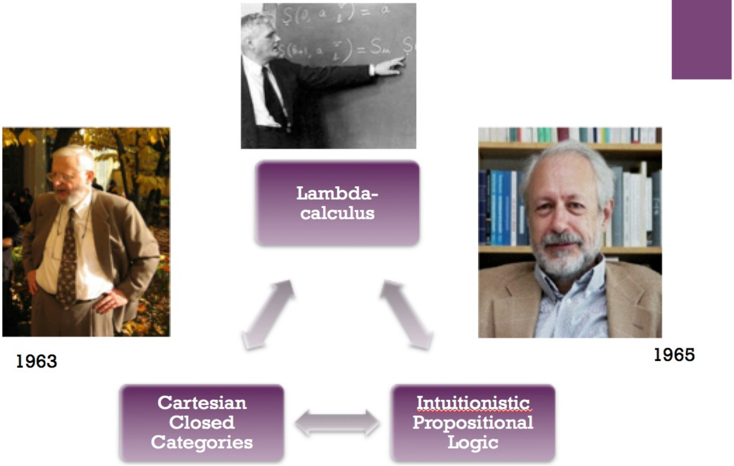
Philosophical discussion of how much it achieves $\Rightarrow$ another talk

Digression: Categorical Proof Theory



Types are formulae/objects in appropriate category,
Terms/programs are proofs/morphisms in the category,
Logical constructors are appropriate categorical constructions.
Most important: Reduction is proof normalization (Tait)
Outcome: Transfer results/tools from logic to CT to CSci

Curry-Howard Correspondence



Linear Logic

A proof theoretic logic described by Jean-Yves Girard in 1986.

Basic idea: assumptions cannot be discarded or duplicated. They must be used exactly once – just like dollar bills

Other approaches to accounting for logical resources before.

Great win of Linear Logic: Account for resources when you want to, otherwise fall back on traditional logic, $A \to B$ iff $!A \multimap B$

Dialectica Categories

Hyland suggested that to provide a categorical model of the Dialectica Interpretation, one should look at the functionals corresponding to the interpretation of logical implication.

I looked and instead of finding a cartesian closed category, found a monoidal closed one

Thus the categories in my thesis proved to be models of Linear Logic

Resources in Linear Logic

In Linear Logic formulas denote resources. Resources are premises, assumptions and conclusions, as they are used in logical proofs. For example:

$\$1 \multimap$ latte

If I have a dollar, I can get a Latte

$\$1 \multimap$ cappuccino

If I have a dollar, I can get a Cappuccino

$\$1$

I have a dollar

Using my dollar premise and one of the premisses above, say '$\$1 \multimap$ latte' gives me a latte but the dollar is gone
Usual logic doesn't pay attention to uses of premisses, A implies B and A gives me B but I still have A

Linear Implication and (Multiplicative) Conjunction

Traditional implication: $A, A \to B \vdash B$

$A, A \to B \vdash A \wedge B$ Re-use A

Linear implication: $A, A \multimap B \vdash B$

$A, A \multimap B \nvdash A \otimes B$ Cannot re-use A

Traditional conjunction: $A \wedge B \vdash A$ Discard B

Linear conjunction: $A \otimes B \nvdash A$ Cannot discard B

Of course: $!A \vdash !A \otimes !A$ Re-use

$!A \otimes B \vdash I \otimes B \cong B$ Discard

Challenges of modeling Linear Logic

Traditional categorical modeling of intuitionistic logic

formula $A \rightsquigarrow$ object A of appropriate category

$A \wedge B \rightsquigarrow A \times B$ (real product)

$A \to B \rightsquigarrow B^A$ (set of functions from A to B)

These are real products, so we have projections $(A \times B \to A, B)$ and diagonals $(A \to A \times A)$ which correspond to deletion and duplication of resources

Not linear!!!

Need to use *tensor products* and *internal homs* in CT

Hard: how to define the "make-everything-usual" operator "!"

How do we do it?

Thesis: The Dialectica Categories (1988)

a categorical (internal) version of the interpretation

Four chapters/Four main theorems:

- Dialectica category DC
- DC cofree monoidal comonad
- Categorical model GC
- GC composite (monoidal) comonad

Category DC

Start with a cat C that is cartesian closed (with some other nice properties) Then build a new category DC.

Objects are relations in C, i.e triples (U, X, α), $\alpha : U \times X \to 2$, so either $u\alpha x$ or not.

Maps are pairs of maps in C. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps in C, $(f : U \to V, F : U \times Y \to X)$ such that an 'semi-adjunction condition' is satisfied: for $u \in U, y \in Y$, $u\alpha F(u, y)$ implies $fu\beta y$. (**Note direction and dependence!**)

Theorem: (de Paiva 1987) [Linear structure]

The category DC has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

Dialectica Categories

Proposition

The data above does provide a category DC

Identities are identity on the first component and projection in the second component $id_{(U,X,\alpha)} = (id_U : U \to U, \pi_2 : U \times Y \to Y)$.
Given $(f, F) : (U, X, \alpha) \to (V, Y, \beta)$ and
$(g, G) : (V, Y, \beta) \to (W, Z, \gamma)$, composition is simple composition $f; g : U \to W$ in the first coordinate. But on the second coordinate
$U \times Z \xrightarrow{\Delta \times Z} U \times U \times Z \xrightarrow{U \times f \times Z} U \times V \times Z \xrightarrow{U \times G} U \times Y \xrightarrow{F} X$

Associativity and unity laws come from the base category C
Need to check the semi-adjunction conditions to get the proposition.

Can we give some intuition for these objects?

Blass makes the case for thinking of problems in computational complexity. Intuitively an object of DC

$$A = (U, X, \alpha)$$

can be seen as representing a problem.
The elements of U are instances of the problem, while the elements of X are possible answers to the problem instances.
The relation α checks whether the answer is correct for that instance of the problem or not.

Examples of objects in DC

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where n is related to m iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where f is α-related to n iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where r_1 and r_2 are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality inequality.

Tensor product in DC

Given objects (U, X, α) and (V, Y, β) it is natural to think of $(U \times V, X \times Y, \alpha \times \beta)$ as a tensor product.

This construction does give us a bifunctor

$$\otimes \colon DC \times DC \to DC$$

with a unit $I = (1, 1, id_1)$.

Note that this is not a product.
There are no projections $(U \times V, X \times Y, \alpha \times \beta) \to (U, X, \alpha)$.
Nor do we have a diagonal functor $\Delta \colon DC \to DC \times DC$, taking $(U, X, \alpha) \to (U \times U, X \times X, \alpha \times \alpha)$

Internal-hom in DC

To “internalize" the notion of map between problems, we need to consider the collection of all maps from U to V, V^U, the collection of all maps from $U \times Y$ to X, $X^{U \times Y}$ and we need to make sure that a pair $f \colon U \to V$ and $F \colon U \times Y \to X$ in that set, satisfies the dialectica condition:

$$\forall u \in U, y \in Y, u\alpha F(u,y) \to fu\beta y$$

This give us an object in DC $(V^U \times X^{U \times Y}, U \times Y, \beta^\alpha)$
The relation $\beta^\alpha \colon V^U \times X^{U \times Y} \times (U \times Y) \to 2$ evaluates a pair (f, F) of maps on the pair of elements (u, y) and checks the dialectica implication between the relations.

Internal-hom in DC

Given objects (U, X, α) and (V, Y, β) we can internalize the notion of morphism of DC as the object $(V^U \times X^{U \times Y}, U \times Y, \beta^\alpha)$

This construction does give us a bifunctor

$$* \multimap *\colon DC \times DC \to DC$$

This bifunctor is contravariant in the first coordinate and covariant in the second, as expected

The kernel of our first main theorem is the adjunction

$$A \otimes B \to C \text{ if and only if } A \to [B \multimap C]$$

where $A = (U, X, \alpha), B = (V, Y, \beta)$ and $C = (W, Z, \gamma)$

Products and Coproducts in DC

Given objects (U, X, α) and (V, Y, β) it is natural to think of $(U \times V, X + Y, \alpha \circ \beta)$ as a categorical product in DC.

Since this is a relation on the set $U \times V \times (X + Y)$, either this relation has a $(x, 0)$ or a $(y, 1)$ element and hence the $\circ$ symbol only 'picks' the correct relation α or β.

We can guess a dual construction. However this does not work as a coproduct. It is only a **weak-coproduct**

Theorem: (de Paiva 1987) [Linear structure]

The category DC has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

What about the Modality?

We need an operation on objects/propositions such that:

$!A \to !A \otimes !A$ (duplication)

$!A \to I$ (erasing)

$!A \to A$ (dereliction)

$!A \to !!A$ (digging)

Also ! should be a functor, i.e $(f, F) : A \to B$ then $!(f, F) : !A \to !B$

Theorem: linear and usual logic together

There is a **monoidal** comonad ! in DC which models exponentials/modalities and recovers Intuitionistic (and Classical) Logic.

Take $!(U, X, \alpha) = (U, X^*, \alpha*)$, where $(-)^*$ is the free commutative monoid in C.

(Cofree) Modality !

To show this works we need to show several propositions:

! is a monoidal comonad: there is a natural transformation $m_(-,-) :!A\otimes !B \to !(A \otimes B)$ and $m_I : I \to !I$ satifying many comm diagrams

! induces a commutative comonoid structure on $!A$

$!A$ also has naturally a coalgebra structure induced by the comonad !

The comonoid and coalgebra structures interact in a nice way.

There are plenty of other ways to phrase these conditions. The more usual way seems to be

Theorem: Linear and non-Linear logic together

There is a **monoidal** adjunction between DC and its **cofree** coKleisli category for the monoidal comonad ! above.

Cofree Modality !

Old way: "There is a monoidal comonad ! on a linear category DC satisfying (lots of conditions)" and

Theorem: Linear and non-Linear logic together

The coKleisli category associated with the comonad ! on DC is cartesian closed.

To show cartesian closedness we need to show:

$$Hom_{Kl!}(A \& B, C) \cong Hom_{Kl!}(A, [B, C]_{Kl!})$$

The proof is then a series of equivalences that were proved before:
$Hom_{Kl!}(A\&B, C) \cong Hom_{DC}(!(A\&B), C) \cong$
$Hom_{DC}(!A \otimes !B, C) \cong Hom_{DC}(!A, [!B, C]_{DC}) \cong$
$Hom_{kl!}(A, [!B, C]_{DC}) \cong Hom_{kl!}(A, [B, C]_{kl!})$
(Seely, 1989; section 2.5 of thesis)

Dialectica Category GC

Girard's sugestion in Boulder 1987: objects of GC are triples, a generic object is $A = (U, X, \alpha)$, where U and X are sets and $\alpha \subseteq U \times X$ is a relation. (Continue to think of C as Sets!). A morphism from A to $B = (V, Y, \beta)$ is a pair of functions $f\colon U \to V$ and $F\colon Y \to X$ such that $u\alpha Fy \to fu\beta y$. (**Simplified maps!**)

Theorem (de Paiva 1989): Linear Structure

The category GC has a symmetric monoidal closed structure, and products and co-products that make it a model of FILL/CLL without modalities.

Girard says this category should be related to Henkin Quantifiers.

Internal-hom in GC

As before we "internalize" the notion of map between objects, considering the collection of all maps from U to V, V^U, the collection of all maps from Y to X, X^Y and we make sure that a pair $f\colon U \to V$ and $F\colon Y \to X$ in that set, satisfies the dialectica condition:

$$\forall u \in U, y \in Y, u\alpha Fy \to fu\beta y$$

This give us an object $(V^U \times X^Y, U \times Y, \beta^\alpha)$
The relation $\beta^\alpha\colon V^U \times X^Y \times (U \times Y) \to 2$ evaluates the pair (f, F) on the pair (u, y) and checks that the dialectica implication between relations holds.

Proposition

The data above does provide an internal-hom in the category GC

Tensor product in GC

While the internal-hom in GC is simpler than the one in DC (after all the morphisms are simpler), the opposite is the case for the tensor product.

Given objects (U, X, α) and (V, Y, β), their GC tensor product is $(U \times V, X^V \times Y^U, \alpha \otimes \beta)$ where the relation $\alpha \otimes \beta \colon U \times V \times X^V \times Y^U \to 2$ evaluates the pair (u, v) with the pair (h_1, h_2) and checks that the dialectica tensor between the relations holds.

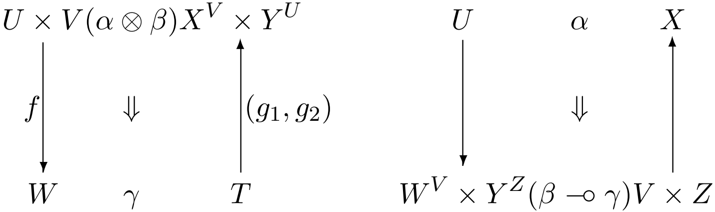
Proposition

The data above does provide an tensor product in the category GC

The Right Structure

Because it's fun, let us calculate the "reverse engineering" necessary for a model of Linear Logic

$$A \otimes B \to C \text{ if and only if } A \to [B \multimap C]$$



Dialectica Category GC

Besides **simplified maps** the whole construction is more symmetric. Problems we had before with a weak-coproduct disappear. However, because of the intuitionistic implication relating relations, morphisms are still unidirectional.

In particular the linear negation $A^{\perp}$ still does not satisfy $A^{\perp^{\perp}} \cong A$
This led to Full Intuitionistic Linear Logic (FILL), Hyland and de Paiva, 1993 – another talk!

Theorem (de Paiva 1989): Linear Structure

The category GC has a symmetric monoidal closed structure, and products and coproducts that make it a model of FILL/CLL without modalities.

The bang operator in GC

As before we want a monoidal comonad that allows us to get back to traditional logic, i.e to a cartesian closed cat.
The previous comonad $!A$ does not work.
We need commutative comonoids wrt the new tensor product, which is complicated.
We can define one comonad in GC that deals with the comonoid structure and one comonad that deals with the coalgebra structure.
The good news is that we can compose them, unlike generic comonads.
They are related by a distributive law and plenty of calculations gets us there

The bang operator !

Take $\mathrm{bang}(U, X, \alpha) = (U, (X^*)^U, (\alpha *)^U)$, where $(-)^*$ is the free commutative monoid in C and $(-)^U : C \to C$ is a monad in C that induces a comonad in GC.

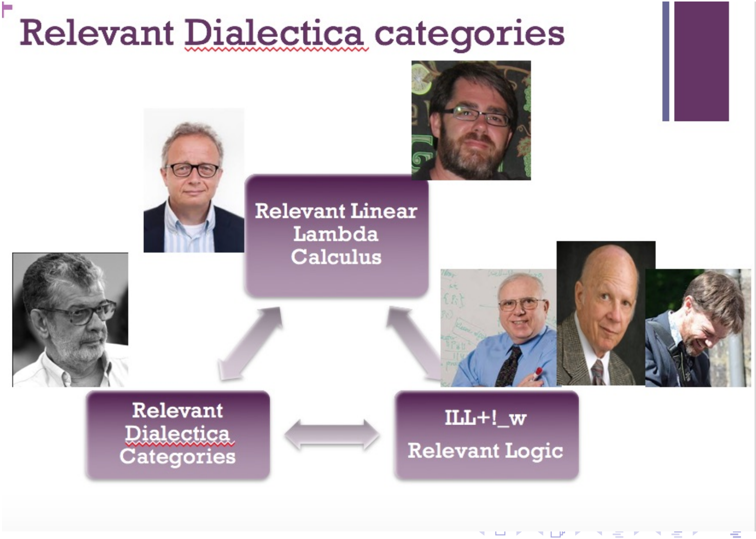
Proposition

There is a comonad bang in GC , a composite of two comonads, which models modalities $!, ?$ in Linear Logic.

Theorem: Linear and non-Linear logic together

There is a **monoidal** adjunction between GC and its coKleisli category for the composite monoidal comonad bang above.

Another Old Application?



Relevance Logic

Precursors: Orlov(1928, Dosen 2018), Ackerman (1958)
Mostly here: Anderson and Belnap I (1975) and II (1992) and M. Dunn
Also Avron (1984-2014).
Connections with paraconsistency important:
Agree with Restall

> *Hilbert systems, with many axioms and few rules, are not so suited to a project of understanding the internal structure of a family of logical systems. (Handbook of the History of Logic (2006))*

Relevance Logic Masterplan

1. Instead of relevance logic R proper, use ILL ($+ \,!_c$)
2. Add modality $!_w$ to get IL
3. Build dual context system DIRL where contexts are relevant and intuitionistic, following DILL (Plotkin/Barber, Barber PhD)
4. Provide Dialectica Models of ILL$+ \,!_c$, Meré PhD (1993)
5. Provide models of DIRL
6. Provide models where modality $!_w$ disappears, but we keep two implications, relevant and intuitionistic, like ILT (Maietti et al, FOSSACS 2000)

Failure of Masterplan

Hoped to do 1-6 for today.

Can only do 1. and 2 badly.

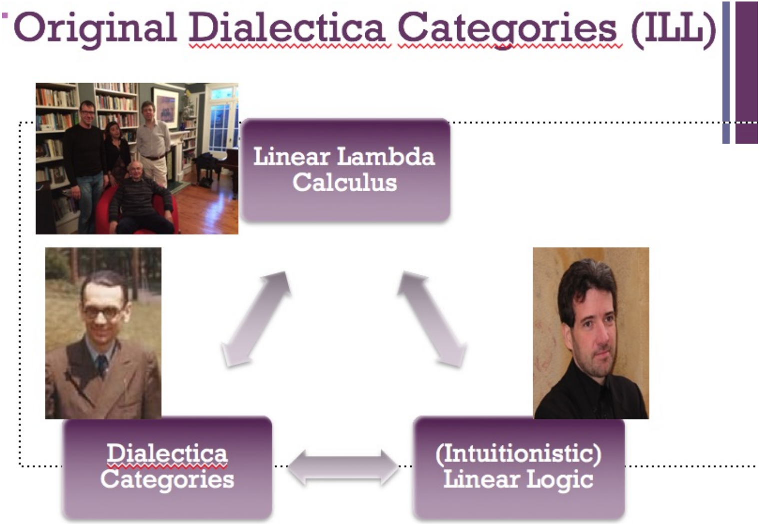
alas 4 does not exist!

Recent IJCAR paper by Kanovich, Kuznetsov, Nigam, and Scedrov points at solutions for relevant systems

This is a "promissory note" for work on relevant logic, from the perspective of Linear Logic.

(Pictorial) Conclusions

Two examples of Dialectica categories
with comonads that compose via distributive laws
Curry-Howard correspondences
Categorical Proof Theory
Much more explaining needed
But pictures help!


Original Dialectica Categories (ILL)
Linear Lambda Calculus
Dialectica Categories
(Intuitionistic) Linear Logic

Girard Dialectica categories (CLL)

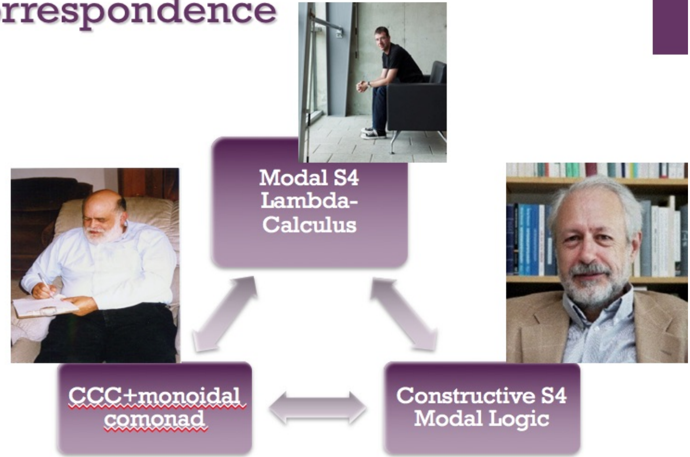


Pattern Linear Lambda Calculus

Girard Dialectica Categories

(FILL/CLL) Linear Logic

Modal (S4) Curry-Howard Correspondence



Lambek calculus Curry-Howard Correspondence

2-sided Linear Lambda-Calculus

Monoidal bi-closed Categories

Lambek Calculus

Thank you!

Some References

(see https://github.com/vcvpaiva/DialecticaCategories)

A.Blass, *Questions and Answers: A Category Arising in Linear Logic, Complexity Theory, and Set Theory*, Advances in Linear Logic (ed. J.-Y. Girard, Y. Lafont, and L. Regnier) London Math. Soc. Lecture Notes 222 (1995).

de Paiva, *A dialectica-like model of linear logic*, Category Theory and Computer Science, Springer, (1989) 341–356.

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Dialectica Categories Applications

Models of Petri nets (more than 2 phds, this year's ACT), non-commutative version for Lambek calculus (linguistics), a model of state (Correa et al based on Reddy)

Generic models of Linear Logic (Schalk2004)
Set Theory work (Samuel Gomes da Silva)

Extensions of Hyland's "Proof Theory in the Abstract", Biering "Copenhagen Interpretation", Hofstra "The dialectica monad and its cousins", Von Glehn "Polynomials"/containers (2015), Sean Moss (2017) thesis.

Also: Oliva et al on games from Dialectica; "The Compiler Forest" Budiu, Galenson and Plotkin (2012); Pedrot LICS 2014, thesis (2015); Pierre Pradic thesis (2020) Constructive aspects of MSO