Kalman Duality as Chu Duality

Contents

1 The basic linear system

Let k be a field and let

$$X,\ U,\ Y \in \mathbf{Vect}_k^{\mathrm{fd}}.$$

A finite-dimensional continuous-time linear system is given by

$$\Sigma = (X, U, Y, A, B, C),$$

with

$$A : X \to X, \qquad B : U \to X, \qquad C : X \to Y,$$

and equations

$$\dot{x} = Ax + Bu, \qquad y = Cx.$$

Its Kalman dual is

$$\Sigma^\vee = (X^*, Y^*, U^*, A^*, C^*, B^*),$$

that is,

$$\dot{\xi} = A^*\xi + C^*\eta, \qquad \zeta = B^*\xi.$$

The local system presentation is stateful: the vector field depends on both the present state and the present input,

$$X \times U \longrightarrow TX, \qquad (x, u) \longmapsto (x, Ax + Bu),$$

together with the readout $C : X \to Y$. This is the lens-like level of the construction.

2 The Hankel Chu object

Let

$$\mathcal{H}_\Sigma^+ = U[z] = \bigoplus_{m\geq 0} U z^m, \qquad \mathcal{H}_\Sigma^- = Y^*[z] = \bigoplus_{n\geq 0} Y^* z^n.$$

Define

$$h_\Sigma : U[z] \times Y^*[z] \longrightarrow k$$

by

$$h_\Sigma(uz^m, \eta z^n) = \eta\big(CA^{m+n}Bu\big), \tag{1}$$

and extend bilinearly.

Definition 1. *The* Hankel Chu object *of* Σ *is*

$$\mathsf{Chu}(\Sigma) = \big(U[z], Y^*[z], h_\Sigma\big) \in \mathrm{Chu}(\mathbf{Vect}_k, k).$$

The coefficients CA^rB are the Markov parameters of the system. The defining relation

$$h_\Sigma(zp, q) = h_\Sigma(p, zq)$$

expresses shift invariance.

3 Factorization through state and costate

Define the reachability map

$$\rho_\Sigma : U[z] \longrightarrow X, \qquad \rho_\Sigma(uz^m) = A^m Bu,$$

and the transpose-observability map

$$\omega_\Sigma^\vee : Y^*[z] \longrightarrow X^*, \qquad \omega_\Sigma^\vee(\eta z^n) = (A^*)^n C^* \eta.$$

Then

$$h_\Sigma(p, q) = \big\langle \omega_\Sigma^\vee(q), \rho_\Sigma(p) \big\rangle. \tag{2}$$

Thus the Hankel relation factors through the canonical evaluation

$$X^* \times X \longrightarrow k.$$

This factorization separates two roles:

$$\boxed{\text{past interventions or reachable histories}} \qquad \text{and} \qquad \boxed{\text{future observations or experiments.}}$$

The state space is an intermediate realization of the external past–future pairing.

4 Kalman duality is Chu duality

The Chu dual of (P, N, e) is

$$(P, N, e)^{\perp} = (N, P, e^{\mathsf{T}}), \qquad e^{\mathsf{T}}(n, p) = e(p, n).$$

Theorem 2 (Kalman–Chu duality)**.** *For every finite-dimensional linear system Σ there is a canonical isomorphism*

$$\mathsf{Chu}(\Sigma^{\vee}) \cong \mathsf{Chu}(\Sigma)^{\perp}.$$

Equivalently, once the carriers are chosen as input jets and output-covector jets, the Chu involution is exactly the Kalman transpose duality.

Proof. The positive carrier of $\Sigma^{\vee}$ is $Y^*[z]$, while its negative carrier is

$$(U^*)^*[z] \cong U[z]$$

by finite-dimensional reflexivity. For generators,

$$\begin{aligned} h_{\Sigma^{\vee}}(\eta z^n, u z^m) &= u\big(B^*(A^*)^{m+n} C^* \eta\big) \\ &= \eta\big(C A^{m+n} B u\big) \\ &= h_{\Sigma}(u z^m, \eta z^n). \end{aligned}$$

Hence $h_{\Sigma^{\vee}} = h_{\Sigma}^{\mathsf{T}}$ under the canonical identifications. □

5 Biextensional collapse and minimal realization

Let

$$\mathcal{R} = \operatorname{span}\{A^m B u \mid m \geq 0,\ u \in U\}$$

be the reachable subspace, and let

$$\mathcal{N} = \bigcap_{n \geq 0} \ker(CA^n)$$

be the unobservable subspace.

The left radical of the Hankel pairing is

$$\operatorname{Rad}_L(h_{\Sigma}) = \rho_{\Sigma}^{-1}(\mathcal{N}),$$

while the right radical is

$$\operatorname{Rad}_R(h_{\Sigma}) = (\omega_{\Sigma}^{\vee})^{-1}(\mathcal{R}^{\perp}).$$

Consequently,

$$U[z]/\operatorname{Rad}_L(h_{\Sigma}) \cong \mathcal{R}/(\mathcal{R} \cap \mathcal{N}).$$

The latter is the minimal reachable-observable state space

$$X_{\min} = \mathcal{R}/(\mathcal{R} \cap \mathcal{N}).$$

Dually,

$$Y^*[z]/\operatorname{Rad}_R(h_{\Sigma}) \cong X_{\min}^*.$$

Proposition 3. *The biextensional collapse of* $\mathsf{Chu}(\Sigma)$ *is, up to the canonical identifications induced by reachability and observability,*

$$\big(X_{\min}, X_{\min}^*, \mathrm{ev}\big).$$

Its dimension is the Hankel rank.

Because $X_{\min}$ is finite-dimensional, the reduced pairing is not merely separated on both sides but perfect.