Open Petri Nets

John C. Baez Jade Master

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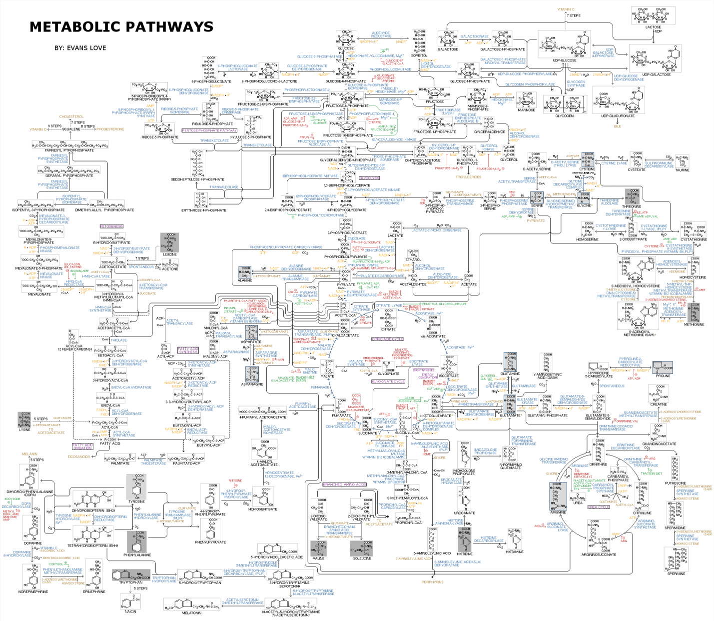
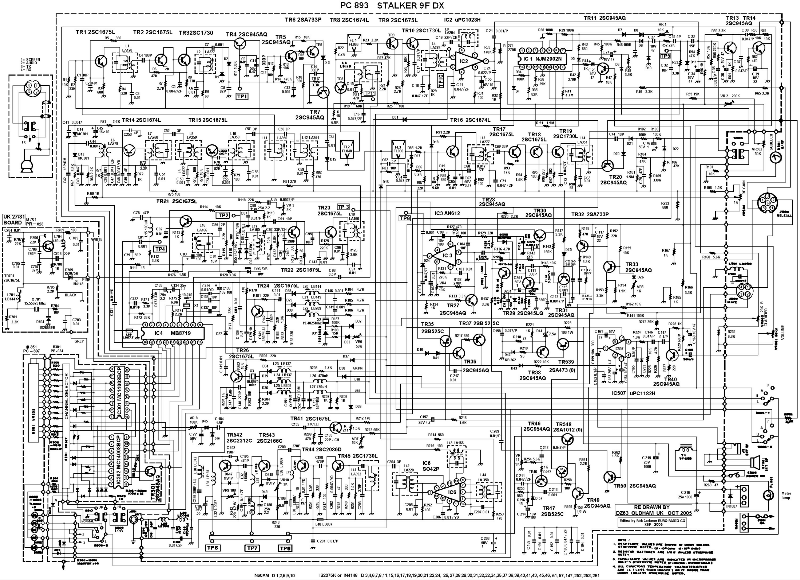
University of California Riverside

Complex networks are everywhere.

The Metabolic System



An Electrical Circuit



Divide and Conquer

Lawvere's *Functorial Semantics of Algebraic Theories* describes notation which people use for systems and quantitative meaning which people attach to this notation. The former is called syntax and the latter is called semantics.

$$\text{Syntax} \xrightarrow{\text{Functor}} \text{Semantics}$$

Traditionally this has been used to study algebra, but applied category theory seeks to apply this strategy to the science and engineering.

These papers describe a general framework for this.

Baez, John C., and Kenny Courser. "Structured cospans." arXiv preprint arXiv:1911.04630 (2019).

Fong, Brendan. "The algebra of open and interconnected systems." arXiv preprint arXiv:1609.05382 (2016).

The key is that composition in the syntax category is gluing.

Therefore, functoriality of

$$\text{Syntax} \xrightarrow{\text{Functor}} \text{Semantics}$$

gives an isomorphism

$$F(g \circ f) \cong F(g) \circ F(f)$$

i.e. a recipe for building the semantics on a composite from the semantics on each component.

"*We make a star as we make a constellation, by putting its parts together and marking off its boundaries*" -N.Goodman, On Starmaking (1980)

The glue which binds things together and makes up our conceptual landscape.

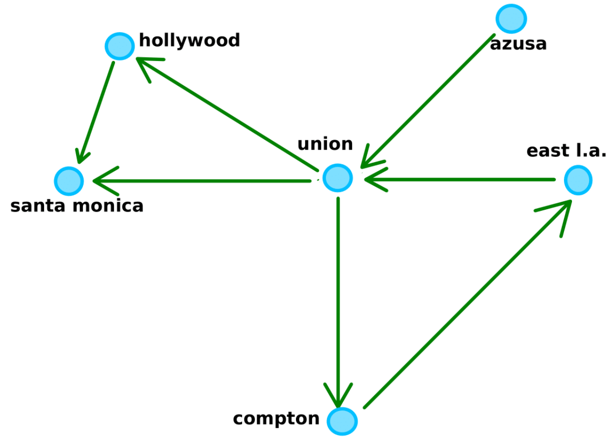
Graphs

A graph has edges, vertices, sources, and targets.

$$E \rightrightarrows V$$

These graphs are directed with multiple edges.
A morphism of graphs is a pair of functions between the edges and vertices commuting suitably with source and target.
This gives a category Grph of graphs and their morphisms.

Some train and bus routes in LA

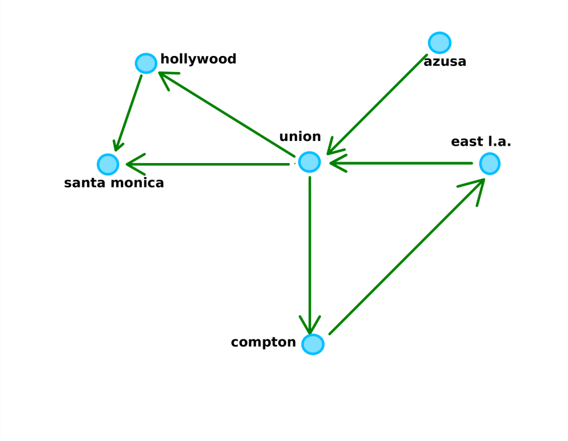


The Free Category on a Graph

Given a graph we can generate it's reflexive, transitive closure. A category where

- objects are vertices and,
- morphisms are either idenitites or finite sequences of composable edges.

Paths in Los Angeles



$$\text{azusa} \to \text{union} \to \text{compton}$$

$$\text{east l.a.} \to \text{union} \to \text{hollywood} \to \text{santa monica}$$

$$1_{\text{union}}$$

This Construction Satisfies Universal Properties

The forgetful functor

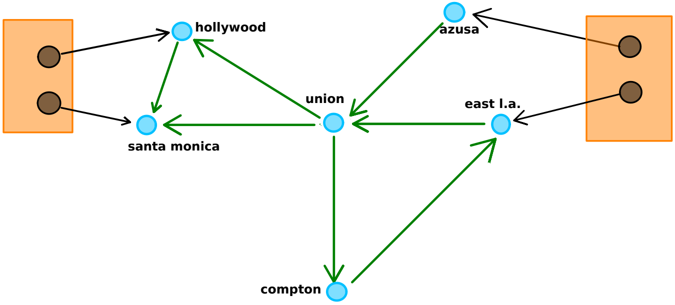
$$U\colon \mathsf{Cat} \to \mathsf{Grph}$$

is right adjoint to the free category functor

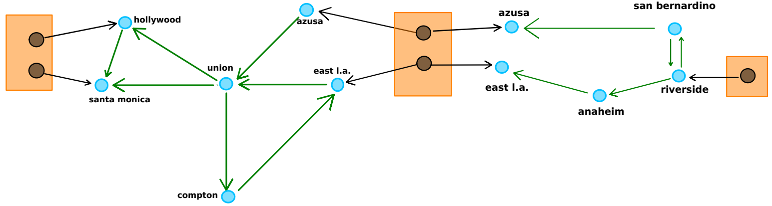
$$F\colon \mathsf{Grph} \to \mathsf{Cat}$$

Open Graphs

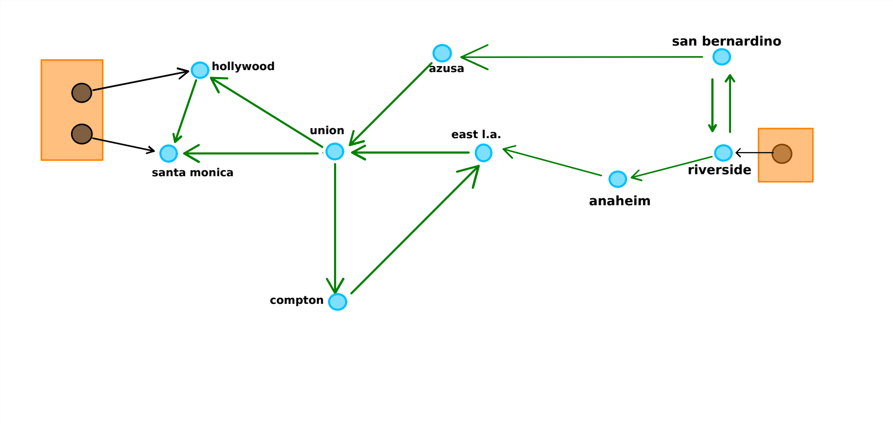
A graph can be opened by equipping input and output sets and functions to the vertices of your graph.



Which can be glued together...



To get...



hollywood
azusa
san bernardino
union
east l.a.
santa monica
riverside
anaheim
compton

This is formalized using pushouts.

$$
\begin{tikzcd}
 & & G +_{LY} H & & \\
 & G \arrow[ur] & & H \arrow[ul] & \\
LX \arrow[ur] & & LY \arrow[ul] \arrow[ur] & & LZ \arrow[ul]
\end{tikzcd}
$$

- LX,LY,LZ are the discrete graphs on the boundary sets X Y and Z.
- This pushout and diagram takes place in the category of graphs and graph morphisms.

Theorem.

The reflexive transitive closure preserves gluing.

Proof: F is a left adjoint so it preserves pushouts.

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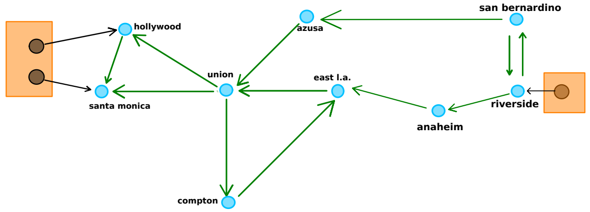
Proof: F is a left adjoint so it preserves pushouts.

Fine print: Gluing/pushout is more complicated for categories.



You need more than just paths that start in LA and go to the inland empire. You need paths which loop around, zig-zagging between both.

This requires taking the free category on each component, gluing together their underlying graphs, and then taking the reflexive transitive closure of the result. In other words,

$$F(G +_{LY} H) \cong F(UF(G) +_{LY} UF(H))$$

Petri Nets

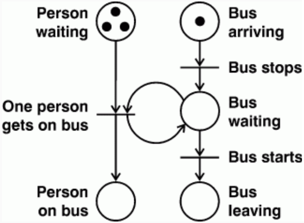
Petri nets are just fancy graphs

Definition: A Petri net is a pair of functions of the following form

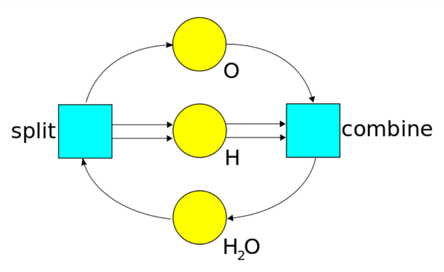
$$T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

where $\mathbb{N} \colon \mathsf{Set} \to \mathsf{Set}$ is the free commutative monoid monad which sends a set X to $\mathbb{N}[X]$ the free commutative monoid on X.

A morphism of Petri nets is a pair of functions between the edges and vertices commuting suitably with the source and target.



Person waiting
Bus arriving
Bus stops
One person gets on bus
Bus waiting
Bus starts
Person on bus
Bus leaving



O
split
H
combine
H_2O

Graphs are to free categories as Petri nets are to free *commutative monoidal categories*.

Commutative monoidal categories are a special sort of symmetric monoidal category. Symmetric monoidal categories are languages for processes that can be performed in sequence and in parallel so this explains the usefulness of Petri nets.

Symmetric monoidal categories are the bread and butter of applied category theory.

Commutative monoidal categories

A commutative monoidal category has morphisms and objects which are commutative monoids.

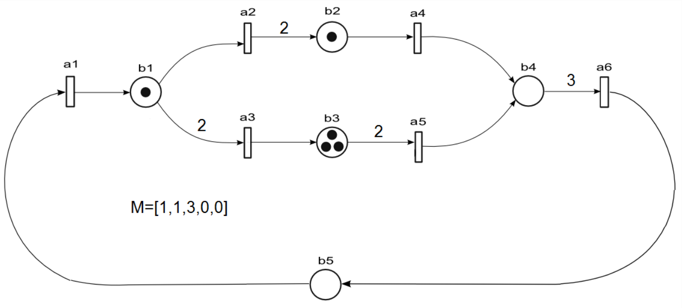
$$\mathrm{Mor}\, C \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}\, C$$

Source, target, composition, and assignment of identities are all commutative monoid homomorphisms.

Like a symmetric monoidal category except the tensor is strictly commutative and the braiding is the identity.

In *Petri Nets are Monoids*, Messeguer and Montanari introduced the idea. We use a variant of this: For a Petri net P, the commutative monoidal category FP has

- objects given by possible markings of P with tokens
- morphisms given by ways that these markings can be shuffled around using sequences of transitions

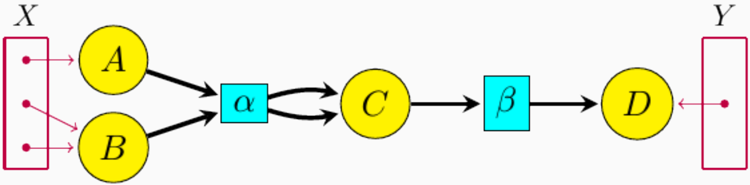


- The "solution space" of a Petri net.
- The reflexive transitive closure under of a Petri net under sequential composition *and parallel composition*
- The operational semantics of a Petri net.

This closure forms half of an adjunction

$$\begin{tikzcd} \mathsf{Petri} \arrow[rr, bend left, "F"] & & \mathsf{CMC} \arrow[ll, bend left, "U"] \end{tikzcd}$$

A Petri net is opened by equipping it with input and output sets and functions from these sets to the places of your Petri net.



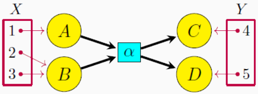
We can think of this as a morphism between two sets.

Definition: An open Petri net $P\colon X \to Y$ is a cospan in Petri of the form

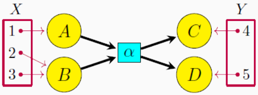
$$\begin{tikzcd} & P & \\ LX \arrow[ur, hook] & & LY \arrow[ul, hook'] \end{tikzcd}$$

Where LX and LY are the Petri nets with no transitions and X or Y as their set of places.

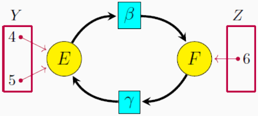
Given an open Petri net from X to Y



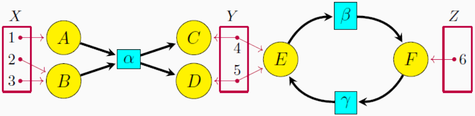
Given an open Petri net from X to Y



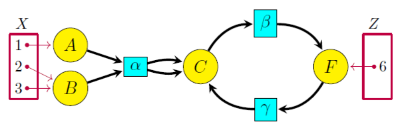
and an open Petri net from Y to Z



To compose them first you place them end to end



and identify the places which come from the same element of Y



This is formalized using pushouts

$$
\begin{tikzcd}
 & & P +_{LY} Q & & \\
 & P \arrow[ur] & & Q \arrow[ul] & \\
LX \arrow[ur] & & LY \arrow[ul] \arrow[ur] & & LZ \arrow[ul]
\end{tikzcd}
$$

which takes the disjoint union and mods out by the equivalence relation described above.

The situation is the same as with graphs.

Theorem.

The category of processes of a Petri net preserves gluing.

Proof: $F\colon \mathsf{Petri} \to \mathsf{CMC}$ is a left adjoint so it preserves pushouts.

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Fine print: Gluing of commutative monoidal categories is a bit more complicated.

The Compositional Formula

$$F(P +_{LY} Q) \cong F(UF(P) +_{LY} UF(Q))$$

where

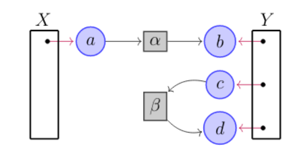
$$F\colon \mathsf{Petri} \to \mathsf{CMC}$$

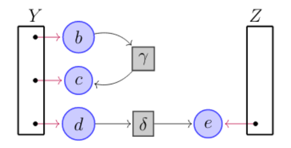
is the free commutative monoidal category on a Petri net functor and

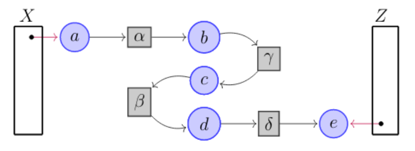
$$U\colon \mathsf{CMC} \to \mathsf{Petri}$$

is its right adjoint.

Extra Processes Occur in Gluing







If we care about the language of this gluing being nice and coherent, we want open Petri nets to live as morphisms in a symmetric monoidal category (or even better double category or bicategory).

This follows from the theory of structured cospans.

More Formally

Theorem. There is a symmetric monoidal category Open(Petri) where

- objects are sets $X, Y, \ldots$
- morphisms are (equivalence classes of) open Petri nets $P\colon X \to Y$,
- composition is given by pushout and,
- monoidal product is given by coproduct on sets and pointwise coproduct on morphisms.

Open(Petri) is more naturally a monoidal bicategory or monoidal double category because composition using pushout is not strictly associative. To make this into a category we need to define open Petri nets *up to isomorphism*.

More Formally

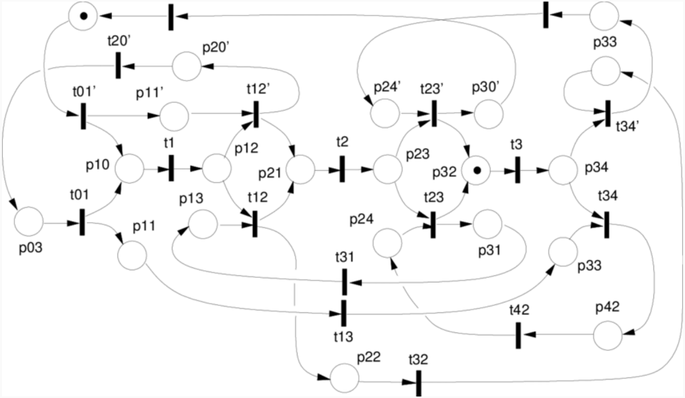
Theorem.

There is a monoidal double functor

$$\mathrm{Open}(\mathrm{F})\colon \mathrm{Open}(\mathrm{Petri}) \to \mathrm{Open}(\mathrm{CMC})$$

which sends an open Petri net to it's open category of processes.

The coherence laws of a symmetric monoidal category ensure that complex networks can be built in a coherent way using open Petri nets.



Reachability Semantics

Some Motivation

The reachability problem asks: given two markings m and n, is there a sequence of transitions which can fire starting at m and ending in n. Reachability is good for formal verification and many other decidability problems can be reduced to Petri net reachability.

- In 1984 Mayr showed that the reachability problem was decidable but . . .

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- In 2018 the time complexity was shown to be greater than *any primitive recursive function*

The analogue of reachability for Turing machines is the halting problem so Petri nets are right on the edge of being Turing complete. This puts them in the sweet spot of expressiveness.

Open Petri nets are a natural setting to discuss reachability.

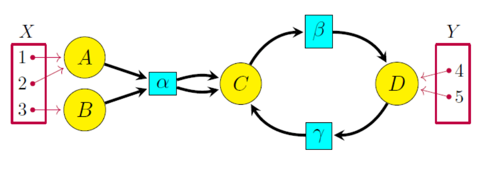
Definition: For an open Petri net $P\colon X \to Y$ its reachability relation

$$\blacksquare(P) \subseteq \mathbb{N}[X] \times \mathbb{N}[Y]$$

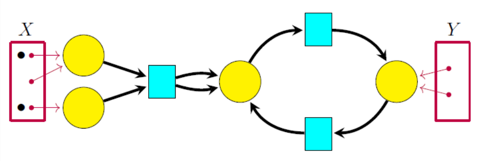
contains an element (x, y) if y is reachable from x.

Example

Let $P \colon X \to Y$ be the following open Petri net:

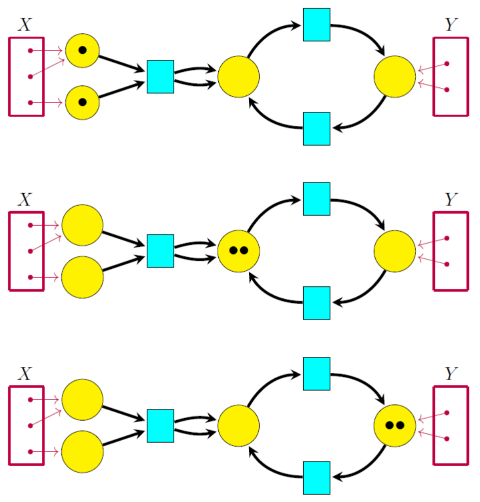


then we can equip X with an initial marking,



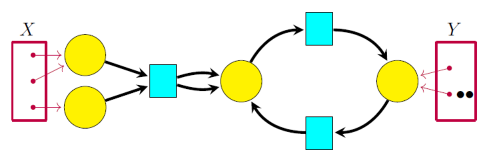
Example

shuffle this marking around using the transitions,



Example

and pop the tokens back into Y leaving no tokens behind.



This can all be made categorical.

Proposition: For a Petri net P, a marking n is reachable from m if and only if there is a morphism $f\colon m \to n$ in the free commutative monoidal category FP.

Definition: For a cospan of categories

$$\begin{tikzcd} & C & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "j"'] \end{tikzcd}$$

its reachability relation

$$\pi_0(C) \subseteq \mathrm{Ob}\, X \times \mathrm{Ob}\, Y$$

contains an element (x, y) if there is a morphism $f\colon i(x) \to j(y)$ in C.

So to get the reachability relation of an open Petri net

$$\begin{tikzcd} & P & \\ LX \arrow[ur] & & LY \arrow[ul] \end{tikzcd}$$

we apply the semantics functor $F\colon \mathsf{Petri} \to \mathsf{CMC}$

$$\begin{tikzcd} & FP & \\ FLX \arrow[ur] & & FLY \arrow[ul] \end{tikzcd}$$

and take the reachability of this.

This process is also an instance of syntax mapping to semantics.

Let Rel be the 2-category where

- objects are sets X, Y, ...
- morphisms are relations $R \subseteq X \times Y$ and
- a 2-morphism from $R \subseteq X \times Y$ to $R' \subseteq X \times Y$ is an inclusion

$$R \subseteq R'$$

And Open(Petri) can be upgraded to a 2-category where the 2-morphisms can only be the identity.

Theorem: There is a lax symmetric monoidal 2-functor

$$\blacksquare \colon \mathsf{Open}(\mathsf{Petri}) \to \mathsf{Rel}$$

which makes the following assignment on morphisms

$$\begin{tikzcd} LX \arrow[d, "i"] \\ P \\ LY \arrow[u, "j"'] \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} \mathbb{N}[X] \arrow[d, "\blacksquare(P)"] \\ \mathbb{N}[Y] \end{tikzcd}$$

Proof: $\blacksquare$ is constructed as the composite:

$$\mathrm{Open}(\mathrm{Petri}) \xrightarrow{\mathrm{Open}(F)} \mathrm{Open}(\mathrm{CMC}) \xrightarrow{\pi_0} \mathbf{Rel}$$

where the reachability of categories gives the second arrow.

This result describes the extent to which we can reason about reachability in compositional way.

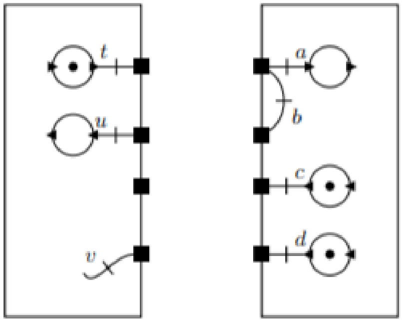
Laxness means that we have an inclusion

$$\blacksquare(P) \circ \blacksquare(Q) \subseteq \blacksquare(P \circ Q)$$

which allows us to break up reachability problems into smaller subproblems...although we don't get everything.

Conclusion

- Penrose, Statebox, and formal verification.
- The compositional formulas seem more general. It shows up in computing compositional solutions to the algebraic path problem.
- More theoretical and experimental work is needed to put this to use.



Petri nets are inherently categorical. Grothendieck said

> *The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months—when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!*

References

- José Meseguer and Ugo Montanari (1990) Petri nets are Monoids
- Ernst Mayr (1981) Persistence of vector replacement systems is decidable
- Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux and Filip Mazowiecki (2018) The reachability problem for Petri nets is not elementary. Available at arXiv:1809.07115.
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