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LIMIT MONADS IN CATEGORIES

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ANDERS JUNGERSEN KOCK

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INTRODUCTION

The content of the present thesis is roughly this: The category of complete categories is <u>monadic</u> over the category CAT of categories.

This means that there is a monad on CAT so that the "Universal resolution" category for the monad, as constructed by Eilenberg and Moore ("the category of algebras over the monad") is the category of complete categories and limit preserving functors. In fact, there will be a whole family of monads, one for each suitable category of indexcategories. "Complete" is then understood relative to this family. For instance, the family of finite discrete indexcategories gives rise to the monad for finite products.

To state the main result in a more precise way, we must define what we mean by "the" category of complete categories, $\mathcal{R}$. We are only dealing with one sided completeness, and for convenience, it is right completeness. First, it is clear from the set up that we do not want $\mathcal{R}$ to be a subcategory of CAT, but that we will consider the objects of $\mathcal{R}$ as categories <u>equipped with some further structure</u>, namely a choice of colimits. Consequently, the morphisms of $\mathcal{R}$ will be functors which preserve the chosen colimits on the nose. From this again follows that we must distinguish carefully between the notions of <u>isomorphic</u> and <u>equal</u> objects in a category, and between the notions of <u>equivalent</u> and <u>isomorphic</u> categories.

Actually, we have not been able to prove the result with such a liberal definition of $\mathcal{R}$. We have rather considered a subcategory consisting of categories, where the chosen colimits satisfy some equations. The objects in this subcategory are called regular colimit algebras. The equations imposed are roughly speaking associativity equations; they hold true up to isomorphism in any category with some choice of colimits. One such equation is

$$(A+B)+C \quad = \quad A+(B+C),$$

'+' denoting binary sum, i.e. colimit over the discrete category with two objects.

A great deal of the work consists in defining the notion of associative and regular associative colimits on a category. We feel that this notion is a very natural one, and that one should try to make colimits associative whenever possible. We have done that for some important categories; in fact, the crucial point of the whole work is the construction of associative (infinitary) sums on a suitable category $\mathfrak{S}$ of sets, since these sums are used in defining the notion of associativity of colimits in general. Here it clearly shows up that equivalent categories are not equally good: the skeletal category of sets cannot be equipped with associative sums. But $\mathfrak{S}$ can.

In the concluding Chapter III, we exhibit some further examples of categories with associative colimits. For an arbitrary ring, a certain category equivalent to tthe category of all modules over it has a natural regular colimit structure. The dual of $\mathfrak{S}$ has a regular colimit structure for finite indexcategories, i.e. $\mathfrak{S}$ itself has associative finite inverse limits. Also we show that $\mathfrak{S}$ has the property that the tensor-product of modules in $\mathfrak{S}$ is strictly associative.

The content of Chapter I is the construction of $\mathfrak{S}$ and its colimits, and the definition of some auxiliary monads, called prelimit monads, on

CAT. These in turn are used to define the notion of associative colimits. Chapter II contains the main construction: the colimit monads, and an analysis of their algebras. It is also shown that a colimit monad to a category assigns "the" free right complete category on it, in a certain weak sense.

Notation and conventions.

The main notational obstacle is that we have to deal partly with composition of morphisms in a category, partly with evaluation of a function on an element in its domain. As a principle, we use 'f.g' for the first (in diagrammatic order), and '(x)f' for the second. In practice, we have used concatenation for both if it causes not too much confusion.

The hom sets of a category $\mathcal{A}$ are denoted $\mathcal{A}(A,A')$ or $\hom(A,A')$.

When colimit formation appears as a functor, we have often written it on the left of its argument,

$$\varinjlim (R).$$

This holds in particular for the sign $\coprod$ for sums:

$$\coprod_{m \in M} X_m .$$

The morphisms constituting a colimit diagram are denoted incl_D, D an object in the indexcategory. In particular, we have

$$X_m \xrightarrow{\mathrm{incl}_m} \coprod_{m \in M} X_m .$$

If this is in a category of sets, and $x \in X_m$, then $(x)\mathrm{incl}_m$ is denoted x_m.

We sometimes use Kronecker δ ; $\delta(x,y)$ is 0 if $x \neq y$, else it is the ordinal number 1.

If A is an element of a set $\mathbb{A}$, a unique mapping $1 \longrightarrow \mathbb{A}$ is defined having A as image. That mapping is denoted $\in_A$. This will apply also if A is an object of a category $\mathbb{A}$. We use the numeral 1 to denote the ordinal number <u>One</u>, i.e. 0 , where 0 is the empty set. Similarly for all other numerals.

CHAPTER I

THE CATEGORY $\mathcal{S}$ AND THE PRELIMIT MONADS

1. Categories and $\mathcal{U}$-categories.

We shall work in a set theory with universes. To be specific, let us agree that we have a model of Zermelo - Fraenkel set theory with axiom of choice, in which there are two sets U, V, satisfying $U \in V$ and being standard transitive models for ZF+AC (see e.g. Cohen [1]). The two sets are fixed throughout; one further assumption about them will be made below.

Let us recall briefly the set theoretical notions of well ordered set and ordinal number. They will play a fundamental role in what follows. A totally ordered set is *well ordered* if every nonempty subset has a first element with respect to the ordering. An *order type* is an order isomorphism class of well ordered sets, "order isomorphism" having the obvious meaning.

A set $X \in U$ which is well ordered by $\in$ (interpreted as $<$), and which satisfies $Y \in X \implies Y \subseteq X$ is by the classical definition of von Neumann called an *ordinal number* in U. There is precisely one ordinal number of U in each order type represented in U ([1], p.61). A *cardinal number* is an ordinal number which, as a set, is isomorphic to no smaller ordinal number.

We have to agree on a meaning for the word "category." Not all definitions suit our purposes. We use a definition where the hom sets are not disjoint, but form a family of sets bi-indexed by the set of objects. This means that the composition is a tri-indexed family of functions

$$\hom(A,B) \times \hom(B,C) \xrightarrow{\text{comp}(A,B,C)} \hom(A,C) .$$

"The" composition will be denoted by juxtaposition or by a dot; it only makes sense if it cannot be misunderstood which composition mapping is applied.

The concept "functor" is defined accordingly; a function $|\mathcal{A}| \to |\mathcal{B}|$ called the object mapping and a bi-indexed set of functions called the hom set mappings.

The category U-Ens of U -sets is defined in the usual way, having as object set U, and hom(X,Y) the set of functions from X to Y (a function being a subset of $X \times Y$). We shall assume that such a function is also a member of U. A similar assumption is made for V. These assumptions are special cases of the general

ASSUMPTION. If $X \in U$, then the absolute power set of X is also a member of U. Similarly for V.

DEFINITION 1.1. Let $\mathcal{S}$ denote the full subcategory of the category of U -sets determined by the ordinal numbers in U.

DEFINITION 1.2. Call a category $\mathcal{A}$ an $\mathcal{O}$-category if $|\mathcal{A}|$ is an ordinal number in U, and each $\mathcal{A}(A,A')$ is an ordinal number in U.

Note that we do not require the composition functions to preserve any ordering.

DEFINITION 1.3. Let $CAT_{W_1}^{W_0}$ denote the category of categories $\mathcal{A}$ with $|\mathcal{A}| \in W_1$ and each $\mathcal{A}(A,A') \in W_0$, and all functors in between.

The definition will apply to the cases where $W_1 = U$ or V. It will also apply to the case $W_1 = |\mathcal{S}|$, i.e. the set of ordinal numbers in U. Let us agree not to write the W_i in this case,aand also in this case replace the letters CAT by the letters Cat. So Cat_W consists of categories $\mathcal{A}$ with $|\mathcal{A}| \in W$ and each hom set an ordinal number in U ($\mathcal{A}$ is then called a <u>local</u> $\mathcal{O}$ -<u>category</u>). Similarly, Cat^W consists of categories $\mathcal{A}$ with $|\mathcal{A}|$ an ordinal number in U and with hom sets in W. Finally, Cat is the category of $\mathcal{O}$ -categories (Definition 1.2).

It is obvious that $\mathcal{S}$ is equivalent to the category of U -sets; and that Cat is equivalent to CAT_U^U. In both cases, they sit as reflexive subcategories. A reflection functor c is gotten by putting a well ordering on each set in U, and reflecting it to the ordinal of the same order type. The reflection isomorphism is then take to be the unique order preserving set isomorphism. Similarly, one gets a reflection

$$\mathrm{CAT}_U^U \xrightarrow{\ c\ } \mathrm{Cat}$$

by well ordering the object set and the hom sets of each category.

Definition 1.3 is not made entirely precise. A precise definition in terms of set theory would require $|\mathrm{CAT}_{W_1}^{W_0}|$ to be described as, say, the set of ordered triples (X,Y,Z), where $X \in W_1$, Y is a function from $X \times X$ to W_0 (the family of hom sets), Z is a function from $X \times X \times X$ defining the family of composition mappings. In particular, $|\mathrm{Cat}|$ could be described precisely as a set in such a way. Similarly, each hom set $\mathrm{Cat}(\mathcal{A}, \mathcal{B})$ could be described as a single set. Using some such precise description will give

$$\mathrm{Cat} \in \mathrm{CAT}_V^U .$$

2. Sums and Grothendieck constructions.

By the well known completeness of U-Ens, for any I -indexed family ($I \in U$) of objects N_i in U-Ens, one may choose a sum diagram in U-Ens

$$N_i \xrightarrow{\text{incl}_i} \coprod_{i \in I} N_i \qquad \forall i \in I.$$

Even in the skeletal category of sets, the sum diagram can be chosen in many ways (even though the sum object itself cannot). The same is true of $\mathbf{S}$. If we, however, restrict ourselves to index sets I which are themselves objects in $\mathbf{S}$, there is a canonical choice of sum diagrams. It is the well known ordinal sum formation of ordinal numbers.

The construction of ordinal sum may be phrased in set theoretic terms as follows. Let N_i ($i \in I \in |\mathbf{S}|$) be the I -indexed family of ordinals in U. Let A be the set

$$A = \{\langle i,n\rangle \mid i \in I \wedge n \in N_i\}.$$

Then the lexicographic ordering

$$(\langle i,n\rangle < \langle i',n'\rangle) \iff (i < i') \vee (i = i' \wedge n < n')$$

is a well ordering; therefore, there is a unique ordinal (which we shall write $\coprod_{i \in I} N_i$) with the property that it is order isomorphic to A. Let

$$\alpha : A \longrightarrow \coprod_{i \in I} N_i$$

be the unique order isomorphism. It is clear that $\coprod_{i \in I} N_i \in U$.

For each $i \in I$, there is a function

$$t_i : N_i \longrightarrow A,$$

namely the one which sends $n \in N_i$ to $\langle i,n \rangle \in A$. Obviously t_i is one - to - one, and A is the disjoint union of the images of the t_i's. Since α is one - to - one, onto, the same is true of the $t_i . \alpha$'s. So the morphisms in $\mathfrak{S}$

$$\mathrm{incl}_i \ : \ N_i \xrightarrow{t_i . \alpha} \coprod_{i \in I} N_i$$

constitute $\coprod_{i \in I} N_i$ as the categorical sum of the N_i's in $\mathfrak{S}$. Obviously, $\coprod_{i \in I} N_i$ is just the ordinal sum of the N_i's. One easily sees that each incl_i maps onto an interval $\{ t \mid x_0 \leq t < x_1 \}$ in $\coprod_{i \in I} N_i$.

The sums chosen in $\mathfrak{S}$ will have nice properties; in fact, they will endow $\mathfrak{S}$ with a certain algebraic structure, to be defined in Section 7. For the moment, let us just notice that the sums have the following three properties

(2.1) $$\coprod_N 1 = N$$

(2.2) $$\coprod_1 N = N$$

(2.3) $$\coprod_{m \in \coprod_{n \in N} M_n} L_m = \coprod_{n \in N} \Big(\coprod_{m \in M_n} L_m \Big) .$$

(N, M_n, and L_m ordinals in U.) Henceforth, $\coprod$ denotes the chosen sums in $\mathfrak{S}$ only.

Having the sum structure $\coprod$ in $\mathfrak{S}$ at our disposal, we can also get a canonical form for a well known construction, due to Grothendieck [5], namely the construction which (speaking informally) to a functor from a small category $\mathbb{D}$ into the category of small categories assigns a small category fibered over $\mathbb{D}^{\mathrm{opp}}$, and another one opfibered over $\mathbb{D}$.

Formally, let $\mathbb{D}$ be an $\mathcal{O}$-category (i.e. $\mathbb{D} \in |\mathrm{Cat}|$), and let

$$\mathbb{D} \xrightarrow{\bar{R}} \mathrm{Cat}$$

be a functor. Define an $\mathcal{O}$-category $\bar{R}\bar{\mu}$ as follows:

set of objects : $\coprod_{D \in |\mathbb{D}|} |D\bar{R}|$;

hom$(X_D, X'_{D'})$: $\coprod_{d \in \mathbb{D}(D,D')} (D'\bar{R})((X)(d\bar{R}), X')$;

composition : let $x_d \in \bar{R}\bar{\mu}(X_D, X'_{D'})$, $x'_{d'} \in \bar{R}\bar{\mu}(X'_{D'}, X''_{D''})$, where $d : D \to D'$ and $d' : D' \to D''$ are morphisms in $\mathbb{D}$. Put

$$x_d . x'_{d'} = (((x)(d'\bar{R})).x')_{d.d'} ;$$

units : put I_{X_D} equal to $(I_X)_{I_D}$.

Also, define a functor $\bar{R}t : \bar{R}\bar{\mu} \to \mathbb{D}$ by

$$(X_D)\bar{R}t = D,$$

$$(x_d)\bar{R}t = d.$$

It is well known ([5]) that (no matter how you chose sums in $\mathcal{S}$), $\bar{R}\bar{\mu}$ is indeed a category, and $\bar{R}t$ is indeed a functor - even a split opfibration with the following opcleavage: to X_D and $d : D \to D'$ assign $(I_{XdR})_d$. (We use the terminology of [4]).

Again, no matter how sums are chosen, the construction is natural, i.e. can be extended to a functor

$$[\mathrm{Cat}, \in_{\mathrm{Cat}}] \xrightarrow{\bar{\mu}} \mathrm{Cat};$$

we define it on morphisms h in $[\mathrm{Cat}, \in_{\mathrm{Cat}}]$

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\ H\ } & \mathbb{E} \\ & \overset{\Longrightarrow}{h} & \\ \bar{R} \searrow & & \swarrow \bar{P} \\ & \mathrm{Cat} & \end{array}$$

as follows. The functor $h\bar{\mu}$ is given by

$$X_D \rightsquigarrow (Xh_D)_{DH} \quad , \; X \in |D\bar{R}| ,$$

$$X_D \xrightarrow{x_d} X'_{D'} \quad \rightsquigarrow \quad (Xh_D)_{DH} \xrightarrow{((x)h_{D'})_{dH}} (X'h_{D'})_{D'H} \; ,$$

where $d: D \to D'$, and $x: X\, d\bar{R} \to X'$, so that

$$(x)h_{D'} \quad : \quad X\, d\bar{R}\, h_{D'} \; = \; (Xh_D)dH\bar{P} \longrightarrow X'h_{D'} \; .$$

In the next section we shall, however, define a monad, depending on the special way the Grothendieck construction was chosen here.

Note that if $\mathbb{D}$ is discrete (i.e., is an ordinal), and $\bar{R}$ has discrete categories as values, we get the set - sum

$$\bar{R}\bar{\mu} \; = \; \coprod_{D \in \mathbb{D}} D\bar{R} \; .$$

3. <u>Subcategories of Cat stable under the Grothendieck construction.</u>

A subcategory $\mathrm{Cat}_o \subseteq \mathrm{Cat}$ (full or not) may happen to be stable under the Grothendieck construction $\bar{\mu}$, t in the sense that if $\mathbb{D} \in |\mathrm{Cat}_o|$, and $\bar{R} : \mathbb{D} \to \mathrm{Cat}$ factors through Cat_o, then $\bar{R}\bar{\mu}$ and $\bar{R}t : \bar{R}\bar{\mu} \longrightarrow \mathbb{D}$ are themselves in Cat_o. For reasons that will become clear, we state the following

DEFINITION 3.1. A subcategory $\mathrm{Cat}_o \subseteq \mathrm{Cat}$ which is stable under the

Grothendieck construction $\bar{\mu}$, t in the sense explained above will be said to admit a prelimit calculus.

One might also call such a subcategory regular, since it is defined by a closedness property analogous to (and generalizing) that of regular cardinals, as is seen in Proposition 3.2 below.

The following are useful examples:

(i) $\mathcal{S} \subseteq$ Cat (the category of $\mathcal{U}$-sets, Definition 1.1);

(ii) $\mathcal{P}o \subseteq$ Cat, the category of preordered $\mathcal{U}$-sets, i.e. the full subcategory of Cat determined by those $\mathbb{A}$ where $\mathbb{A}(A,A')$ equals 0 or 1 for all $A, A' \in |\mathbb{A}|$;

(iii) $\mathrm{Cat}_{\mathrm{fin}} \subseteq$ Cat, the category of $\mathcal{U}$-categories $\mathbb{A}$ with $|\mathbb{A}|$ and $\mathbb{A}(A,A')$ finite for all $A, A' \in |\mathbb{A}|$;

(iv) the category of $\mathcal{U}$-monoids, i.e. $\mathcal{U}$-categories $\mathbb{A}$ with $|\mathbb{A}| = 1$.

The listed subcategories of Cat are all of the type described in the following obvious

PROPOSITION 3.2. Let ε and μ be regular cardinals. Then the full subcategory of Cat determined by those $\mathbb{A}$ with

$$|\mathbb{A}| < \varepsilon,$$

$$\mathbb{A}(A,A') < \mu \quad \text{for all } A \neq A' \in |\mathbb{A}|,$$

$$\mathbb{A}(A,A') < \max(\mu,2) \quad \text{for all } A \in |\mathbb{A}|$$

admits a calculus of prelimits.

If ν denotes the cardinality of U, the four examples listed correspond to $(\varepsilon, \mu) = (\nu,1), (\nu,2), (\aleph_0,\aleph_0), (2,\nu)$, respectively.

A subcategory of Cat admitting a calculus of prelimits is not necessarily of this form; let us exhibit two examples:

(v) the category of $\mathcal{U}$ -grupoids, i.e. $\mathcal{U}$ -categories with all morphisms invertible,

(vi) the category of well ordered $\mathcal{U}$ -sets; the well ordering of such a set A is not assumed to be related to the well ordering A has qua $\mathcal{U}$ -set.

4. Commacategories and categories of diagrams.

Here we recall Lawvere's definition of commacategories and fix our conventions for them. We also define certain extended commacategories, related to Bénabou's cylindercategories.

Recall [8] that if F, G are functors with common codomain

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C} \tag{4.1}$$

then the commacategory (F,G) has as set of objects the set of triples (A,b,C) with $A \in |\mathcal{A}|$, $C \in |\mathcal{C}|$, and $b : AF \longrightarrow CG$ in $\mathcal{B}$. The hom set $(F,G)((A,b,C),(A',b',C'))$ is the set of such pairs (a,c) so that

$$a \in \mathcal{A}(A,A'),\ c \in \mathcal{C}(C,C'),$$

and so that the diagram

$$\begin{array}{ccc} AF & \xrightarrow{(a)F} & A'F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ CG & \xrightarrow[(c)G]{} & C'G \end{array}$$

commutes.

Composition in (F,G) is obvious.

The only commacategories we shall use are those where $\mathcal{A}$ or $\mathcal{C}$ is the category 1. If $\mathcal{A}$ is 1, F in (4.1) is determined by an object $B \in |\mathcal{B}|$, so $F = \epsilon_B$, and the commacategory consequently denoted (ϵ_B, G). The objects of this category we give a special notation; we write

$$B \xrightarrow{b} \underline{C}G \quad \text{for} \quad (0,b,C).$$

If G is a full inclusion, we leave out the underbar of C. If $|\mathcal{C}|$ is well ordered and each hom set of $\mathcal{B}$ is well ordered, we can well order $|(\epsilon_B, G)|$ by the obvious isomorphism

$$|(\epsilon_B,G)| \cong \coprod_{C \in |\mathcal{C}|} \mathcal{A}(B,CG) \quad ;$$

we also use (ϵ_B,G) to denote the category obtained by replacing $|(\epsilon_B,G)|$ by the corresponding ordinal number.

Turning back to the general case, we shall write $(\mathcal{A},G)$ instead of (F,G), provided F is a full inclusion of categories (in particular, if F is an identity functor).

In the case where $\mathcal{B}$ is "the" category of categories, or any other category with a 2-dimensional structure, one can throw in more morphisms by letting morphisms be non - commutative diagrams (4.2) but with a specified "2-cell" (in the category case: functortransformation) from the counterclockwise composite to the other. (Or one could do the dual thing.)

Doing this in the special case where $\mathcal{B}$ is CAT^V_V, $\mathcal{A}$ a subcategory Cat_o of CAT^V_V, and $\mathcal{C}$ is 1 leads to the following definition.

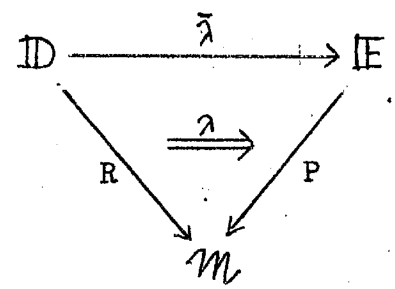
DEFINITION 4.1. Let $\mathcal{M} \in |\mathrm{CAT}^{\mathrm{V}}_{\mathrm{V}}|$. Then the <u>category of</u> Cat_o -<u>diagrams over</u> $\mathcal{M}$, denoted $[\mathrm{Cat}_o, \in_{\mathcal{M}}]$, is the category having object set

$$|[\mathrm{Cat}_o, \in_{\mathcal{M}}]| = |(\mathrm{Cat}_o, \in_{\mathcal{M}})|.$$

A morphism

$$(\mathbb{D} \xrightarrow{R} \mathcal{M}) \xrightarrow{\lambda} (\mathbb{E} \xrightarrow{P} \mathcal{M})$$

is a diagram in $\mathrm{CAT}^{\mathrm{V}}_{\mathrm{V}}$



i.e. a functor $\bar{\lambda} : \mathbb{D} \to \mathbb{E}$ and a transformation of functors $R \xrightarrow{\lambda} \bar{\lambda}.P$.
It is obvious how to compose morphisms.

This definition is well known. In the next section we show that if Cat_o admits a calculus of prelimits

$$\mathcal{M} \rightsquigarrow [\mathrm{Cat}_o, \in_{\mathcal{M}}]$$

is an endofunctor on $\mathrm{CAT}^{\mathrm{V}}_{\mathrm{V}}$ that carries the structure of a monad.

5. <u>The prelimit monads</u>.

We use the Grothendieck construction (in the specific form given in Section 2) to produce a monad T, η, μ on $\mathbf{CAT}^{\mathrm{V}}_{\mathrm{V}}$. We produce not just <u>one</u> monad, but rather one monad for each subcategory $\mathrm{Cat}_o \subseteq \mathrm{Cat}$ admitting a calculus of prelimits. Throughout **this section**, Cat_o is a fixed such **subcategory.**

DEFINITION 5.1. Let $\mathcal{A} \in |\mathrm{CAT}_V^V|$. Define $(\mathcal{A})T$ to be the category

$$[\mathrm{Cat}_o, \epsilon_{\mathcal{A}}].$$

If $F: \mathcal{A} \longrightarrow \mathcal{B}$ is a functor between categories in CAT_V^V, define $(F)T$ to be the functor "composing with F".

Clearly, T thus defined is an endofunctor on CAT_V^V. There is an obvious transformation d_o from the functor T to the functor with constant value Cat_o, given by

$$(d_o)_{\mathcal{A}} \ : [\mathrm{Cat}_o, \epsilon_{\mathcal{A}}] \xrightarrow{\partial_o} \mathrm{Cat}_o ,$$

where ∂_o is the usual "taking domain" functor. It will be convenient to use the following convention: If $F: \mathbb{A} \longrightarrow [\mathrm{Cat}_o, \epsilon_{\mathcal{A}}]$ is a functor, denote the composite functor

$$\mathbb{A} \xrightarrow{F} [\mathrm{Cat}_o, \epsilon_{\mathcal{A}}] \xrightarrow{\partial_o} \mathrm{Cat}_o$$

by $\mathbf{F}$.

DEFINITION 5.2. Let $\mathcal{A} \in |\mathrm{CAT}_V^V|$. Define a functor

$$\eta_{\mathcal{A}} \ : \mathcal{A} \longrightarrow \mathcal{A}T$$

by

$$(A)\eta_{\mathcal{A}} = (1 \xrightarrow{\epsilon_A} \mathcal{A})$$

and

$$(A \xrightarrow{a} A')\eta_{\mathcal{A}} \quad = \quad \begin{array}{ccc} 1 & \xrightarrow{I_1} & 1 \\ & \epsilon_A \searrow \overset{\epsilon_{[a]}}{\Longrightarrow} \swarrow \epsilon_{A'} & \\ & \mathcal{A} & \end{array} ,$$

$\epsilon_{[a]}$ being the obvious transformation.

Recall the Grothendieck construction $\bar{\mu}$ of Section 2.

DEFINITION 5.3. Let $\mathcal{A} \in |\mathrm{CAT}_V^V|$. Define a functor

$$\mu_{\mathcal{A}} : \mathcal{A}T^2 \longrightarrow \mathcal{A}T$$

as follows. Let $\mathbb{D} \xrightarrow{R} \mathcal{A}T$ be an object in $\mathcal{A}T^2$. Put $\overline{R\mu_{\mathcal{A}}}$ equal to $(\overline{R})\bar{\mu}$, and put $R\mu_{\mathcal{A}}$ equal to the functor

$$\overline{R\mu_{\mathcal{A}}} \;=\; (\overline{R})\bar{\mu} \xrightarrow{R\mu_{\mathcal{A}}} \mathcal{A}$$

given on objects by

$$(X_D)(R\mu_{\mathcal{A}}) \;=\; (X)DR ,$$

where

$$X_D \in (\overline{R})\bar{\mu} \;=\; \coprod_{D \in |\mathbb{D}|} |D\overline{R}| ,$$

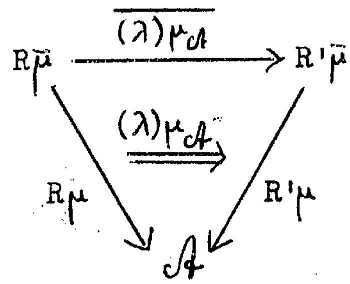
and on morphisms by sending $x_d : X_D \longrightarrow X'_{D'}$ (where $d : D \longrightarrow D'$ and $x : (X)(d\overline{R}) \longrightarrow X'$) to

$$(X)(DR) \xrightarrow{(dR)_X} (X)(d\overline{R})(D'R) \xrightarrow{(x)(D'R)} (X')(D'R) .$$

On morphisms, $\mu_{\mathcal{A}}$ is given as follows. A morphism $\lambda : R \longrightarrow R'$, i.e. a triangle

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\quad L \quad} & \mathbb{D}' \\ & \overset{\lambda}{\Longrightarrow} & \\ {\scriptstyle R}\searrow & & \swarrow{\scriptstyle R'} \\ & [\mathrm{Cat}_0, \epsilon_{\mathcal{A}}] & \end{array}$$

goes by $\mu_{\mathcal{A}}$ to the triangle



where $\overline{(\lambda)\mu_{\mathcal{A}}}$ and $(\lambda)\mu_{\mathcal{A}}$ are given by (i) and (ii), respectively:

(i) $\overline{(\lambda)\mu_{\mathcal{A}}} = \lambda\bar{\mu}$ is the functor given on objects by

$$X_D \rightsquigarrow ((X)\overline{\lambda_D})_{DL} \quad , X \in |D\bar{R}|$$

and on morphisms by

$$X_D \xrightarrow{x_d} X'_{D'} \rightsquigarrow (X\,\overline{\lambda_D})_{DL} \xrightarrow{((x)\,\overline{\lambda_{D'}})_{dL}} (X'\,\overline{\lambda_{D'}})_{D'L} \, ,$$

where $d : D \longrightarrow D'$, $x : X\,d\bar{R} \longrightarrow X'$ in $D'\bar{R}$, and therefore

$$(X)\,\overline{\lambda}_D(dL\overline{R'}) = (X)(d\bar{R})\,\overline{\lambda_{D'}} \xrightarrow{(x)\overline{\lambda_{D'}}} X'\,\overline{\lambda_{D'}} \, ;$$

(ii) $(\lambda)\mu_{\mathcal{A}}$ is the transformation given by

$$((\lambda)\mu_{\mathcal{A}})_{X_D} = (X)DR \xrightarrow{(\lambda_D)_X} (X)\,\overline{\lambda_D}\,(DLR') \, .$$

It is straightforward to check that $\eta_{\mathcal{A}}$ and $\mu_{\mathcal{A}}$ are functors, likewise that they are natural in $\mathcal{A} \in |CAT^V_V|$, so that

$$\eta : I_{CAT^V_V} \Rightarrow T \quad \text{and} \quad \mu : T^2 \Rightarrow T$$

are transformations between the indicated functors.

THEOREM 5.4. The T, η, μ from Definitions 5.1, 5.2, and 5.3, respectively, form a monad on CAT^V_V.

PROOF. We are required to prove that for $\mathcal{A} \in |CAT^V_V|$ arbitrary, we have the following equalities of functors:

(5.1) $$\eta_{\mathcal{A}T} \cdot \mu_{\mathcal{A}} = (\eta_{\mathcal{A}})T \cdot \mu_{\mathcal{A}} = I_{\mathcal{A}T}$$

(5.2) $$\mu_{\mathcal{A}T} \cdot \mu_{\mathcal{A}} = (\mu_{\mathcal{A}})T \cdot \mu_{\mathcal{A}} .$$

For each of the three equations, it suffices to prove that the two functors involved agree on objects, and then to produce a natural transformation φ between the functors with each φ_R the appropriate identity morphism.

The fact that

(5.3) $$|\eta_{\mathcal{A}T} \cdot \mu_{\mathcal{A}}| = |I_{\mathcal{A}T}|$$

follows immediately from the definition and from the particular choice of sums in $\mathbb{S}$ making (2.2) hold. Also, the obvious transformation

$$\eta_{\mathcal{A}T} \cdot \mu_{\mathcal{A}} \xrightarrow{\varphi} I_{\mathcal{A}T} ,$$

(which can of course be constructed for any choice of sums) has here $\varphi_R = I_R$ $(R \in |\mathcal{A}T|)$.

Next

(5.4) $$|(\eta_{\mathcal{A}})T \cdot \mu_{\mathcal{A}}| = |I_{\mathcal{A}T}| .$$

For, given an object $R : \mathbb{D} \longrightarrow \mathcal{A}$ in $\mathcal{A}T$, we have

$$(R)(\eta_{\mathcal{A}}T) : \mathbb{D} \longrightarrow [\mathrm{Cat}_0, \epsilon_{\mathcal{A}}] = \mathcal{A}T$$

given by

$$D \rightsquigarrow (1 \xrightarrow{\epsilon_D} \mathcal{A}); \quad (d : D \longrightarrow D') \rightsquigarrow \epsilon_{[d]} .$$

Now it follows easily from the definition of μ and from (2.1) that (5.4) holds. And again, there is no trouble in constructing a natural φ with

$\varphi_R = I_R$ for all $R \in |\mathcal{A}T|$.

One might briefly recapitulate the arguments so far by saying: The unit laws for the monad T, η, μ come from the unit laws (2.1) and (2.2) of the sum formation in $\mathfrak{S}$. As one would expect then, the associativity law of the monad (5.2) is derived from associativity of sums in $\mathfrak{S}$, i.e. equation (2.3). - The proof of (5.2) is just as complicated as straightforward. We commence by giving a list of where the different elements occurring in the proof belong. (Strictly speaking, since we do not work with disjoint sets, elements do not belong in unique places. One should rather say, then, that the list tells us which composition mapping we apply to an element and its neighbor on the paper.

<u>Address list</u>.

$\mathbb{D} \xrightarrow{R} [Cat_o, \in_{[Cat_o, \in_{\mathcal{A}}]}] \in |\mathcal{A}T^3|$

$D \xrightarrow{d} D'$, a morphism in $\mathbb{D}$

$X \in |D\bar{R}|$

$X' \in |D'\bar{R}|$

$x : (X)(d)\bar{R} \longrightarrow X'$, a morphism in $D'\bar{R}$

$Y \in |X\overline{D\bar{R}}|$

$Y' \in |X'\overline{D'\bar{R}}|$

$y : (Y)(\overline{d\bar{R}})_X(x)\overline{D'\bar{R}} \longrightarrow Y'$, a morphism in $(X')\overline{D'\bar{R}}$.

With this $(R)\bar{\mu}_{\mathcal{A}T} \xrightarrow{(R)\mu_{\mathcal{A}T}} [Cat_o, \in_{\mathcal{A}}]$ is by definition described as follows: $(R)\bar{\mu}_{\mathcal{A}T}$ =

$$(5.5)\quad \begin{cases} \text{object set} = \coprod_{D \in |\mathbb{D}|} |D\bar{R}| \\ \hom(X_D, X'_{D'}) = \coprod_{d \in \mathbb{D}(D,D')} (D'\bar{R})(X\, d\bar{R}, X'), \end{cases}$$

with $R\mu_{\mathcal{A}T}$ given by

$$(5.6)\quad \begin{array}{l} X_D \leadsto (X)DR \\ x_d \leadsto (X)DR \xrightarrow{(dR)_X} (X)d\bar{R}\, D'R \xrightarrow{(x)(D'R)} (X')D'R\,. \end{array}$$

Thus $\overline{R\mu_{\mathcal{A}T}} : R\bar{\mu}_{\mathcal{A}T} \longrightarrow cat_o$ is given by

$$(5.7)\quad \begin{array}{l} X_D \leadsto (X)\overline{DR} \\ x_d \leadsto (X)\overline{DR} \xrightarrow{\overline{(dR)_X}} (X)\, d\bar{R}\, \overline{D'R} \xrightarrow{(x)\overline{D'R}} (X')\overline{D'R}. \end{array}$$

Applying $\mu_{\mathcal{A}}$ to $(R)\mu_{\mathcal{A}T}$ (given in (5.6)) gives by definition of μ a functor $(R)\mu_{\mathcal{A}T}\,\mu_{\mathcal{A}}$:

$$\overline{(R\mu_{\mathcal{A}T})}\bar{\mu} = (R\mu_{\mathcal{A}T})\,\bar{\mu}_{\mathcal{A}} \longrightarrow \mathcal{A}.$$

Applying the definition of $\bar{\mu}$ to (5.7) describes then $R\mu_{\mathcal{A}T}\,\bar{\mu}_{\mathcal{A}}$ as the following $\mathcal{O}$-category:

$$\begin{cases} \text{object set} = \coprod_{X_D \in |R\bar{\mu}_{\mathcal{A}T}|} |(X_D)\overline{R\mu_{\mathcal{A}T}}| \\ \hom(Y_{X_D}, Y'_{X'_{D'}}) = \\ \qquad \coprod_{x_d \in R\bar{\mu}_{\mathcal{A}T}(X_D, X'_{D'})} ((X'_{D'})\overline{R\mu_{\mathcal{A}T}})(Y(x_d\,\overline{R\mu_{\mathcal{A}T}}), Y')\,. \end{cases}$$

Using (5.5) and (5.7), this may be written as

$$(5.8)\quad \begin{cases} \text{object set} = \coprod_{X_D \in \coprod_{D \in |\mathbb{D}|} |D\bar{R}|} |(X)\overline{DR}| \\ \hom(Y_{X_D}, Y'_{X'_{D'}}) = \coprod_{x_d \in \coprod_{d \in \mathbb{D}(D,D')} (D'\bar{R})(Xd\bar{R}, X')} ((X')\overline{D'R})(Y\ \overline{dR_X}\ (x)(\overline{D'R}), Y') . \end{cases}$$

It is equipped with a functor $R\mu_{\mathcal{A}T}\mu_{\mathcal{A}}$ to $\mathcal{A}$, which (using (5.6)) is

$$Y_{X_D} \rightsquigarrow (Y)((X_D)(R\mu_{\mathcal{A}T})) = (Y)((X)DR)$$

$$y_{x_d} \rightsquigarrow$$

$$(5.9)\quad (Y)((X_D)(R\mu_{\mathcal{A}T})) \xrightarrow{((x_d)(R\mu_{\mathcal{A}T}))_Y} (Y)((x_d)(R\mu_{\mathcal{A}T}))((X_D)R\mu_{\mathcal{A}T}) \longrightarrow$$
$$\xrightarrow{(y)((X'_{D'})(R\mu_{\mathcal{A}T}))} (Y')((X'_{D'})(R\mu_{\mathcal{A}T}))$$

$$= (Y)((X)DR) \xrightarrow{((dR)_X)_Y} (Y(\overline{dR}_X))((X\ d\bar{R})D'R) \longrightarrow$$
$$\xrightarrow{((x)(D'R))_{(Y)(\overline{dR}_X)}} ((Y)(\overline{dR}_X))((x)\overline{D'R}((X')D'R) \longrightarrow$$
$$\xrightarrow{(y)((X')(D'R))} (Y')((X')D'R) .$$

On the other hand $(R)(\mu_{\mathcal{A}})T$ is the composite functor

$$(5.10)\quad \mathbb{D} \xrightarrow{R} [Cat_o, \epsilon_{[Cat_o, \epsilon_{\mathcal{A}}]}] \xrightarrow{\mu_{\mathcal{A}}} [Cat_o, \epsilon_{\mathcal{A}}] .$$

Taking the value of $\mu_{\mathcal{A}}$ on this gives a functor to $\mathcal{A}$ from the category $(R.\bar{\mu}_{\mathcal{A}})\bar{\mu}$. According to the definitions, this category has

(5.11) $$\begin{cases} \text{object set} = \coprod_{D \in |\mathbb{D}|} |(DR)\bar{\mu}_{\mathcal{A}}| \\ \hom(Z_D, Z'_{D'}) = \coprod_{d \in \mathbb{D}(D,D')} ((D'R)\bar{\mu}_{\mathcal{A}})(Z_D\, dR\, \bar{\mu}_{\mathcal{A}}, Z'_{D'}) \end{cases},$$

and again using the definition of $\bar{\mu}_{\mathcal{A}}$ in terms of $\bar{\mu}$, and taking Z_D, $Z'_{D'}$ to be $(Y_X)_D$, $(Y'_{X'})_{D'}$, respectively, (5.11) becomes equal to the category

(5.12) $$\begin{cases} \text{object set} = \coprod_{D \in |\mathbb{D}|} \coprod_{X \in |DR|} |X\overline{DR}| \\ \hom((Y_X)_D, (Y'_{X'})_{D'}) = \coprod_{d \in \mathbb{D}(D,D')} (\overline{D'R}\,\bar{\mu})((Y\, \overline{dR}_X)_{Xd\bar{R}}, Y'_{X'}) \\ = \coprod_{d \in \mathbb{D}(D,D')} \coprod_{x \in (D'\bar{R})(X\, d\bar{R},\, X')} ((X')\overline{D'R})(Y\, \overline{dR}_X\, (x)(\overline{D'R}), Y'). \end{cases}$$

Also, $\mu_{\mathcal{A}}$ applied to (5.10) gives by definition the following functor from (5.11) to $\mathcal{A}$

$$Z_D \rightsquigarrow (Z)((DR)\mu_{\mathcal{A}})$$

$$z_d \rightsquigarrow (Z)((DR)\mu_{\mathcal{A}}) \xrightarrow{((dR)\mu_{\mathcal{A}})_Z} (Z)(dR\,\bar{\mu}_{\mathcal{A}})((D'R)\mu_{\mathcal{A}}) \longrightarrow$$

$$\xrightarrow{(z)((D'R)\mu_{\mathcal{A}})} (Z')((D'R)\mu_{\mathcal{A}}) .$$

If $Z = Y_X$, $z = y_x$ this is easily seen to be

$$(Y_X)_D \rightsquigarrow (Y)((X)DR)$$

(5.13) $$(y_x)_d \rightsquigarrow (Y)((X)DR) \xrightarrow{((dR)_X)_Y} (Y\,\overline{dR}_X)((Xd\bar{R})D'R) \longrightarrow$$

$$\xrightarrow{((x)D'R)_{(Y)(\overline{dR})_X}} Y\ \overline{dR}_X\ ((x)\overline{D'R})(X'D'R) \xrightarrow{(y)((X')D'R)} (Y')(X'D'R)$$

Now no matter how sums were chosen, one would get a functor φ_R from $(R)\mu_{\mathcal{A}T}\bar{\mu}_{\mathcal{A}}$ (given by (5.8)) to $(R)(\mu_{\mathcal{A}})T\bar{\mu}_{\mathcal{A}}$ by sending Y_{X_D} to $(Y_X)_D$, y_{x_d} to $(y_x)_d$; and using (5.9) and (5.12) one immediately gets

$$\varphi_R \circ (R)(\mu_{\mathcal{A}})T\mu_{\mathcal{A}} = (R)\mu_{\mathcal{A}T}\mu_{\mathcal{A}} .$$

Clearly, φ_R is an isomorphism in $[Cat_o, \in_{\mathcal{A}}]$. But using the associativity of sums in $\mathcal{S}$ (equation (2.3)), one immediately gets that φ_R is the identity of $(R)\mu_{\mathcal{A}T}\mu_{\mathcal{A}} = (R)(\mu_{\mathcal{A}})T\mu_{\mathcal{A}}$ in $[Cat_o, \in_{\mathcal{A}}]$. But the existence of a natural transformation φ whose instances are equalities implies that the domain and codomain of φ are equal functors.

The following proposition is easily seen to hold; it just says something about the size of hom sets in $[Cat_o, \in_{\mathcal{A}}]$.

PROPOSITION 5.5. The monad T on CAT_V^V defines by restriction a monad (also called T) on CAT_V^U.

Recall [3] that a monad T, η, μ on a category $\mathcal{C}$ defines a category $\mathcal{C}^T$ with objects pairs (C, ξ) $(C \in |\mathcal{C}|, \xi : CT \longrightarrow C)$ such that $\xi T \circ \xi = \mu_C \circ \xi$ and $\eta_C \circ \xi = I_C$, and with

$$(\mathcal{C}^T)((C, \xi), (C', \xi')) \subseteq \mathcal{C}(C, C') ,$$

namely those $g : C \longrightarrow C'$ which satisfy $gT \circ \xi' = \xi \circ g$. Beck and Lawvere used the term <u>algebras over T</u> for the objects of $\mathcal{C}^T$; ξ they called the <u>structure</u> of the algebra; the g's are called <u>homomorphisms</u>. Using this terminology, we state

DEFINITION 5.6. An algebra for the monad T, η, μ is called a prelimit algebra (with respect to Cat_o). A homomorphism for the monad is called a prelimit homomorphism.

6. The π_0 -functor.

Recall that a quotient object q of an object B in a category is an equivalence class of epimorphisms with domain B, the equivalence relation being

$$B \xrightarrow{e} X \sim B \xrightarrow{e'} X'$$

iff there exists an isomorphism $j : X \longrightarrow X'$ with $e \cdot j = e'$.

We shall say that a family E of epimorphisms in $\mathcal{B}$ form a (full) choice of quotient objects if

(i) $I_B \in E$ for all $B \in |\mathcal{B}|$.

(ii) E is stable under composition

(iii) in each quotient object, there is at most (precisely) one element from E.

(If the hom sets of $\mathcal{B}$ are not disjoint, the definition must be modified in an obvious way).

In the category $\mathcal{S}$ there is a canonical full choice $\mathcal{O}_e$ of quotient objects:

DEFINITION 6.1. If $A \xrightarrow{f} B$ is an epimorphism in $\mathcal{S}$, i.e. an onto map between ordinal numbers, we take $f \in \mathcal{O}_e$ iff the following holds:

$$b < b' \text{ in } B \Longleftrightarrow \min(f^{-1}(b)) < \min(f^{-1}(b')) \text{ in } A.$$

If this holds, we shall also say: f gives B the quotient well ordering.

Defining <u>(**full**) choice of subobjects in a category</u> as the notion dual to (full) choice of quotient objects, we also have a canonical choice $\mathcal{O}_m$ of subobjects in $\mathcal{S}$,

DEFINITION 6.2. If

$$A \xrightarrow{f} B$$

is a monomorphism in $\mathcal{S}$, i.e. a 1-1 map of ordinal numbers, we take $f \in \mathcal{O}_m$ iff f is orderpreserving.

The choices $\mathcal{O}_e$ and $\mathcal{O}_m$, together with all identity maps of U-Ens, form a (non-full) choice of quotient- and subobjects in U-Ens.

It is well known that the category of sets is a reflective subcategory of the category of categories (as well as a coreflective one). We are going to construct a specific reflection functor $\pi_0 : \mathrm{Cat}^U \to \mathcal{S}$ for the specific categories Cat^U and $\mathcal{S}$.

Recall that a <u>connected component</u> of a category $\mathcal{A}$ is a subclass $C \subsetneq |\mathcal{A}|$ such that any two objects A, A' in the subclass can be connected by a finite string of morphisms in $\mathcal{A}$ with alternating direction

$$A \longrightarrow B_1 \longleftarrow B_2 \longrightarrow B_3 \longleftarrow \cdots \longrightarrow B_n \longleftarrow A' \ ,$$

and so that C is maximal with respect to this property. Then $|\mathcal{A}|$ is the disjoint union of its connected components.

Now let $\mathbb{A} \in |\mathrm{Cat}^U|$. The set of connected components of $\mathbb{A}$ determines a quotient object of $|\mathbb{A}|$, and since $|\mathbb{A}| \in |\mathcal{S}|$, there is a unique $\mathcal{O}_e$ -chosen quotient, denoted

$$|\mathbb{A}| \xrightarrow{p_{\mathbb{A}}} (\mathbb{A})\pi_0 \ .$$

The map $p_{\mathbb{A}}$ can uniquely be extended to a functor $\mathbb{A} \xrightarrow{p_{\mathbb{A}}} (\mathbb{A})\pi_o i$ (i : the inclusion functor of discrete $\mathcal{O}$ -categories into Cat^U). Finally there is a unique way of extending $\pi_o : |Cat^U| \longrightarrow |\mathcal{S}|$ to a functor

$$\pi_o : Cat^U \longrightarrow \mathcal{S},$$

in such a way that the $p_{\mathbb{A}}$'s form a natural transformation

$$\pi_o \circ i \longrightarrow I_{Cat^U}.$$

We shall usually write $\{A\}$ for $(A)p_{\mathbb{A}}$, $A \in |\mathbb{A}|$. The restriction of π_o to Cat will also be denoted π_o.

In the next section we shall as a main lemma need the following result establishing a connection between the monad T, η, μ of Section 4 and the π_o just defined. By composing with the obvious isomorphism

$$[Cat, \in_1] \cong Cat$$

the functors $\pi_o : Cat \longrightarrow \mathcal{S}$ and $i : \mathcal{S} \longrightarrow Cat$ give functors

$$[Cat, \in_1] \underset{\bar{i}}{\overset{\bar{\pi}_o}{\rightleftarrows}} \mathcal{S}.$$

LEMMA 6.3. The following diagram of functors commutes

$$\begin{array}{ccccc}
[Cat, \in_{[Cat,\in_1]}] & & \xrightarrow{\mu_1} & & [Cat, \in_1] \\
\Big\downarrow (\bar{\pi}_o\,\bar{i})T & & & & \Big\downarrow \bar{\pi}_o \\
[Cat, \in_{[Cat,\in_1]}] & \xrightarrow[\mu_1]{} & [Cat, \in_1] & \xrightarrow[\bar{\pi}_o]{} & \mathcal{S}
\end{array}$$

PROOF. We shall produce a natural transformation Φ from the clockwise composite functor to the counterclockwise composite; Φ will turn out to be the identity. Let R be an object in the upper left category, i.e.,

$$R : \mathbb{A} \longrightarrow [\mathrm{Cat}, \epsilon_1] .$$

There is an obvious functor, natural in R,

$$\varphi_R : R\mu_1 \longrightarrow R\,\bar{\pi}_0\,\bar{i}\,\mu_1$$

given by

$$X_A \overset{\varphi_R}{\rightsquigarrow} (Xp)_A \ , \ X \in |A\bar{R}|,$$

$$x_a \overset{\varphi_R}{\rightsquigarrow} 0_a \ ,$$

where in the last relation $a : A \longrightarrow A'$ in $\mathbb{A}$, $x : (X)(a\bar{R}) \longrightarrow X'$ in $A'\bar{R}$ and where

$$0_a \in \coprod_{\mathbb{A}(A,A')} \delta(\{X\ a\bar{R}\}, \{X'\}) .$$

We then put

$$(6.1) \qquad \Phi_R = (\varphi_R)\bar{\pi}_0 : R\,\mu_1\bar{\pi}_0 \longrightarrow R\,(\bar{\pi}_0\,\bar{i})T\,\mu_1\bar{\pi}_0 .$$

We claim that Φ_R is an isomorphism. To see this, it suffices to examine $|\varphi_R|$. First, $|\varphi_R|$ is onto, so Φ_R is onto. Secondly, let $(Xp)_A$ be connected to $(X'p)_{A'}$ in $R\,\bar{\pi}_0\,\bar{i}\,\mu_1$, e.g. by

$$(Xp)_A \xrightarrow{0_{a_1}} (X_1p)_{A_1} \xleftarrow{0_{a_2}} (X_2p)_{A_2} \longrightarrow \cdots \xleftarrow{0_{a_n}} (X'p)_{A'} ;$$

this means by definition that $(X)(a_1\bar{R})$ can be connected to X_1 in $A_1\bar{R}$; thus X_A can be connected to $(X_1)_{A_1}$ in $R\tilde{\mu}_1$; and it means that $(X_2)(a_2\bar{R})$ can be connected to X_1 in $A_1\bar{R}$; thus $(X_2)_{A_2}$ can be connected to $(X_1)_{A_1}$ in $R\,\tilde{\mu}_1$; and so on.

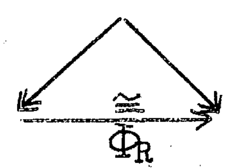
This proves that Φ_R is 1-1 and onto. We note that Φ_R fits into a commutative diagram

$$\begin{array}{ccc} \coprod_{A \in |\mathbb{A}|} |A\bar{R}| & \longrightarrow & \coprod_{A \in |\mathbb{A}|} (A\bar{R})\pi_o \\ {\scriptstyle x_A \rightsquigarrow \{x_A\}} \Big\downarrow & & \Big\downarrow {\scriptstyle \{x\}_A \rightsquigarrow \{\{x\}_A\}} \\ (R\,\mu_1)\bar{\pi}_o & \xrightarrow[\Phi_R]{\cong} & (R)(\bar{\pi}_o\bar{i})\mu_1\bar{\pi}_o \end{array} ,$$

where the two vertical maps are the $\mathcal{O}_e$ -chosen quotient maps, and where the upper horizontal map is the sum over $|\mathbb{A}|$ of the $\mathcal{O}_e$ -chosen quotient maps

$$|A\bar{R}| \xrightarrow{p_A} (A\bar{R})\pi_o ,$$

$A \in |\mathbb{A}|$. But now it is easy to see that by the special choice $\mathcal{O}_e$ of quotients in $\mathcal{S}$, and the definition of sums in $\mathcal{S}$, a sum of $\mathcal{O}_e$ -chosen quotient maps will itself be an $\mathcal{O}_e$ -chosen quotient map. And since $\mathcal{O}_e$ is stable under composition of maps, Φ_R fits in a diagram



,

where the arrows $\twoheadrightarrow$ are $\mathcal{O}_e$ -chosen. Since there is only one $\mathcal{O}_e$ -chosen map in a given quotient object, we conclude that Φ_R is the identity. Since it is also natural in R, the lemma follows.

Because of an application in Chapter II, we shall also define a π_o -functor for arbitrary small categories

(6.6) $$\pi_o : CAT_U^U \longrightarrow U\text{-Ens} .$$

Let $\mathcal{A} \in \mathrm{CAT}_U^U$. Then connectedness defines an equivalence relation on $|\mathcal{A}|$. Take $\mathcal{A}\pi_o$ to be the set of equivalence classes.

We shall not notationally distinguish between the two different π_o - functors defined. In each case it will be clear which is meant.

7. $\mathcal{S}$ as a colimit algebra.

We shall endow $\mathcal{S}$ with the structure of a prelimit algebra with respect to all of Cat (Definition 5.6).

DEFINITION 7.1. Let $\mathcal{B} \in |\mathrm{CAT}_V^V|$ be arbitrary. We say that a functor φ

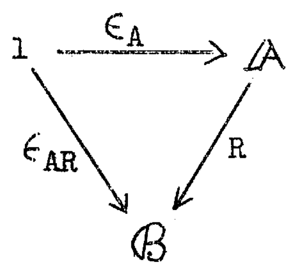
$$\mathcal{B}T = [\mathrm{Cat}, \epsilon_{\mathcal{A}}] \xrightarrow{\varphi} \mathcal{B}$$

is a <u>colimit</u> <u>assignment</u> on $\mathcal{B}$ if

(i) $\eta_{\mathcal{B}} \cdot \varphi = I_{\mathcal{B}}$, and

(ii) for arbitrary $\mathbb{A} \xrightarrow{R} \mathcal{B}$ $(\mathbb{A} \in |\mathrm{Cat}|)$, $R\varphi$ is a colimit of R with φ of the arrows

(A-R):



being mappings in $\mathcal{B}$ constituting $R\varphi$ as $\underrightarrow{\lim}(R)$. We shall call $\mathcal{B}, \varphi$ a <u>colimit</u> <u>algebra</u> or φ <u>an</u> <u>associative</u> <u>colimit</u> <u>assignment</u> if φ at the same time is a prelimit algebra structure and a colimit assignment.

Note that condition (i) in the definition is automatically satisfied if φ is a prelimit structure. It should already now be clear that a colimit algebra has colimits which fit together in a very simple way.

Recall the functors $\bar{\pi}_o$, $\bar{1}$ of the preceding section:

$$\mathcal{S} \underset{\bar{1}}{\overset{\bar{\pi}_o}{\leftrightarrows}} [\mathrm{Cat}, \epsilon_1] = 1T .$$

With the terminology of Definition 7.1, we shall prove

THEOREM 7.2. Let ξ be the composite functor

$$\mathcal{S}T \xrightarrow{\bar{1}T} 1T^2 \xrightarrow{\mu_1} 1T \xrightarrow{\bar{\pi}_o} \mathcal{S} .$$

Then $\mathcal{S}, \xi$ is a colimit algebra, and $\bar{\pi}_o$ is a homomorphism.

PROOF. We first prove that ξ is a prelimit structure. Actually, most of the work for this has been done in Section 6. First

$$\eta_{\mathcal{S}} \cdot \xi = \eta_{\mathcal{S}} \cdot \bar{1}T \cdot \mu_1 \cdot \bar{\pi}_o = \bar{1} \cdot \eta_{1T} \cdot \mu_1 \ \bar{\pi}_o$$
$$= \bar{1} \cdot \bar{\pi}_o = I .$$

Secondly

$$\xi T \cdot \xi = \bar{1}T^2 \cdot \mu_1 T \cdot (\bar{\pi}_o \cdot \bar{1})T \cdot \mu_1 \cdot \bar{\pi}_o ,$$

and by Lemma 6.3, the equation continues

$$= \bar{1}T^2 \cdot \mu_1 T \cdot \mu_1 \cdot \bar{\pi}_o = \bar{1}T^2 \cdot \mu_{1T} \cdot \mu_1 \cdot \bar{\pi}_o$$
$$= \mu_{\mathcal{S}} \cdot \bar{1}T \cdot \mu_1 \cdot \bar{\pi}_o = \mu_{\mathcal{S}} \cdot \xi .$$

This proves that ξ is a prelimit structure.

To prove that it is a colimit assignment, consider a functor

$$\mathbb{A} \xrightarrow{R} \mathcal{S},$$

$R \in |\mathcal{S}T|$; in particular, $\mathbb{A} \in |\mathrm{Cat}|$. Also consider the morphisms in $\mathcal{S}T$

$$(A\text{-}R):\qquad \begin{array}{ccc} 1 & \xrightarrow{\ \epsilon_A\ } & \mathbb{A} \\ & {\scriptstyle \epsilon_{AR}}\searrow \quad \cdot \quad \swarrow{\scriptstyle R} & \\ & \mathfrak{S} & \end{array}$$

$A \in |\mathbb{A}|$. Now $R\xi$ is π_0 of an $\mathcal{O}$-category $(R)\bar{\imath}T\mu_1$ with object set

$$\coprod_{A \in |\mathbb{A}|} AR \ ;$$

the morphisms $(A\text{-}R)$ are easily seen to go to the set mappings

$$AR \xrightarrow{\ incl_A\ } R\xi$$

given by

$$X \rightsquigarrow X_A p \ .$$

The $incl_A$'s constitute $R\xi$ as a colimit of R. First, it is clear from the definition of morphisms in $R\bar{\imath}T\mu_1$ that given $a : A \longrightarrow A'$ in $\mathbb{A}$ and $X \in AR$, we can connect X_A to $((X)(aR))_{A'}$ in $R\bar{\imath}T\mu_1$, so that

$$(X_A)p \ = \ ((X)(aR))_{A'}p \ .$$

This gives $aR \circ incl_{A'} = incl_A$. Next, let a family of mappings f_A in $\mathfrak{S}$ be given : $f_A : AR \longrightarrow M$, such that for each $a : A \longrightarrow A'$ in $\mathbb{A}$

(7.1) $$f_A \ = \ aR \circ f_{A'} \ .$$

Define $f_\infty : R\xi \longrightarrow M$ by

$$X_A p \overset{f_\infty}{\rightsquigarrow} (X)f_A \ ,$$

$X \in AR$. This is well defined, for if $X_A p = X'_{A'}p$, i.e. $\{X_A\} = \{X'_{A'}\}$, then it is easy, using the equations (7.1), to prove $(X)f_A \ = \ (X')f_{A'}$.

It is clear that $\mathrm{incl}_A \circ f_\infty = f_A$, and that f_∞ is the unique mapping $R\xi \longrightarrow M$ having this property. This proves that ξ has property (ii) of Definition 7.1, so it is a colimit assignment. Finally, π_c's being a homomorphism follows directly from Lemma 6.1. The proof is complete.

DEFINITION 7.3. Given a functor $\mathbb{A} \xrightarrow{R} \mathcal{S}$ with $\mathbb{A} \in |\mathrm{Cat}|$, so that $R\xi$ is defined. Call $R\xi$ the <u>natural</u> <u>colimit</u> of R, and denote it

$$R\xi = \varinjlim(R) .$$

REMARK 7.4. Let $\mathcal{B}$, $\mathcal{B}T \xrightarrow{\varphi} \mathcal{B}$ be a colimit algebra (or just a finite - sum algebra, i.e. with T defined with respect to the subcategory $\mathcal{S}_{fin} \subseteq \mathrm{Cat}$). Denote $\coprod_2 A_i$ by $A_0 + A_1$. Then

$$(A_0+A_1)+A_2 = \coprod_3 A_i = A_0+(A_1+A_2),$$

and the following diagrams commute

$$\begin{array}{ccccc} A_0 & \xrightarrow{\mathrm{incl}_0} & A_0+A_1 & \xrightarrow{\mathrm{incl}_0} & (A_0+A_1)+A_2 \\ \| & & & & \| \\ A_0 & & \xrightarrow[\mathrm{incl}_0]{\hspace{6em}} & & \coprod_3 A_i \end{array}$$

$$\begin{array}{ccccc} A_1 & \xrightarrow{\mathrm{incl}_1} & A_0+A_1 & \xrightarrow{\mathrm{incl}_0} & (A_0+A_1)+A_2 \\ \| & & & & \| \\ A_1 & & \xrightarrow[\mathrm{incl}_1]{\hspace{6em}} & & \coprod_3 A_i \end{array}$$

$$\begin{array}{ccc} A_2 & \xrightarrow{\mathrm{incl}_1} & (A_0+A_1)+A_2 \\ \| & & \| \\ A_2 & \xrightarrow[\mathrm{incl}_2]{} & \coprod_3 A_i \end{array} \quad ;$$

similarly for the three analogous diagrams formed by considering $A_0+(A_1+A_2)$. One might express this by saying: binary sums are associative in a colimit algebra. This is justification for using "associative colimit assignment" synonymous with "colimit algebra structure".

We use this remark to show that $\mathfrak{S}$ is in some sense the smallest "category of sets" which can carry associative colimits - at least if one admits that the only natural category of sets smaller than $\mathfrak{S}$ is the skeletal one. Precisely

PROPOSITION 7.5. The skeletal category $\overline{\mathfrak{S}}$ of sets in U does not admit an associative colimit assignment, not even a finite associative sum assignment.

PROOF. Since the category is skeletal, the chosen sum diagram for $\aleph_0$, $\aleph_0$ is

$$\aleph_0 \xrightarrow{\text{incl}_0} \aleph_0 \xleftarrow{\text{incl}_1} \aleph_0 . \tag{7.3}$$

Assume that the binary sum formation comes from a structure φ as in Remark 7.4 (with $\overline{\mathfrak{S}}$ for $\mathcal{B}$). Putting $A_0 = A_1 = A_2 = \aleph_0$ in the commutative diagrams there, one gets (with incl_0, incl_1 the maps in (7.3):

$$\text{incl}_0 \,.\, \text{incl}_1 = \text{incl}_1 \,.\, \text{incl}_0 : \aleph_0 \to \aleph_0$$

which is contrary to the fact that incl_0 and incl_1 have disjoint images. This proves the proposition.

8. Further properties of $\mathfrak{S}$, $\mathfrak{T}$.

For future reference, we collect some useful facts about $\mathfrak{S}$. We have already in Definition 6.1 put a choice of quotient objects on $\mathfrak{S}$. These are related to the natural colimit structure $\mathfrak{F}$.

PROPOSITION 8.1. Let $q_i : A_i \longrightarrow B_i$ be $\mathcal{O}_e$ -chosen quotients in $\mathfrak{S}$ for $i \in I \in |\mathfrak{S}|$. Then

$$\coprod_I A_i \xrightarrow{\coprod q_i} \coprod_I B_i$$

is an $\mathcal{O}_e$ -chosen quotient.

The proof is easy.

Let $\mathcal{E} \in |\mathrm{Cat}|$ be given by

$$|\mathcal{E}| = 2$$

$$\mathcal{E}(0, 1) = 0$$

(8.1)

$$\mathcal{E}(0, 0) = \mathcal{E}(1, 1) = 1$$

$$\mathcal{E}(1, 0) = 2 ,$$

i.e. it looks like

$$\overset{\circ}{1} \rightrightarrows \overset{\circ}{0} \;;$$

let $\mathcal{E} \xrightarrow{R} \mathfrak{S}$ give the diagram

$$(1)R \underset{g}{\overset{f}{\rightrightarrows}} (0)R$$

in $\mathfrak{S}$. Then there is a smallest equivalence relation $\sim$ on $(0)R$ with

$$(a)f \sim (a)g \quad \forall a \in (1)R .$$

So a quotient of $(0)R$ is determined. It is now easy to see that if q is the $\mathcal{O}_e$ -chosen quotient, then

$$(1)R \underset{g}{\overset{f}{\rightrightarrows}} (0)R \xrightarrow{q} Q$$

is also the colimit diagram according to ξ, in particular (0-R):

(8.2)
$$\begin{array}{ccc} 1 & \xrightarrow{\epsilon_0} & \xi \\ & \searrow^{\epsilon_{(0)R}} \quad \swarrow_{R} & \\ & \mathcal{S} & \end{array}$$

goes by ξ to q. Conversely, any chosen quotient map in $\mathcal{S}$ can be gotten as ξ acting on (8.2) for suitable R.

The proof of these facts is easy.

PROPOSITION 8.2. Let $\mathbb{D} \xrightarrow{R} \mathcal{S}$ be arbitrary. Then the canonical map q

(8.3)
$$\coprod_{D \in |\mathbb{D}|} DR \xrightarrow{q} (R)\xi$$

given by $\mathrm{incl}_D \cdot q = (D\text{-}R)\xi$ is an $\mathcal{O}_e$ -chosen quotient mapping, (D-R) denoting the morphism

$$\begin{array}{ccc} 1 & \xrightarrow{\epsilon_D} & \mathbb{D} \\ & \searrow^{\epsilon_{DR}} \quad \swarrow_{R} & \\ & \mathcal{S} & \end{array}.$$

PROOF. This is essentially just to observe that $(R)\xi$ was defined as π_0 of an $\mathcal{O}$ -category with object set the sum in (8.3).

CHAPTER II

THE COLIMIT MONADS

1. The functors $\mathrm{Gray}.\varphi_o$.

For $\mathcal{A} \in \mathrm{CAT}_V^U$, we shall produce a functor

$$(1.1) \qquad [\mathrm{Cat}^U, \epsilon_{\mathcal{A}}] \xrightarrow{\mathrm{gray}.\varphi_o} (\text{U-Ens})^{\mathcal{A}^{opp}} .$$

Likewise, for $\mathcal{A} \in \mathrm{Cat}_V$, we shall produce

$$(1.2) \qquad [\mathrm{Cat}^U, \epsilon_{\mathcal{A}}] \xrightarrow{\mathrm{Gray}.\varphi_o} \mathcal{S}^{\mathcal{A}^{opp}} .$$

We are prevented from saying that these functors are natural in $\mathcal{A}$ by the fact that the codomain categories do not depend functorial in $\mathcal{A}$.

In [4], Gray constructs a pair of adjoint functors between 1): the category of categories over $\mathcal{A}$, $(\mathrm{CAT}, \epsilon_{\mathcal{A}})$, and 2): the categories split fibered over $\mathcal{A}$. His definition of the functor from 1) to 2) is easily extended to $[\mathrm{CAT}, \epsilon_{\mathcal{A}}]$. Applying "$\pi_o$ fiberwise" (the φ_o to be made precise below) sends us into the category of categories split fibered over $\mathcal{A}$ and with discrete fibres; this category is equivalent to the category of contravariant set valued functors on $\mathcal{A}$.

The composite (1.1) can be described explicitely as follows.

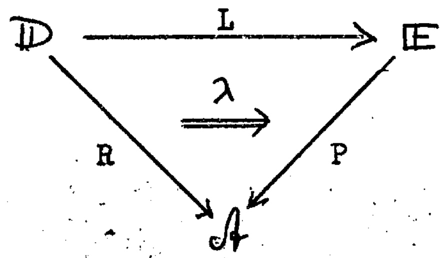
Let $R : \mathbb{D} \longrightarrow \mathcal{A}$ be an object of $[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]$. Then $(R)\mathrm{gray}.\varphi_o$ is the functor $\mathcal{A}^{opp} \longrightarrow$ U-Ens given by (with π_o as in (I.6.6))

$$(1.3) \qquad \begin{array}{l} A \rightsquigarrow (\epsilon_A, R)\pi_o \\ (A' \xrightarrow{a} A) \rightsquigarrow (\{A \xrightarrow{b} \underline{D}R\} \rightsquigarrow \{A' \xrightarrow{a} A \xrightarrow{b} \underline{D}R\}) . \end{array}$$

Here $\{-\}$ denotes π_0 -classes in $(\in_A, R)$, $(\in_{A'}, R)$, respectively, and the notation $A \xrightarrow{b} \underline{D}R$ is obvious for an object in $(\in_A, R)$, and is explained in Section 4 of Chapter I.

Next, let there be given a morphism in $[\mathrm{Cat}^U, \in_{\mathcal{A}}]$

(1.4)



then $(\lambda)\mathrm{gray}.\varphi_0$ is the transformation whose A -component is

(1.5) $$\{A \xrightarrow{b} \underline{D}R\} \rightsquigarrow \{A \xrightarrow{b} DR \xrightarrow{\lambda_D} \underline{\underline{D}L}P\} .$$

If $\mathcal{A} \in \mathrm{Cat}_V$, i.e. is locally well ordered, then $|(\in_A, R)|$ is well ordered by the canonical isomorphism

$$|(\in_A, R)| \cong \coprod_{D \in |\mathbb{D}|} \mathcal{A}(A, DR) .$$

Let $(\in_A, R)\pi_0$ have the quotient well ordering (Section 6 of Chapter I) and use $(\in_A, R)\pi_0$ to denote also the corresponding ordinal number. Then (1.3) and (1.5) defines the functor $\mathrm{Gray}.\varphi_0$ of (1.2).

Restricting $\mathrm{Gray}.\varphi_0$ to

$$\mathcal{A}T = [\mathrm{Cat}, \in_{\mathcal{A}}] \subseteq [\mathrm{Cat}^U, \in_{\mathcal{A}}]$$

gives a functor

(1.6) $$\mathcal{A}T \xrightarrow{\mathrm{Gray}.\varphi_0} \mathcal{S}^{\mathcal{A}^{opp}} .$$

Now $\mathcal{S}^{\mathcal{A}^{opp}}$ is a colimit algebra with the structure inherited by the structure on $\mathcal{S}$ (I.7); and $\mathcal{A}T$ carries the prelimit structure $\mu_{\mathcal{A}}$. Relative to these we have

PROPOSITION 1.1. The functor Gray.φ_0 in (1.6) is a prelimit homomorphism

The proof is omitted, since we will later factor Gray.φ_0 into two functors, each of which will be proved a homomorphism.

For $\mathcal{A} \in \mathrm{CAT}_V^U$ we will in general have the hom sets in $(\text{U-Ens})^{\mathcal{A}^{\mathrm{opp}}}$ are big, i.e. in V. We have however,

PROPOSITION 1.2. The full image of the functor gray.φ_0 in (1.1) has hom sets of the size of sets in U.

PROOF. By a theorem of Isbell [6], it suffices to prove that every object (R)gray.φ_0 is proper in the sense of [6]: A functor $F : \mathcal{A}^{\mathrm{opp}} \longrightarrow$ $\longrightarrow$ U-Ens is proper, if there is an $M \in U$ and an M-indexed family of pairs (A_m, x_m), $A_m \in |\mathcal{A}|$, $x_m \in (A_m)F$, so that for every A and every $x \in (A)F$ there is an m and an $a \in \mathcal{A}(A, A_m)$ so that

$$x = (x_m)(a)F .$$

It is easily seen that for $F = (R)\mathrm{gray}.\varphi_0$ we can get a $|\mathbb{D}|$-indexed family with this property, ($\mathbb{D}$ being the domain of R), namely

$$(DR, \{ DR \xrightarrow{I} \underline{DR} \}) \qquad D \in |\mathbb{D}|.$$

2. The endofunctor Ψ on CAT_V^U.

Let

$$c : \mathrm{CAT}_V^U \longrightarrow \mathrm{Cat}_V$$

denote a fixed equivalence, having the inclusion $\mathrm{Cat}_V \subseteq \mathrm{CAT}_V^U$ as a left inverse, $i . c = I_{\mathrm{Cat}_V}$.

Let for fixed $\mathcal{A} \in CAT_V^U$ $\mathcal{A}\Theta$ and $\mathcal{A}\tilde{\Theta}$ denote the full images of the functors $gray.\varphi_o$, $c.Gray.\varphi_o$, respectively, i.e.

$$[Cat^U, \in_{\mathcal{A}}] \xrightarrow{gray.\varphi_o} \mathcal{A}\Theta \subseteq U\text{-}Ens^{\mathcal{A}^{opp}}$$

$$[Cat^U, \in_{\mathcal{A}c}] \xrightarrow{Gray.\varphi_o} \mathcal{A}\tilde{\Theta} \subseteq \mathfrak{S}^{\mathcal{A}^{opp}}.$$

They will satisfy

(2.1)
(i) $\mathcal{A}\Theta \simeq \mathcal{A}\tilde{\Theta}$
(ii) Θ is an endofunctor $CAT_V^U \longrightarrow CAT_V^U$
(iii) $\mathcal{A}\tilde{\Theta}$ admits associative colimits.

One would like (ii) and (iii) satisfied simultaneously. We shall construct a functor Ψ so that

(2.2)
(i) $\mathcal{A}\Psi \simeq \mathcal{A}\Theta \simeq \mathcal{A}\tilde{\Theta}$
(ii) Ψ is an endofunctor $CAT_V^U \longrightarrow CAT_V^U$
(iii) $\mathcal{A}\Psi$ admits associative colimits.

Let $\mathcal{A}$ and $\mathcal{B}$ denote categories in CAT_V^U. We define an equivalence relation $\equiv$ on $|[Cat^U, \in_{\mathcal{A}}]|$.

DEFINITION 2.1. Put

$$\mathbb{D} \xrightarrow{R} \mathcal{A} \equiv \mathbb{D}' \xrightarrow{R'} \mathcal{A}$$

iff

(2.3) $$|\mathbb{D}| = |\mathbb{D}'| \quad \wedge \quad |R| = |R'|$$

and

(2.4) $$(R)gray.\varphi_o = (R')gray.\varphi_o .$$

This is clearly equivalent to saying: (2.3) holds, and for all $A \in |\mathcal{A}|$

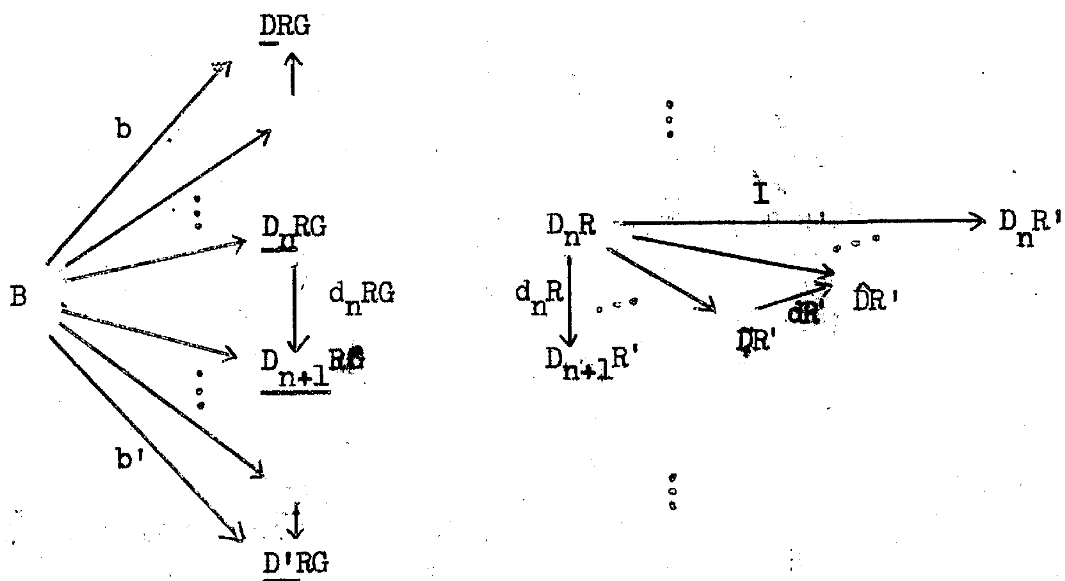
$$\{A \xrightarrow{a} \underline{D}R\} \rightsquigarrow \{A \xrightarrow{a} \underline{D}R'\}$$

is well defined and defines the identity mapping.

LEMMA 2.2. Let $\mathcal{A} \xrightarrow{G} \mathcal{B}$ be an arbitrary functor in CAT_V^U. Then with R, R' as in the Definition

$$R \equiv R' \Rightarrow R.G \equiv R'.G .$$

PROOF. Let $B \xrightarrow{b} \underline{D}RG \sim B \xrightarrow{b'} \underline{D'}RG$ in $(\in_B, R)$. This means that there exists a chain connecting them, as displayed in the left half of the diagram

(2.5) 

But $D_nR \xrightarrow{d_nR} \underline{D_{n+1}}R \sim D_nR \xrightarrow{I} D_nR$ in $(\in_{D_nR}, R)$. Therefore by assumption $d_nR : D_nR \to \underline{D_{n+1}}R'$ and $D_nR \xrightarrow{I} \underline{D_n}R'$ can be connected in $(\in_{D_nR}, R')$; a connection is displayed in the right half of the diagram above. For each triangle appearing on the left hand side, form such a connection. Take G on all the diagrams on the right hand side and patch them together with the left hand diagram and get a connection establishing

$$B \xrightarrow{b} \underline{D}R'G \sim B \xrightarrow{b'} \underline{D'}R'G.$$

The lemma now is immediate.

It will be convenient to introduce the following notation. Let

$$(2.6) \qquad \mathbb{D} \xrightarrow{R} \mathcal{A} \qquad \mathbb{E} \xrightarrow{P} \mathcal{A}$$

be objects in $[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]$, and let m be a morphism in $U\text{-}\mathrm{Ens}^{\mathcal{A}^{opp}}$

$$(2.7) \qquad (R)\mathrm{gray}.\varphi_o \xrightarrow{m} (P)\mathrm{gray}.\varphi_o \, .$$

For $D \in |\mathbb{D}|$, we get an element

$$\{DR \xrightarrow{I} \underline{DR}\}m_{DR} \in (\epsilon_{DR}, P)\pi_o \, .$$

Chose an object in (ϵ_{DR}, P) representing this element

$$DR \xrightarrow{{}_*m^D} \underline{E}P \, .$$

(This use of the axiom of choice is absolutely unessential, it is just a notational convenience).

It is easy to see the following: let in addition to (2.6) and (2.7) $S : \mathbb{C} \longrightarrow \mathcal{A}$ be given together with

$$(S)\mathrm{gray}.\varphi_o \xrightarrow{n} (R)\mathrm{gray}.\varphi_o \, .$$

Then for $C \in |\mathbb{C}|$, ${}_*n^C : CS \longrightarrow DR$, we have

$$(2.8) \qquad {}_*(n \, . \, m)^C \sim {}_*n^C \, . \, {}_*m^D$$

in $(\in_{CS}, P)$. Observe likewise that if $d : D \longrightarrow D'$ is a morphism in $\mathbb{D}$, then

$$(2.9) \qquad dR \circ {}_*m^{D'} \sim {}_*m^{D}$$

in $(\in_{DR}, R)$.

DEFINITION 2.3. Let $\mathcal{A} \in CAT_V^U$. Let $\mathcal{A}\Psi$ denote the category with

$$|\mathcal{A}\Psi| = |[Cat^U, \in_{\mathcal{A}}]| / \equiv ;$$

if $R, P \in |[Cat^U, \in_{\mathcal{A}}]|$ and $\{R\}, \{P\}$ denotes their $\equiv$ classes, then

$$(\mathcal{A}\Psi)(\{R\}, \{P\}) = (U\text{-Ens}^{\mathcal{A}^{opp}})(R\ gray.\varphi_o, P\ gray.\varphi_o) .$$

Composition in the category $\mathcal{A}\Psi$ is defined by means of the composition in $(U\text{-Ens})^{\mathcal{A}^{opp}}$. - Further:

Let $\mathcal{A} \xrightarrow{G} \mathcal{B}$ be a morphism in CAT_V^U. Let $(G)\Psi$ denote the functor

$$(G)\Psi \quad : \quad \mathcal{A}\Psi \longrightarrow \mathcal{B}\Psi,$$

given on objects by

$$(2.10) \qquad \{R\} \overset{(G)\Psi}{\rightsquigarrow} \{R \circ G\}$$

and on maps by

$$(2.11) \qquad \left((R)gray.\varphi_o \xrightarrow{m} (P)gray.\varphi_o\right) \rightsquigarrow \left(\{B \xrightarrow{b} \underline{DRG}\} \overset{(m)(G)\Psi}{\rightsquigarrow} \{B \xrightarrow{b} DRG \xrightarrow{({}_*m^D)G} \underline{EPG}\}\right)$$

where R, P, and m are as in (2.6), (2.7), respectively.

It is clear from the definition that $\mathcal{A}\Psi \simeq \mathcal{A}\Theta$, so that we immediately have (i) and (ii) of (2.2) satisfied. Of course, one has to check various points of the definition. By Lemma 2.2, (2.10) does not depend on the choice of R in $\{R\}$. Also, it is easily checked that (2.11) does not depend on the choice of R, P, ${}_{*}m^{D}$ and $B \xrightarrow{b} \underline{DRG}$ in their respective classes. Finally, the same arguments will prove that Ψ commutes with composition, $(G \cdot G')\Psi = (G)\Psi \cdot (G')\Psi$.

Also, by proposition 1.2, the hom sets of $\mathcal{A}\Psi$ are (isomorphic to) sets in U, so that $\mathcal{A}\Psi$ may be considered as being in CAT^{U}_{V}.

One reason why it is natural to consider the relation $\equiv$ is given by

PROPOSITION 2.4. Let $\mathbb{D} \xrightarrow{R} \mathcal{A} \equiv \mathbb{D}' \xrightarrow{R'} \mathcal{A}$, and let

$$DR \xrightarrow{incl_D} \underrightarrow{lim}\,(R)\ ,$$

$D \in |\mathbb{D}| = |\mathbb{D}'|$, be a colimit diagram for R. Then it is also a colimit diagram for R'.

PROOF. It suffices tp prove that for any choice of colimit diagrams for R and R', there exists an isomorphism δ making the diagrams

$$\begin{array}{ccc} \underrightarrow{lim}\,(R') & \xrightarrow[\simeq]{\delta} & \underrightarrow{lim}\,(R) \\ {\scriptstyle incl'_D}\uparrow & & \uparrow{\scriptstyle incl_D} \\ DR' & = & DR \end{array}$$

commute for all $D \in |\mathbb{D}| = |\mathbb{D}'|$. For this, it suffices to prove that $D \xrightarrow{d} D'$ in $\mathbb{D}'$ implies commutativity of

$$(2.12)\qquad \begin{array}{ccccc} & & \underrightarrow{lim}\,(R) & & \\ & {\scriptstyle incl_D}\nearrow & & \nwarrow{\scriptstyle incl_{D'}} & \\ DR & & \xrightarrow{(d)R'} & & D'R \end{array}$$

(and a converse statement). Since in $(\in_{DR}, R')$

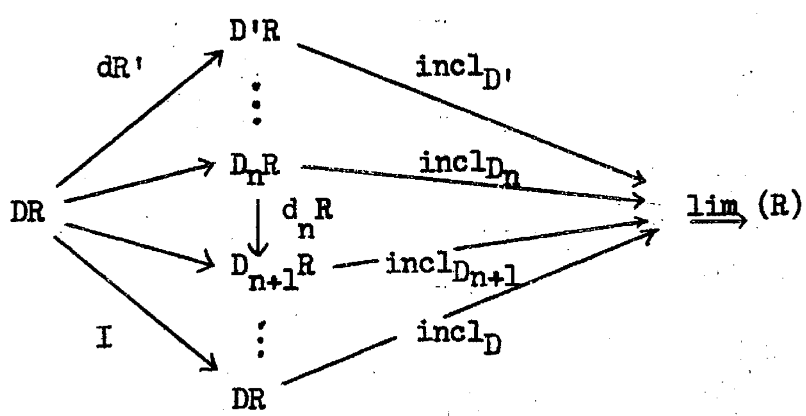
$$DR = DR' \xrightarrow{I} \underline{DR}' \sim DR' \xrightarrow{(d)R'} \underline{D}'R' ,$$

we have by assumption

$$DR \xrightarrow{I} \underline{DR} \sim DR \xrightarrow{(d)R'} \underline{D}'R$$

in $(\in_{DR}, R)$. Let a connection be as in the left side of the diagram

(2.13)



Now commutativity of the total diagram in (2.13) (i.e. of (2.12)) follows from the commutativity of the small triangles in (2.12). The proposition follows.

3. <u>The monad structure on Ψ</u>.

We still have to equip $\mathcal{A}\Psi$ as a colimit algebra. We get a slightly more precise information by making Ψ into a monad and produce a morphism of monads $T \xrightarrow{\Gamma} \Psi$, i.e.

$$\mathcal{A}T = [\mathrm{Cat}, \in_{\mathcal{A}}] \xrightarrow{\Gamma_{\mathcal{A}}} \mathcal{A}\Psi .$$

DEFINITION 3.1. Let $\mathcal{A} \in \mathrm{CAT}_V^U$. Then

$$\mathcal{A} \xrightarrow{\eta_{\mathcal{A}}} \mathcal{A}\Psi$$

is the functor given by

$$A \rightsquigarrow \{1 \xrightarrow{\epsilon_A} \mathcal{A}\} \qquad (A \in |\mathcal{A}|)$$

$$A \xrightarrow{a} A' \rightsquigarrow \left(\begin{array}{ccc} 1 & \longrightarrow & 1 \\ \epsilon_A \searrow & \overset{\epsilon_{[a]}}{\Longrightarrow} & \swarrow \epsilon_{A'} \\ & \mathcal{A} & \end{array} \right) \text{gray}.\varphi_0 \, .$$

Clearly $\eta_{\mathcal{A}}$ is natural in $\mathcal{A}$.

DEFINITION 3.2. Let $\mathcal{A} \in \mathrm{CAT}_V^U$. Then

$$\mathcal{A}\Psi^2 \xrightarrow{\mu_{\mathcal{A}}} \mathcal{A}\Psi$$

is the functor given as follows. Let $\{\mathbb{D} \xrightarrow{R} \mathcal{A}\Psi\}$ be an object in $\mathcal{A}\Psi^2$. Let r be a lifting of $|R|$:

(3.1)
$$\begin{array}{ccccc} & & r & & |[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]| \\ & \nearrow & & & \downarrow \{\cdot\} \\ |\mathbb{D}| & \xrightarrow{|R|} & |\mathcal{A}\Psi| = & & |[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]|/_{\equiv} \, . \end{array}$$

Then $\{R\}\mu_{\mathcal{A}} \in |\mathcal{A}\Psi|$ is $\{\cdot\}$ of the following object $R\bar{\nu} \xrightarrow{R\nu} \mathcal{A}$ in $[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]$:

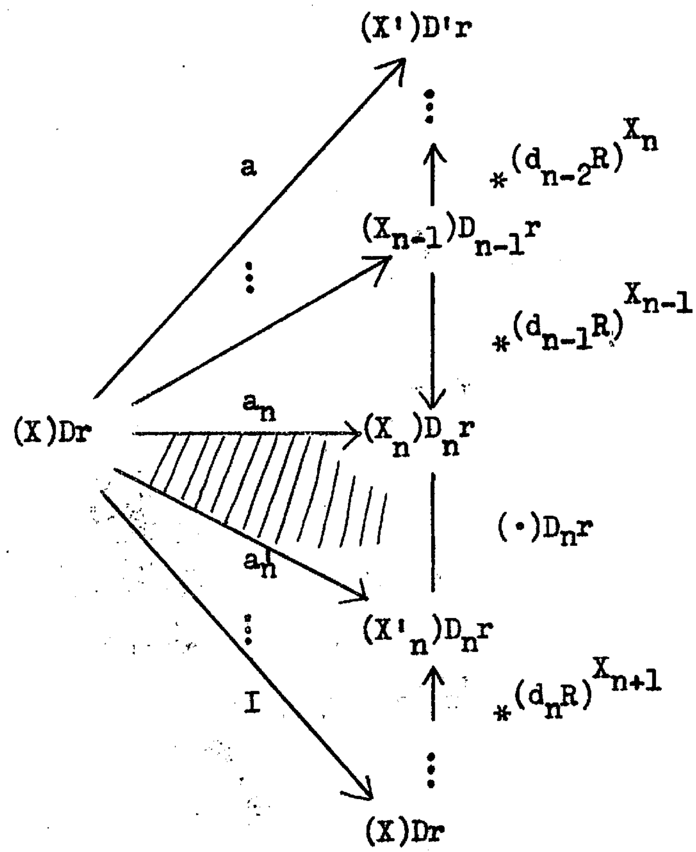
$$|R\bar{\nu}| = \coprod_{D \in |\mathbb{D}|} |D\bar{r}|$$

$$(R\bar{\nu})(X_D, X'_{D'}) \subseteq \mathcal{A}((X)Dr, (X')D'r) \; ,$$

namely consisting of those a for which there exists a connection in $\mathbb{D}$ from D' to D

$$D' \ldots \xleftarrow{d_{n-2}} D_{n-1} \xrightarrow{d_{n-1}} D_n \xleftarrow{d_n} \ldots D$$

and a commutative diagram of the form

(3.2) 

;

the shaded diagram designates a connection

$$(X)Dr \xrightarrow{a_n} (\underline{X_n})D_nr \sim (X)Dr \xrightarrow{a'_n} (\underline{X'_n})D_nr$$

in $(\in_{XDr}, D_nr)$; X_n is determined since $_*(d_{n-1}R)^{X_{n-1}}$ strictly is an object of a comma category, not just a morphism in $\mathcal{A}$.

Composition in $R\overline{\mathcal{V}}$ is inherited from the composition in $\mathcal{A}$.

The functor $R\nu$ assigns to X_D $(X)Dr$, and to $a \in (R\overline{\mathcal{V}})(X_D, X'_{D'})$ as above it assigns $a \in \mathcal{A}((X)Dr, (X')D'r)$.

On morphisms in $\mathcal{A}\Psi^2$, $\mu_{\mathcal{A}}$ is defined as follows. Let

$$\{\mathbb{D} \xrightarrow{R} \mathcal{A}\Psi\} \xrightarrow{m} \{\mathbb{E} \xrightarrow{P} \mathcal{A}\Psi\},$$

and let r be a lifting of $|R|$ as before, and similarly p a lifting of $|P|$. Then

$$(m)\mu_{\mathcal{A}} \ : (R\nu)\mathrm{gray}.\varphi_0 \longrightarrow (P\nu)\mathrm{gray}.\varphi_0$$

in $U\text{-}Ens^{\mathcal{A}^{opp}}$ is given by

$$(3.3)\ \left\{A \xrightarrow{\ b\ } (\underline{X_D})R\nu\right\} \ \rightsquigarrow\ \left\{A \xrightarrow{\ b\ } (X)Dr \xrightarrow{\ {}_*({}_*m^D)^X\ } (Y)Ep \ = \ (\underline{Y_E})P\nu\right\}.$$

Here ends Definition 3.2. Note how in (3.3) ${}_*({}_*m^D)^X$ uniquely makes sense according to the convention of Section 2: m goes from $(R)\mathrm{gray}.\varphi_0$ to $(P)\mathrm{gray}.\varphi_0$, so we have for $D \in |\mathbb{D}|$

$$DR \xrightarrow{\ {}_*m^D\ } EP$$

in $\mathcal{A}\Psi$, that is, a morphism in $U\text{-}Ens^{\mathcal{A}^{opp}}$

$$(Dr)\mathrm{gray}.\varphi_0 \longrightarrow (Ep)\mathrm{gray}.\varphi_0 \ ;$$

so for $X \in |D\check{\bar{r}}|$, ${}_*({}_*m^D)^X$ makes sense and goes as indicated.

We have to justify the definition. Call the triangles in the shaded areas of (3.2) <u>inessential</u>. (They are of the form

$$\begin{array}{ccc} & & (X^m)D_n r \\ & \nearrow^{a^m} & \\ (X)Dr & & \Big\downarrow (x)D_n r \\ & \searrow_{a^{m+1}} & \\ & & (X^{m+1})D_n r \end{array}$$

.)

Because of the inessential triangles in (3.2), one easily sees that for fixed $R \in \{R\}$ and fixed lifting r (3.1), $R\bar{\nu} \xrightarrow{\ R\ } \mathcal{A}$ is independent of the choice of the elements ${}_*(dR)^X$ in their respective components. Next we prove that for fixed $R \in \{R\}$, any choice of lifting r (3.1) gives the same $R\nu$. For, for $d : D \longrightarrow D'$ in $\mathbb{D}$ and $X \in |D\bar{r}_0| = |D\bar{r}_1|$ (r_0 and r_1 being the two liftings), we can chose $_*(dR)_0^X = {}_*(dR)_1^X$. Finally $R\nu$ does not depend on the choice of $R \in \{R\}$. (A fortiori, $\{R\nu\} = \{R\}\mu_{\mathcal{A}}$ is well defined.) Let namely $R, P \in \{R\}$

$$\mathbb{D} \xrightarrow{R} \mathcal{A} \qquad \mathbb{E} \xrightarrow{P} \mathcal{A}$$

$|\mathbb{D}| = |\mathbb{E}|$, $|R| = |P|$. Choose a common lifting r (3.1) for $|R|$ and $|P|$. Then clearly $|R\nu| = |P\nu|$. We just have to prove that the two conditions for

$$a : (X)Dr \longrightarrow (X')D'r$$

to be in $(R\bar{\nu})(X_D, X'_{D'})$, respectively $(P\bar{\nu})(X_D, X'_{D'})$ are equivalent. So let there exist a diagram (3.2). We produce a diagram of the same type proving $a \in (P\bar{\nu})(X_D, X'_{D'})$. The inessential triangles we leave as they are. Consider now an essential triangle, e.g. the (only) one displayed in (3.2). Since $(R)\text{gray}.\varphi_0 = (P)\text{gray}.\varphi_0$, we have

$$D_{n-1}P = D_{n-1}R \xrightarrow{(d_{n-1})R} D_nR = \underline{D_nP}$$

$$\sim \quad D_{n-1}P \xrightarrow{I} \underline{D_{n-1}P}$$

in $(P)\text{gray}.\varphi_0$. If a connecting chain with k links exist between these two objects, a typical link being

$$\begin{array}{ccc} & & E_iP \\ & \overset{f_i}{\nearrow} & \big\downarrow e_iP \\ D_{n-1}P & & \\ & \underset{f_{i+1}}{\searrow} & E_{i+1}P \end{array} ,$$

then using (2.8) it is easily seen that the displayed essential triangle can be replaced by a chain consisting of k essential triangles of the form

$$\begin{array}{ccc} (X)Dr & \xrightarrow{\;b_i\;} & (Z_i)E_ip \\ & \searrow & \Big\downarrow {}_*(e_iP)^{Z_i} \\ & & (Z_{i+1})E_{i+1}p \end{array} \quad ,$$

where $Z_i \in |E_i\bar{p}|$ and $b_i = a_{n-1} \cdot {}_*f_i^{X_{n-1}}$, plus some inessential triangels.

To prove that $\mu_{\mathcal{A}}$ is well defined on morphisms, we need

LEMMA 3.3. Let $\mathbb{E} \xrightarrow{\;P\;} \mathcal{A}\mathcal{Y}$, and let

$$B \xrightarrow{\;n\;} \underline{E}P \;\sim\; B \xrightarrow{\;\tilde{n}\;} \underline{\tilde{E}}P$$

in $(\in_B, P)$. Then if B is represented by $\mathbb{A} \xrightarrow{\;a\;} \mathcal{A}$, and $A \in |\mathbb{A}|$,

$$(3.5) \qquad Aa \xrightarrow{\;{}_*n^A\;} \underline{Y_E}\, P\nu \;\sim\; Aa \xrightarrow{\;{}_*\tilde{n}^A\;} \underline{\tilde{Y}_{\tilde{E}}}P\nu$$

in $(\in_A, P\nu)$.

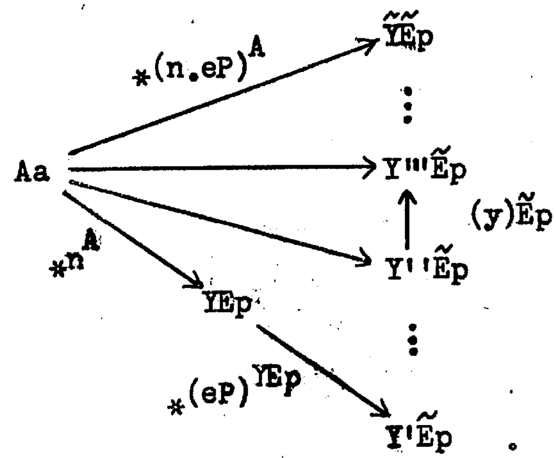
PROOF. Clearly it suffices to assume that n and $\tilde{n}$ can be connected in just one step

$$\begin{array}{ccc} B & \xrightarrow{\;n\;} & EP \\ & \underset{\tilde{n}}{\searrow} & \Big\downarrow (e)P \\ & & \tilde{E}P \end{array} \qquad\qquad E \xrightarrow{\;e\;} \tilde{E} \text{ in } \mathbb{E}.$$

Let ${}_*n^A$ be $A \longrightarrow \underline{Y}Ep$. Then by (2.8)

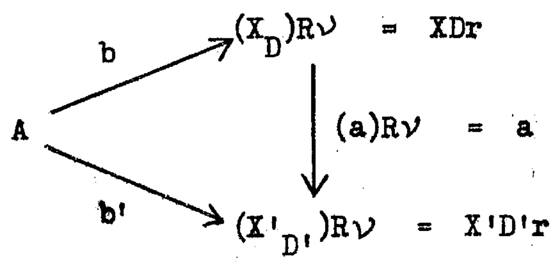
$${}_*\tilde{n}^A \sim {}_*n^A \cdot {}_*((e)P)^Y$$

in $(\epsilon_A, \tilde{E}p)$, where $\widetilde{\tilde{E}p} \xrightarrow{\tilde{E}p} \mathcal{A}$ represents $\tilde{E}P$, and similarly for Ep. Let a connection be given, having k links, say:



Now $_*(eP)^{YEp} \in (P\,\check{\nu})(Y_E, Y'_{\tilde{E}})$, and $(y)\tilde{E}p \in (P\check{\nu})(Y''_{\tilde{E}}, Y'''_{\tilde{E}})$. The diagram can now easily be reinterpreted as a connection (of length $k+1$) required for (3.5).

Now we can prove that the definition (3.3) does not depend on the c choice of b in its component of $(\epsilon_A, R\nu)$. Clearly it suffices to prove that b and $b' = b.a$



give same value. The condition on a is that it fits into a diagram (3.2). So it again suffices to prove that we get the same right hand side of (3.3) for

$$XDr \xrightarrow{a_{n-1}} ((X_{n-1})_{D_{n-1}})R\nu \qquad \text{and} \qquad XDr \xrightarrow{a_n} ((X_n)_{D_n})R\nu \ ,$$

and in turn for this and

$$(X)Dr \xrightarrow{a'_n} (X'_n)_{D_n} R\nu .$$

For the essential triangle, note that $d_{n-1}R \sim I_{D_{n-1}R}$ in $(\epsilon_{D_{n-1}R}, P)$. Apply Lemma 3.3 with Dr for a, X for A, and $d_{n-1}R$ and $I_{D_{n-1}R}$ for n and $\tilde{n}$, respectively. Then (3.5) gives the desired equality. For the second equality, use (2.9) together with the obvious fact that

$$(3.6) \qquad A \xrightarrow{c} \underline{Y}Ep \sim A \xrightarrow{c'} \underline{Y'}Ep$$

in (ϵ_A, Ep) implies

$$(3.7) \qquad A \xrightarrow{c} \underline{Y_E}P\nu \sim A \xrightarrow{c'} \underline{Y'_E}P\nu .$$

This shows (3.3) independent of choice of a.

The implication (3.6) $\Longrightarrow$ (3.7) also shows that the (3.3) for fixed choice of ${}_*m^D$ does not depend on choice of ${}_*({}_*m^D)^X$. And the Lemma 3.3 shows that it does not depend on the choice of ${}_*m^D$ either. Finally, the choice of R and P in their respective classes are immaterial; for, we can chose the same ${}_*m^D$ for R, P and R', P'.

The proof that Ψ, η, μ constitutes a monad is contained in the next section.

4. Proof of the monad laws for Ψ, η, μ.

THEOREM 4.1. Let Ψ, η, μ be as in the Definitions 2.3, 3.1. and 3.2, respectively. Then Ψ, η, μ is a monad on CAT_V^U.

PROOF. We are required to prove commutativity of the diagram

(4.1)
$$\begin{array}{ccc} \mathcal{A}\Psi^3 & \xrightarrow{\mu_{\mathcal{A}\Psi}} & \mathcal{A}\Psi^2 \\ {\scriptstyle (\mu_{\mathcal{A}})\Psi}\downarrow & & \downarrow{\scriptstyle \mu_{\mathcal{A}}} \\ \mathcal{A}\Psi^2 & \xrightarrow[\mu_{\mathcal{A}}]{} & \mathcal{A}\Psi \end{array}$$

Let $\mathbb{D} \xrightarrow{R} \mathcal{A}\Psi^2$ represent an object of $\mathcal{A}\Psi^3$, and let

(4.2)
$$\begin{array}{ccc} & & |[\mathrm{Cat}^U, \epsilon_{\mathcal{A}\Psi}]| \\ & \nearrow^{r} & \downarrow\{\cdot\} \\ |\mathbb{D}| & \xrightarrow[|R|]{} & |\mathcal{A}\Psi^2| \end{array} \qquad \begin{array}{ccc} & & |[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]| \\ & \nearrow^{Ds} & \downarrow\{\cdot\} \\ |D\bar{r}| & \xrightarrow[|Dr|]{} & |\mathcal{A}\Psi| \end{array}$$

(the last for every $D \in |\mathbb{D}|$) be liftings as indicated. In particular, for $X \in |D\bar{r}|$, XDs is a functor

$$XD\bar{s} \xrightarrow{XDs} \mathcal{A},$$

and for $Y \in |XD\bar{s}|$, $YXDs \in |\mathcal{A}|$. Construct $(R)\bar{\nu} \xrightarrow{(R)\nu} \mathcal{A}\Psi$ according to Definition 2.3 with $\mathcal{A}\Psi$ in place of $\mathcal{A}$. The liftings (4.2) give a lifting

$$\begin{array}{ccc} & & |[\mathrm{Cat}^U, \epsilon_{\mathcal{A}}]| \\ & \nearrow & \downarrow\{\cdot\} \\ |(R)\bar{\nu}| & \xrightarrow[|(R)\nu|]{} & |\mathcal{A}\Psi| \end{array}$$

sending X_D to XDs. Use this lifting to construct $(R)\nu\nu$,

(4.3)
$$(R)\nu\bar{\nu} \xrightarrow{(R)\nu\nu} \mathcal{A}$$

representing $(\{R\})\mu_{\mathcal{A}\Psi}\cdot\mu_{\mathcal{A}}$.

On the other hand, $\{R\}(\mu_{\mathcal{A}})\Psi$ is represented by $R \circ \mu_{\mathcal{A}}$, for which we have a lifting t

$$\begin{array}{ccc} & & [\mathrm{Cat}^U, \in_{\mathcal{A}}] \\ & \overset{t}{\nearrow} & \downarrow \\ |\mathbb{D}| & \xrightarrow[|R\ .\ \mu_{\mathcal{A}}|]{} & |\mathcal{A}\psi| \end{array}$$

given by

$$D\bar{t} \xrightarrow{Dt} \mathcal{A} \quad = \quad (Dr)\bar{\nu} \xrightarrow{(Dr)\nu} \mathcal{A} .$$

Use this lifting to compute

$$(4.4) \qquad (R\ .\ \mu_{\mathcal{A}})\bar{\nu} \xrightarrow{(R\ .\ \mu_{\mathcal{A}})\nu} \mathcal{A}$$

representing $(\{R\})(\mu_{\mathcal{A}}\psi)\mu_{\mathcal{A}}$. We claim that (4.3) = (4.4). The sets of objects of the two categories are the ordinal numbers

$$\coprod_{X_D \in \coprod_{D \in |\mathbb{D}|} |D\bar{r}|} |XD\bar{s}| \quad , \qquad \coprod_{D \in |\mathbb{D}|} \Big(\coprod_{X \in |D\bar{r}|} |XD\bar{s}| \Big),$$

which are equal by associativity of ordinal sum (I.2.3). Also, the two object mappings in (4.3) and (4.4) are equal:

$$Y_{X_D} = (Y_X)_D \rightsquigarrow YXDs \in |\mathcal{A}| .$$

We have to prove

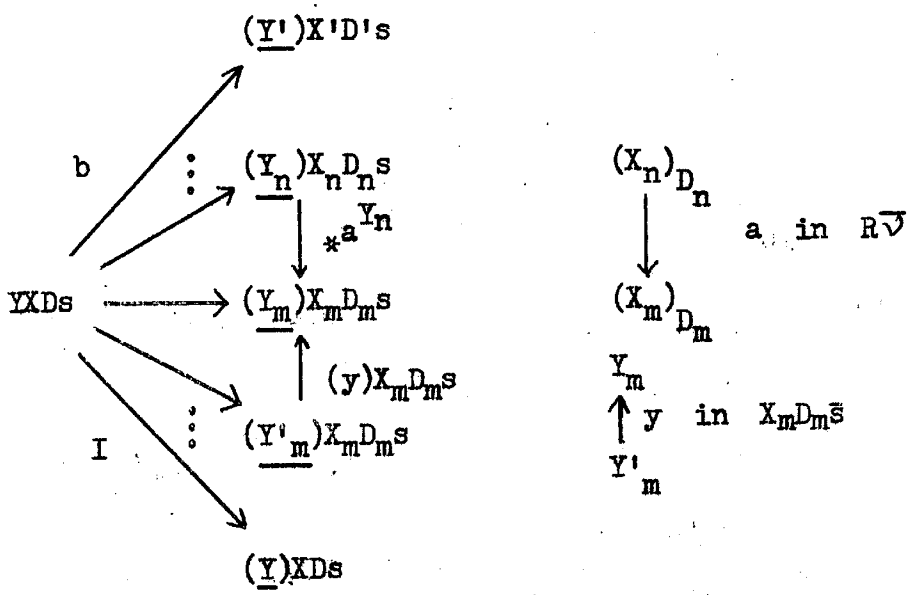
$$(4.5) \qquad ((R)\nu\bar{\nu})(Y_{X_D}, Y'_{X'_{D'}}) = ((R\ .\ \mu_{\mathcal{A}})\bar{\nu})((Y_X)_D, (Y'_{X'})_{D'}) .$$

Both sides in (4.5) are subsets of $\mathcal{A}(YXDs, Y'X'D's)$, and since the functors (4.3) and (4.4) have the inclusions as hom set mappings, (4.5) will prove (4.3) = (4.4).

The condition for

$$(4.6) \qquad b : YXDs \longrightarrow Y'X'D's$$

being in the left hand side of (4.5) is the existence of a diagram analogous to (3.2)

(4.7) 

(one essential and one inessential triangle displayed). Now a being a morphism in $R\nabla$ sits in a diagram (3.2) (with X_n, D_n for X, D, and X_m, D_m for X', D'). Let it have k links, say. Then it is easily seen that the essential triangle in (4.7) can be replaced by k essential triangles having as vertical part either

(4.8) $$_*({}_*(dR)^{\tilde{X}})^{\tilde{Y}}$$

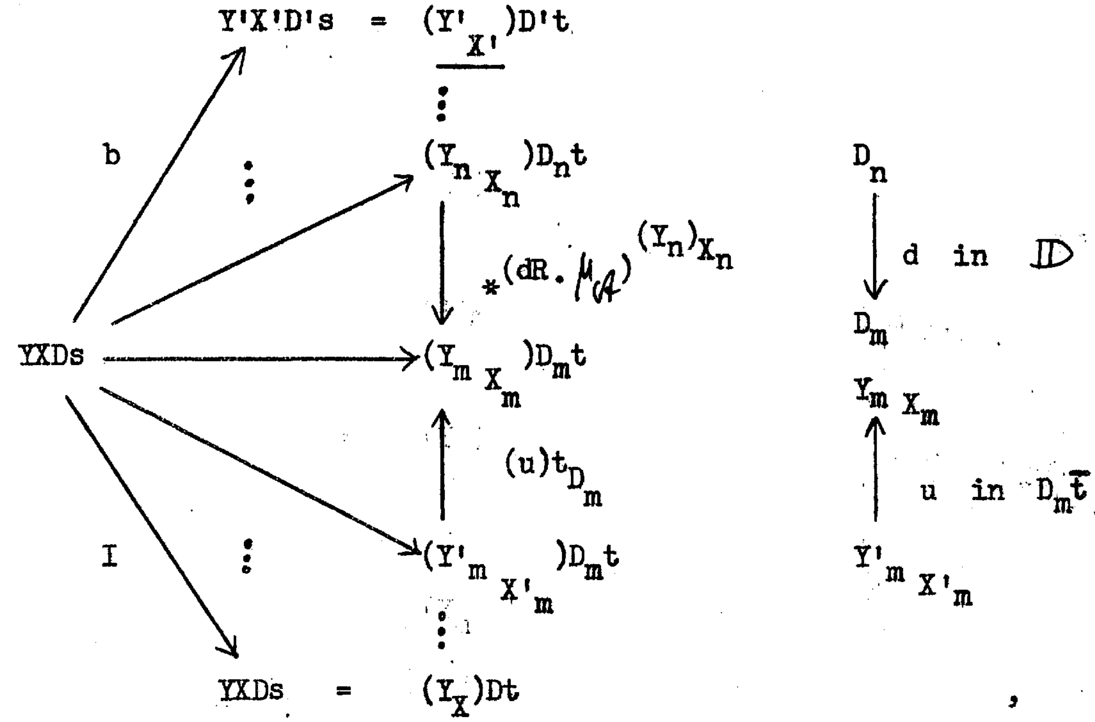
or

(4.9) $$_*((x)\tilde{D}r)^{\hat{Y}}$$

(d a morphism in $\mathbb{D}$, x a morphism in $\tilde{D}\bar{r}$, $\tilde{X}$ in some $|\hat{D}\bar{r}|$, $\tilde{Y}$ in $\tilde{X}\hat{D}s$, $\hat{Y}$ in some $\hat{X}\tilde{D}\bar{s}$) interspersed with inessential triangles like the one already displayed:

(4.10) $$(y)\check{X}\check{D}s \; .$$

The condition for b in (4.6) to be in the right hand side of (4.5) **is the existence of a diagram**

(4.11) 

again with an essential and an inessential triangle displayed. By the very definition of $\mu_{\mathcal{A}}$ on a morphism, the essential triangle in (4.11) is of the form (4.8). Now consider the inessential triangle in (4.11); u, being in $D_m\overline{t} = (D_m r)\overline{\nu}$, sits as an a in a diagram analogous to (3.2) (with k triangles, say), its essential and inessential triangles having as vertical part morphisms (4.9) and (4.10) (with X_m in place of $\tilde{X}$). The functor t being an inclusion then easily gives that the inessential triangle in (4.11) can be replaced by k triangles of form (4.9) and (4.10).

Conversely, a connection from I_{YXDs} to b using as connecting morphisms morphisms of the form (4.8), (4.9), and (4.10) can, by the same analysis of the involved categories of the form $(P)\overline{\nu}$, be interpreted either as a diagram defining b to be in $R\,\nu\overline{\nu}$, or as one defining b to be in $(R\,.\,\mu_{\mathcal{A}})\,\overline{\nu}$.

Given next a morphism

$$m : \{R\} \longrightarrow \{P\}$$

in $\mathcal{A}\Psi^3$; it is not difficult to trace it the two ways round in (4.1) and see that the resulting morphism in $\mathcal{A}\Psi$, i.e. in $U\text{-Ens}^{\mathcal{A}^{opp}}$ in both cases have A -component ($A \in |\mathcal{A}|$):

$$\left\{ A \xrightarrow{\ a\ } YXDs \;=\; \underline{(Y_{X_D})} R\nu\nu \right\}$$

$$\rightsquigarrow \quad \left\{ A \xrightarrow{\ a\ } YXDs \xrightarrow{\ {}_*({}_*({}_*m^D)^X)^Y\ } Y'X'D's \;=\; \underline{(Y'_{X'_{D'}})} P\nu\nu \right\} .$$

The theorem now follows.

5. The relation between the monads T and Ψ.

Recall that if $T, \tilde{\eta}, \tilde{\mu}$ and Ψ, η, μ are monads on a category $\mathcal{C}$, then a morphism of monads $\gamma : (T, \tilde{\eta}, \tilde{\mu}) \longrightarrow (\Psi, \eta, \mu)$ is a functor ~~trans~~-formation $T \xrightarrow{\ \gamma\ } \Psi$ satisfying: For each $\mathcal{A} \in |\mathcal{C}|$, the following diagrams commute

$$(5.1) \qquad \begin{array}{ccc} & & \mathcal{A}T \\ & \overset{\tilde{\eta}_{\mathcal{A}}}{\nearrow} & \\ \mathcal{A} & & \downarrow \gamma_{\mathcal{A}} \\ & \underset{\eta_{\mathcal{A}}}{\searrow} & \\ & & \mathcal{A}\Psi \end{array} \qquad \begin{array}{ccccc} \mathcal{A}T^2 & & \xrightarrow{\quad \tilde{\mu}_{\mathcal{A}} \quad} & & \mathcal{A}T \\ {\scriptstyle (\gamma_{\mathcal{A}})T} \swarrow & \searrow {\scriptstyle \gamma_{\mathcal{A}T}} & & & \\ \mathcal{A}\Psi T & & \mathcal{A}T\Psi & & \downarrow \gamma_{\mathcal{A}} \\ {\scriptstyle \gamma_{\mathcal{A}\Psi}} \searrow & \nearrow {\scriptstyle (\gamma_{\mathcal{A}})\Psi} & & & \\ \mathcal{A}\Psi^2 & & \xrightarrow[\quad \mu_{\mathcal{A}} \quad]{} & & \mathcal{A}\Psi \end{array}$$

It follows that a Ψ -algebra structure on an object $\mathcal{A} \in |\mathcal{C}|$, i.e. a morphism $\mathcal{A}\Psi \xrightarrow{\ \xi\ } \mathcal{A}$ satisfying $\eta_{\mathcal{A}} \circ \xi = I_{\mathcal{A}}$, $\mu_{\mathcal{A}} \circ \xi = \xi\Psi \circ \xi$, gives rise to a T -algebra structure on $\mathcal{A}$, namely

$$\mathcal{A}T \xrightarrow{\ \gamma_{\mathcal{A}}\ } \mathcal{A}\Psi \xrightarrow{\ \xi\ } \mathcal{A} .$$

Also, $\gamma_{\mathcal{A}}$ is a homomorphism of T-algebras, $\mathcal{A}T$ and $\mathcal{A}\Psi$ having the T-algebra structure $\tilde{\mu}_{\mathcal{A}}$ and $\gamma_{\mathcal{A}} \cdot \mu_{\mathcal{A}}$, respectively.

Letting now $T, \tilde{\eta}, \tilde{\mu}$ denote the prelimit monad of Chapter I (with respect to $\mathrm{Cat}_o = \mathrm{Cat}$), and Ψ, η, μ the monad of Section 3, we produce a morphism of monads on CAT_V^U:

$$(T, \tilde{\eta}, \tilde{\mu}) \xrightarrow{\gamma} (\Psi, \eta, \mu) .$$

It will follow from the remarks above that a Ψ-algebra canonically gives rise to a prelimit algebra. The morphism γ is given in the obvious way:

DEFINITION 5.1. Let $\mathcal{A} \in |\mathrm{CAT}_V^U|$. Define

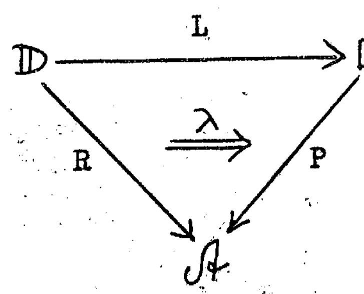
(5.2) $$\mathcal{A}T \xrightarrow{\gamma_{\mathcal{A}}} \mathcal{A}\Psi$$

to be the obvious functor given on objects by

$$\mathbb{D} \xrightarrow{R} \mathcal{A} \rightsquigarrow \{\mathbb{D} \xrightarrow{R} \mathcal{A}\} .$$

($\mathbb{D}$ being in Cat is a fortiori in Cat^U, so that the class of R modulo $\equiv$ (Section 2) is defined.) On morphisms, $\gamma_{\mathcal{A}}$ is given by sending a morphism

(5.3)



in $\mathcal{A}T$ to the morphism $(R)\mathrm{gray}.\varphi_o \longrightarrow (P)\mathrm{gray}.\varphi_o$ in $\text{U-Ens}^{\mathcal{A}^{opp}}$, whose A-component ($A \in |\mathcal{A}|$) is

$$\{A \xrightarrow{a} (\underline{D})R\} \rightsquigarrow \{A \xrightarrow{a} DR \xrightarrow{\lambda_D} (\underline{DL})P\} .$$

Obviously $\gamma_{\mathcal{A}}$ is a functor, and it is natural in $\mathcal{A}$.

PROPOSITION 5.2. The transformation $\gamma : T \longrightarrow \psi$ given by (5.2) is a morphism of monads.

PROOF. It is clear from the definitions that the first diagram of (5.1) commutes. Let

$$\mathbb{D} \xrightarrow{R} \mathcal{A}T$$

be an object in $\mathcal{A}T^2$. Tracing it counterclockwise in the diagram gives an object represented by the obvious functor from the category having as set of objects

$$\coprod_{D \in |\mathbb{D}|} |D\bar{R}|$$

and with $\hom(X_D, X'_{D'}) \subseteq \mathcal{A}(XDR, X'D'R)$, namely those a for which there exists a diagram (3.2). But clearly we may choose

$$_*(d_{n-1}R)^{X_{n-1}} = (d_{n-1}R)X_{n-1}$$

in that diagram. It is now immediate from the definitions that the object in $[\mathrm{Cat}^U, \in_{\mathcal{A}}]$ described by this is equivalent under $\equiv$ to $(R)\tilde{\mu}_{\mathcal{A}}$ (Definition I.5.3, except for a minor change in notation).

Consider next a morphism λ in $\mathcal{A}T^2$, say (5.3) with $\mathcal{A}$ replaced by $\mathcal{A}T$. It is then easy to see that we can choose

$$_*((\lambda)(\gamma_{\mathcal{A}}T)\gamma_{\mathcal{A}\psi})^D = (\lambda_D)\gamma_{\mathcal{A}}$$

for $D \in |\mathbb{D}|$ and

$$_*((\lambda_D)\gamma_{\mathcal{A}})^X = (\lambda_D)_X$$

for $X \in |D\bar{R}|$. Now it is obvious that chasing λ the two ways round in the second diagram of (5.2) gives the following morphism in $\mathcal{A}\Psi$:

$$\{A \xrightarrow{a} (\underline{X}_D)R\tilde{\mu}_{\mathcal{A}}\}$$

$$\rightsquigarrow \{A \xrightarrow{a} (X)DR \xrightarrow{(\lambda_D)_X} (X)\bar{\lambda}_D(DL)P = \underline{((X)\bar{\lambda}_D)_{DL}}\, P\, \tilde{\mu}_{\mathcal{A}}\}.$$

This proves the proposition.

DEFINITION 5.3. A pair $(\mathcal{A}, \xi)$, where ξ is a Ψ-algebra structure on $\mathcal{A}$ is called a <u>regular colimit algebra</u>.

This name will be justified in Section 9, where it is proved that such a ξ automatically will have the property that the corresponding T -algebra structure

$$\mathcal{A}T \xrightarrow{\gamma_{\mathcal{A}}} \mathcal{A}\Psi \xrightarrow{\xi} \mathcal{A}$$

is a colimit assignment and therefore $(\mathcal{A}, \gamma_{\mathcal{A}}\xi)$ a colimit algebra. (Definition I.7.1.)

6. The relation between $\mathcal{A}\Psi$ and $\mathcal{A}\tilde{\Theta} \subseteq \mathcal{S}^{\mathcal{A}^{opp}}$.

The present section is not essential for the main results in Sections 7 and 9.

Let $\mathcal{A} \in |Cat_V|$, i.e. be locally an $\mathcal{O}$ -category. There is an obvious equivalence of categories

$$(6.1) \qquad \sigma : \mathcal{A}\Psi \longrightarrow \mathcal{A}\tilde{\Theta} \subseteq \mathcal{S}^{\mathcal{A}^{opp}}$$

($\mathcal{A}\tilde{\Theta}$ being, as in Section 2, the full image of

$$Gray.\phi_0 : [Cat^U, \epsilon_{\mathcal{A}}] \longrightarrow \mathcal{S}^{\mathcal{A}^{opp}}).$$

It is given by the

DEFINITION 6.1. Let ϑ in (6.1) be the functor given on objects by

$$\{\mathbb{D} \xrightarrow{R_i} \mathcal{A}\} \rightsquigarrow (R_i)\mathrm{Gray}.\varphi_0$$

and on morphisms by

$$((R_0)\mathrm{gray}.\varphi_0 \xrightarrow{m} (R_1)\mathrm{gray}.\varphi_0) \rightsquigarrow \alpha_0^{-1} . m . \alpha_1 ,$$

where α_i is the (unique) isomorphism in U -$\mathrm{Ens}^{\mathcal{A}^{opp}}$ from $(R_i)\mathrm{gray}.\varphi_0$ to $(R_i)\mathrm{Gray}.\varphi_0$ which for each $A \in |\mathcal{A}|$ is orderpreseving, $(\epsilon_A, R_i)\pi_0$ being well ordered as quotient set of the well ordered set

$$|(\epsilon_A, R_i)| \cong \coprod_{D \in |\mathbb{D}|} \mathcal{A}(A, DR_i) .$$

It is obvious that

$$\gamma_{\mathcal{A}} . \vartheta = \mathrm{Gray}.\varphi_0 : [\mathrm{Cat}^U, \epsilon_{\mathcal{A}}] \longrightarrow \mathfrak{S}^{\mathcal{A}^{opp}}, \tag{6.2}$$

$\gamma_{\mathcal{A}}$ as in Definition 5.1.

To state the following lemma, it will be convenient to extend Ψ to an endofunctor on all of CAT_V^V. This is done by replacing U in Section 2 everywhere by V. In particular, the hom sets of an $\mathfrak{X}\Psi$ may now be hom sets of V -$\mathrm{Ens}^{\mathfrak{X}^{opp}}$.

Recall the canonical colimit algebra structure ξ on $\mathfrak{S}$ (I.7), which by the usual "pointwise limit" construction immediately gives a colimit algebra structure (also denoted ξ or $\underrightarrow{\lim}$) on any category of the form $\mathfrak{S}^{\mathcal{B}}$. It is possible to factor this ξ as follows:

$$\begin{array}{ccc} \mathfrak{S}^{\mathcal{A}^{opp}}{}_T = [\mathrm{Cat}, \epsilon_{\mathfrak{S}^{\mathcal{A}^{opp}}}] & \xrightarrow{\ \xi\ } & \mathfrak{S}^{\mathcal{A}^{opp}} \\ \gamma \big\downarrow & \nearrow_{\bar{\rho}} & \\ (\mathfrak{S}^{\mathcal{A}^{opp}})\Psi & & \end{array}$$

where $\bar{\rho}$ is given on objects by the commutativity of this diagram, and on morphisms by sending

$$(\mathbb{D} \xrightarrow{R} \mathcal{S}^{\mathcal{A}^{opp}})\ \text{gray}.\varphi_o \xrightarrow{m} (\mathbb{E} \xrightarrow{P} \mathcal{S}^{\mathcal{A}^{opp}})\ \text{gray}.\varphi_o$$

to the morphism $(m)\bar{\rho} : (R)\xi \longrightarrow (P)\xi$ given by commutativity of the diagrams

$$\begin{array}{ccccc} & (R)\xi & \xrightarrow{(m)\bar{\rho}} & (P)\xi & \\ (D\text{-}R)\xi = \text{incl}_D & \uparrow & & \uparrow & \text{incl}_E = (E\text{-}P)\xi \\ & DR & \xrightarrow[{*}m^D]{} & EP & \end{array}$$

where $(D\text{-}R)$ is the morphism in $\mathcal{S}^{\mathcal{A}^{opp}}{}_T$

$$\begin{array}{ccc} 1 & \xrightarrow{\epsilon_D} & \mathbb{D} \\ & {\scriptstyle \epsilon_{DR}} \searrow \quad \swarrow {\scriptstyle R} & \\ & \mathcal{S}^{\mathcal{A}^{opp}} & \end{array}$$

and $(E\text{-}P)$ is defined in a similar way.

The only non trivial thing to prove to see that this actually well - defines $\bar{\rho}$, is:

$$(\mathbb{D} \xrightarrow{R} \mathcal{S}^{\mathcal{A}^{opp}}) \equiv (\mathbb{D}' \xrightarrow{R'} \mathcal{S}^{\mathcal{A}^{opp}}) \implies (R)\xi = (R')\xi .$$

This follows easily from the special property (Proposition I.8.3) of ξ which makes both $(R)\xi$ and $(R')\xi$ chosen quotients of $\coprod_{|\mathbb{D}|} DR = \coprod_{|\mathbb{D}'|} DR'$, - together with Proposition 2.4.

LEMMA 6.2. Let $\mathcal{A} \in \text{Cat}_V$. Then with σ and $\bar{\rho}$ as above, we have a commutative diagram

$$(6.3)\qquad \begin{array}{ccc} \mathcal{A}\Psi^2 & \xrightarrow{(\sigma)\Psi} & (\mathcal{S}^{\mathcal{A}^{opp}})\Psi \\ \downarrow{\scriptstyle \mu_{\mathcal{A}}} & & \downarrow{\scriptstyle \bar{\rho}} \\ \mathcal{A}\Psi & \xrightarrow[\sigma]{} & \mathcal{S}^{\mathcal{A}^{opp}} \end{array}$$

PROOF. Let an object $\{R\} \in |\mathcal{A}\Psi^2|$ be given. It is easy to prove, just using the fact that ξ is a colimit assignment on $\mathcal{S}^{\mathcal{A}^{opp}}$ that an isomorphism $j : (R.\sigma)\xi \longrightarrow (R\nu)\ \mathrm{Gray}.\varphi_o$ is defined by $\mathrm{incl}_D \circ j = j_D$, where

$$\{A \xrightarrow{a} \underline{X}\ Dr\} \quad \overset{j_D}{\rightsquigarrow} \quad \{A \xrightarrow{a} \underline{X_D}\ R\nu\}$$

(all notation being that of Definition 3.2). For each $D \in |\mathbb{D}|$, let $K_D \in |\mathcal{S}^{\mathcal{A}^{opp}}|$ be

$$A \rightsquigarrow |(\in_A, Dr)| = \coprod_{X \in |D\bar{r}|} \mathcal{A}(A, XDr) ,$$

and let $k_D : K_D \longrightarrow (Dr)\ \mathrm{Gray}.\varphi_o$ be the obvious map. The definition of j can now be stated by commutativity of the diagram

$$\begin{array}{ccccc} & & & & (R.\sigma)\xi \\ & & & \nearrow{\scriptstyle \{j_D\}} & \\ & & \coprod_{D\in|\mathbb{D}|} (Dr)\mathrm{Gray}.\varphi_o & & \cong \Big\downarrow\ j \\ & \nearrow{\scriptstyle \coprod k_D} & & & \\ \coprod_{D\in|\mathbb{D}|} Z_D & & \xrightarrow{\qquad\{q_D\}\qquad} & & (R\nu)\ \mathrm{Gray}.\varphi_o \end{array}$$

$q_D : Z_D \longrightarrow (R\nu)\ \mathrm{Gray}.\varphi_o$ given by

$$A \xrightarrow{a} \underline{X}\ Dr \rightsquigarrow \{A \xrightarrow{a} \underline{X_D}\ R\nu\}.$$

Evaluate the whole diagram on a fixed $A \in |\mathcal{A}|$. Clearly $\{q_D\}_A$ and each $(k_D)_A$ become $\mathcal{O}_e$-chosen epimorphisms in $\mathcal{S}$. So by Proposition I.8.1 $(\coprod k_D)_A$ is a chosen epimorphism in $\mathcal{S}$. Finally, by Proposition I.8.2 $\{j_D\}_A$ is a chosen epimorphism in $\mathcal{S}$. This implies that each j_A and therefore j must be the identity. But $(R.\sigma)\xi$ and $(R\,\nu)$ Gray.φ_o are the results of carrying $\{R\}$ clockwise and counterclockwise around in the diagram (6.3).

It is now trivial to check commutativity of the diagram (6.3) on morphisms in $\mathcal{A}\psi^2$. The lemma is proved.

It is clear from (6.2) and the fact that $|\gamma_{\mathcal{A}}|$ is an onto mapping that the full image $\mathcal{A}\tilde{\Theta}$ of Gray.φ_o equals the full image of σ. So

$$\sigma : \mathcal{A}\psi \longrightarrow \mathcal{A}\tilde{\Theta}$$

is an equivalence and has $|\sigma|$ an onto map. We can deduce

LEMMA 6.3. For any $n \geq 0$

$$(\sigma)\psi^n : \mathcal{A}\psi^{n+1} \longrightarrow \mathcal{A}\tilde{\Theta}\psi^n$$

is an equivalence of categories and $|(\sigma)\psi^n|$ is an onto map.

PROOF. First, ψ always carries equivalences to equivalences. This follows from the fact that ψ can be extended to a strong functor with respect to the enrichment of CAT_V^U over CAT_V^V. Next let

$$|(\sigma)\psi^{n-1}| : |\mathcal{A}\psi^n| \longrightarrow |\mathcal{A}\tilde{\Theta}\psi^{n-1}|$$

be onto, and let $\{\mathbb{D} \xrightarrow{R} \mathcal{A}\tilde{\Theta}\psi^{n-1}\}$ be an object in $\mathcal{A}\tilde{\Theta}\psi^n$. Since $(\sigma)\psi^{n-1}$ is an equivalence and onto on object stage, there exists a lifting

$$\begin{array}{ccc} & & \mathcal{A}\Psi^{n} \\ & \overset{R'}{\nearrow} & \Big\downarrow (\sigma)\Psi^{n-1} \\ \mathbb{D} & \xrightarrow[R]{} & \mathcal{A}\tilde{\Theta}\Psi^{n-1} \end{array}$$

Now $\{R\} = \{R'\}(\sigma)\Psi^{n}$.

In view of Lemma 6.3, we can now get more information about the diagram (6.3). Namely

PROPOSITION 6.4. The diagram (6.3) can be factored as follows

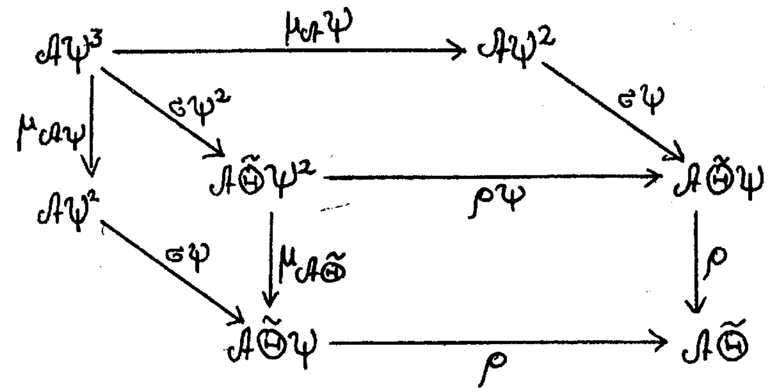
$$(6.4) \qquad \begin{array}{ccccc} \mathcal{A}\Psi^{2} & \xrightarrow{\sigma\Psi} & \mathcal{A}\tilde{\Theta}\Psi & \xrightarrow{i\Psi} & (\mathcal{S}^{\mathcal{A}^{opp}})\Psi \\ \Big\downarrow \mu_{\mathcal{A}} & & \Big\downarrow \rho & & \Big\downarrow \bar{\rho} \\ \mathcal{A}\Psi & \xrightarrow[\sigma]{} & \mathcal{A}\tilde{\Theta} & \xrightarrow[i]{} & \mathcal{S}^{\mathcal{A}^{opp}} \end{array}$$

for a (necessarily unique) ρ.

PROOF. Immediate from (i): commutativity of the outer diagram (Lemma 6.2), (ii): $\sigma\Psi$ being onto as well objects as morphisms (Lemma 6.3), and (iii): i being a full and faithful inclusion.

PROPOSITION 6.5. The functor ρ makes $\mathcal{A}\tilde{\Theta}$ into a regular colimit algebra (Definition 5.3).

PROOF. We must see that it is a structure for the monad Ψ, η, μ. Obviously $\eta_{\mathcal{A}\tilde{\Theta}} \cdot \rho = I_{\mathcal{A}\tilde{\Theta}}$. The other equation for being a structure is commutativity of the lower right hand diagram in



By Lemma 6.3, $(\sigma)\Psi^2$ is an epimorphism in CAT_V^U, so it suffices to prove commutativity of the total diagram. Apply the equation of Proposition 6.4

$$\sigma\Psi \cdot \rho = \mu_{\mathcal{A}} \cdot \sigma$$

twice; then commutativity follows immediately from the associativity law for μ.

The fact that $\gamma_{\mathcal{A}\tilde{\Theta}} \cdot \rho : \mathcal{A}\tilde{\Theta}T \longrightarrow \mathcal{A}\tilde{\Theta}$ is a colimit assignment is immediate from the fact that $\xi = \gamma_{\mathfrak{G}^{\mathcal{A}^{\mathrm{opp}}}} . \bar{\rho}$ is a colimit assignment on $\mathfrak{G}^{\mathcal{A}^{\mathrm{opp}}}$. And since $\sigma : \mathcal{A}\Psi \longrightarrow \mathcal{A}\tilde{\Theta}$ is an equivalence and by (6.4) a homomorphism with respect to the T-structures $\gamma_{\mathcal{A}\Psi} \cdot \mu_{\mathcal{A}}$, $\gamma_{\mathcal{A}\tilde{\Theta}} \cdot \rho$, we immediately have that $\gamma_{\mathcal{A}\Psi} \cdot \mu_{\mathcal{A}}$ is a colimit assignment. Let us summarize this in

PROPOSITION 6.6. The structure $\mu_{\mathcal{A}}$ makes $\mathcal{A}\Psi$ into a regular colimit algebra; and $\gamma_{\mathcal{A}\Psi} \cdot \mu_{\mathcal{A}}$ is a colimit assignment.

The Theorem of Section 9 will show that the last statement in this proposition follows from the first. Note how the Propositions 5.2, 6.4, 6.5, and 6.6 analyze the constituents of the composite functor $\mathrm{Gray}.\Phi_0$:

$$[\mathrm{Cat}, \epsilon_{\mathcal{A}}] \longrightarrow \mathcal{A}\Psi \longrightarrow \mathcal{A}\tilde{\Theta} \longrightarrow \mathfrak{G}^{\mathcal{A}^{\mathrm{opp}}}.$$

In particular, we may conclude that the composite is a prelimit homomorphism. Note also that Proposition 6.6 holds also for any $\mathcal{A}' \in CAT_V^U$; this follows from the fact that such an $\mathcal{A}'$ is isomorphic to an $\mathcal{A}$ in the subcategory $Cat_V \subseteq CAT_V^U$.

7. The main properties of Ψ.

One main property (the other one comes in Section 9) of $\Psi : CAT_V^U \longrightarrow$ $\longrightarrow CAT_V^U$ is that it is a functor and assigns to a category a free right complete category on it, in the following sense.

DEFINITION 7.1. A free right complete category on a category $\mathcal{A}$ is a right complete category $\hat{\mathcal{A}}$ together with a functor

$$\mathcal{A} \xrightarrow{\ y\ } \hat{\mathcal{A}}$$

so that if $\mathcal{A} \xrightarrow{h} \mathcal{B}$ is any functor to a right complete category, there exists a right continuous functor $H : \hat{\mathcal{A}} \to \mathcal{B}$, unique up to isomorphism, with $y.H = h$.

"Right complete" means here: having colimits over indexcategories in CAT_U^U; "right continuous" is understood relative to these.

Note that H is not required to be unique, only unique up to isomorphism. For the same reason, $\hat{\mathcal{A}}$ is only determined by $\mathcal{A}$ up to equivalence of categories; in particular, chosing an $\hat{\mathcal{A}}$, y for each $\mathcal{A}$ does not in general determine a functor $\wedge$. Recently, Ulmer [10] has shown that a free right complete category $\hat{\mathcal{A}} \in CAT_V^U$ exists for any $\mathcal{A} \in CAT_V^U$. He takes $\hat{\mathcal{A}}$ to be a suitable subcategory of $U\text{-Ens}^{\mathcal{A}^{opp}}$. For $\mathcal{A}$ small, i.e. $\mathcal{A} \in CAT_U^U$, his $\hat{\mathcal{A}}$ reduces to $U\text{-Ens}^{\mathcal{A}^{opp}}$, with y the Yoneda embedding. However, no way is known to make $U\text{-Ens}^{\mathcal{A}^{opp}}$ depend strictly functorial on $\mathcal{A} \in CAT^U_U$. (The same applies to $\mathcal{S}^{\mathcal{A}^{opp}}$.)

The present Ψ , however, solves the problem in a functorial way. (And at the same time, we get an associative choice of colimits on the solution:)

THEOREM 7.2. The monad Ψ, η, μ on CAT^U_V has the property: for each $\mathcal{A} \in |CAT^U_V|$, $\mathcal{A}\Psi$, $\eta_{\mathcal{A}}$ is a free right complete category on $\mathcal{A}$.

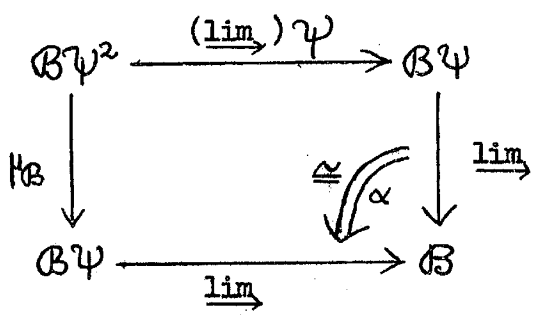
PROOF. By Proposition 6.6, $\mathcal{A}\Psi$ has colimits for indexcategories in Cat. Therefore colimits exist in $\mathcal{A}$ for all indexcategories in CAT^U_U. We just have to prove the "almost" -universality. To do this, we need some notions.

DEFINITION 7.3. A colimit assignment on a category $\mathcal{B}$

(7.1) $$\mathcal{B}T \xrightarrow{\gamma_{\mathcal{B}}} \mathcal{B}\Psi \xrightarrow{\underrightarrow{\lim}} \mathcal{B}$$

which factors through $\gamma_{\mathcal{B}}$ is called <u>regular</u> if the following diagram commutes up to an isomorphism α .

(7.2)

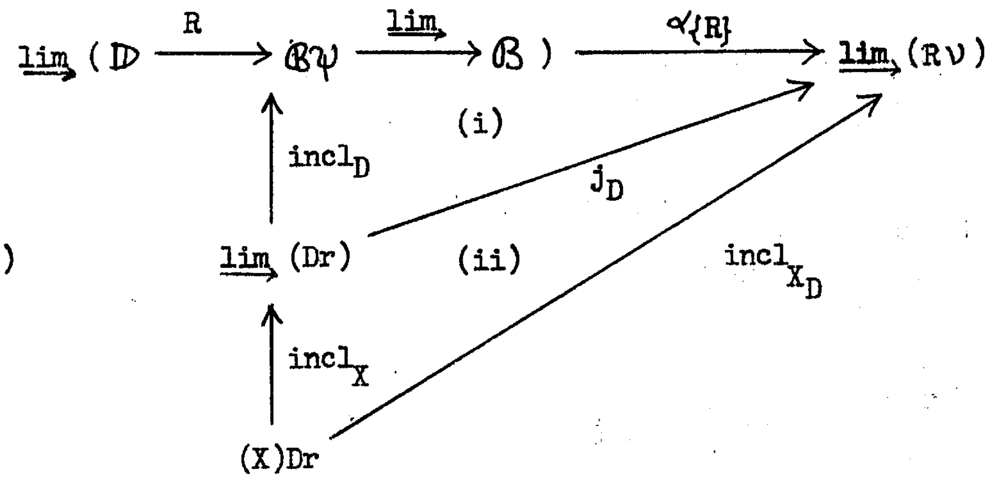


PROPOSITION 7.4. Any colimit assignment of the form (7.1) is regular.

PROPOSITION 7.5. Any right complete category admits a regular colimit assignment (7.1). (Not necessarily associative !)

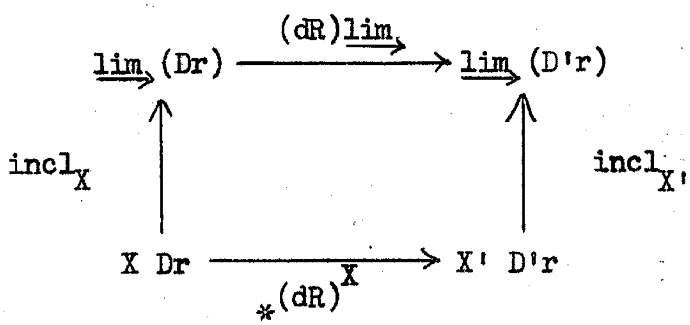
PROOF of Proposition 7.4. Consider an object $\{\mathbb{D} \xrightarrow{R} \mathcal{B}\Psi\}$ in $\mathcal{B}\Psi^2$, and let r and $R\nu$ be as in Definition 3.2 (with $\mathcal{B}$ in place of $\mathcal{A}$). We produce an isomorphism $\alpha_{\{R\}}$ (upper line in the diagram):

(7.3)

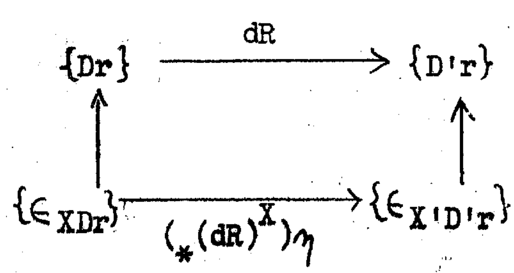


by requiring the triangles (i) to commute for all $D \in |\mathbb{D}|$; here j_D is determined by requiring the triangles (ii) to commute for all $X \in |\overline{Dr}|$. The fact that each j_D is well determined is easy. To see that $(dR)\varinjlim \circ j_D = j_D$ for $d : D \longrightarrow D'$ in $\mathbb{D}$, we observe that for all $X \in |\overline{Dr}|$, the diagrams

(7.4)



commute; for, this diagram is the functor $\varinjlim$ acting on the diagram in $\mathcal{B}\Psi$:



(the vertical arrows being the obvious ones). The desired commutativity now follows because each ${}_*(dR)^X$ is a morphism $X_D \longrightarrow X'_{D'}$ in $R\bar{\nabla}$.

The inverse $\alpha^{-1}_{\{R\}}$ is given by

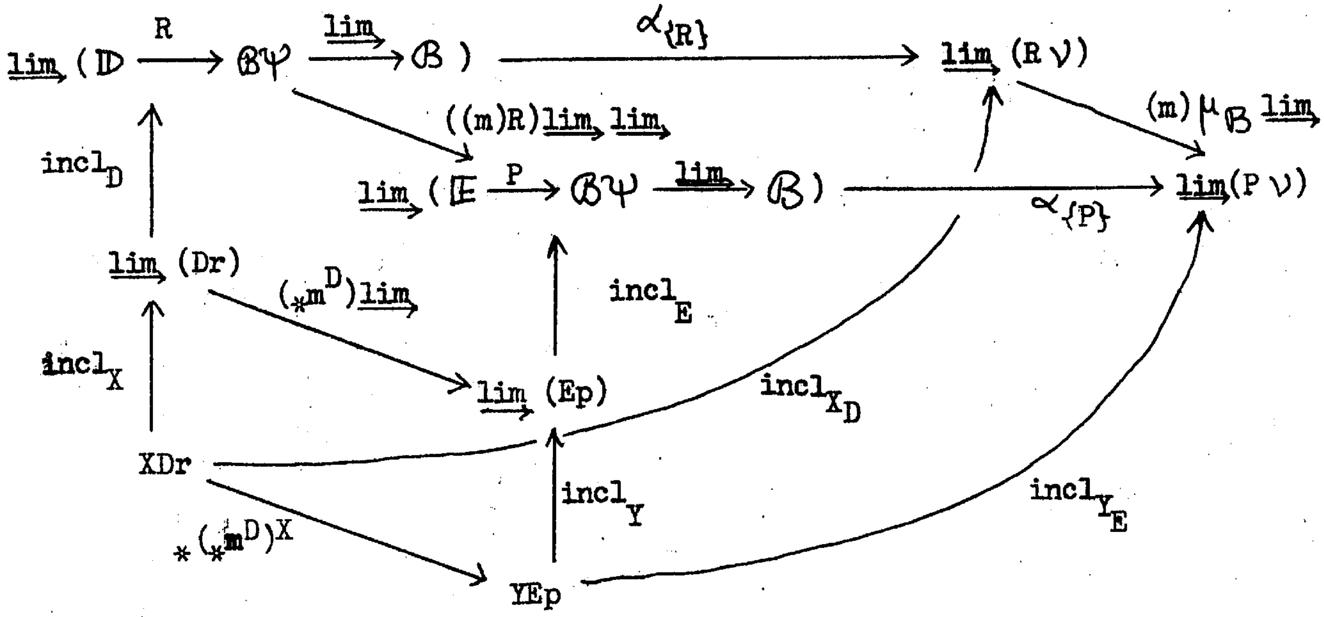
$$\mathrm{incl}_{X_D} \cdot \alpha^{-1}_{\{R\}} = \mathrm{incl}_X \cdot \mathrm{incl}_D \; ;$$

to prove that this defines a morphism between the two limits, consider a morphism $a : X_D \longrightarrow X'_{D'}$ in $R\bar{\nabla}$, a as in the diagram (3.2). Place $\underrightarrow{\lim}\,(\mathbb{D} \xrightarrow{R} \mathcal{B}\psi \xrightarrow{\underrightarrow{\lim}} \mathcal{B})$ to the right of that diagram. It is then obvious that it suffices to prove commutativity of diagrams of the form (i) or (ii):

$$\begin{array}{ccccc}
 & (X_{n-1})D_{n-1}r & & & \\
{}_*(d_{n-1}R)^{X_{n-1}} & \downarrow & \text{(i)} & \searrow \; \mathrm{incl}_{X_{n-1}}\circ\mathrm{incl}_{D_{n-1}} & \\
 & (X_n)D_nr & \xrightarrow{\mathrm{incl}_{X_n}\circ\mathrm{incl}_{D_n}} & \underrightarrow{\lim}\,(\mathbb{D} \xrightarrow{R} \mathcal{B}\psi \xrightarrow{\underrightarrow{\lim}} \mathcal{B}). & \\
(x)D_nr & \uparrow & \text{(ii)} & \nearrow \; \mathrm{incl}_{X'_n}\circ\mathrm{incl}_{D_n} & \\
 & (X'_n)D_nr & & &
\end{array}$$

For type (ii) (corresponding to the inessential triangles of (3.2)), this is obvious; for type (i), it is an immediate consequence of commutativity of (7.4).

To prove naturality in $\{R\}$, let an $m: \{R\} \longrightarrow \{P\}$ be given, as in Section 3, with $\mathcal{B}$ instead of $\mathcal{A}$. Then the definition (3.3) of $(m)\mu_{\mathcal{B}}$, together with an argument similar to that establishing commutativity of (7.4), gives commutativity of the curved diagram in



Two applications of the said argument, on the other hand, give the commutativity of the two left hand squares in the diagram. This proves naturality of α, thus the Proposition 7.4.

PROOF of Proposition 7.5. By the Proposition 7.4, we just have to produce a colimit assignment of the form (7.1). We first make a choice of colimit diagrams in $\mathcal{B}$, i.e. construct a functor

$$[\mathrm{Cat}^U, \epsilon_{\mathcal{B}}] \xrightarrow{\underrightarrow{\lim}} \mathcal{B}$$

so that

(i) $\quad \bar{\eta}_{\mathcal{B}} \circ \underrightarrow{\lim} = I_{\mathcal{B}}$

(ii) $\quad (\mathbb{D} \xrightarrow{R} \mathcal{B}) \equiv (\mathbb{D}' \xrightarrow{R'} \mathcal{B}) \implies$

the colimit diagrams for R and R' are identical.

That this can be done follows from $(B)\bar{\eta}_{\mathcal{B}} \equiv (B')\bar{\eta}_{\mathcal{B}} \implies B = B'$, and from Proposition 2.4.

We then automatically get a functor $\underrightarrow{\lim}$, whose object function fac-

tors over $|\gamma_{\mathcal{B}}|$. To produce a functor $\underrightarrow{\lim} : \mathcal{B}\Psi \to \mathcal{B}$ we take on objects:

$$\underrightarrow{\lim}\,(\{\mathbb{D} \xrightarrow{R} \mathcal{B}\}) = \underrightarrow{\lim}\,(\mathbb{D} \xrightarrow{R} \mathcal{B}) \,.$$

On morphisms, say

$$(R)\ \text{gray}.\varphi_0 \xrightarrow{m} (P)\ \text{gray}.\varphi_0$$

(with $P : \mathbb{E} \to \mathcal{B}$) we take $(m)\,\underrightarrow{\lim}$ so that the diagrams

(7.5)
$$\begin{array}{ccc} \underrightarrow{\lim}\,(R) & \xrightarrow{(m)\,\underrightarrow{\lim}} & \underrightarrow{\lim}\,(P) \\ \text{incl}_D \uparrow & & \uparrow \text{incl}_E \\ DR & \xrightarrow[{}_*m^D]{} & EP \end{array}$$

commute for all $D \in |\mathbb{D}|$, with ${}_*m^D$ as in Section 2. That the diagrams (7.5) determine $(m)\,\underrightarrow{\lim}$ follows from universality of $\underrightarrow{\lim}\,(R)$, together with (2.9) and the equations

$$\text{incl}_E = (e)P \,.\, \text{incl}_{E'}$$

for $E \xrightarrow{e} E'$ in $\mathbb{E}$. Functorality of $\underrightarrow{\lim}$ follows similarly, using (2.8) instead of (2.9). Proposition 7.5 is proved.

The following proposition is obvious.

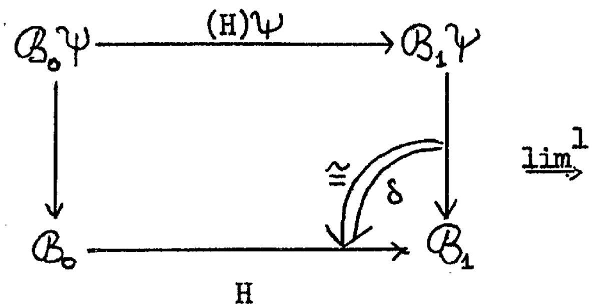
PROPOSITION 7.6. Let $\mathcal{B}_i\Psi \xrightarrow{\underrightarrow{\lim}^i} \mathcal{B}_i$ be regular colimit assignments, $i = 0, 1$. Then a functor

$$H : \mathcal{B}_0 \to \mathcal{B}_1$$

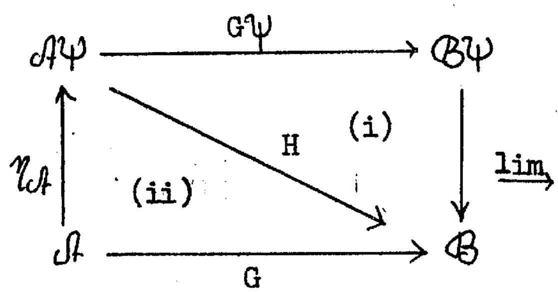
is right continuous (with respect to colimits with indexcategories in

CAT^U_U) if and only if the following diagram commutes up to an isomorphism δ:

(7.6)



We can now complete the proof of Theorem 7.2; it will really just be a repetition in the 2 -dimensional case of the familiar construction of free algebras for the algebras over a monad. Let $G : \mathcal{A} \longrightarrow \mathcal{B}$ be given, where $\mathcal{B}$ is right complete. By Proposition 7.5, we can put a regular colimit assignment $\underrightarrow{\lim}$ on $\mathcal{B}$. We define a functor H by commutativity of the triangle (i) in



By naturality of η and by $\eta_{\mathcal{B}} \circ \underrightarrow{\lim} = I_{\mathcal{B}}$, we get commutativity of (ii). To prove that H is right continuous, we must by Proposition 7.6 prove the existence of a diagram (7.6) with $\mathcal{B}$, $\underrightarrow{\lim}$ for $\mathcal{B}_1$, $\underrightarrow{\lim}^1$ and $\mathcal{A}\psi$, $\mu_{\mathcal{A}}$ for $\mathcal{B}_0$, $\underrightarrow{\lim}^0$. But

$$\mu_{\mathcal{A}} \circ H = \mu_{\mathcal{A}} \circ G\psi \circ \underrightarrow{\lim} = G\psi^2 \circ \mu_{\mathcal{B}} \circ \underrightarrow{\lim}$$

$$\cong G\psi^2 \circ (\underrightarrow{\lim})\psi \circ \underrightarrow{\lim} = (G\psi \circ \underrightarrow{\lim})\psi \circ \underrightarrow{\lim} = H\psi \circ \underrightarrow{\lim},$$

the isomorphism sign by (7.2). This proves H right continuous. If also H' is right continuous and has $\eta_{\mathcal{A}} \circ H' = G$, we get

$$H' = (\eta_{\mathcal{A}})\Psi \cdot \mu_{\mathcal{A}} \cdot H' \cong (\eta_{\mathcal{A}})\Psi \cdot (H')\Psi \cdot \underrightarrow{\lim}$$

$$= (G)\Psi \circ \underrightarrow{\lim} = H,$$

the isomorphism sign by right continuity (7.6). So a right continuous extension of G over $\eta_{\mathcal{A}}$ exists and is unique up to isomorphism, and the theorem is proved.

The theorem and proof is immediately "relativizable" in the sense that one may construct also the "free- category - with - colimits - over - indexcategories - in - Cat_o^U," where Cat_o^U is any <u>suitable</u> subcategory of Cat^U. We briefly sketch this in the next section, in particular explain what we mean by "suitable" in this context. (It will of course mean something related to "admitting a calculus of prelimits.")

8. <u>Submonads of Ψ</u>.

Just as we in Section 5 constructed not just one prelimit monad T but one T for each subcategory $Cat_o \subseteq Cat$ admitting a calculus of prelimits, we can construct the Ψ_o relative to certain of these subcategories.

Consider $[Cat_o, \in_{\mathcal{A}}] \subseteq [Cat^U, \in_{\mathcal{A}}]$. The equivalence relation $\equiv$ on $|[Cat^U, \in_{\mathcal{A}}]|$ restricts to an equivalence relation $\equiv$ on $|[Cat_o, \in_{\mathcal{A}}]|$. In this way, a full subcategory

(8.1) $$\mathcal{A}\Psi_o \subseteq \mathcal{A}\Psi$$

is created. If $1 \in |Cat_o|$, $\eta_{\mathcal{A}}$ factors through $\mathcal{A}\Psi_o$.

DEFINITION 8.1. We say that a full subcategory $Cat_o \subseteq Cat^U$ <u>admits</u> **<u>a calculus of regular prelimits</u> if $1 \in |Cat_o|$, and if, whenever $\mathbb{D} \in |Cat_o|$,** the following holds for all $\mathcal{A} \in |CAT_V^U|$:

$$\mathbb{D} \xrightarrow{R} \mathcal{A}\psi \text{ factors through } \mathcal{A}\psi_o$$

implies

$$\{R\}\mu_{\mathcal{A}} \in |\mathcal{A}\psi_o|.$$

The definition says, briefly, that ψ_o must be a submonad of ψ.

The arguments of the preceding section immediately carries over to give e.g. that $\mathcal{A}\psi_o$ is a free Cat_o complete category on the category $\mathcal{A}$.

The proof of the following proposition is obvious.

PROPOSITION 8.2. If ε is a regular cardinal, then $\mathcal{S}_\varepsilon \subseteq Cat^U$ (the full subcategory of Cat consisting of discrete $\mathcal{O}$-categories of cardinality $< \varepsilon$) admits a calculus of regular prelimits; the corresponding monad ψ_o is isomorphic to the prelimit monad for $Cat_o = \mathcal{S}_\varepsilon$ (I.5).

In particular, the free category with associative sums of cardinality $< \varepsilon$ on a category $\mathcal{A}$ can be described simply as $[\mathcal{S}_\varepsilon, \epsilon_{\mathcal{A}}]$.

PROPOSITION 8.3. Let ε be an uncountable regular cardinal. The category $Cat_\varepsilon^{(\varepsilon)}$ of categories $\mathbb{D}$ with $|\mathbb{D}|$ an ordinal number of cardinality $< \varepsilon$, and each $\mathbb{D}(D,D')$ a set in U of cardinality $< \varepsilon$, admits a calculus of regular prelimits.

PROOF. Let $\mathbb{D} \xrightarrow{R} \mathcal{A}\psi_o \subseteq \mathcal{A}\psi$ be given with $\mathbb{D} \in Cat_\varepsilon^{(\varepsilon)}$, and with $\mathcal{A}\psi_o$ as in (8.1) with $Cat_\varepsilon^{(\varepsilon)}$ for Cat_o. If we construct $R\bar{\nu}$ as in Definition 3.2, we will arrive at a category with less than ε objects; but we cannot control the size of the hom sets, since we have assumed nothing about the hom sets of $\mathcal{A}$. But it is easy to see that

$R\overline{\mathcal{V}} \xrightarrow{R\nu} \mathcal{A}$ stands in the relation $\equiv$ to the following functor

$$R\underline{\mathcal{V}} \xrightarrow{R\underline{\nu}} \mathcal{A} ,$$

where $|R\underline{\mathcal{V}}| = |R\overline{\mathcal{V}}|$, and whose structure as a category is the free category on the graph having as edges symbols of the form (i) and (ii)

(i) $$X_D \xrightarrow{[_*(dR)^X]} X'_{D'}$$

for $D \xrightarrow{d} D'$ in $\mathbb{D}$ and $_*(dR)^X = (XDr \longrightarrow \underline{X'}D'r) \in (\mathcal{E}_{XDr}, D'r)$ as in Section 2;

(ii) $$X_D \xrightarrow{[(x)Dr]} X'_D$$

for $x : X \longrightarrow X'$ in $D\overline{r}$.

This graph has less than $\mathcal{E}$ edges. Since $\mathcal{E}$ was uncountable, we get the same bound on the size of the hom sets in the free category on the graph, and the Proposition is proved.

By putting a well ordering on the hom sets of the free category constructed, one obtains an $\mathcal{O}$-category isomorphic to $R\underline{\mathcal{V}}$, and therefore also $\equiv$ to it. So the Proposition also holds true if we read $Cat_{\mathcal{E}}^{(\mathcal{E})}$ as the category of $\mathcal{O}$-categories with the same cardinality limitations.

9. Algebras for the monad Ψ.

We prove here that an algebra for the monad Ψ is automatically a colimit algebra.

Let $Cat_o \subseteq Cat \subseteq Cat^U$ be a subcategory admitting a calculus of prelimits and of regular prelimits (Definitions I.3.1 and 8.1). Thus we get the monads corresponding to Cat_o:

$$\Psi, \eta, \mu \quad \text{and} \quad T, \tilde{\eta}, \tilde{\mu},$$

and a morphism of monads

$$\gamma_{\mathcal{A}} \ : \ \mathcal{A}T \longrightarrow \mathcal{A}\Psi,$$

given by Definition 5.1. The monads and γ are fixed in this section. If $R \in |\mathcal{A}T|$, we shall as usual often write $\{R\}$ for $(R)\gamma_{\mathcal{A}}$.

THEOREM 9.1. Let $\mathcal{A}\Psi \xrightarrow{\xi} \mathcal{A}$ be a structure for the monad Ψ, i.e.

$$\text{(9.1)} \qquad \begin{aligned} \xi T \cdot \xi &= \mu_{\mathcal{A}} \cdot \xi \\ \eta_{\mathcal{A}} \cdot \xi &= I_{\mathcal{A}} \end{aligned}$$

Then $\mathcal{A}T \xrightarrow{\gamma_{\mathcal{A}}} \mathcal{A}\Psi \xrightarrow{\xi} \mathcal{A}$ is a colimit assignment (relative to Cat_o).

PROOF. We produce a transformation

$$I_{\mathcal{A}\Psi} \xrightarrow{\quad \tau \quad} \xi \cdot \eta_{\mathcal{A}}$$

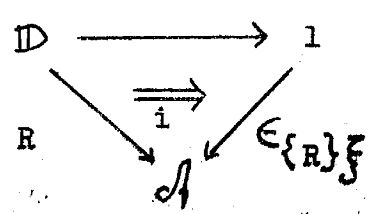
(we shall not need naturality); each instance

$$(R)\gamma_{\mathcal{A}} \xrightarrow{\quad \tau_{R\gamma_{\mathcal{A}}} \quad} (R)\gamma_{\mathcal{A}}\,\xi\,\eta_{\mathcal{A}}$$

will be of the form $(i)\gamma_{\mathcal{A}}$, where

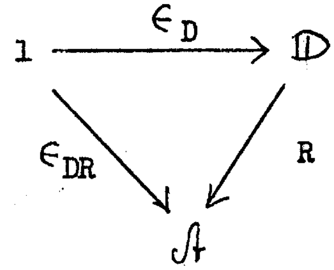
$$R \xrightarrow{\ i \} \{R\}\,\xi\,\tilde{\eta}_{\mathcal{A}}$$

is a morphism in $\mathcal{A}T$



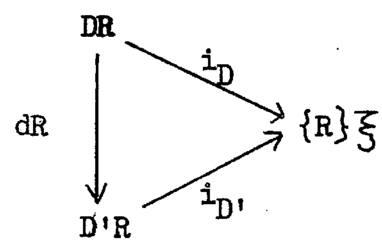
with $i_D : DR \longrightarrow \{R\}\xi$ in turn given as the value of $\gamma_{\mathcal{A}} \cdot \xi$ at the morphism

(9.2)



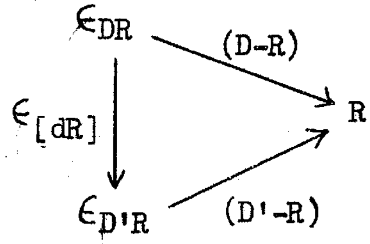
in $\mathcal{A}T$. The morphism (9.2) we denote (D-R). To see that i is natural, note that for $d : D \longrightarrow D'$ a morphism in $\mathbb{D}$, the diagram

(9.3)



is the value of $\gamma_{\mathcal{A}} \cdot \xi$ at the diagram in $\mathcal{A}T$

(9.4)



In general, (9.4) is not commutative in $\mathcal{A}T$, but $\gamma_{\mathcal{A}}$ of it is easily seen to be commutative in $\mathcal{A}\Psi$. The commutativity of (9.3) then follows.

To prove that $\gamma_{\mathcal{A}} \cdot \xi$ is a colimit assignment, let $R : \mathbb{D} \longrightarrow \mathcal{A}$ be an object in $\mathcal{A}T$. We must show that the morphisms in $\mathcal{A}$

$$i_D = (D\text{-}R)\ \gamma_{\mathcal{A}} \cdot \xi : DR \longrightarrow \{R\}\xi,$$

$D \in |\mathbb{D}|$, form a colimit diagram for R. Commutativity of the diagrams (9.3) is proved already. Next assume that

$$(9.5)\qquad \begin{array}{ccc} \mathbb{D} & \longrightarrow & 1 \\ & {\scriptstyle R}\searrow \quad \overset{f}{\Longrightarrow} \quad \swarrow{\scriptstyle \in_A} & \\ & \mathcal{A} & \end{array}$$

is given. We must find a unique

$$\{R\}\xi \xrightarrow{\ f_\infty\ } A$$

with

$$(9.6)\qquad i_D \circ f_\infty = f_D \qquad \forall D \in |\mathbb{D}|.$$

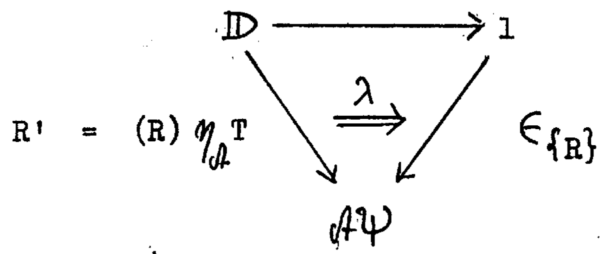
The existence of such an f_∞ is easy: simply take $\gamma_{\mathcal{A}} \circ \xi$ on the morphism (9.5) in $\mathcal{A}T$. To prove uniqueness, let f_∞ satisfy (9.6). This means that we have a commutative diagram in $\mathcal{A}T$

$$\begin{array}{ccc} R & \xrightarrow{\ i\ } & \in_{\{R\}\xi} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \in_{[f_\infty]}} \\ & & \in_A \end{array} \quad .$$

Apply $\gamma_{\mathcal{A}} \circ \xi$ to this diagram and get

$$\begin{array}{ccccc} \{R\}\xi & \xrightarrow{(i)\gamma_{\mathcal{A}}\xi} & (\in_{\{R\}\xi})\,\gamma_{\mathcal{A}}\circ\xi & = & \{R\}\xi \\ & {\scriptstyle (f)\gamma_{\mathcal{A}}\xi}\searrow & & & \downarrow{\scriptstyle f_\infty} \\ & & (\in_A)\,\gamma_{\mathcal{A}}\xi & = & A \end{array} \; .$$

If we can prove the top arrow to be $I_{\{R\}\xi}$, uniqueness of f_∞ is immediate. Consider the morphism in $\mathcal{A}\Psi T$



given by

$$\lambda_D = (D\text{-}R)\,\gamma_{\mathcal{A}} .$$

Naturality of λ follows again from commutativity of $\gamma_{\mathcal{A}}$ of the diagrams (9.4). It is now easy to see that

$$(\lambda)\,\gamma_{\mathcal{A}\Psi} \cdot \xi\Psi = (\lambda)\,\xi T \cdot \gamma_{\mathcal{A}} = (i)\,\gamma_{\mathcal{A}} ,$$

and so by (9.1)

$$(9.7) \qquad (i)\,\gamma_{\mathcal{A}} \cdot \xi = (\lambda)\,\gamma_{\mathcal{A}\Psi} \cdot \mu_{\mathcal{A}} \cdot \xi .$$

But $(\lambda)\,\gamma_{\mathcal{A}\Psi} \cdot \mu_{\mathcal{A}} = I_{\{R'\}\mu_{\mathcal{A}}}$, as is easily seen using the definition (3.3) of $\mu_{\mathcal{A}}$'s value on a morphism. (In (3.3), ${}_*m^D$ can now be chosen to be $\lambda_D = (D\text{-}R)\,\gamma_{\mathcal{A}}$, and ${}_*({}_*m^D)^0$ can be chosen to be $I_{DR} : DR \longrightarrow \underline{D}R$.) So (9.7) must be the identity morphism, and the theorem follows.

The theorem proved can be expressed: The concept of a category - with - regular - colimit - structure (relative to Cat_o) is <u>monadic</u> or <u>doctrinal</u> over CAT_V^U. This is the terminology of Beck and Lawvere. A special case of the theorem was proved by them, namely: The concept of category - with - chosen - initial - object is doctrinal. It appears here by taking Cat_o to consist of discrete $\mathcal{O}$-categories $\mathbb{D}$ with $|\mathbb{D}| = 0$ or $= 1$.

CHAPTER III

ASSOCIATIVITIES ARISING FROM $\mathcal{S}$

1. A general construction of regular associative colimits.

It is well known [7] that coequalizers and sums in a category suffice to construct arbitrary colimits. There is a simple condition which will insure that the constructed colimits form a regular colimit structure (in particular, the colimit formation will be associative).

Let ε be a regular uncountable cardinal, and let Ψ_o be the submonad corresponding to $\mathrm{Cat}_{\varepsilon}^{(\varepsilon)}$. (II, Section 8.)

Assume that a category $\mathcal{B}$ has associative sums of cardinality $< \varepsilon$ satisfying $\coprod_1 X = X$, a full choice of quotient objects, and that coequalizers exist. Further assume that if $q_x : A_x \longrightarrow B_x$ are chosen quotients for all $x \in X$, $\overline{\overline{X}} < \varepsilon$, then

$$\coprod_X A_x \xrightarrow{\coprod q_x} \coprod_X B_x$$

is a chosen quotient. In this case we can construct a functor $\mathcal{B}\Psi_o \xrightarrow{\varinjlim} \mathcal{B}$ which turns $\mathcal{B}$ into a regular colimit algebra relative to Ψ_o.

Let $\mathbb{D} \xrightarrow{R} \mathcal{B}$ represent an object in $\mathcal{B}\Psi_o$, $\mathbb{D} \in \mathrm{Cat}_{\varepsilon}^{(\varepsilon)}$. Form the usual coequalizer diagram, q_R being the chosen quotient in the class determined by the coequalizer:

$$\coprod_{D \in |\mathbb{D}|} \ \coprod_{D' \in |\mathbb{D}|} \ \coprod_{\mathbb{D}(D,D')} DR \ \underset{f_1}{\overset{f_0}{\rightrightarrows}} \ \coprod_{D \in |\mathbb{D}|} DR \xrightarrow{q_R} \varinjlim (R) \ ;$$

$$\mathrm{incl}_D \cdot \mathrm{incl}_{D'} \cdot \mathrm{incl}_{D \xrightarrow{d} D'} \cdot f_i = \begin{cases} \mathrm{incl}_D & i = 0 \\ (dR) . \mathrm{incl}_{D'} & i = 1 \end{cases}$$

(Henceforth, arrows $\twoheadrightarrow$ denote chosen quotient morphisms.) Strictly, the index set $\mathbb{D}(D,D')$ in the left hand sum is not in $\mathbb{S}$; the sum shall then just mean <u>some</u> sum with that index set. Denote by in_D the composite

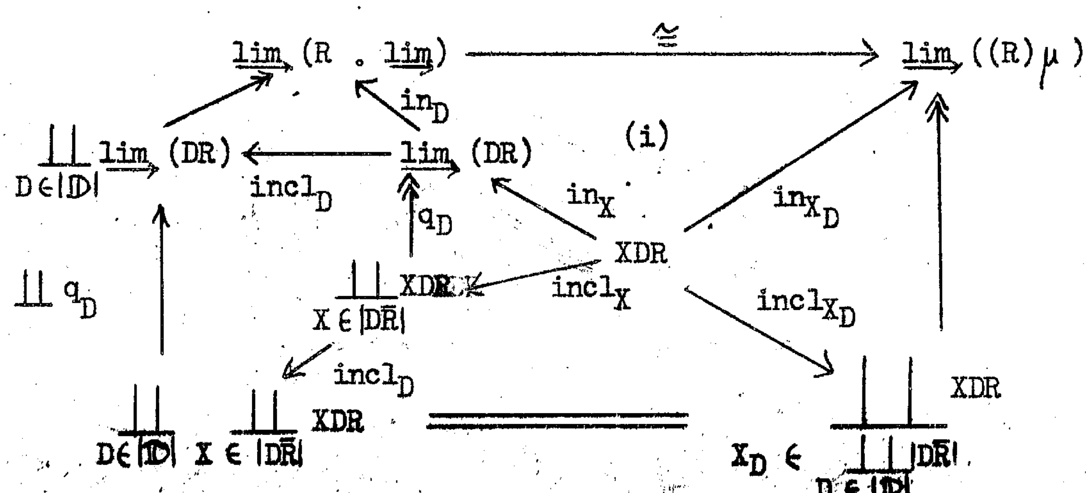
$$in_D \; : \; DR \xrightarrow{incl_D} \coprod_{D \in |\mathbb{D}|} DR \xrightarrow{q_R} \!\!\!\!\to \underrightarrow{lim}\,(R) \; .$$

These in_D form a colimit diagram for R, as is well known. It is easily seen that $\underrightarrow{lim}\,(R)$ and the in_D do not depend on the choice of $R \in \{R\} \in |\mathcal{B}\psi_o|$. On morphisms, say $m : \{R\} \to \{P\}$, $\underrightarrow{lim}$ is given the value $(m)\underrightarrow{lim}$ determined by commutativity of the diagrams

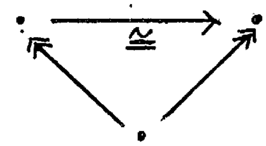
$$\begin{array}{ccc} DR & \xrightarrow{in_D} & \underrightarrow{lim}\,(R) \\ {\scriptstyle {}_*m^D}\downarrow & & \downarrow{\scriptstyle (m)\underrightarrow{lim}} \\ EP & \xrightarrow[in_E]{} & \underrightarrow{lim}\,(P) \end{array} \quad ,$$

notation as in II.2.

We can then easily show that $\underrightarrow{lim}$ is a structure for the monad ψ_o. The unit law obviously holds. So we just have to prove that the natural isomorphism constucted in Proposition II.2.4 is the identity. The isomorphism is given by commutativity of the triangle (i) below for all X, D ; hence the outer diagram commutes:



All morphisms which come as chosen quotients by definition, are marked $\twoheadrightarrow$. But now by assumption, $\coprod q_D$ is a chosen quotient mapping. Since chosen quotients are stable under composition, the diagram is of the form



This implies that the isomorphism is an identity.

An unpleasant defect of the present work is that we have been unable to show that every category with colimits is equivalent to a category with a regular associative colimit structure. Bénabou has a construction which out of a multiplicative category produces a strictly associative multiplicative category. Lawvere has pointed out to the author that the method can be applied here to manufacture associative sums.

THEOREM 1.1. Let $\mathcal{A}$ be a category with chosen sum diagrams for index sets of cardinality $< \varkappa$, where $\varkappa$ is a regular cardinal. Then $\mathcal{A}$ is equivalent to a category $\bar{\mathcal{A}}$ with associative sums of cardinality less than $\varkappa$, i.e. $\bar{\mathcal{A}}$ can be equipped with a structure for the monad T, η, μ defined in I.5 (with respect to $Cat_0 = \mathcal{S}_\varkappa$, the category of discrete $\mathcal{O}$-categories of cardinality $< \varkappa$).

PROOF. The choice of sums in $\mathcal{A}$ defines a functor

$$\mathcal{A}T = [\mathcal{S}_\varkappa, \in_{\mathcal{A}}] \xrightarrow{\Sigma} \mathcal{A} .$$

Define $\bar{\mathcal{A}}$ by

$$|\bar{\mathcal{A}}| = |\mathcal{A}T| ,$$

$$\bar{\mathcal{A}}(R,P) = \mathcal{A}((R)\Sigma, (P)\Sigma)$$

with composition in $\bar{\mathcal{A}}$ defined by means of composition in $\mathcal{A}$. We shall define a structure

$$\bar{\mathcal{A}}\,T \xrightarrow{\ \xi\ } \bar{\mathcal{A}}\ .$$

The object mapping of ξ is defined by $|\xi| = |\mu_{\mathcal{A}}|$; this makes sense, since $|\bar{\mathcal{A}}| = |\mathcal{A}T|$, and for all categories $\mathcal{B}$

$$|\mathcal{B}\,T| = \big||\mathcal{B}|\,T\big|\ .$$

Writing this definition of $|\xi|$ out in terms of elements, it will be obvious how to define ξ on morphisms; and the proof of $\xi T . \xi = \mu_{\mathcal{A}} \circ \xi$ is essentially the same as the proof of associativity of μ (and is anyway simple, since all indexcategories are discrete).

2. Finite left limits in $\mathcal{S}$.

Let $A_0, \ldots , A_{n-1}$ be ordinal numbers (n finite). Then the set theoretic product

(2.1) $$\bigtimes_{i \in n} A_i$$

can be well ordered by means of the orderings of the A_i's: namely by the lexicographic ordering. We can also well order it antilexicographically, i.e. lexicographically from the right. This is the traditional well ordering of a product of ordinal numbers, and it will become clear in the next section that it is better. So let $\prod_{i \in n} A_i$ denote the unique ordinal number of order type (2.1) ordered lexicographically from the right, and let α denote the (unique) order isomorphism

$$\prod_{i \in n} A_i \xrightarrow{\ \alpha\ } \bigtimes_{i \in n} A_i\ .$$

Composing α with the n projection mappings, we get n morphisms in $\mathcal{S}$:

$$\prod_{i \in n} A_i \xrightarrow{\text{proj}_i} A_i ,$$

$i \in n$, and it is clear that they constitute $\prod_{i \in n} A_i$ as the categorical product (discrete (left) limit) of the A_i's. The following morphisms, natural in the A_y's, are easily seen to be order preserving isomorphisms, i.e. identities, for X and Y_x finite ordinals:

(2.2) $$\prod_1 A \xrightarrow{\text{proj}_0} A$$

(2.3) $$\prod_{x \in X} \prod_{y \in Y_x} (A_y)_x \xrightarrow{e} \prod_{y_x \in \coprod_{x \in X} Y_x} A_{y_x}$$

e given by

(2.4) $$e \cdot \text{proj}_{y_x} = \text{proj}_x \cdot \text{proj}_y .$$

This says that $\mathcal{S}$ has finite, associative products $\prod$ or that $\mathcal{S}^{\text{opp}}$ has finite associative sums. Furthermore, in I.6 we constructed a full choice of subobjects on $\mathcal{S}$, i.e. a full choice of quotient objects on $\mathcal{S}^{\text{opp}}$. We can now apply the technique of Section 1 to $\mathcal{B} = \mathcal{S}^{\text{opp}}$ to construct all finite colimits in $\mathcal{S}^{\text{opp}}$. It will be a regular colimit structure

(2.4) $$\mathcal{S}^{\text{opp}}\psi_0 \xrightarrow{\xi} \mathcal{S}^{\text{opp}} ;$$

ψ_0 the submonad (II.8) corresponding to $\text{Cat}_0 \subseteq \text{Cat}$, $\mathcal{O}$-categories with finite object set. We of course interpret ξ as an associative (left) limit structure on $\mathcal{S}$.

It is easily seen that the products distribute from the left over colimits in $\mathcal{S}$, e.g.

$$A \times (B + C) = (A \times B) + (A \times C) ,$$

the equality being the usual natural distributivity isomorphism. The corresponding distributivity - from - the - right holds only up to isomorphism. But this must be so. For

PROPOSITION 3.1. Let $\mathcal{E}$ be a category $\simeq$ U-Ens with sum $+$ and product $\times$. Then the two natural distributivity isomorphisms

$$(A\times B)+(A\times C) \xrightarrow{\lambda_{A,B,C}} A\times(B+C)$$

$$(B\times A)+(C\times A) \xrightarrow{\rho_{B,C,A}} (B+C)\times A$$

cannot both have all their instances to be identities.

PROOF. Consider $(A+B)\times(C+D)$. Then by the definition of the distributivity isomorphisms λ and ρ, one gets a commutative diagram

$$\begin{array}{ccc}
 & & (A\times C + A\times D) + (B\times C + B\times D) \\
 & \nearrow^{\mathrm{incl}_0 \circ \mathrm{incl}_1} & \downarrow \lambda+\lambda \\
 & & (A\times(C+D)) + (B\times(C+D)) \\
 & & \downarrow \rho \\
A\times D & & (A+B)\times(C+D) \\
 & & \uparrow \lambda \\
 & & ((A+B)\times C) + ((A+B)\times D) \\
 & \searrow_{\mathrm{incl}_1 \circ \mathrm{incl}_0} & \uparrow \rho+\rho \\
 & & (A\times C + B\times C) + (A\times D + B\times D)\,.
\end{array}$$

But now put $A = B = C = D \neq 0$. If the vertical maps were all identities, the images of the two maps $\mathrm{incl}_0 \circ \mathrm{incl}_1$ and $\mathrm{incl}_1 \circ \mathrm{incl}_0$ would be non - disjoint. But in $\mathcal{E} \simeq$ U-Ens, this is false.

3. Colimits in module categories.

For an arbitrary ring Λ, we put a regular colimit structure on a category equivalent to the category of all Λ -modules in U.

We use same notation for a module and its underlying set.

DEFINITION 3.1. Let ${}_{\Lambda}\mathcal{M}$ be the category of all those left Λ -modules A, where $A \in |\mathcal{S}|$ and the zero element of A is the first element in A, i.e. the ordinal number 0.

THEOREM 3.2. For any ring Λ , ${}_{\Lambda}\mathcal{M}$ can be equipped with a regular colimit structure

$$({}_{\Lambda}\mathcal{M})\Psi \xrightarrow{\ \xi\ } {}_{\Lambda}\mathcal{M}$$

PROOF. We shall construct specific sums and quotient objects and use Section 1. First, since an epimorphism in the category of modules also is an epimorphism in the category of sets, we take the chosen quotient morphisms of ${}_{\Lambda}\mathcal{M}$ to be those epimorphisms which considered as mappings in $\mathcal{S}$ are $\mathcal{O}_e$ -chosen quotient morphisms.

The more delicate thing is the construction of sums. To do this, we need transfinite products of ordinal numbers (Sierpiński [9], XIV §17), generalizing the product of Section 2.

Let A_x be an X -indexed family of ordinal numbers, where X is an ordinal number. Let

$$\widetilde{\bigoplus_{x \in X}} A_x \tag{3.1}$$

denote the set of all X -sequences of ordinal numbers satisfying

(i) $(x)f \in A_x$ for all $x \in X$

(ii) $(x)f \neq 0$ for at most finitely many x .

The set is ordered lexicographically from the right, i.e.: $f < g$ iff

$x_0 f < x_0 g$ where $x_0 \in X$ is the largest element where f and g have different values. (A largest one exists because of condition (ii).) It is now well known that (3.1) is well ordered by $<$. (The proof of [9], XIV §15, carries over to our case where the A_x may be different, simply by replacing the A_x's by a common upper bound K, and get (3.1) as a subset of what Sierpiński calls K^X.)

For each $x \in X$, let

$$i_x : A_x \longrightarrow \widetilde{\bigoplus_{x \in X}} A_x$$

be given by

$$(x')(a)i_x = \begin{cases} a & \text{if } x = x' \\ 0 & \text{if not.} \end{cases}$$

Let the ordinal number corresponding to the well ordered set (3.1) be denoted by $\bigoplus_{x \in X} A_x$, and let the unique order isomorphism be α; let incl_x denote the composite

$$\mathrm{incl}_x : A_x \xrightarrow{i_x} \widetilde{\bigoplus_{x \in X}} A_x \xrightarrow{\alpha} \bigoplus_{x \in X} A_x .$$

Suppose now that each A_x is in ${}_\Lambda\mathcal{M}$; so it is equipped with a Λ-module structure $+_x$ and $\lambda\cdot_x$ for all $\lambda \in \Lambda$. We can put a Λ-module structure on $\bigoplus_{x \in X} A_x$ by putting

$$(x)(f + g) = (x)f +_x (x)g ;$$

similarly for scalar multiplication. Then with this, $\bigoplus_{x \in X} A_x$ is in ${}_\Lambda\mathcal{M}$, and each incl_x is a morphism in ${}_\Lambda\mathcal{M}$. Furthermore, it is clear that what we have constructed is nothing but a usual categorical sum of modules.

We must show associativity of the sum, i.e. that the canonical isomorphisms

(3.2)
$$A \xrightarrow{\mathrm{incl}_0} \bigoplus_{1} A$$

and

(3.3)
$$\bigoplus_{y_x \in \coprod_{x \in X} Y_x} A_{y_x} \xrightarrow{\varepsilon} \bigoplus_{x \in X} \bigoplus_{y \in Y_x} A_{y_x}$$

are identities, i.e. order preserving isomorphisms. Here ε is given by

$$\mathrm{incl}_{y_x} \cdot \varepsilon = \mathrm{incl}_x \cdot \mathrm{incl}_y .$$

It is well known to be an isomorphism; its inverse is in fact nothing but an appropriate generalization of (2.4): replace the signs $\prod$ by $\oplus$, and note that (2.4) makes sense if we let proj_i mean the composite

$$\bigoplus_{i \in I} A_i \xrightarrow{\cong} \tilde{\bigoplus_{i \in I}} A_i \subseteq \bigtimes_{i \in I} A_i \xrightarrow{\mathrm{proj}_i} A_i ,$$

where I is an ordinal number and $\bigtimes_{i \in I} A_i$ is just the (infinite) set theoretic product of the I-indexed family A_i. But it is easy to prove that e in (2.3), reinterpreted in this way, is orderpreserving. It follows that (3.3) is orderpreserving. This establishes unit and associative laws for the sum $\oplus$ in $_\wedge\mathcal{M}$.

To apply the construction of Section 1, we have to prove that if $q_x : A_x \longrightarrow B_x$ is a chosen quotient in $_\wedge\mathcal{M}$ for all $x \in X$, then so is

(3.4)
$$\bigoplus_{x \in X} A_x \xrightarrow{\oplus q_x} \bigoplus_{x \in X} B_x .$$

Let $f_o < f_1$ be elements in $\oplus B_x$. Define $g_i \in \oplus A_x$ by

$$x \rightsquigarrow \min(q_x^{-1}(xf_i)) \in A_x$$

$i = 0, 1$. Clearly

$$(3.5) \qquad g_i = \min((\oplus q_x)^{-1}(f_i)) .$$

Let $x \in X$ be the last element where the f_i's differ. Then $xf_o < xf_1$. But

$$xf_o < xf_1 \Rightarrow \min(q_x^{-1}(xf_o)) < \min(q_x^{-1}(xf_1)) ,$$

since q_x is a chosen quotient. This says that

$$xg_o < xg_1 .$$

But clearly x is the last element where $g_o \neq g_1$; so $g_o < g_1$. This and (3.5) shows that (3.4) is a chosen quotient.

Now Section 1 applies, and the theorem is proved.

Note that the proof depended heavily on the peculiar properties of the categorical sum in module categories. So it cannot be carried over to the category of (non abelian) groups, say.

4. Tensorproducts over commutative rings.

Let Λ be a commutative ring. We show here that the category $\mathcal{S}^{(\Lambda)}$ of Λ-modules in $\mathcal{S}$ admits an associative tensorproduct. The proof would also apply to the subcategory $_\Lambda\mathcal{M} \subseteq \mathcal{S}^{(\Lambda)}$ considered in Section 3.

We may assume that the underlying set of Λ is in $\mathcal{S}$, and that the zero of the ring is the number 0, and that the unit of the ring is the number 1. This is not just for notational convenience.

We first construct a monad F in $\mathcal{S}$, assigning to A (the underlying of) the free Λ-module on A. Put

$$AF = \bigoplus_A \Lambda \quad ;$$

$$A \xrightarrow{\eta_A} AF$$

is given by $(a')(a)\eta_A = \delta(a,a')$, $a, a' \in A$. By the last Section, AF carries a canonical Λ-module structure. So we don't have to describe μ further. Functorality of F in A follows from the fact that $\bigoplus_A \Lambda$ is a categorical sum in $\mathcal{S}^{(\Lambda)}$ with index set A.

If A and B are Λ-modules, it is well known that $A \otimes B$ can be constructed as a quotient module of the free Λ-module on the set $A \times B$. Letting $\times$ denote the binary version of the products introduced on $\mathcal{S}$ in Section 2, we can thus completely describe a specific construction of tensorproducts in $\mathcal{S}^{(\Lambda)}$ by requiring $A \otimes B$ to be the chosen quotient of $(A \times B)F$

(4.1) $$(A \times B)F \xrightarrow{q_{A,B}} A \otimes B ,$$

$A, B \in \mathcal{S}^{(\Lambda)}$.

THEOREM 4.1. The tensorproduct described by (4.1) is associative.

Unfortunately, the tensorproduct has no strict units.

PROOF of the theorem. We compare $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ with $A \otimes B \otimes C$ (defined as the correct, chosen quotient of $(A \times B \times C)F$), and show that the natural isomorphisms are in both cases identities. Consider the diagram

(4.2) $$\begin{array}{ccc} (A \otimes B) \otimes C & \underset{h'}{\overset{q'}{\rightleftarrows}} & ((A \otimes B) \times C)F \\ k \uparrow \cong & & S \uparrow \ \downarrow R \\ A \otimes B \otimes C & \underset{h}{\overset{q}{\rightleftarrows}} & (A \times B \times C)F \end{array} ,$$

where q and q' are the chosen quotient mappings, h and h' left inverses - with - minimal - values for q and q' respectively (which determines h and h' uniquely), and where finally S and R are constructed as follows: S is (s)F, where s is

$$s : \quad A \times B \times C \xrightarrow{\eta \times C} (A \times B)F \times C \xrightarrow{\bar{q} \times C} (A \otimes B) \times C \; ;$$

$\bar{q}$ is the chosen quotient mapping defining $A \otimes B$; S is a Λ-module homomorphism. And R is the Λ-module homomorphism determined by the set mapping r

$$(4.3) \quad r : \quad (A \otimes B) \times C \xrightarrow{\bar{h} \times C} (A \times B)F \times C \xrightarrow{\rho} (A \times B \times C)F \, ,$$

where $\bar{h}$ is the minimal left inverse for $\bar{q} : (A \times B)F \longrightarrow A \otimes B$ and ρ is given by

$$(f,c) \overset{\rho}{\leadsto} ((a,b,c') \leadsto (a,b)f \cdot \delta(c,c')),$$

$f : A \times B \longrightarrow \Lambda$ an element in $(A \times B)F$.

Then it is easily seen that

$$S \circ q' = q \circ k$$

$$R \circ q \circ k = q' \quad .$$

In particular, for $f \in (A \times B \times C)F$,

$$(4.4) \qquad (f)SR \equiv f \, ,$$

where $\equiv$ is the congruence relation defining q.

LEMMA 4.2. We have $h \circ S = k \circ h'$.

PROOF. Define for $f \in ZF$ the <u>support</u> of f, supp(f), to be the set $Z' \subseteq Z$ where $f : Z \longrightarrow \Lambda$ takes non zero values. We need a sublemma:

LEMMA 4.3. Let $\ell \in (A \times B \times C)F$ be in the image of h, and let $(a,b,c) \in \mathrm{supp}(\ell)$. Then $(a,b)\eta \in (A \times B)F$ is in the image of $\bar{h} : A \otimes B \longrightarrow$ $\longrightarrow (A \times B)F$, $\bar{h}$ as in (4.3).

PROOF. Let $g \equiv (a,b)\eta$ with g the minimal element in $(A \times B)F$ having this property. It suffices to prove $g = (a,b)\eta$. Suppose $g < (a,b)\eta$. Then by definition

$$(4.5) \qquad (a,b,c)\eta \overset{S.R}{\rightsquigarrow} (g,c)\rho + (a,b,c)\eta$$

since ρ for fixed c is 1-1. Since $1 \in \Lambda$ is the smallest non zero element in Λ, we get $(a',b') \in \mathrm{supp}(g) \Rightarrow (a',b') < (a,b)$, and so also

$$(4.6) \qquad (a',b',c') \in \mathrm{supp}((g,c)\rho) \Rightarrow (a',b',c') < (a,b,c) .$$

By (4.4) and (4.5) we get on the other hand

$$(g,c)\rho \equiv (a,b,c)\eta .$$

Then if $(a,b,c)\ell = n$

$$(4.7) \qquad \ell \equiv \ell - (a,b,c)\eta .n + (g,c)\rho .n = \ell'$$

and ℓ' is like ℓ for $(a',b',c') > (a,b,c)$ but has value 0 on (a,b,c) by (4.6). So $\ell' < \ell$, $\ell' \equiv \ell$, contradicting the assumption that ℓ was in the image of h and thus minimal in its class.

We can now return to the proof of Lemma 4.2. It suffices to prove that if ℓ is in the image of h, then $(\ell)S$ is in the image of h'. Suppose this were not so; then there would exist a $t \in ((A \otimes B) \times C)F$ with

$$(4.8) \qquad t \equiv (\ell)S$$

$$(4.9) \qquad t < (\ell)S .$$

The support of S consists entirely of elements of the form $((a,b)\eta\bar{q}, c)$, so by (4.9) the last point of difference between t and $(\ell)S$ is of this form, say that one displayed. Let r be like t on elements $\geq ((a,b)\eta\bar{q}, c)$, and 0 elsewhere. So supp r $\subseteq$ supp ℓ, and therefore one easily gets, using Lemma 4.3, that the last difference between (r)S and ℓ is (a,b,c), and on that, ℓ is bigger. Consider next t - r ; we show

$$(4.10) \qquad (a',b',c') \in \text{supp}\,((t-r)R) \Rightarrow (a',b',c') < (a,b,c) .$$

For, the condition implies the existence of an $f \in A \otimes B$ so that

$$(4.11) \qquad (f,c') \in \text{supp}\,(t-r) \text{ and } (a',b') \in \text{supp}((f)\bar{h}) .$$

But we constructed t - r so that

$$(f,c') \in \text{supp}\,(t-r) \Longrightarrow (f,c') < ((a,b)\eta\bar{q}, c) .$$

If $c' < c$, the conclusion of (4.10) automatically holds. If $c' = c$, $f < (a,b)\eta\bar{q}$, which implies

$$(f)\bar{h} < (a,b)\eta\overline{qh} = (a,b)\eta ,$$

the last equality sign again because of Lemma 4.3. Since $1 \in \Lambda$ is the smallest non zero element, we conclude that the supporting elements of

$(f)\overline{h}$ are smaller than (a,b); this then holds in particular for (a',b') by (4.11). So we have proved (4.1o). Since ℓ has larger value on (a,b,c) than (r)R has, and same value on larger triples, we conclude by (4.1o) that $(t)R = (t - r)R + (r)R < \ell$. On the other hand by (4.8), (t)R

$$(t)R \equiv (\ell)SR \equiv \ell .$$

This contradicts the minimality of ℓ and proves Lemma 4.2.

LEMMA 4.4. The mapping $h \cdot S$ is orderpreserving.

PROOF. This is an easy consequence of Lemma 4.3.

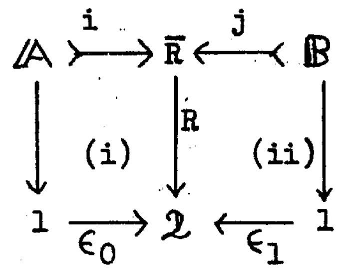
It is now easy to use the Lemmas 4.2 and 4.4 to prove that k in (4.2) is orderpreserving, hence the identity. The proof that the other isomorphism $A \otimes B \otimes C \longrightarrow A \otimes (B \otimes C)$ is likewise the identity, is similar (we have not used that $A \times B \times C$ was ordered lexicographically <u>from the right</u> in an essential way). This then proves the theorem.

5. Associative composition of profunctors.

We shall recall Bénabou's definition of profunctors and their composition. We restrict to profunctors relative to the categories Cat and $\mathfrak{S}$ of Chapter I. It will be fairly obvious that the definition we give is equivalent to Bénabou's.

Note first that since Cat is 'based' on $\mathfrak{S}$, there is an obvious full choice of subobjects in Cat. A chosen subobject will be denoted $\rightarrowtail$. Let $\mathfrak{2}$ denote the category with $|\mathfrak{2}| = 2$, $\mathfrak{2}(0,1) = 1$, $\mathfrak{2}(1,0) = 0$, and $\mathfrak{2}(i,i) = 1$ for $i = 0, 1$.

DEFINITION 5.1. Let $\mathbb{A}, \mathbb{B} \in |\mathrm{Cat}|$. A <u>profunctor</u> $R : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B}$ is a category $\bar{R} \in |\mathrm{Cat}|$, together with a functor $R : \bar{R} \longrightarrow \mathbf{2}$, so that the two diagrams (i) and (ii) in



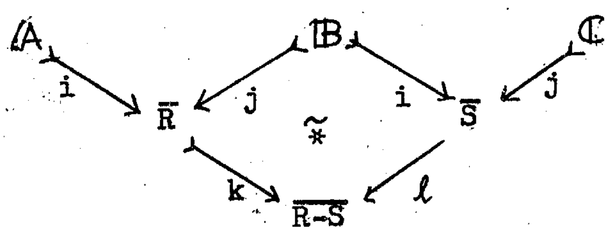
are meet (pull back) diagrams.

We shall usually omit i and j from the notation.

If $S : \mathbb{B} \nrightarrow \mathbb{C}$ is another profunctor, we define the composite profunctor

$$R * S : \mathbb{A} \nrightarrow \mathbb{C}$$

as follows. Construct a comeet diagram $\tilde{*}$ in Cat (<u>which</u> one is described below)



and take $\overline{R * S}$ to be the full subcategory generated by the images of $i.k$ and $j.\ell$. It will be clear from the description of $\overline{R\text{-}S}$ below that the images of $i.k$ and $j.\ell$ are disjoint and that no morphisms go from the latter to the former, wherefore a functor $\overline{R*S} \xrightarrow{R*S} \mathbf{2}$ is defined.

The category $\overline{R\text{-}S}$ is described by

$$|\overline{R\text{-}S}| = |\mathbb{A}| + |\mathbb{B}| + |\mathbb{C}|$$

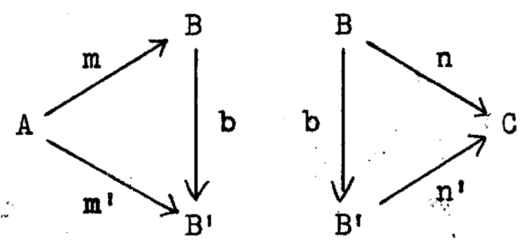
$$(\overline{R\text{-}S})(A_0, B_1) = \bar{R}(A,B)$$

$$(\overline{R-S})(B_1,C_2) = \overline{S}(B,C)$$

and $(\overline{R-S})(A_0,C_2)$ the chosen quotient of the (lexicographically ordered) set $W^{R,S}_{A,C}$ of triples (m,B,n) with $m \in \overline{R}(A,B)$, $n \in \overline{S}(B,C)$, under the equivalence relation

$$(m,B,n) \sim (m',B',n')$$

if there exists $b : B \longrightarrow B'$ in $\mathbb{B}$ so that both the diagrams



commute (in $\overline{R}$ and $\overline{S}$, respectively).

Composition in $\overline{R-S}$ is obvious.

Note that composition of profunctors is expressed entirely in terms of certain finite limits and colimits (certain meets and comeets in Cat).

We can now show that the composition described is associative. The method is very much the same as in Section 4. Let a third profunctor $T : \mathbb{C} \mapsto \mathbb{D}$ be given. We shall compare $(R*S)*T$ and $R*(S*T)$ with an obvious $R*S*T$, having $(R*S*T)(A,D)$ the chosen quotient of the set $W^{R,S,T}_{A,D}$ of quintuples

(5.1) $$(m,B,n,C,\sigma)$$

where

$$A \xrightarrow{m} B \text{ in } \overline{R}\ ; \quad B \xrightarrow{n} C \text{ in } \overline{S}$$

$$C \xrightarrow{\sigma} D \text{ in } \overline{T}\ .$$

Again we use lexicographic ordering for $W_{A,D}^{R,S,T}$. One may now find a commutative diagram

$$\begin{array}{ccc} R*(S*T) & \twoheadleftarrow & W_{A,D}^{R,(S*T)} \\ a \uparrow \cong & & \uparrow t \\ R*S*T & \twoheadleftarrow & W_{A,D}^{R,S,T} \end{array} \qquad (5.2)$$

t given by sending (5.1) to

$$(m,B,\{(n,C,\sigma)\}) .$$

It is then easily seen that t is a chosen quotient. So (5.2) gives that a is the identity. The other identity $(R*S)*T = R*S*T$ comes by the same method.

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