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Cut-elimination Theorems for Some Infinitary Modal Logics

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Abstract. In this article, a cut-free system TLM_{ω_1} for infinitary propositional modal logic is proposed which is complete with respect to the class of all Kripke frames. The system TLM_{ω_1} is a kind of Gentzen style sequent calculus, but a sequent of TLM_{ω_1} is defined as a finite tree of sequents in a standard sense. We prove the cut-elimination theorem for TLM_{ω_1} via its Kripke completeness.

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1 Introduction

Let LM be the formal system for propositional modal logic which consists of the propositional fragment of Gentzen's sequent calculus LK (see, e. g., [2]) and the following inference rule for the modal operator $\Box$:

$$\frac{\Gamma \to \varphi}{\Box\Gamma \to \Box\varphi}\,(\mathrm{nec}), \quad \text{where } \Box\Gamma = \{\Box\gamma : \gamma \in \Gamma\}.$$

It is known that the logic **K**, that is, the propositional modal logic characterized by the class of all Kripke frames, is axiomatized by LM. It is also well-known that since the cut-elimination algorithm for **LK** (see, e. g., [14, 2]) also works for LM, LM is cut-free and **K** has a formal system which satisfies the subformula property. Now, we discuss infinitary propositional modal logic in the same way. Let LK_{ω_1} be the formal system which consists of the propositional fragment of LK and the following inference rules for infinitary connectives (the formal definition of LK_{ω_1} is given in Section 3, see also [4]):

$$\frac{\Gamma \to \Delta, \varphi \quad (\text{for all } \varphi \in \Theta)}{\Gamma \to \Delta, \bigwedge\Theta}\,(\to \textstyle\bigwedge), \qquad \frac{\varphi, \Gamma \to \Delta \quad (\text{for some } \varphi \in \Theta)}{\bigwedge\Theta, \Gamma \to \Delta}\,(\textstyle\bigwedge \to),$$

$$\frac{\Gamma \to \Delta, \varphi \quad (\text{for some } \varphi \in \Theta)}{\Gamma \to \Delta, \bigvee\Theta}\,(\to \textstyle\bigvee), \qquad \frac{\varphi, \Gamma \to \Delta \quad (\text{for all } \varphi \in \Theta)}{\bigvee\Theta, \Gamma \to \Delta}\,(\textstyle\bigvee \to).$$

Here, Θ is a countable set of formulas and the upper sequents of $(\to \bigwedge)$ and $(\bigvee \to)$ are countable. We write LM_{ω_1} for the system which consists of LK_{ω_1} and the inference rule (nec) for the modal operator, and write $\mathbf{K}_{\omega_1}$ for the logic axiomatized by LM_{ω_1}. Since the cut-elimination algorithm for LK_{ω_1} in [4] also works for LM_{ω_1}, LM_{ω_1} is cut-free.

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However, LM_{ω_1} does not axiomatize the infinitary modal logic characterized by the class of all Kripke frames. Let BF_{ω_1} be the formula $\bigwedge_{i\in\omega}\Box p_i \supset \Box\bigwedge_{i\in\omega} p_i$ of infinitary modal logic which corresponds to the *Barcan formula* BF, that is, the formula $\forall x\,\Box\varphi \supset \Box\forall x\,\varphi$ of predicate modal logic. It is easy to see that BF_{ω_1} is valid in every Kripke model. Hence, BF_{ω_1} is necessary to axiomatize the infinitary modal logic characterized by the class of all Kripke frames (see [11, 3, 16, 15]). However, BF_{ω_1} is not derivable in LM_{ω_1}, hence, LM_{ω_1} is Kripke incomplete, though it is a natural extension of LM. On the other hand, the system which consists of LM_{ω_1} and the axiom schema $\rightarrow \mathrm{BF}_{\omega_1}$ axiomatizes the logic characterized by the class of all Kripke frames ([16, 15]). We write $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ for the logic axiomatized by LM_{ω_1} and additional axiom schema BF_{ω_1} (the formal definition of the symbol $\oplus$ is given in Section 3).

Now, we consider to give a formal system for $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ which satisfies the subformula property. Obviously, when we have an additional axiom schema BF_{ω_1}, the cut-elimination algorithm in [4] does not work, and although Kaneko and Nagashima proposed a cut-free system for $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ in [6], their system does not satisfy the subformula property, because of the following inference rule for BF_{ω_1}:

$$\frac{\Gamma \rightarrow \Delta,\ \Box\theta \quad (\text{for all } \theta \in \Theta) \quad \Box\bigwedge\Theta,\ \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,.$$

In Section 4, we propose a cut-free system TLM_{ω_1} for $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ which satisfies the subformula property. The system TLM_{ω_1} is a kind of Gentzen style sequent calculus, but a sequent of TLM_{ω_1}, which is called a *tree sequent*, is defined as a finite tree of sequents in a standard sense. In [7], Kashima and Shimura introduced a notion of *connection* into sequent systems, and proved the cut-elimination theorem for the logic $\mathbf{H}_* + \mathrm{D}$, the intermediate logic with the axiom $\mathrm{D} = \forall x\,(\varphi(x) \vee p) \supset \forall x\,\varphi(x) \vee p$. Although [7] did not deal with the tree sequents explicitly, the idea is essentially equivalent (see also [1]). However, to derive the formula BF_{ω_1}, we need more. A tree sequent of TLM_{ω_1} is a finite tree of sequents $\Gamma \rightarrow \Delta$, but Γ and Δ are *countable* sets of formulas, instead of finite sets. In Section 5, we prove the cut-elimination theorem for TLM_{ω_1} via Kripke completeness.

2 The syntax and semantics for infinitary propositional modal logic

The *language* of infinitary propositional modal logic consists of a countable set V of propositional variables, the symbols $\bigwedge$ and $\bigvee$ for infinite conjunction and disjunction, respectively, the symbol $\supset$ for implication, the symbol $\neg$ for negation, and the symbol $\Box$ for necessity. The set of *formulas* of the infinitary propositional modal logic is the least set which satisfies the following conditions:

1. each propositional variable in V is a formula;
2. if Θ is a countable set of formulas, then $(\bigwedge\Theta)$ is a formula;
3. if Θ is a countable set of formulas, then $(\bigvee\Theta)$ is a formula;
4. if φ and ψ are formulas, then $(\varphi \supset \psi)$ is a formula;
5. if φ is a formula, then $(\neg\varphi)$ is a formula;
6. if φ is a formula, then $(\Box\varphi)$ is a formula.

We write $\varphi \wedge \psi$ for $(\bigwedge\{\varphi,\psi\})$ and $\varphi \vee \psi$ for $(\bigvee\{\varphi,\psi\})$. Also, we sometimes write

$\top$ for $\bigwedge\emptyset$ and $\bot$ for $\bigvee\emptyset$. To satisfy the above closure conditions, we need recursive construction of formulas up to ω_1 as follows:

1. define the set Φ_0 of formulas by $\Phi_0 = V$;
2. for any $\alpha \in \omega_1$, define the set $\Phi_{\alpha+1}$ of formulas by
$$\Phi_{\alpha+1} = \Phi_\alpha \cup \{\bigwedge\Theta : \Theta \subset \Phi_\alpha \text{ and } |\Theta| \in \omega_1\} \cup \{\bigvee\Theta : \Theta \subset \Phi_\alpha \text{ and } |\Theta| \in \omega_1\} \cup \{\varphi \supset \psi : \varphi,\ \psi \in \Phi_\alpha\} \cup \{\neg\varphi : \varphi \in \Phi_\alpha\} \cup \{\Box\varphi : \varphi \in \Phi_\alpha\};$$
3. for any limit ordinal $\alpha \in \omega_1$, define the set Φ_α by $\Phi_\alpha = \bigcup_{\beta\in\alpha}\Phi_\beta$;
4. the set of all formulas of infinitary modal logic is $\bigcup_{\alpha\in\omega_1}\Phi_\alpha$.

A *Kripke frame* is a pair $\langle W, R\rangle$ such that W is a non-empty set and R is a binary relation on W. A *Kripke model* $\mathcal{M}$ is a triple $\langle W, R, v\rangle$, where $\langle W, R\rangle$ is a Kripke frame and v is a function from V to $\mathcal{P}(W)$. For any formula φ, we say that φ is *valid at* $w \in W$ if one of the following holds, according to the construction of φ, and write $w \vDash \varphi$ if this holds:

(1) $w \vDash p$ if $w \in v(p)$;
(2) $w \vDash \bigwedge\Theta$ if every θ in Θ satisfies $w \vDash \theta$;
(3) $w \vDash \bigvee\Theta$ if there exists θ in Θ such that $w \vDash \theta$;
(4) $w \vDash \chi \supset \psi$ if $w \nvDash \chi$ or $w \vDash \psi$;
(5) $w \vDash \neg\psi$ if $w \nvDash \psi$;
(6) $w \vDash \Box\psi$ if, for any w' in W, $w\,R\,w'$ implies $w' \vDash \psi$.

Let φ be a formula and $\mathcal{M} = \langle W, R, v\rangle$ be a Kripke model. We say that φ is *valid in* $\mathcal{M}$ if $w \vDash \varphi$ for any $w \in W$. Let $F = \langle W, R\rangle$ be a Kripke frame. We say that φ is *valid in* F and write $F \vDash \varphi$ if, for any valuation v, φ is valid in the model $\langle W, R, v\rangle$. Let $\mathbf{L}$ be a logic and C be a class of Kripke frames. A logic $\mathbf{L}$ is said to be *complete* with respect to the class C of Kripke frames if, for any formula φ, $\varphi \in \mathbf{L}$ whenever $F \vDash \varphi$ for any $F \in C$. A logic $\mathbf{L}$ is said to be *sound* with respect to the class C of Kripke frames if, for any formula φ, $\varphi \in \mathbf{L}$ implies $F \vDash \varphi$ for any $F \in C$. A logic $\mathbf{L}$ is said to be *characterized* by the class C of Kripke frames if, $\varphi \in \mathbf{L}$ if and only if $F \vDash \varphi$ for any $F \in C$.

3 Some formal systems

First, we define the system LK_{ω_1} for the classical infinitary logic (see [4]). A *sequent* $\Gamma \to \Delta$ of LK_{ω_1} is a pair of finite sets Γ and Δ of formulas. The axiom schemata of LK_{ω_1} are $p \to p$, $\to \bigwedge\emptyset$, and $\bigvee\emptyset \to$. Below, we list the inference rules of LK_{ω_1}, where Γ, Δ denotes the set $\Gamma \cup \Delta$ of formulas and Γ, φ and φ, Γ denote $\Gamma, \{\varphi\}$:

set $\dfrac{\Gamma \to \Delta}{\Gamma' \to \Delta'}$ (set) $\quad (\Gamma \subset \Gamma'$ and $\Delta \subset \Delta')$;

cut $\dfrac{\Gamma \to \Delta, \varphi \quad \varphi, \Lambda \to \Xi}{\Gamma, \Lambda \to \Delta, \Xi}$ (cut);

conjunction $\dfrac{\Gamma \to \Delta, \varphi \text{ (for all } \varphi \in \Theta)}{\Gamma \to \Delta, \bigwedge\Theta}$ $(\to \bigwedge)$, $\quad \dfrac{\varphi, \Gamma \to \Delta \text{ (for some } \varphi \in \Theta)}{\bigwedge\Theta, \Gamma \to \Delta}$ $(\bigwedge \to)$;

disjunction $\dfrac{\Gamma \to \Delta, \varphi \text{ (for some } \varphi \in \Theta)}{\Gamma \to \Delta, \bigvee\Theta}$ $(\to \bigvee)$, $\quad \dfrac{\varphi, \Gamma \to \Delta \text{ (for all } \varphi \in \Theta)}{\bigvee\Theta, \Gamma \to \Delta}$ $(\bigvee \to)$;

implication $\dfrac{\varphi, \Gamma \to \Delta, \psi}{\Gamma \to \Delta, \varphi \supset \psi}\,(\to\supset)$, $\dfrac{\Gamma \to \Delta, \varphi \quad \psi, \Lambda \to \Xi}{\varphi \supset \psi, \Gamma, \Lambda \to \Delta, \Xi}\,(\supset\to)$;

negation $\dfrac{\varphi, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\varphi}\,(\to\neg)$, $\dfrac{\Gamma \to \Delta, \varphi}{\neg\varphi, \Gamma \to \Delta}\,(\neg\to)$.

The system LK_{ω_1} axiomatizes the infinitary propositional classical logic characterized by the class of all complete Boolean algebras (see [12, 4]). The formal system LM_{ω_1} for infinitary modal logic consists of LK_{ω_1} and the derivation rule (nec).

Let LX be any Gentzen style sequent calculus. A sequent $\Gamma \to \Delta$ is said to be *derivable* in LX from a set S of sequents, and $\mathcal{D}$ is called a *derivation* of $\Gamma \to \Delta$ from S, if one of the following conditions holds (see [4]):

1. $\Gamma \to \Delta$ is an axiom of LX or a member of S. $\mathcal{D} = (S, \Gamma \to \Delta, \emptyset)$;

2. there exist a set $\{\Gamma_i \to \Delta_i : i \in I\}$ of sequents, a set $\{\mathcal{D}_i : i \in I\}$ of derivations, and an inference rule (R) in LX such that for any $i \in I$ the sequent $\Gamma_i \to \Delta_i$ is derivable from S and $\mathcal{D}_i$ is a derivation of $\Gamma_i \to \Delta_i$ from S, and $\Gamma \to \Delta$ is derivable from $\{\Gamma_i \to \Delta_i : i \in I\}$ by (R), as follows:

$$\frac{\Gamma_i \to \Delta_i \quad (i \in I)}{\Gamma \to \Delta}\,(\mathrm{R}).$$

$\mathcal{D} = ((\mathrm{R}), \Gamma \to \Delta, (\mathcal{D}_i)_{i\in I})$.

A formula φ is said to be *derivable in* LX *from a set* Γ *of formulas* if the sequent $\to \varphi$ is derivable from the set $\{\to \psi : \psi \in \Gamma\}$ of sequents. If a formula φ is derivable from the empty set, φ is said to be *derivable*. Let $\mathcal{D} = ((\mathrm{R}), \Gamma \to \Delta, (\mathcal{D}_i)_{i\in I})$ be a derivation. Then, the rule (R) is called the *last rule* of $\mathcal{D}$. The set of *subderivations* of $\mathcal{D}$ consists of $\mathcal{D}$ and all subderivations of $\mathcal{D}_i$ for all $i \in I$. An inference rule is said to be *included* in $\mathcal{D}$ if it is the last rule of some subderivations of $\mathcal{D}$. A derivation $\mathcal{D}$ is said to be *cut-free* if $\mathcal{D}$ does not include the cut-rule. A logic $\mathbf{L}$ is said to be *axiomatized* by a formal system LX if $\mathbf{L}$ is the set of all formulas which are derivable in LX. A formal system is said to be *complete* (resp. *sound*) with respect to the class C of Kripke frames if the logic axiomatized by the system is complete (resp. sound) with respect to C.

The *least infinitary modal logic* $\mathbf{K}_{\omega_1}$ is the logic which is axiomatized by LM_{ω_1}. Let (R) be any additional axiom schema or inference rule. We write $\mathrm{LM}_{\omega_1} \oplus (\mathrm{R})$ for the system which consists of LM_{ω_1} and (R), and write $\mathbf{K}_{\omega_1} \oplus (\mathrm{R})$ for the logic axiomatized by $\mathrm{LM}_{\omega_1} \oplus (\mathrm{R})$. For any formula φ, we write $\mathrm{LM}_{\omega_1} \oplus \varphi$ for the system which consists of LM_{ω_1} and axiom schema $\to \varphi$, and write $\mathbf{K}_{\omega_1} \oplus \varphi$ for the logic axiomatized by $\mathrm{LM}_{\omega_1} \oplus \varphi$.

In [4], Feferman proved the cut-elimination theorem for LK_{ω_1}. Since the cut-elimination algorithm in [4] also works for LM_{ω_1}, the cut-elimination theorem for LM_{ω_1} is obtained immediately:

Theorem 3.1. *If a formula* φ *is derivable in* LM_{ω_1}, *there exists a derivation of* φ *in* LM_{ω_1} *which does not include the rule* (cut).

Now, we introduce a pair of new inference rules for modal operator (see [9, 10]):

$$\frac{\Box\Gamma \to \varphi}{\Box\Gamma \to \Box\varphi}\,(\to\Box), \qquad \frac{\varphi, \Gamma \to \Delta}{\Box\varphi, \Gamma \to \Delta}\,(\Box\to).$$

It is easy to see that the following relations hold:

1. $\mathrm{LM}_{\omega_1} \oplus (\rightarrow \Box) = \mathrm{LM}_{\omega_1} \oplus (\Box \mathrm{p} \supset \Box\Box \mathrm{p})$;
2. $\mathrm{LM}_{\omega_1} \oplus (\Box \rightarrow) = \mathrm{LM}_{\omega_1} \oplus (\Box \mathrm{p} \supset \mathrm{p})$;
3. $\mathrm{LM}_{\omega_1} \oplus (\rightarrow \Box) \oplus (\Box \rightarrow) = \mathrm{LM}_{\omega_1} \oplus (\Box \mathrm{p} \supset \Box\Box \mathrm{p}) \oplus (\Box \mathrm{p} \supset \mathrm{p})$.

Then, for each of the above systems, the cut-elimination theorem is obtained by the cut-elimination algorithm in [4], immediately.

Let BF_{ω_1} be the formula $\bigwedge_{i\in\omega} \Box p_i \supset \Box \bigwedge_{i\in\omega} p_i$ of infinitary modal logic. It is easy to see that BF_{ω_1} is valid in any Kripke model. Now, from the cut-elimination theorems, it follows that BF_{ω_1} is not derivable in any of the above systems. Hence, we have

Theorem 3.2. *The following logics are Kripke incomplete:*

$$\mathbf{K}_{\omega_1}, \quad \mathbf{K}_{\omega_1} \oplus (\Box p \supset \Box\Box p), \quad \mathbf{K}_{\omega_1} \oplus (\Box p \supset p), \quad \mathbf{K}_{\omega_1} \oplus (\Box p \supset \Box\Box p) \oplus (\Box p \supset p).$$

On the other hand, the logic $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ is characterized by the class of all Kripke frames (see [16, 15], also [11, 3]), and the logics $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1} \oplus (\Box \mathrm{p} \supset \Box\Box \mathrm{p})$, $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1} \oplus (\Box \mathrm{p} \supset \mathrm{p})$, and $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1} \oplus (\Box \mathrm{p} \supset \Box\Box \mathrm{p}) \oplus (\Box \mathrm{p} \supset \mathrm{p})$ are characterized by the class of transitive, reflexive, and transitive and reflexive Kripke frames, respectively (see [16, 15]).

4 The system $\mathbf{TLM}_{\omega_1}$

In this section, we introduce the system TLM_{ω_1} for the logic $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$. A sequent of TLM_{ω_1}, which is called a *tree sequent*, is a finite tree of sequents in a standard sense (see Figure 1). However, for each sequent $\Gamma \rightarrow \Delta$ in a node of a tree sequent, Γ and Δ are countable sets of formulas. We assume that each node N of a tree sequent has an address (ξ, n), where ξ is the address of the immediate predecessor N' of N and n denotes that N is the nth immediate successor of N'. The address of the root is 0.

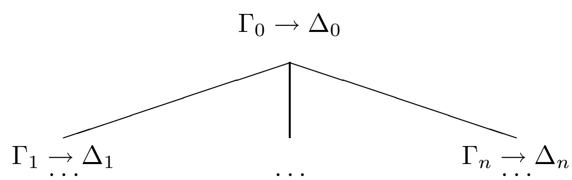

Figure 1. A tree sequent

For each i, Γ_i and Δ_i are countable sets of formulas

We write $\xi : \Gamma \rightarrow \Delta$ for the node of a tree sequent whose address is ξ and attached sequent is $\Gamma \rightarrow \Delta$. We sometimes identify an address ξ with the node $\xi : \Gamma \rightarrow \Delta$ at ξ, and the set of all addresses of a tree sequent T with T itself. In this manner, for any n nodes $\xi_1, \ldots, \xi_n$ in T, $T \setminus \{\xi_1, \ldots, \xi_n\}$ denotes the graph obtained from T by removing n nodes at $\xi_1, \ldots, \xi_n$, and

$$T \setminus \{\xi_1, \ldots, \xi_n\} \cup \{\xi_1 : \Gamma_1 \rightarrow \Delta_1\} \cup \cdots \cup \{\xi_n : \Gamma_n \rightarrow \Delta_n\}$$

denotes the tree sequent which is obtained from T by replacing the nodes at $\xi_1, \ldots, \xi_n$ with $\xi_1 : \Gamma_1 \rightarrow \Delta_1, \ldots, \xi_n : \Gamma_n \rightarrow \Delta_n$, respectively. For any tree sequent T and a node ξ of T, we write $\downarrow\xi$ for the subtree of T which consists of ξ and all descendants of ξ.

An axiom of TLM_{ω_1} is a tree sequent which includes one of the sequent of the shape $\varphi \to \varphi$, $\to \bigwedge \emptyset$, and $\bigvee \emptyset \to$ in some nodes. The inference rules of TLM_{ω_1} are the following:

set $$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta\}}{T \setminus \{\eta\} \cup \{\eta : \Gamma' \to \Delta'\}}\ (\mathrm{set})_+,$$ where $\Gamma \subset \Gamma'$ and $\Delta \subset \Delta'$ (see Figure 2);

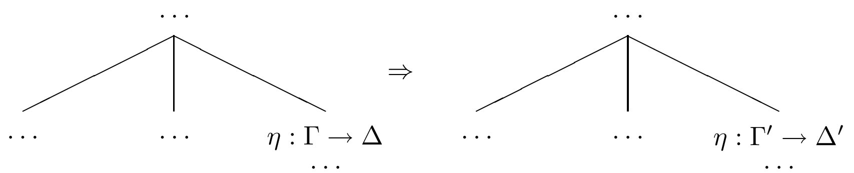

Figure 2. $(\mathrm{set})_+$

The rule $(\mathrm{set})_+$ of TLM_{ω_1} corresponds to the rule (set) of LM_{ω_1} applied to a node η of a tree sequent. The other nodes are not changed and the choice of η is arbitrary

cut $$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\} \quad T \setminus \{\eta\} \cup \{\eta : \varphi, \Lambda \to \Xi\}}{T \setminus \{\eta\} \cup \{\eta : \Gamma, \Lambda \to \Delta, \Xi\}}\ (\mathrm{cut})_+;$$

conjunction $$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \theta\} \quad (\text{for all } \theta \in \Theta)}{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \bigwedge \Theta\}}\ (\to \bigwedge)_+,$$

$$\frac{T \setminus \{\eta\} \cup \{\eta : \Lambda, \Gamma \to \Delta\} \quad (\Lambda \subset \Theta)}{T \setminus \{\eta\} \cup \{\eta : \bigwedge \Theta, \Gamma \to \Delta\}}\ (\bigwedge \to)_+,$$

where Θ is a countable set of formulas and Λ is a subset of Θ (see Figure 3);

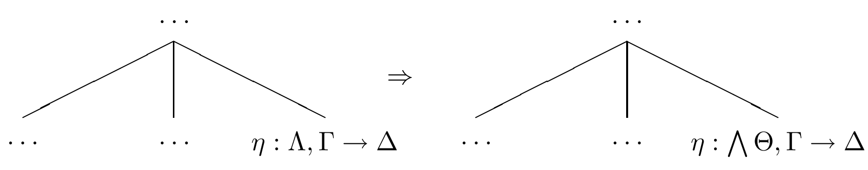

Figure 3. $(\mathrm{set})_+$

The rule $(\bigwedge \to)_+$ of TLM_{ω_1} corresponds to the rule $(\bigwedge \to)$ of LM_{ω_1} applied to a node η of a tree sequent. However, in application of $(\bigwedge \to)_+$, the set Λ is allowed to be coutable, and $(\bigwedge \to)_+$ can connect all formulas in Λ in one application

disjunction $$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \Lambda\} \quad (\Lambda \subset \Theta)}{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \bigvee \Theta\}}\ (\to \bigvee)_+,$$

$$\frac{T \setminus \{\eta\} \cup \{\eta : \theta, \Gamma \to \Delta\} \quad (\text{for all } \theta \in \Theta)}{T \setminus \{\eta\} \cup \{\eta : \bigvee \Theta, \Gamma \to \Delta\}}\ (\bigvee \to)_+,$$

where Θ is a countable set of formulas and Λ is a subset of Θ;

implication $$\frac{T \setminus \{\eta\} \cup \{\eta : \varphi, \Gamma \to \Delta, \psi\}}{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi \supset \psi\}}\ (\to \supset)_+,$$

$$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\} \quad T \setminus \{\eta\} \cup \{\eta : \psi, \Gamma \to \Delta\}}{T \setminus \{\eta\} \cup \{\eta : \varphi \supset \psi, \Gamma \to \Delta\}}\ (\supset \to)_+;$$

negation
$$\frac{T \setminus \{\eta\} \cup \{\eta : \varphi, \Gamma \to \Delta\}}{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \neg\varphi\}}\,(\to \neg)_+,$$
$$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\}}{T \setminus \{\eta\} \cup \{\eta : \neg\varphi, \Gamma \to \Delta\}}\,(\neg \to)_+;$$

necessitation 1
$$\frac{T \setminus \{\eta, (\eta, n)\} \cup \{\eta : \Gamma \to \Delta\} \cup \{(\eta, n) : \to \varphi\}}{T \setminus \{\eta, (\eta, n)\} \cup \{\eta : \Gamma \to \Delta, \Box\varphi\}}\,(\to \Box)_+,$$
where $(\eta, n) : \to \varphi$ is a leaf of the upper tree sequent (see Figure 4);

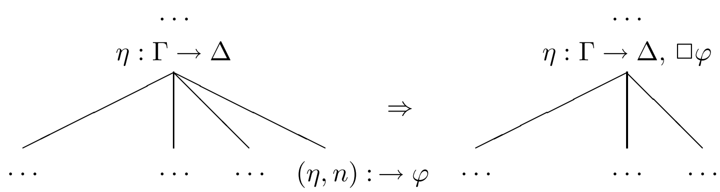

Figure 4. $(\to \Box)_+$

The rule $(\to \Box)_+$ can be applied to a node η of a tree sequent only when η has an immediate successor of the shape $(\eta, n) : \to \varphi$ which is a leaf of the tree. The node $(\eta, n) : \to \varphi$ will disappear after the application of the rule $(\to \Box)_+$

necessitation 2
$$\frac{T \setminus \{\eta, (\eta, 0), \ldots, (\eta, n)\} \cup \{\eta : \Gamma \to \Delta\} \cup \bigcup_{k=0}^{n}\{(\eta, k) : \varphi, \Gamma_k \to \Delta_k\}}{T \setminus \{\eta, (\eta, 0), \ldots, (\eta, n)\} \cup \{\eta : \Box\varphi, \Gamma \to \Delta\} \cup \bigcup_{k=0}^{n}\{(\eta, k) : \Gamma_k \to \Delta_k\}}\,(\Box \to)_+,$$
where $\{(\eta, 0), \ldots, (\eta, n)\}$ is the set of all immediate successors of η, and any immediate successor of η has the formula φ in the left hand side (see Figure 5).

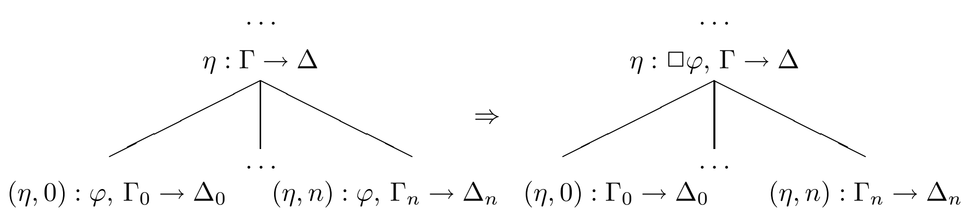

Figure 5. $(\Box \to)_+$

A formula $\Box\varphi$ can be introduced to the left hand side of a node $\eta : \Gamma \to \Delta$ of a tree sequent by the rule $(\Box \to)_+$ only when all of the immediate successors $(\eta, 0), \ldots, (\eta, n)$ of η have the formula φ in their left hand sides

A formula φ of infinitary propositional modal logic is said to be *derivable in* TLM_{ω_1} if the tree sequent $\to \varphi$, which consists only of the root, is derivable in TLM_{ω_1}.

Theorem 4.1. *The formula* BF_{ω_1} *is derivable in* TLM_{ω_1}.

Proof. Since a sequent in a node of a tree sequent is a pair of countable sets of formulas, the following proof figure is a well-defined derivation of BF_{ω_1}:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{0:\{\Box p_j\}_{j\neq i}\rightarrow\}\cup\{(0,1):p_i\rightarrow p_i\}}{\{0:\{\Box p_i\}_{i\in\omega}\rightarrow\}\cup\{(0,1):\rightarrow p_i\}}(\Box\rightarrow)_+}{\{0:\bigwedge_{i\in\omega}\Box p_i\rightarrow\}\cup\{(0,1):\rightarrow p_i\}\quad(\text{for all } i\in\omega)}(\bigwedge\rightarrow)}{\{0:\bigwedge_{i\in\omega}\Box p_i\rightarrow\}\cup\{(0.1):\rightarrow\bigwedge_{i\in\omega}p_i\}}(\rightarrow\bigwedge)}{\{0:\bigwedge_{i\in\omega}\Box p_i\rightarrow\Box\bigwedge_{i\in\omega}p_i\}}(\rightarrow\ \Box)_+}{\{0:\rightarrow\bigwedge_{i\in\omega}\Box p_i\supset\Box\bigwedge_{i\in\omega}p_i\}}(\rightarrow\supset)_+$$

□

5 The cut-elimination theorem for $\mathbf{TLM}_{\omega_1}$

The embedding * from the set of all tree sequents of TLM_{ω_1} to the set of all formulas is defined inductively as follows:

1. If $T=\{0:\Gamma\rightarrow\Delta\}$, then $T^*:=\bigwedge\Gamma\supset\bigvee\Delta$;

2. if $0:\Gamma\rightarrow\Delta$ is the root of T and A is the set of all immediate successors of the root, then $T^*:=\bigwedge\Gamma\supset\bigvee\Delta\vee\bigvee_{\xi\in A}\Box(\downarrow\xi)^*$.

Theorem 5.1. *If a formula* φ *is derivable in* TLM_{ω_1}*, then it is valid in any Kripke model.*

Proof. Suppose a tree sequent T has a derivation $\mathcal{D}$ in TLM_{ω_1}. Then, easy induction on $\mathcal{D}$ shows that T^* is valid in any Kripke model. □

Now, we prove that the cut-free fragment of TLM_{ω_1} is complete with respect to the class of Kripke frames. First, we show the following lemma.

Lemma 5.1. *Let* T *be a tree sequent of* TLM_{ω_1} *and* $\eta:\Gamma\rightarrow\Delta$ *be a node of* T. *Suppose* T *has no cut-free derivation. Then the following statements* $1-5$ *hold:*

1. *If* $\bigwedge\Theta\in\Gamma$*, then the tree sequent* $T\setminus\{\eta\}\cup\{\eta:\Theta,\Gamma\rightarrow\Delta\}$ *has no cut-free derivation* (Figure 6). *If* $\bigwedge\Theta\in\Delta$*, then there exists a formula* $\theta\in\Theta$ *such that the tree sequent* $T\setminus\{\eta\}\cup\{\eta:\Gamma\rightarrow\Delta,\theta\}$ *has no cut-free derivation.*

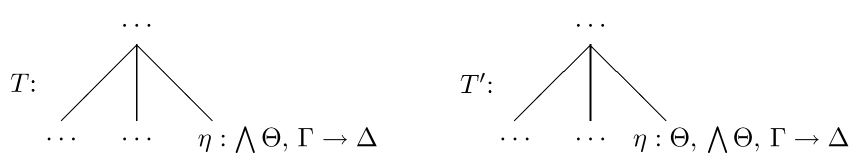

Figure 6. Lemma 5.1, conjunction
The tree sequent T' is obtained from T by adding the set Θ of formulas to the left hand side of η

2. *If* $\bigvee\Theta\in\Gamma$*, then there exists a formula* $\theta\in\Theta$ *such that the tree sequent* $T\setminus\{\eta\}\cup\{\eta:\theta,\Gamma\rightarrow\Delta\}$ *has no cut-free derivation. If* $\bigvee\Theta\in\Delta$*, then the tree sequent* $T\setminus\{\eta\}\cup\{\eta:\Gamma\rightarrow\Delta,\Theta\}$ *has no cut-free derivation.*

3. *If* $\varphi \supset \psi \in \Gamma$, *then one of the tree sequents* $T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\}$ *and* $T \setminus \{\eta\} \cup \{\eta : \psi, \Gamma \to \Delta\}$ *has no cut-free derivation. If* $\varphi \supset \psi \in \Delta$, *then the tree sequent* $T \setminus \{\eta\} \cup \{\eta : \varphi, \Gamma \to \Delta, \psi\}$ *has no cut-free derivation.*

4. *If* $\neg\varphi \in \Gamma$, *then the tree sequent* $T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\}$ *has no cut-free derivation. If* $\neg\varphi \in \Delta$, *then the tree sequent* $T \setminus \{\eta\} \cup \{\eta : \varphi, \Gamma \to \Delta\}$ *has no cut-free derivation.*

5. *If* $\Box\varphi \in \Gamma$ *and* $(\eta, i) : \Gamma_{(\eta,i)} \to \Delta_{(\eta,i)}$ $(i = 0, \dots, n)$ *is the list of all immediate successors of* η *in* T, *then the tree sequent*

$$T \setminus \{(\eta, 0), \dots, (\eta, n)\} \cup \bigcup_{i=0,\dots,n} \{(\eta, i) : \varphi, \Gamma_{(\eta,i)} \to \Delta_{(\eta,i)}\}$$

has no cut-free derivation (Figure 7). *If* $\Box\varphi \in \Delta$ *and* $(\eta, i) : \Gamma_{(\eta,i)} \to \Delta_{(\eta,i)}$ *for* $i = 0, \dots, n$ *is the list of all immediate successors of* η *in* T, *then the tree sequent* $T \cup \{(\eta, n+1) :\to \varphi\}$ *has no cut-free derivation* (Figure 8).

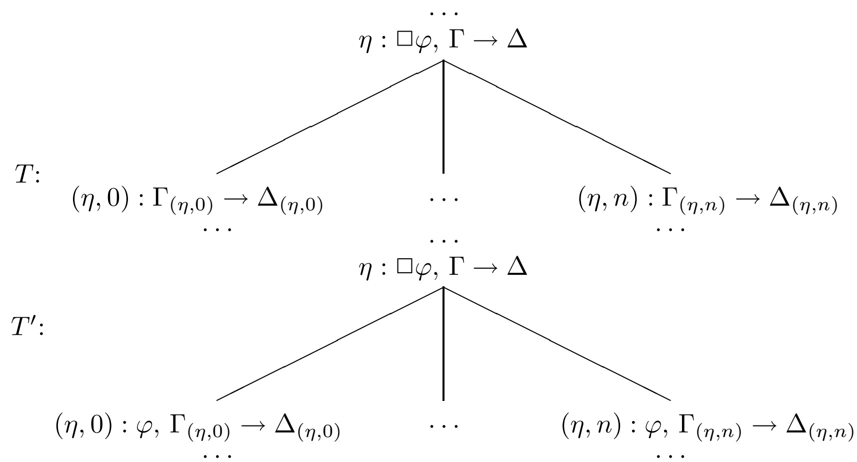

Figure 7. Lemma 5.1, necessitation left

The tree sequent T' is obtained from T by adding the formula φ to the left hand sides of the immediate successors (η, i) $(i = 0, \dots, n)$

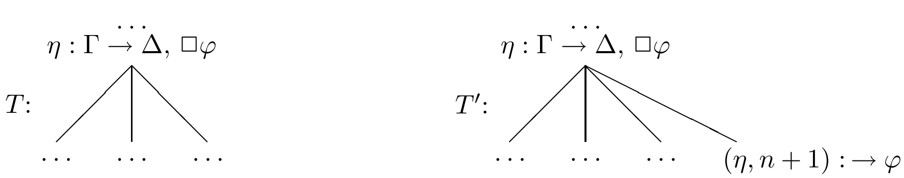

Figure 8. Lemma 5.1, necessitation right

The tree sequent T' is obtained from T by adding the new node $(\eta, n+1) :\to \varphi$

P r o o f. Straightforward from the definitions of inference rules of TLM_{ω_1}. □

T h e o r e m 5.2. *If a formula* φ *of infinitary propositional modal logic is valid in any Kripke model, then there exists a derivation of* φ *in* TLM_{ω_1} *which does not include* $(\mathrm{cut})_+$.

P r o o f. We show that if φ has no cut-free derivation, then there exists a Kripke model which refutes φ. Let $(\varphi_i)_{i\in\omega}$ be an enumeration of all subformulas of φ and $(\psi_i)_{i\in\omega}$ be the sequence $\varphi_0, \varphi_0, \varphi_1, \varphi_0, \varphi_1, \varphi_2 \ldots$, in which each φ_i occurs infinitary many times. Now, we define a sequence $(T_i)_{i\in\omega}$ of tree sequents such that each T_i has no cut-free derivation. Let $T_0 := \{0 :\rightarrow \varphi\}$. Suppose T_k is defined. For each address η in T_k, let $\{\eta : \Gamma_\eta \rightarrow \Delta_\eta\}$ be the node of T_k at η. We define T_{k+1} according as the construction of ψ_k. By Lemma 5.1, the construction of T_{k+1} is well-defined.

1. $\psi_k = p$: define $T_{k+1} := T_k$;

2. $\psi_k = \bigwedge\Theta$: for each η in T_k, if $\bigwedge\Theta \in \Gamma_\eta$, then add Θ to Γ_η, and if $\bigwedge\Theta \in \Delta_\eta$, then add a formula $\theta \in \Theta$ chosen in Lemma 5.1 to Δ_η;

3. $\psi_k = \bigvee\Theta$: dual to the previous case;

4. $\psi_k = \chi \supset \psi$: for each η in T_k, if $\chi \supset \psi \in \Gamma_\eta$, then add χ to Δ_η or ψ to Γ_η so that T_{k+1} does not have any cut-free derivation, and if $\chi \supset \psi \in \Delta_\eta$, then add χ to Γ_η and ψ to Δ_η;

5. $\psi_k = \neg\psi$: for each η in T_k, if $\neg\psi \in \Gamma_\eta$, then add ψ to Δ_η, and if $\neg\psi \in \Delta_\eta$, then add ψ to Γ_η;

6. $\psi_k = \Box\psi$: for each η in T_k, if $\Box\psi \in \Gamma_\eta$, then add ψ to the left hand sides of all immediate successors $(\eta, 0), \ldots, (\eta, n)$ of η, and if $\Box\psi \in \Delta_\eta$, then add a new node $(\eta, n+1) :\rightarrow \psi$ under η.

Now, we define the Kripke frame $\langle W, R\rangle$ as the *limit* of the sequence $(T_i)_{i\in\omega}$. For each $i \in \omega$, suppose the node of T_i at an address η is denoted by $\eta : \Gamma^i_\eta \rightarrow \Delta^i_\eta$. For each η, define the sets $\lim\Gamma_\eta$ and $\lim\Delta_\eta$ by $\lim\Gamma_\eta = \bigcup_{i\in\omega}\Gamma^i_\eta$, $\lim\Delta_\eta = \bigcup_{i\in\omega}\Delta^i_\eta$. We write $\lim\eta$ for the pair $(\lim\Gamma_\eta, \lim\Delta_\eta)$. Then, define W as the collection of all $\lim\eta$, and define the relation R on W by $\lim\eta\, R\, \lim\zeta$ iff ζ is an immediate successor of η. We define a valuation v by $v(p) = \{\lim\eta : p \in \lim\Gamma_\eta\}$ for each propositional variable p. Now, a simple induction shows that for any $i \in \omega$ and any address η, if $\varphi_i \in \lim\Gamma_\eta$, then $\lim\eta \vDash \varphi_i$, and if $\varphi_i \in \lim\Delta_\eta$, then $\lim\eta \nvDash \varphi_i$. Since φ is a member of $(\varphi_i)_{i\in\omega}$ and $\varphi \in \lim\Delta_0$, we complete the proof. □

From Theorem 5.1 and Theorem 5.2, the following theorems hold immediately.

T h e o r e m 5.3. *A formula φ has a cut-free derivation in* TLM_{ω_1} *if and only if* $F \vDash \varphi$ *for every Kripke frame* F.

T h e o r e m 5.4. *If a formula φ is derivable in* TLM_{ω_1}*, then there exists a derivation of φ in* TLM_{ω_1} *which does not include* $(\mathrm{cut})_+$.

Since any rule in TLM_{ω_1} but $(\mathrm{cut})_+$ satisfies the subformula property, the system TLM_{ω_1} minus $(\mathrm{cut})_+$ is a formal system for $\mathbf{K}_{\omega_1} \oplus \mathrm{BF}_{\omega_1}$ with the subformula property.

Now, consider the systems which are obtained from TLM_{ω_1} by replacing $(\Box\rightarrow)_+$ with one of the rules listed below. Then, by the same argument, it follows that each of these systems is cut-free and axiomatizes the logic which is characterized by the class of reflexive, transitive, and reflexive and transitive frames, respectively:

$\Box p \supset p$:

$$\frac{T \setminus \{\eta, (\eta,0), \ldots, (\eta,n)\} \cup \{\eta : \varphi, \Gamma \rightarrow \Delta\} \cup \bigcup_{k=0}^{n}\{(\eta,k) : \varphi, \Gamma_k \rightarrow \Delta_k\}}{T \setminus \{\eta, (\eta,0), \ldots, (\eta,n)\} \cup \{\eta : \Box\varphi, \Gamma \rightarrow \Delta\} \cup \bigcup_{k=0}^{n}\{(\eta,k) : \Gamma_k \rightarrow \Delta_k\}}\,(\Box\rightarrow)_{\mathrm{T}},$$

where $\{(\eta,0), \ldots, (\eta,n)\}$ is the set of all immediate successors of η;

$\Box p \supset \Box\Box p$:

$$\frac{T\backslash \downarrow\eta \cup \{\eta : \Gamma \to \Delta\} \cup \bigcup_{\xi\in\downarrow\eta,\, \xi\neq\eta}\{\xi : \varphi, \Gamma_\xi \to \Delta_\xi\}}{T\backslash \downarrow\eta \cup \{\eta : \Box\varphi, \Gamma \to \Delta\} \cup \bigcup_{\xi\in\downarrow\eta,\, \xi\neq\eta}\{\xi : \Gamma_\xi \to \Delta_\xi\}}\,(\Box\to)_4;$$

$\Box p \supset p \oplus \Box p \supset \Box\Box p$:

$$\frac{T\backslash \downarrow\eta \cup \bigcup_{\xi\in\downarrow\eta}\{\xi : \varphi,\, \Gamma_\xi \to \Delta_\xi\}}{T\backslash \downarrow\eta \cup \{\eta : \Box\varphi,\, \Gamma_\eta \to \Delta_\eta\} \cup \bigcup_{\xi\in\downarrow\eta,\, \xi\neq\eta}\{\xi : \Gamma_\xi \to \Delta_\xi\}}\,(\Box\to)_{\mathrm{S4}}.$$

6 Infinitary intuitionistic logic

In this section, we give a formal system for infinitary intuitionistic logic which satisfies the subformula property.

Let D be the formula $\forall x\,(\varphi(x) \vee q) \supset \forall x\varphi(x) \vee q$ of predicate logic which is known as the axiom of constant domain. Let D_{ω_1} be the infinitary translation of D, that is, the formula $\bigwedge_{i\in\omega}(p_i \vee q) \supset \bigwedge_{i\in\omega} p_i \vee q$ of infinitary logic. It is known that the axiom D_{ω_1} is necessary to axiomatize the infinitary intuitionistic logic characterized by the class of all Kripke frames ([8], see also [5, 13]). We write $\mathbf{H}_{\omega_1} + \mathrm{D}_{\omega_1}$ for the infinitary intuitionistic logic characterized by the class of all Kripke frames.

The language we consider in this section consists of a countable set of propositional variables, the symbols $\bigwedge$ and $\bigvee$ for countable conjunction and disjunction, respectively, and $\supset$ for implication. The set of formulas is defined in the same way as in Section 2, but $\neg\varphi$ is defined as an abbreviation of $\varphi \supset \bot$. The system TLJ_{ω_1} is a tree type sequent calculus for infinitary intuitionistic logic. A tree sequent of TLJ_{ω_1} is a finite tree of sequents of the shape $\Gamma \to \Delta$, where Γ and Δ are countable sets of formulas. Note that the cardinality of the right hand side of a sequent of a node is also countable. The axiom and inference rules for $\bigwedge$ and $\bigvee$ of TLJ_{ω_1} are same as for TLM_{ω_1}. Inference rules for implication are the following:

$$\frac{T \setminus \{\eta, (\eta,k)\} \cup \{\eta : \Gamma \to \Delta\} \cup \{(\eta,k) : \varphi \to \psi\}}{T \setminus \{\eta, (\eta,k)\} \cup \{\eta : \Gamma \to \Delta, \varphi \supset \psi\}}\,(\to\supset)_{\mathrm{I}},$$

$$\frac{T \setminus \{\eta\} \cup \{\eta : \Gamma \to \Delta, \varphi\} \quad T \setminus \{\eta\} \cup \{\eta : \psi, \Gamma \to \Delta\}}{T \setminus \{\eta\} \cup \{\eta : \varphi \supset \psi, \Gamma \to \Delta\}}\,(\supset\to)_{\mathrm{I}}\,.$$

Here, in $(\to\supset)_{\mathrm{I}}$, the node $(\eta,k) : \varphi \to \psi$ is a leaf of the upper sequent and disappears after the application of the rule $(\to\supset)_{\mathrm{I}}$. On the other hand, $(\supset\to)_{\mathrm{I}}$ is the same as $(\supset\to)_+$ in TLM_{ω_1}. The structural rules of TLJ_{ω_1} are $(\mathrm{set})_{\mathrm{I}}$, $(\mathrm{cut})_{\mathrm{I}}$, and $(\mathrm{M})_{\mathrm{I}}$, where $(\mathrm{set})_{\mathrm{I}}$ and $(\mathrm{cut})_{\mathrm{I}}$ are the same as $(\mathrm{set})_+$ and $(\mathrm{cut})_+$ in TLM_{ω_1}, respectively, and $(\mathrm{M})_{\mathrm{I}}$ is the following rule:

$$\frac{T \setminus \{\eta, (\eta,k)\} \cup \{\eta : \Gamma \to \Delta\} \cup \{(\eta,k) : \Theta, \Lambda \to \Sigma\}}{T \setminus \{\eta, (\eta,k)\} \cup \{\eta : \Theta, \Gamma \to \Delta\} \cup \{(\eta,k) : \Lambda \to \Sigma\}}\,(\mathrm{M})_{\mathrm{I}}\,.$$

It is easy to see that the formula D_{ω_1} is derivable in TLJ_{ω_1}.

We define the embedding * from the set of tree sequents of TLJ_{ω_1} to the set of formulas of infinitary logic. Let T be any tree sequent of TLJ_{ω_1}. Then the formula T^* is defined inductiviely as follows:

1. If $T = \{0 : \Gamma \to \Delta\}$, then $T^* := \bigwedge\Gamma \supset \bigvee\Delta$;

2. if $0 : \Gamma \to \Delta$ is the root of T and A is the set of all immediate successors of the root 0, then $T^* := \bigwedge\Gamma \supset (\bigvee\Delta \vee \bigvee_{\xi\in A}(\downarrow\xi)^*)$.

It is easy to check that if a tree sequent T is derivable in TLJ_{ω_1}, then T^* is valid in every Kripke model. Hence, we have the following

Theorem 6.1. *If a formula φ is derivable in TLJ_{ω_1}, then φ is valid in every Kripke frame.*

By the definitions of the derivation rules for Boolean connectives and $(\mathrm{M})_{\mathrm{I}}$, the following lemma holds immediately.

Lemma 6.1. *Let T be a tree sequent of TLJ_{ω_1} which has no cut-free derivation. For any address η in T, suppose that the node at η is denoted by $\eta : \Gamma_\eta \to \Delta_\eta$. Then the following holds for any node $\eta : \Gamma_\eta \to \Delta_\eta$ in T:*

1. *If $p \in \Gamma_\eta$, then the tree sequent $T \setminus \downarrow\eta \cup \bigcup_{\xi \in \downarrow\eta} \{\xi : p, \Gamma_\xi \to \Delta_\xi\}$ has no cut-free derivation* (Figure 9).

2. *If $\bigwedge\Theta \in \Gamma_\eta$, then the tree sequent $T \setminus \downarrow\eta \cup \bigcup_{\xi \in \downarrow\eta} \{\xi : \Theta, \Gamma_\xi \to \Delta_\xi\}$ has no cut-free derivation. If $\bigwedge\Theta \in \Delta_\eta$, then there exists a formula $\theta \in \Theta$ such that the tree sequent $T \setminus \{\eta\} \cup \{\eta : \Gamma_\eta \to \Delta_\eta, \theta\}$ has no cut-free derivation.*

3. *If $\bigvee\Theta \in \Gamma_\eta$, then there exists a formula $\theta \in \Theta$ such that the tree sequent $T \setminus \downarrow\eta \cup \bigcup_{\xi \in \downarrow\eta} \{\xi : \theta, \Gamma_\xi \to \Delta_\xi\}$ has no cut-free derivation. If $\bigvee\Theta \in \Delta_\eta$, then the tree sequent $T \setminus \{\eta\} \cup \{\eta : \Gamma_\eta \to \Delta_\eta, \Theta\}$ has no cut-free derivation.*

4. *If $\varphi \supset \psi \in \Gamma_\eta$, then one of the two tree sequents $T \setminus \{\eta\} \cup \{\eta : \Gamma_\eta \to \Delta_\eta, \varphi\}$, $T \setminus \downarrow\eta \cup \bigcup_{\xi \in \downarrow\eta} \{\xi : \psi, \Gamma_\xi \to \Delta_\xi\}$ has no cut-free derivation. If $\varphi \supset \psi \in \Delta_\eta$, then the tree sequent $T \cup \{(\eta, n+1) : \varphi \to \psi\}$ has no cut-free derivation, where $(\eta, n+1)$ is a new node.*

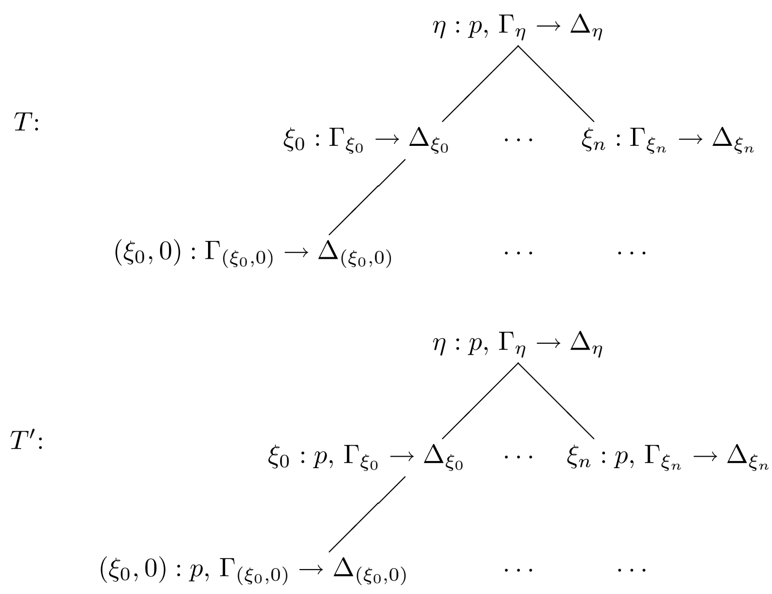

Figure 9. Lemma 6.1, propositional variable

The tree sequent T' is obtained from T by adding the propositional variable p to the left hand sides of the members of $\downarrow\eta$

Now, in the same way as in Section 5, we can show that the cut-free fragment of TLJ_{ω_1} is complete with respect to the class of Kripke frames. In this case, we refer Lemma 6.1 for the construction of the counter model, instead of Lemma 5.1. Then, the cut-elimination theorem for TLJ_{ω_1} holds immediately.

Theorem 6.2. *A formula φ has a cut-free derivation in* TLJ_{ω_1} *if and only if* $F \models \varphi$ *for any Kripke frame* F.

Theorem 6.3. *If a formula φ is derivable in* TLJ_{ω_1}, *then there exists a derivation of φ in* TLJ_{ω_1} *which does not include* $(\mathrm{cut})_{\mathrm{I}}$.

References

[1] Avron, A., The method of hypersequents in the proof theory of propositional non-classical logics. In: Logic: From Foundations to Applications, European Logic Colloquium (W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, eds.), Oxford 1996, pp. 1 – 32.

[2] Buss, S. R., An introduction to proof theory. In: Handbook of proof theory (S. R. Buss, ed.), Elsevier, Amsterdam et al. 1998, pp. 1 – 78.

[3] Fattorosi-Barnaba, M., and S. Grassotti, An infinitary graded modal logic (Graded modalities VI). Math. Logic Quarterly **41** (1995), 547 – 563.

[4] Feferman, S., Lectures on proof theory. In: Proceedings of the Summer School in Logic (M. H. Löb, ed.), Lecture Notes in Mathematics 70, Springer-Verlag, Berlin-Heidelberg 1967, pp. 1 – 107.

[5] Görnemann, S., A logic stronger than intuitionism. J. Symbolic Logic **36** (1971), 249 – 261.

[6] Kaneko, M., and T. Nagashima, Game logic and its applications II. Studia Logica **58** (1997), 273 – 303.

[7] Kashima, R., and T. Shimura, Cut-elimination theorem for the logic of constant domains. Math. Logic Quarterly **40** (1994), 153 – 172.

[8] Nadel, M. E., $\mathcal{L}_{\omega_1\omega}$ and admissible fragments. In: Model-Theoretic Logic (J. Barwise, and S. Feferman, eds.), Springer-Verlag, Berlin-Heidelberg-New York 1985, pp. 271 – 316.

[9] Ohnishi, M., and K. Matsumoto, Gentzen method in modal calculi. Osaka J. Math. **9** (1957), 113 – 130.

[10] Ohnishi, M., and K. Matsumoto, Gentzen method in modal calculi II. Osaka J. Math. **11** (1959), 115 – 120.

[11] Radev, S., Infinitary propositional normal modal logic. Studia Logica **46** (1987), 291 – 309.

[12] Rasiowa, H., and R. Sikorski, The Mathematics of Metamathematics. PWN-Polish Scientific Publishers, Warszawa 1963.

[13] Rauszer, C., and B. Sabalski, Remarks on distributive pseudo-Boolean algebra. Bull. Acad. Polon. Sci., Serie Math., Astr., Phys. **23** (1975), 123 – 129.

[14] Schwichtenberg, H., Proof theory: Some applications of cut-elimination. In: Handbook of Mathematical Logic (J. Barwise, ed.), North-Holland Publ. Comp., Amsterdam-New York-Oxford 1977, pp. 867 – 895.

[15] Tanaka, Y., Kripke completeness of infinitary predicate multi-modal logics. Notre Dame J. Formal Logic (to appear).

[16] Tanaka, Y., and H. Ono, The Rasiowa-Sikorski Lemma and Kripke completeness of predicate and infinitary modal logics. In: Advances in Modal Logic '98, Vol. 2 (M. Zakharyaschew, K. Segerberg, M. de Rijk, and H. Wansing, eds.), CSLI Publications, Stanford 2000, pp. 419 – 437.

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