In our paper, we defined a λ-ring to be a coalgebra for a comonad W : $\mathsf{CRing} \to \mathsf{CRing}$ defined by

$$W(R) = \mathsf{CRing}(\Lambda, R)$$

where the plethory $\Lambda = K(\mathsf{Schur})$ is descended from a 2-plethory Schur, or more exactly $\overline{kS}$ which is the free 2-rig on one variable with its tautological 2-plethory structure.

Our goal in these notes is to tie in this abstract conceptual description with the (complicated) traditional description of λ-rings. We pretend we know nothing about the ring structure of Λ, and develop the classical facts in due course.

1 Ideals in 2-rigs

The category $\overline{kS}$ is finitely complete and cocomplete. By Maschke's theorem, all short exact sequences split, and $\overline{kS}$ is a 2-Hilbert space with an orthonormal basis given by irreps of symmetric groups.

Definition 1: Let $\mathcal{R}$ be a 2-rig with finite limits and finite colimits, with the monoidal product distributing over both. An *ideal* of $\mathcal{R}$ is a full replete subcategory $\mathcal{I}$ such that

1. If $x \in \mathcal{R}$ and $y \in \mathcal{I}$, then $x \otimes y \in \mathcal{I}$;
2. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $A, C \in \mathcal{I}$ iff $B \in \mathcal{I}$. $\Box$

If all exact sequences in R split, then condition 2. is equivalent to I being closed under finite coproducts and idempotent splittings.

Definition 2: If a 2-rig $\mathcal{R}$ has an ideal $\mathcal{I}$, then $\mathcal{R}/\mathcal{I}$ is the 2-rig whose objects are those of $\mathcal{R}$, and whose morphisms are equivalence classes $[f]$ of morphisms f in $\mathcal{R}$, where $[f] = [g]$ for $f, g : A \to B$ means the image of $f - g$ belongs to $\mathcal{I}$.

It may be checked that R/I is indeed a 2-rig (in the sense of our paper). By definition, the evident "quotient" map $R \to R/I$ is eso (essentially surjective on objects) and full.

Proposition 1: If $\mathcal{R}$ is a 2-rig which as an additive category is a 2-Hilbert space, then the same is true of $\mathcal{R}/\mathcal{I}$ for any ideal $\mathcal{I}$ of $\mathcal{R}$.

Proof: We claim that if D is an indecomposable object of $\mathcal{R}$, then the same is true of $[D] = D \mod \mathcal{I}$. Indeed, $[D]$ has no nontrivial idempotents since the map $k = \mathcal{R}(D, D) \to \mathcal{R}/\mathcal{I}([D], [D])$ is onto, by fullness of $\mathcal{R} \to \mathcal{R}/\mathcal{I}$. $\Box$

If A is an object of $\mathcal{R}$, then (A) denotes the smallest ideal containing A. In the beginning, we will be particularly interested in $\mathcal{R} = \overline{kS}/(\Lambda^{n+1})$ where Λ^{n+1} is the sign representation of S_{n+1}.

Finally, let $\phi : \mathcal{R} \to \mathcal{S}$ be a 2-rig map, and factorize ϕ by a full eso functor $\mathcal{R} \to \overline{\mathcal{R}}$ followed by a faithful functor $\overline{\mathcal{R}} \to \mathcal{S}$. We define $\ker \phi$ to be the ideal of $\mathcal{R}$ consisting of objects A such that $\phi(A) \cong 0$. It is straightforward to check that $\overline{\mathcal{R}}$ is canonically equivalent to $\mathcal{R}/\ker \phi$.

2 Symmetric and exterior power operations

Let $(R, \eta : R \to \mathsf{CRing}(\Lambda, R))$ be a W-coalgebra. The composition

$$R \xrightarrow{\eta} \mathsf{CRing}(\Lambda, R) \xrightarrow{\mathrm{ev}_{[S^n]}} R$$

defines an operation $\sigma^n : R \to R$. These operations collate into a single map

$$\sigma : R \to \prod_{n \geq 0} R \cdot t^n = R[[t]]$$

where t^n is regarded as a formal placeholder to indicates the degree of a homogeneous element in a $\mathbb{N}$-graded ring. This map takes $r \in R$ to $\sum_{n \geq 0} \sigma^n(r) t^n$.

For 2-rigs $\mathcal{R}$, the σ^n-operations on $K(\mathcal{R})$ lift to Schur functors $S^n : \mathcal{R} \to \mathcal{R}$. For example, let $\mathcal{R}[\mathbb{N}]_+$ be the 2-rig of $\mathbb{N}$-graded Schur objects (here the subscript $+$ indicates *unsigned* symmetry). Provided that the grade 0 component of an object A is trivial, it is possible to construct the free commutative monoid $\exp(A)$, because in that case

$$\exp(A) = \sum_{n \geq 0} A^{\otimes n} / S_n$$

will be finitary in every grade. Because the coproduct of commutative monoids is given by tensor product on the underlying objects, and because the free construction preserves coproducts, we deduce in such cases an isomorphism (exponential law)

$$\exp(A \oplus B) \cong \exp(A) \otimes \exp(B).$$

In particular, for A (and likewise B) concentrated in grade 1, the grade n component of $\exp(A)$ is the symmetric power $S^n(A)$, and we deduce

$$S^n(A \oplus B) \cong (\exp(A) \otimes \exp(B))_n = \sum_{j+k=n} S^j(A) \otimes S^k(B).$$

This Schur functor isomorphism, which is pseudonatural in $\mathcal{R}$, may be formalized as an isomorphism

$$S^n(x \oplus y) \cong \sum_{j+k=n} S^j(x) \otimes S^k(y)$$

in the free 2-rig $\overline{kS[x, y]}$ on two generators. Descending to the Grothendieck group, we therefore have an identity in the ring $\Lambda[x, y] \cong \Lambda[x] \otimes \Lambda[y]$:

$$\sigma^n(x+y) = \sum_{j+k=n} \sigma^j(x) \cdot \sigma^k(y).$$

This is an instance of the coaddition law on the biring Λ in explicit form. Let us restate this in terms of what it means for λ-rings R. For an element $r \in R$, let $\lceil r \rceil : \Lambda[x] \to R$ denotes the unique λ-ring map that sends x to r. Then a sum $r+s$ in R is obtained as the composite

$$1 \xrightarrow{x} \Lambda[x] \xrightarrow{\alpha} \Lambda[x] \otimes \Lambda[y] \xrightarrow{\lceil r \rceil \otimes \lceil s \rceil} R \otimes R \xrightarrow{m} R$$

and $\sigma^n(r+s)$ is obtained as the composite

$$1 \xrightarrow{[S^n]} \Lambda[x] \xrightarrow{\alpha} \Lambda[x] \otimes \Lambda[y] \xrightarrow{\lceil r \rceil \otimes \lceil s \rceil} R \otimes R \xrightarrow{m} R.$$

It then follows from the coaddition identity above that

$$\sigma^n(r+s) = \sum_{j+k=n} \sigma^j(r)\sigma^k(s).$$

This may be expressed more compactly as the equation

$$\sigma(r+s) = \sigma(r) \cdot \sigma(s)$$

which follows from simple algebra:

$$\begin{aligned}
\sigma(r+s) &= \textstyle\sum_{n \geq 0} \sigma^n(r+s)t^n \\
&= \textstyle\sum_{n \geq 0} \left(\sum_{j+k=n} \sigma^j(r)\sigma^k(s) \right) t^n \\
&= \left(\textstyle\sum_{j \geq 0} \sigma^j(r)t^j \right) \cdot \left(\sum_{k \geq 0} \sigma^k(s)t^k \right) \\
&= \sigma(r) \cdot \sigma(s).
\end{aligned}$$

An essentially identical story may be told for the exterior power operations in place of the symmetric power operations. Again let $(R, \eta : R \to \mathsf{CRing}(\Lambda, R))$ be a W-coalgebra. The composition

$$R \xrightarrow{\eta} \mathsf{CRing}(\Lambda, R) \xrightarrow{\mathrm{ev}_{[\Lambda^n]}} R$$

defines an operation $\lambda^n : R \to R$. These operations collate into a single map

$$\lambda : R \to \prod_{n \geq 0} R \cdot t^n = R[[t]]$$

This map takes $r \in R$ to $\sum_{n \geq 0} \lambda^n(r)t^n$.

Continuing the analogous story, let $\mathcal{R}$ be a 2-rig and let $\mathcal{R}[\mathbb{N}]_-$ be the 2-rig of $\mathbb{N}$-graded Schur objects, but this time with *signed* symmetry). It follows from Joyal's rule of signs that for a graded object A concentrated in grade 1, we have

$$\exp(A)_n = \Lambda^n(A).$$

Then, from the exponential law, follows another Schur isomorphism

$$\Lambda^n(A \oplus B) \cong \sum_{j+k=n} \Lambda^j(A) \otimes \Lambda^k(B).$$

This allows us us to deduce another instance of coaddition: in $\Lambda[x, y]$ we have the formal equation

$$\lambda^n(x+y) = \sum_{j+k=n} \lambda^j(x) \cdot \lambda^k(y)$$

which allows us to deduce that in any λ-ring R we have

$$\lambda^n(r+s) = \sum_{j+k=n} \lambda^j(r) \cdot \lambda^k(s)$$

or more compactly, $\lambda(r+s) = \lambda(r) \cdot \lambda(s)$ as formal power series in $R[[t]]$.

Observe that we could have derived the identities involving σ^n by working in $\mathcal{R}[\mathbb{N}]_-$, just by considering objects A, B (or generators x, y) as $\mathbb{N}$-graded objects concentrated in grade 2 instead of grade 1.

Here is one more formal identity, one that interlocks the operations σ^j, λ^k. Again, letting x be the generator of $\overline{kS}$, put $\sigma_t(x) = \sum_{n\geq 0} \sigma^n(x)t^n$ and $\lambda_t(x) = \sum_{n\geq 0} \lambda^n(x)t^n$, as expressions in $\Lambda[[t]]$.

Theorem 1: We have $\sigma_t(x) \cdot \lambda_{-t}(x) = 1$.

Proof: Let us abbreviate $\overline{kS}[\mathbb{N}]_-$ to G (for graded Schur objects, with the signed symmetry), and let DG be the category of differential $\mathbb{N}$-graded Schur objects, again with the signed symmetry. Thus DG consists of chain complexes of Schur objects. The forgetful functor $U : \mathsf{DG} \to \mathsf{G}$ is a 2-rig map, and so is the homology functor $H : \mathsf{DG} \to \mathsf{G}$.

Consider the chain complex $M_x = (x \xrightarrow{1_x} x \to 0 \to 0 \to \ldots)$ with copies of x in degrees $0, 1$. Since M_x is exact, $H(M_x) = 0$ is a trivial graded object. $U(M_x)$ may be written as $x_0 \oplus x_1$ where x_i denotes a graded object supported on grade i with a copy of x there. Since the 2-rig maps H, U commute with symmetric powers, we have

$$H(S^n(M_x)) \cong S^n(H(M_x)) \cong S^n(0)$$

which vanishes for $n > 0$, and for $n = 0$ is just k concentrated in degree 0. On the other hand,

$$U(S^n(M_x)) \cong S^n(U(M_x)) \cong S^n(x_1 \oplus x_2) \cong \sum_{j+k=n} S^j(x_0) \otimes S^k(x_1)$$

where the underlying Schur object of $S^j(x_0)$ in grade 0 is $S^j(x)$, and the underlying Schur object of $S^k(x_1)$ in grade k is $\Lambda^k(x)$, again by Joyal's rule of signs.

By the categorified Euler formula, we have an equality in the Grothendieck group Λ,

$$[H_0(S^n(M_x))]-[H_1(S^n(M_x))]+[H_2(S^n(M_x))]-\ldots = [S^n(M_x)_0]-[S^n(M_x)_1]+[S^n(M_x)_2]-\ldots,$$

where we saw the left side is 0 for $n > 0$, and 1 if $n = 0$. On the right-hand side, we have

$$[S^n(x) \otimes \Lambda^0(x)] - [S^{n-1}(x) \otimes \Lambda^1(x)] + [S^{n-2}(x) \otimes \Lambda^2(x)] - \ldots.$$

Putting this together in the form of power series with Λ-coefficients, this yields

$$1 + 0t + 0t^2 + \ldots = \left(\sum_{j \geq 0} \sigma^j(x)t^j\right) \cdot \left(\sum_{k \geq 0} (-1)^k \lambda^k(x)t^k\right)$$

which is summarized by the equation $\sigma_t(x) \cdot \lambda_{-t}(x) = 1$. $\Box$

Bear in mind that we have not yet used any of the theory of symmetric group representations: Young symmetrizers, Specht modules, the theory of symmetric functions, etc., or at least nothing beyond Maschke's theorem. In particular, we are pretending that we are unaware of (or at best have heard gossip of) the identification of Λ with a polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$ that would enable an identification between $W(R)$ and $1 + tR[[t]]$ that is implicit in the standard accounts. That will come later in our account. And yet our conceptual methods show for example that for a λ-ring R, the standard map

$$\lambda : R \to 1 + tR[[t]] \quad r \mapsto \sum_{n \geq 0} \lambda^n(r)t^n$$

indeed takes sums to products of formal power series.

Definition 3: For a ring R, the *lowering operator* D is the derivation

$$D = t\frac{d}{dt} : R[[t]] \to R[[t]]$$

that takes a sequence of coefficients $(a_n)_{n \geq 0}$ of a power series to the sequence $(na_n)_{n \geq 0}$.

When we apply the lowering operator to the equation $\lambda_{-t} \cdot \sigma_t = 1$ in $\Lambda[[t]]$, we obtain

$$-t\lambda'_{-t} \cdot \sigma_t + t\lambda_{-t} \cdot \sigma'_t = 0$$

and so we obtain

$$t\lambda_{-t}\sigma'_t = t\lambda'_{-t}\sigma_t. \qquad (1)$$

Either side of this equation defines a power series we will denote by ψ_t. Because λ_{-t} and σ_t are reciprocals, the following is the exact same equation:

$$t\frac{\sigma_t'}{\sigma_t} = t\frac{\lambda_{-t}'}{\lambda_{-t}}. \qquad (2)$$

Equation (2) may be more recognizable to the experts than equation (1); particularly the expression on the right is one way of defining the power series ψ_t whose coefficients are Adams operations. The presence of logarithmic derivatives is of course suggestive as well. However, it seems to us that the expression

$$t\lambda_{-t}'\sigma_t$$

is simpler and more direct, and results in fewer formal manipulations in what we do.

Proposition 2: For formal variables $x, y \in \Lambda[x, y]$, we have $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$.

Proof: We have

$$\begin{aligned} D[\sigma_t(x+y)] \cdot \lambda_{-t}(x+y) &= D[\sigma_t(x)\sigma_t(y)] \cdot \lambda_{-t}(x)\lambda_{-t}(y) \\ &= [D\sigma_t(x) \cdot \sigma_t(y) + D\sigma_t(y) \cdot \sigma_t(x)] \cdot \lambda_{-t}(x)\lambda_{-t}(y) \\ &= D\sigma_t(x)\lambda_{-t}(x) + D\sigma_t(y)\lambda_{-t}(y) \\ &= \psi_t(x) + \psi_t(y) \end{aligned}$$

which completes the proof. □

3 Line objects

In the doctrine of symmetric monoidal categories, a line object (or invertible object) is an object L together with an object L^* and an isomorphism $\varepsilon : L \otimes L^* \cong 1$, where 1 denotes the monoidal unit. Line objects are closed under the monoidal product.

As is well-known, we can arrange for an isomorphism $\eta : 1 \cong L^* \otimes L$ so that η and ε are the unit and counit of a monoidal dual pair $L \dashv L^*$.

A line object L is *even* if the symmetry $\sigma : L \otimes L \to L \otimes L$ equals $1_{L \otimes L}$. It is *odd* if σ equals $-1_{L \otimes L}$.

In the literature, an even line object is called a *1-dimensional object*. Odd line objects typically get short shrift if they are mentioned at all. There is no really compelling reason this should be so, since the following propositions show that there is a duality between even and odd line objects.

Proposition 3: If L is an even line object in a 2-rig $\mathcal{R}$, then in the 2-rig of super $\mathcal{R}$-objects $\mathcal{R}[\mathbb{Z}_2]$, the object $(C_0, C_1) = (L, 0)$ is an even line object, and the object $(0, L)$ is an odd line object. If L is odd, then $(L, 0)$ is odd and $(0, L)$ is even.

Proposition 4: If L is an even line object in a 2-rig, then $S^n(L) \cong L^{\otimes n}$ for all $n \geq 0$, and $\Lambda^n(L) = 0$ for all $n > 1$. If L is an odd line object, then $\Lambda^n(L) \cong L^{\otimes n}$ for all $n \geq 0$, and $S^n(L) = 0$ for all $n > 1$.

Another way of stating Proposition 4 is that if L is even in $\mathcal{R}$, then in the λ-ring $K(\mathcal{R})$ we have

$$\sigma_t([L]) = \sum_{n \geq 0} [L^{\otimes n}] t^n, \qquad \lambda_t([L]) = 1 + [L]t.$$

If L is odd, we have

$$\lambda_t([L]) = \sum_{n \geq 0} [L^{\otimes n}] t^n, \qquad \sigma_t([L]) = 1 + [L]t.$$

For the moment let us focus on even line objects L, letting duality take care of the rest. We have

$$\lambda'_t(L) = [L], \qquad \lambda'_{-t}(L) = [L]$$

so that

$$\psi_t([L]) = t\lambda'_{-t}(L)\sigma_t(L) = t[L] \cdot \sum_{n \geq 0} [L^{\otimes n}] t^n = \sum_{n \geq 1} [L^{\otimes n}] t^n$$

Proposition 5: If in a 2-rig $\mathcal{R}$ every object can be expressed as a coproduct of even line objects, then the Adams operations $\psi^n : K(\mathcal{R}) \to K(\mathcal{R})$, defined by the power series $\psi_t(r) = \sum_{n \geq 1} \psi^n(r) t^n$, are commuting ring homomorphisms.

Proof: Proposition 2 shows that the operations ψ^n preserve addition. If we write

$$A = L_1 \oplus \ldots \oplus L_m, \qquad B = L'_1 \oplus \ldots \oplus L'_n$$

as coproducts of even line objects, then $A \otimes B = \sum_{i,j} L_i \otimes L'_j$ is also a sum of even line objects. Notice also that $\psi^n([L]) = [L^{\otimes n}]$ is multiplicative on even line objects. This implies full multiplicativity by a routine argument:

$$\begin{array}{rcl} \psi^n([A \otimes B]) & = & \psi^n(\sum_{i,j} [L_i \otimes L'_j]) \\ & = & \sum_{i,j} \psi^n([L_i \otimes L'_j]) \\ & = & \sum_{i,j} \psi^n([L_i]) \cdot \psi^n([L'_j]) \\ & = & (\sum_i \psi^n([L_i])) \cdot \left(\sum_j \psi^n([L'_j])\right) \\ & = & \psi^n(\sum_i [L_i]) \cdot \psi^n(\sum_j [L'_j]) \\ & = & \psi^n([A]) \cdot \psi^n([B]). \end{array}$$

Similarly, one proves $\psi^m \psi^n = \psi^n \psi^m$ by observing that this equation holds on even line objects: $\psi^m \psi^n([L]) = [L^{\otimes(mn)}] = \psi^n \psi^m([L])$. $\Box$