1 A natural weak factorization system on the arrow category $\mathbf{SSet}^{\to}$

1.1 Background and informal definition

Natural weak factorization systems were introduced by Grandis and Tholen in [5]. Recall that the definition of a model category (see [2], [6]) involves (among other things) two classes of maps, the cofibrations $\mathbf{L}$ and trivial fibrations $\mathbf{R}$, such that

- if i is a cofibration and p is a trivial fibration, every morphism $(f, g) : i \to p$ in $\mathcal{C}^{\to}$ has a lift, and
- every morphism f factors as $f = pi$, where i is a cofibration and p is a trivial fibration.

We say that the maps $(\mathbf{L}, \mathbf{R})$ is a *weak factorization system*. Actually, [6] requires that the factorization is *functorial*, so that $f = \alpha(f)\beta(f)$, where $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ are functors, and [2] points out that in most examples the factorization can be chosen to be functorial.

A natural weak factorization system is, to begin with, a weak factorization system together with a functorial factorization; however, in a natural weak factorization system, being an $\mathbf{L}$-map or an $\mathbf{R}$-map is an algebraic structure on the morphism rather than a mere property, and given i an $\mathbf{L}$-map and p an $\mathbf{R}$-map, the two structures on i and p combine to determine a lifting function which associates to each map $(f, g) : i \to p$ in $\mathcal{C}^{\to}$ a specific lift.

In more detail, a natural weak factorization system on a category $\mathcal{C}^{\to}$ is a functorial factorization $(\mathbf{L}, \mathbf{R})$, with both $\mathbf{L}, \mathbf{R} : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$, such that $\mathbf{R}$ is a monad and $\mathbf{L}$ is a comonad. For any functorial factorization system we can define an $\mathbf{L}$-map as a coalgebra for the endofunctor $\mathbf{L}$, and an $\mathbf{R}$-map as an algebra for the endofunctor $\mathbf{R}$; with these definitions, it follows that an $\mathbf{L}$-map has the left lifting property with respect to an $\mathbf{R}$-map. When $\mathbf{L}$ is a comonad, the comultiplication ensures that $\mathbf{L}(f)$ is an $\mathbf{L}$-map, and dually when $\mathbf{R}$ is a monad, the multiplication of $\mathbf{R}$ ensures that $\mathbf{R}(f)$ is an $\mathbf{R}$-map for each f. Thus, each natural weak factorization system determines a weak factorization system into a class of maps together with a functorial factorization.

1.2 Definition

We adopt Garner's definition of a natural weak factorization system in [4] which strengthens the definition of Grandis and Tholen with an additional distributivity law.

In the following, letting $\mathrm{dom}, \mathrm{cod} : \mathcal{C}^2 \to \mathcal{C}$ for a category $\mathcal{C}$, $\kappa : \mathrm{dom} \Rightarrow \mathrm{cod}$ will denote the canonical natural transformation.

Definition 1 (Natural weak factorization system)**.** A **natural weak factorization system** on $\mathcal{C}$ is given by

- a comonad, $\mathsf{L} = (L, \Phi, \Sigma)$ on $\mathcal{C}^2$
- a monad $\mathsf{R} = (R, \Lambda, \Pi)$ on $\mathcal{C}^2$
- a distributive law $\Delta : LR \to RL$

satisfying the following equalities:

$$\operatorname{dom} \circ L = \operatorname{dom} \qquad \operatorname{cod} \circ L = \operatorname{dom} \circ R \qquad \operatorname{cod} \circ R = \operatorname{cod}; \tag{1}$$
$$\operatorname{dom} \circ \Phi = 1_{\operatorname{dom}} \qquad \operatorname{cod} \circ \Phi = \kappa \operatorname{dom} \qquad \operatorname{dom} \Lambda = \kappa\Lambda \qquad \operatorname{cod} \Lambda = 1_{\operatorname{cod}} \tag{2}$$
$$\operatorname{dom} \Sigma = 1_{\operatorname{dom}} \qquad \operatorname{cod} \Sigma = \operatorname{dom} \Delta \qquad \operatorname{dom} \Pi = \operatorname{cod} \Delta \qquad \operatorname{cod} \Pi = 1_{\operatorname{cod}} \tag{3}$$

This is equivalent (see [4], Def. 2) to the following, more concise definition.

Definition 2 (Reduced natural weak factorization system)**.** A **reduced natural weak factorization system** on $\mathcal{C}$ is given by:

- A functorial factorization (E, λ, ρ) on $\mathcal{C}$.
- Natural transformations $\sigma : E \Rightarrow EL$ and $\pi : ER \Rightarrow E$, where $L : \mathcal{C}^2 \to \mathcal{C}^2$ is defined uniquely by the requirement that $\kappa L = \lambda$, and R is determined uniquely by the requirement that $\kappa R = \rho$

such that $\sigma \cdot \lambda = \lambda L$, $\rho \cdot \pi = \rho R$, and for all morphisms $f : X \to Y$ in $\mathcal{C}$,

$$\rho_{Lf} \circ \sigma_f = \mathrm{id}_{Ef} \qquad \pi_f \circ \lambda_{Rf} = \mathrm{id}_{Ef} \tag{4}$$
$$E(1_X, \rho_f) \circ \sigma_f = \mathrm{id}_{Ef} \qquad \pi_f \circ E(\lambda_f, 1_Y) = \mathrm{id}_{Ef} \tag{5}$$
$$E(1_X, \sigma_f) \circ \sigma_f = \sigma_{Lf} \circ \sigma_f \qquad \pi_f \circ E(\pi_f, 1_Y) = \pi_f \circ \pi_{Rf} \tag{6}$$

and lastly

$$\sigma_f \circ \pi_f = \pi_{Lf} \circ E(\sigma_f, \pi_f) \circ \sigma_{Rf} \tag{7}$$

In what follows, we will give two ways of associating, to a category $\mathcal{C}$ equipped with a monad (T, η, μ), an induced monad R on the slice category $\mathcal{C}/Y$ for any object Y. We would like to think of the algebras of the monad R as carrying a fiberwise T-algebra structure.

When the category $\mathcal{C}$ is taken to be **SSet**, the category of simplicial sets, and T is taken to be the cone monad, both of these two constructions for a monad R on **SSet**$/Y$ give the same result. Just as the cone monad has for its algebras the contractible spaces, the induced monad R on **SSet**$/Y$ has for its algebras those bundles $p : X \to Y$ those spaces which are fiber homotopy equivalent to the terminal object id_Y in a strong way; i.e., there is a coherent global contraction of each of the fibers. We will refer to R as the *shrinkability monad*, to borrow a beautiful term due to Dold.

In the category of simplicial sets specifically, we get something stronger. The monad R has a complementary comonad L such that (L, R) forms a natural weak factorization system. Of course the object part of L and the counit are already determined by the monad R, so the nontrivial part of this claim is the existence of a comultiplication making the copointed endofunctor L into a comonad compatible with R and satisfying the distributivity law 7. If R-algebras are "acyclic fibrations" then L-coalgebras are our "cofibrations."

An interesting problem for further research would be to extend one or both of the analyses below with further hypotheses so that the complementary comonad L can be constructed, as it can be in **SSet**. Again, one already has a copointed endofunctor L, so the problem is to construct a comultiplication for L satisfying the laws described above. We have been unable to identify reasonably general and well-studied categorical hypotheses which would allow us to construct a comultiplication for L. Nevertheless, it seems likely that this can be done, based on the apparently simple logical form of the expression that defines the comultiplication.

1.3 A monad on a slice category arising from a monad in the underlying category

Let $\mathcal{C}$ be a category with Cartesian products, and (T, μ, η) a monad on $\mathcal{C}$. Recall that T is said to be strong with respect to the Cartesian product if there is given a natural transformation $\alpha_{X,Y} : T(X) \times Y \to T(X \times Y)$ such that the following diagrams commute:

$$\begin{array}{ccc} TA & & \\ \downarrow{\cong} & \searrow^{\cong} & \\ TA \times 1 & \xrightarrow{\alpha_{A,1}} & T(A \times 1) \end{array} \tag{8}$$

$$\begin{array}{ccccc} TA \times (B \times C) & & \xrightarrow{\alpha_{A,B \times C}} & & T(A \times (B \times C)) \\ \downarrow{\cong} & & & & \downarrow{\cong} \\ (TA \times B) \times C & \xrightarrow{\alpha_{A,B} \times C} & T(A \times B) \times C & \xrightarrow{\alpha_{A \times B,C}} & T((A \times B) \times C) \end{array} \tag{9}$$

$$\begin{array}{ccc} A \times B & & \\ \downarrow{\eta_A \times B} & \searrow^{\eta_{A \times B}} & \\ TA \times B & \xrightarrow{\alpha_{A,B}} & T(A \times B) \end{array} \tag{10}$$

$$\begin{array}{ccccc} TTA \times B & \xrightarrow{\alpha_{TA,B}} & T(TA \times B) & \xrightarrow{T(\alpha_{A,B})} & TT(A \times B) \\ \downarrow{\mu \times B} & & & & \downarrow{\mu} \\ TA \times B & & \xrightarrow{\alpha_{A,B}} & & T(A \times B) \end{array} \tag{11}$$

These diagrams are taken from Definition 6.3.3., [1],

It will be sometimes be convenient to write 8 in this equivalent form:

$$T(\pi_A) \circ \alpha_{A,1} = \pi_{TA} : TA \times 1 \to TA \tag{12}$$

Strong monads were introduced by Kock in [8], [9], [7]. They have been used by Moggi ([10], [11]) to develop semantics of sequenced computations.

Fix a category $\mathcal{C}$ with finite limits, and let (T, μ, η) be a strong monad on $\mathcal{C}$.

Fix Y an object in $\mathcal{C}$. In this section we will construct a monad (R, μ^R, η^R) on $\mathcal{C}/Y$.

Lemma 1. *The following diagram commutes:*

$$\begin{array}{ccc} T(X) \times Y & \xrightarrow{\alpha_{X,Y}} & T(X \times Y) \\ & \searrow^{\pi_{TX}} & \downarrow{\scriptstyle T(\pi_X)} \\ & & TX \end{array} \tag{13}$$

Proof. Let $!_Y$ denote the unique morphism $Y \to 1$. Apply naturality of α to $(\mathrm{id}_X, !_Y)$ and use 8 with $A := X$. $\square$

For $f : X \to Y$, we write $E(f)$ for $\mathrm{dom}\, R(f)$.

In what follows, for $f : X \to Y$, $\mathrm{gr}\, f$ denotes the *graph* of f, the map $(\mathrm{id}_X, f) : X \to X \times Y$.

We first define R on objects. We define $E(f)$ by the following pullback square:

$$\begin{array}{ccc} E(f) & \longrightarrow & T(X) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle T\,\mathrm{gr}\, f} \\ TX \times Y & \xrightarrow{\alpha_{X,Y}} & T(X \times Y) \end{array} \tag{14}$$

and $R(f)$ is defined to be the second component of the left map of 14, the composition

$$R(f) := E(f) \to TX \times Y \xrightarrow{\pi_Y} Y \tag{15}$$

If $f : X \to Y, f' : X \to Y$ are two objects in $\mathcal{C}/Y$, and $g : X \to X'$ a morphism in $\mathcal{C}/Y$, it is clear how to use the universal property of the pullback $E(f')$ to construct a map $R(g) : E(f) \to E(f')$, and immediate that it commutes with the maps $R(f)$ and $R(f')$. We omit routine verification of the identity and composition laws for R.

Lemma 2. *Let $t(f) : E(f) \to T(X)$ be the top map in 14 and $s(f) : E(f) \to T(X) \times Y$ the left map in 14. Then $t(f) = \pi_{TX} \circ s(f)$, or equivalently $s(f) = (t(f), R(f))$.*

Proof. By Lemma 1, $\pi_{TX} = T(\pi_X) \circ \alpha_{X,Y}$, so

$$\begin{aligned} \pi_{TX} \circ s(f) &= T(\pi_X) \circ \alpha_{X,Y} \circ s(f) \\ &= T(\pi_X) \circ T(\mathrm{gr}(f)) \circ t(f) \\ &= T(\pi_X \circ \mathrm{gr}(f)) \circ t(f) \\ &= t(f) \end{aligned}$$

$\square$

We will use the notation $t(f)$ for the canonical map $E(f) \to T(X)$ again in what follows.

We describe the unit $\eta^R_f : f \to R(f)$. To construct a map $X \to E(f)$, by the universal property of the pullback it is necessary to give maps $a_0 : X \to TX, a_1 : X \to Y, a_2 : X \to TX$, subject to the requirement that

$$\alpha_{X,Y} \circ (a_0, a_1) = T(\mathrm{gr}\, f) \circ a_2 \tag{16}$$

We will take $a_0 = a_2 = \eta_X$ and $a_1 = f$.

Let us verify that the necessary coherence condition 16 is satisfied:

$$\begin{aligned} T(\mathrm{gr}\, f) \circ \eta_X &= \eta_{X\times Y} \circ \mathrm{gr}\, f \\ &= \alpha_{X,Y} \circ (\eta_X \times \mathrm{id}_Y) \circ \mathrm{gr}\, f \\ &= \alpha_{X,Y} \circ (\eta_X, f) \end{aligned}$$

as desired.

It is clear by the choice of $a_1 := f$ that η^R_f is a morphism in the slice category $\mathcal{C}/Y$.

Let us turn to the multiplication μ^R.

In the following, $T(f)^*\alpha_{1,Y} : E(f) \to T(X)$ denotes the leg of the pullback cone defining $E(f)$ over $T(f)$ in 14.

To give a map $E(R(f)) \to E(f)$, it suffices to give maps $b_0 : ERf \to TX$ and $b_1 : ERf \to TX \times Y$ with

$$\alpha_{X\times Y} \circ b_1 = T(\mathrm{gr}\, f) \circ b_0 \tag{17}$$

. We will take b_0 to be

$$b_0 := ERf \xrightarrow{t(Rf)} T(Ef) \xrightarrow{T(t(f))} T^2X \xrightarrow{\mu} TX \tag{18}$$

and we will take $b_1 = (b_0, R^2(f))$.

Let us prove that the necessary coherence condition 17 is satisfied.

$$\begin{aligned} &T(\mathrm{gr}\, f) \circ b_0 \\ &= T(\mathrm{gr}\, f) \circ \mu_X \circ T(t(f)) \circ t(Rf) \\ &= \mu_{X\times Y} \circ T^2(\mathrm{gr}\, f) \circ T(t(f)) \circ t(Rf) \\ &= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T((t(f), R(f))) \circ t(Rf) \\ &= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T(t(f) \times 1_Y) \circ T(\mathrm{gr}(Rf)) \circ t(Rf) \\ &= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T(t(f) \times 1_Y) \circ \alpha_{Ef,Y} \circ (t(Rf), R^2 f) \\ &= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ \alpha_{TX,Y} \circ (T(tf) \times 1_Y) \circ (t(Rf), R^2 f) \\ &= \alpha_{X,Y} \circ (\mu_X \times 1_Y) \circ (T(tf) \times 1_Y) \circ (t(Rf), R^2 f) \\ &= \alpha_{X,Y} \circ (b_0, R^2 f) \end{aligned}$$

Therefore we have constructed families of maps $\eta^R_f X \to E(f)$, $\mu^R_f : E(R(f)) \to E(f)$. It is immediate by construction that the maps η^R_f and μ^R_f are morphisms in the slice category $\mathcal{C}/Y$, so these are maps $\eta^R_f : f \to R(f)$, $\mu^R_f : R^2(f) \to R(f)$.

Let us now turn to verifying the naturality of η^R and μ^R as well as the unit and associativity laws for a monad.

Proposition 1. *Let $f : X \to Y, g : Z \to Y$. Let $h, k : g \to R(f)$ be morphisms in the slice category $\mathcal{C}/Y$. To show $h = k$ it is sufficient to show that $t(f) \circ h = t(f) \circ k$.*

Proof. This is immediate by the properties of the pullback, because we have assumed that h, k are morphisms in the slice category, and Lemma 2. □

For naturality of η^R, let (X, f) and (X', f') be objects in $\mathcal{C}/Y$ and let $g : (X, f) \to (X', f')$ be a morphism in the slice category. To prove that $R(g) \circ \eta^R_f = \eta^R_{f'} \circ g$, it is sufficient to show that $t(f') \circ R(g) \circ \eta^R_f = t(f') \circ \eta^R_{f'} \circ g$. $t(f') \circ \eta^R_{f'} = \eta_{X'}$ by definition, and $\eta_{X'} \circ g = T(g) \circ \eta_X$. Similarly, $t(f') \circ R(g) = T(g) \circ t(f)$ by definition of $R(g)$ and $t(f) \circ \eta^R_f = \eta_X$ by definition of η^R_f. Thus $t(f') \circ R(g) \circ \eta^R_f = T(g) \circ \eta_X = t(f') \circ \eta^R_{f'} \circ g$, as desired.

For naturality of μ^R, again introduce $(X, f), (X', f')$, and $g : (X, f) \to (X', f')$ in $\mathcal{C}/Y$. It is sufficient to show that $t(f') \circ \mu^R_{f'} \circ R^2(g) = t(f') \circ R(g) \circ \mu^R_f$. Thus:

$$
\begin{aligned}
&t(f') \circ \mu^R_{f'} \circ R^2(g) \\
&= \mu_{X'} \circ T(tf') \circ t(Rf') \circ R^2(g) \\
&= \mu_{X'} \circ T(tf') \circ T(Rg) \circ t(Rf) \\
&= \mu_{X'} \circ T^2 g \circ T(tf) \circ t(Rf) \\
&= T(g) \circ \mu_X \circ T(tf) \circ t(Rf) \\
&= T(g) \circ t(f) \circ \mu^R_f \\
&= t(f') \circ R(g) \circ \mu^R_f
\end{aligned}
$$

as desired.

Let us verify the left unit law $\mu^R_f \eta^R_{R(f)} = 1_{Rf}$. By 1 it suffices to check that $t(f) = t(f) \circ \mu^R_f \circ \eta^R_{R(f)}$. Therefore:

$$
\begin{aligned}
&t(f) \circ \mu^R_f \circ \eta_{R(f)} \\
&= \mu_X \circ T(tf) \circ t(Rf) \circ \eta_{R(f)} \\
&= \mu_X \circ T(tf) \circ \eta_{E(f)} \\
&= \mu_X \circ \eta_{T(X)} \circ t(f) \\
&= t(f)
\end{aligned}
$$

as desired.

Now we verify the right unit law $\mu^R_f(R(\eta^R_f) = 1_{Rf}$. By 1 it suffices to check that $t(f) = t(f) \circ \mu^R_f \circ R(\eta^R_f)$. Therefore:

$$\begin{aligned}
&t(f) \circ \mu^R_f \circ R(\eta^R_f) \\
&= \mu_X \circ T(tf) \circ t(Rf) \circ R(\eta^R_f) \\
&= \mu_X \circ T(tf) \circ T(\eta^R_X) \circ t(f) \\
&= \mu_X \circ T(\eta_X) \circ t(f) \\
&= t(f)
\end{aligned}$$

as desired.

Last, we will verify the associativity of multiplication. By 1 it suffices to check that $t(f) \circ \mu^R_f \circ R(\mu^R_f) = t(f) \circ \mu^R_f \circ \mu^R_{Rf}$. Therefore:

$$\begin{aligned}
&t(f) \circ \mu^R_f \circ R(\mu^R_f) \\
&= \mu_X \circ T(t(f)) \circ t(Rf) \circ R(\mu^R_f) \\
&= \mu_X \circ T(t(f)) \circ T(\mu^R_f \circ t(R^2 f) \\
&= \mu_X \circ T(\mu_X \circ T(tf) \circ t(Rf)) \circ t(R^2 f) \\
&= \mu_X \circ \mu_{TX} \circ T(T(tf) \circ t(Rf)) \circ t(R^2 f) \\
&= \mu_X \circ T(tf) \circ \mu_{Ef} \circ T(t(Rf)) \circ t(R^2 f) \\
&= \mu_X \circ T(tf) \circ t(Rf) \circ \mu^R_{Rf} \\
&= t(f) \circ \mu^R_f \circ \mu^R_{Rf}
\end{aligned}$$

as desired.

This completes the verification of the monad properties.

As we mentioned earlier, a natural problem is to identify sufficient hypotheses on the monad T which would allow us to construct a comultiplication on the copointed endofunctor $f \mapsto \eta^R_f$ on the coslice category $X \setminus \mathcal{C}$. To construct the comultiplication, a plausible choice of cone to construct the desired map $E(f) \to E(\eta^R_f)$ is given by

$$\begin{array}{ccc}
Ef & \xrightarrow{\quad t(f) \quad} & T(X) \\
\big\downarrow{\scriptstyle (t(f), 1_{Ef})} & & \big\downarrow{\scriptstyle T(\mathrm{gr}(\eta^R_f))} \\
T(X) \times Ef & \xrightarrow{\alpha_{X,Ef}} & T(X \times Ef)
\end{array} \tag{19}$$

but it is not clear what hypotheses can be chosen on T to force this diagram to commute. For example, the cone monad $C : \mathbf{SSet} \to \mathbf{SSet}$ is a polynomial monad with Cartesian unit and counit, and it has a right adjoint P, the path space comonad; the underlying polynomial diagram for C is of a certain distinguished form $1 \to I \to 1$. All polynomial functors preserve all connected limits.

1.4 Another monad in a slice category resulting from a monad in the underlying category

In the previous section we worked in the level of generality of an arbitrary strong monad in a category with finite limits. However, as there is limited information and structure to work with in this context, it seems harder to make progress. Furthermore, pulling back along a strength is an unusual operation. In this section we will carry out an analogous construction but in the context of a polynomial monad.

Let $\mathcal{C}$ be a locally closed Cartesian category, with terminal object 1.

In this section, let (T, η, μ) be a monad $\mathcal{C} \to \mathcal{C}$ with Cartesian multiplication and unit natural transformations, preserving connected limits, and such that η_X is a monic morphism for all X.

We will borrow notation from [3], where for any map $f : A \to B$, Δ_f is the pullback functor $\mathcal{C}/B \to \mathcal{C}/A$, $\Sigma_f : \mathcal{C}/A \to \mathcal{C}/B$ is the postcomposition functor $g \mapsto f \circ g$, and Π_f is a right adjoint to Δ_f.

We further assume T has the following property: for any $f : X \to Y$, $T(f)$ has the universal property of $\Pi_{\eta_Y}(f)$, that is, for any object $(Z, p : Z \to TY)$ in $\mathcal{C}/TY$, there is a bijection

$$\mathrm{Hom}_{\mathcal{C}/Y}(\Delta_f(p), f) \cong \mathrm{Hom}_{\mathcal{C}/TY}(p, T(f)) \tag{20}$$

natural in (Z, p).

This very strong requirement determines the functor T in terms of a small amount of data. If Y is chosen to be the terminal object, then we see that for any object X, $T(X)$ can be defined as $\mathrm{dom}\,\Pi_{\eta_1}(!_X)$, where $!_X$ is the unique map $X \to 1$. It follows that T is, up to isomorphism, the polynomial functor induced by the diagram

$$1 \leftarrow 1 \xrightarrow{\eta_1} T(1) \to 1 \tag{21}$$

Note that because η was assumed to be a Cartesian natural transformation, the below pullback diagram indicates that $\Delta_{\eta_Y}\Pi_{\eta_Y} \to \mathrm{id}_{\mathcal{C}/Y}$ is an isomorphism.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{\eta_Y} & TY \end{array} \tag{22}$$

Although our requirements on T seem quite strong, any choice of object I in $\mathcal{C}$ and any $e : 1 \to I$ gives rise to a polynomial diagram

$$1 \leftarrow 1 \to I \to 1 \tag{23}$$

and the resulting polynomial functor T will be equipped with a Cartesian unit natural transformation $\eta_Y : Y \to TY$; moreover if T is any polynomial endofunctor so defined, then for any $f : X \to Y$, Tf has the universal property of $\Pi_{\eta_Y}(f)$. (Proof omitted to cut down on verbosity.) Therefore, given any object I and any morphism $e : 1 \to I$, one has the polynomial endofunctor T defined by 23 and a Cartesian unit transformation, componentwise monic. Because Π_e is a right adjoint, it preserves the terminal object, so $\Pi_e(\mathrm{id}_1)$ is an isomorphism $T(1) \cong I$, and we can identify them. All that remains is to give a Cartesian multiplication such that the associativity and unit laws are satisfied.

Let Y be an arbitrary object in $\mathcal{C}$ and let $R : \mathcal{C}/Y \to \mathcal{C}/Y$ be the polynomial functor arising from the diagram

$$Y \leftarrow Y \xrightarrow{(1_Y,e)} Y \times I \xrightarrow{\pi_Y} Y \tag{24}$$

We will prove that R inherits a monad structure from T.

Proposition 2. *If $f : A \to B$ is monic, then the counit of the adjunction $\Delta_f \dashv \Pi_f$ is an isomorphism.*

Proof. It is easy to see that if f is monic, the unit of the adjunction $\Sigma_f \dashv \Delta_f$ is an isomorphism. Since $\Delta_f\Sigma_f \dashv \Delta_f\Pi_f$, $\Delta_f\Sigma_f$ is naturally isomorphic to the identity iff $\Delta_f\Pi_f$ is. □

Definition 3 (Unit for R)**.** There is a natural transformation $\eta^R : \mathrm{id}_{\mathcal{C}/Y} \to R$.

Proof. Let $f : X \to Y$. We will construct a map $\eta^R : f \to \Sigma_{\pi_Y}\Pi_{(1,e)}f$. The counit of the $\Sigma_{(1,e)} \dashv \Delta_{(1,e)}$ adjunction evaluated at $\Pi_{(1,e)}(f)$ determines a map $\Sigma_{(1,e)}\Delta_{(1,e)}\Pi_{(1,e)}(f) \to \Pi_{(1,e)}f$. Because $(1,e)$ is monic, by the previous proposition $\Delta_{(1,e)}\Pi_{(1,e)}(f) \cong f$ and this simplifies to a map $\Sigma_{(1,e)}(f) \to \Pi_{(1,e)}$. Applying Σ_{Π_Y} to both sides and recognizing that $\pi_Y \circ (1,e) = 1_Y$, this gives the desired natural transformation $\eta^R_f : f \to \Sigma_{\pi_Y}\Pi_{(1,e)}f$. □

The following lemma indicates that we can see R as a fibered version of T, as at least for the case of trivial bundles it acts on the fibers in the expected way.

Lemma 3. *Let X be arbitrary, and let $\pi_Y : Y \times X \to Y$. Then $R(\pi_Y) = \pi_Y : Y \times T(X) \to Y$, and $\eta^R_{\pi_Y} = 1_Y \times \eta_X$.*

Proof. Apply Beck-Chevalley to the pullback square

$$\begin{array}{ccc} Y & \xrightarrow{(1\,e)} & Y \times I \\ \downarrow & & \downarrow{\scriptstyle \pi_I} \\ 1 & \xrightarrow{\quad e \quad} & I \end{array} \tag{25}$$

to see that $\Pi_{(1,e)}(\pi_Y) = \Pi_{(1,e)}\Delta_{!_Y}(X) = 1_Y \times T(!_X) : Y \times T(X) \to Y \times T(1)$. Then compose with the projection π_Y. □

Definition 4. There is a natural transformation $\mu^R : R^2 \to R$.

Proof. First we need to recharacterize R^2 in a more convenient form. By definition, $R^2 = \Sigma_{\pi_Y}\Pi_{(1,e)}\Sigma_{\pi_Y}\Pi_{(1,e)}$. However, we can restructure this using the distributivity law of Π over Σ (see [3], pg. 9) to write this as $\Sigma_{\pi_Y}\Sigma_{1_Y \times T(!_X)}\Pi_{1_Y \times \eta_I}\Pi_{(1,e)}$.

We can rewrite this as $\Sigma_{\pi_Y}\Sigma_{1_Y\times\mu_1}\Pi_{1_Y\times\eta_I}\Pi_{(1,e)}$, where we have changed $T(!_X)$ with μ_1, because obviously the compositions are the same as 1 is terminal. Thus we have to give a natural transformation

$$\Sigma_{\pi_Y}\Sigma_{1_Y\times\mu_1}\Pi_{1_Y\times\eta_I}\Pi_{1,e} \to \Sigma_{\pi_Y}\Pi_{1,e} \tag{26}$$

Now it suffices to give a natural transformation

$$\Sigma_{1_Y\times\mu_1}\Pi_{1_Y\times\eta_I} \to 1_{\mathcal{C}/Y\times I} \cong \Delta_{1_Y\times\eta_I}\Pi_{1_Y\times\eta_I} \tag{27}$$

because we can whisker with $\Pi_{1,e}$ and Σ_{π_Y}, so we take the transpose

$$\Pi_{1_Y\times\eta_I} \to \Delta_{1_Y\times\mu_1}\Delta_{1_Y\times\eta_I}\Pi_{1_Y\times\eta_I} \tag{28}$$

But $\mu_1 \circ \eta_I = 1$ and so we can just take the identity. □

References

[1] Martin Brandenburg. *Tensor Categorical Foundations of Algebraic Geometry*. Oct. 7, 2014. arXiv: 1410.1716 [math]. URL: http://arxiv.org/abs/1410.1716 (visited on 02/04/2023).

[2] William G. Dwyer and J. Spalinski. "Homotopy Theories and Model Categories". In: *Handbook of Algebraic Topology*. Ed. by I. M. James. Amsterdam ; New York: Elsevier Science B.V, 1995. ISBN: 978-0-444-81779-2. URL: https://math.jhu.edu/~eriehl/616-s16/DwyerSpalinski.pdf.

[3] Nicola Gambino and Joachim Kock. "Polynomial Functors and Polynomial Monads". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 154.1 (Jan. 2013), pp. 153–192. ISSN: 0305-0041, 1469-8064. DOI: 10.1017/S0305004112000394. URL: https://www.cambridge.org/core/product/identifier/S0305004112000394/type/journal_article (visited on 02/13/2023).

[4] Richard Garner. *Cofibrantly Generated Natural Weak Factorisation Systems*. Feb. 12, 2007. arXiv: math/0702290. URL: http://arxiv.org/abs/math/0702290 (visited on 01/03/2023).

[5] Marco Grandis and Walter Tholen. "Natural Weak Factorization Systems". In: *ARCHIVUM MATHEMATICUM (BRNO)* 42 (2006), pp. 397–408.

[6] Mark Hovey. *Model Categories*. Mathematical Surveys and Monographs v. 63. Providence, R.I: American Mathematical Society, 1999. 209 pp. ISBN: 978-0-8218-1359-1.

[7] Anders Kock. "Closed Categories Generated by Commutative Monads". In: *Journal of the Australian Mathematical Society* 12.4 (Nov. 1971), pp. 405–424. ISSN: 0004-9735. DOI: 10.1017/S1446788700010272. URL: https://www.cambridge.org/core/product/identifier/S1446788700010272/type/journal_article (visited on 02/04/2023).

[8] Anders Kock. "Monads on Symmetric Monoidal Closed Categories". In: *Archiv der Mathematik* 21.1 (Dec. 1970), pp. 1–10. ISSN: 0003-889X, 1420-8938. DOI: 10.1007/BF01220868. URL: http://link.springer.com/10.1007/BF01220868 (visited on 02/04/2023).

[9] Anders Kock. "Strong Functors and Monoidal Monads". In: *Archiv der Mathematik* 23.1 (Dec. 1972), pp. 113–120. ISSN: 0003-889X, 1420-8938. DOI: 10.1007/BF01304852. URL: http://link.springer.com/10.1007/BF01304852 (visited on 02/04/2023).

[10] E. Moggi. "Computational Lambda-Calculus and Monads". In: *[1989] Proceedings. Fourth Annual Symposium on Logic in Computer Science*. [1989] Proceedings. Fourth Annual Symposium on Logic in Computer Science. Pacific Grove, CA, USA: IEEE Comput. Soc. Press, 1989, pp. 14–23. ISBN: 978-0-8186-1954-0. DOI: 10.1109/LICS.1989.39155. URL: http://ieeexplore.ieee.org/document/39155/ (visited on 02/04/2023).

[11] Eugenio Moggi. "Notions of Computation and Monads". In: *Information and Computation* 93.1 (July 1991), pp. 55–92. ISSN: 08905401. DOI: 10.1016/0890-5401(91)90052-4. URL: https://linkinghub.elsevier.com/retrieve/pii/0890540191900524 (visited on 02/04/2023).