logic in color

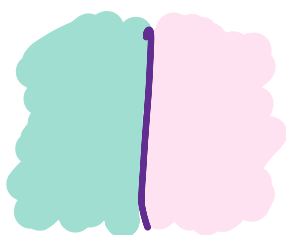
#2 : duality & equipments

A 2-category is [Rel) [data]

- a collection of objects — type

set — A

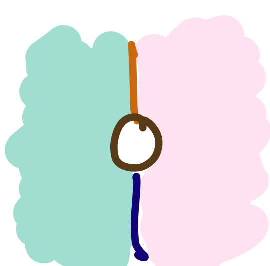
- a collection of morphisms — judgement
  each with source & target object



relation — $A\ a \xrightarrow{R} b\ B$

$R : A \times B \to \{T, F\}$

- a collection of 2-morphisms — inference
  each with source & target morphism



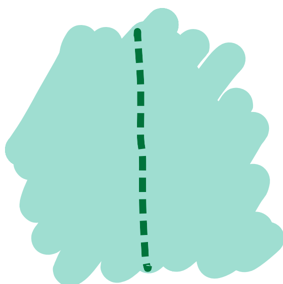
implication

$$\frac{A\ a \xrightarrow{R} b\ B}{A\ a \xrightarrow{S} b\ B}\ \gamma$$

$\forall a \in A, b \in B.$
$R[a,b] \Rightarrow S[a,b]$

[note: colors]

- for each object an identity morphism

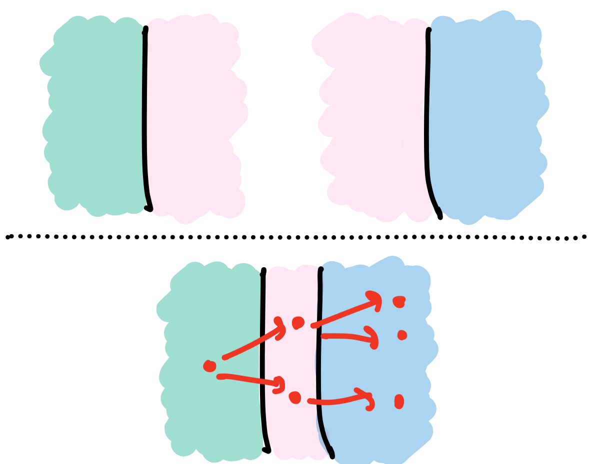


equality relation

$A\,a \vdash a'\,A$

$a = a'$

- on morphisms a composition



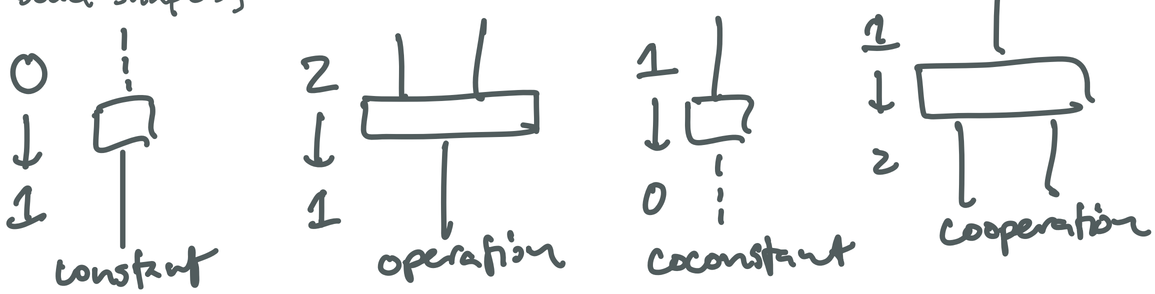
$Aa \xrightarrow{R} bB \quad Bb \xrightarrow{U} cC$

$Aa \xrightarrow{R \circ U} cC$

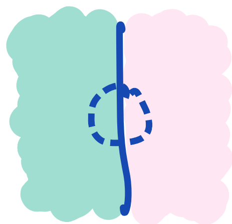
$= \exists b \in B.\ aRb \wedge bUc$

which is associative and unital

[note: bead shapes]

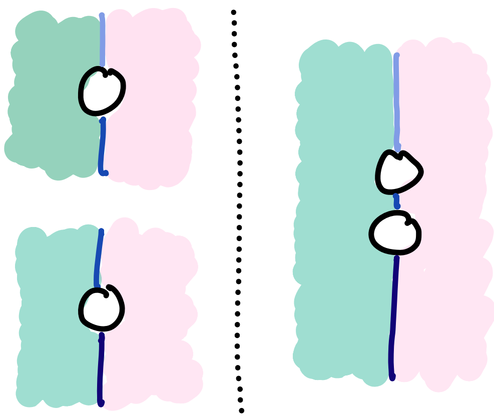


- for each morphism an identity 2-morphism



$$\frac{Aa \xrightarrow{S} bB}{Aa \xrightarrow{S} bB}$$

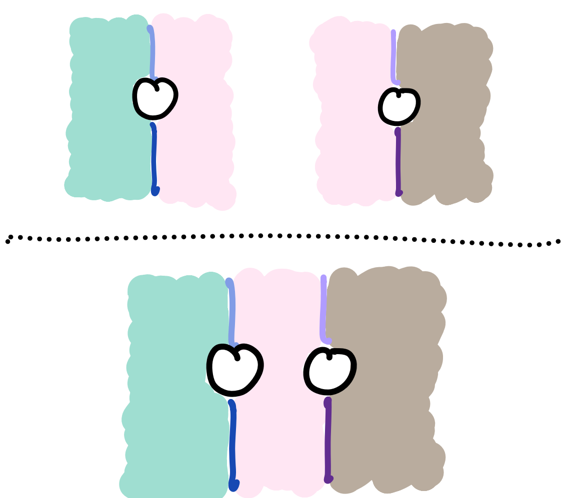
- on 2-morphisms a sequence composition



$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{S} bB}\ \gamma \qquad \frac{Aa \xrightarrow{S} bB}{Aa \xrightarrow{T} bB}\ \delta$$

$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{T} bB}\ \gamma\delta$$

- on 2-morphisms a parallel composition

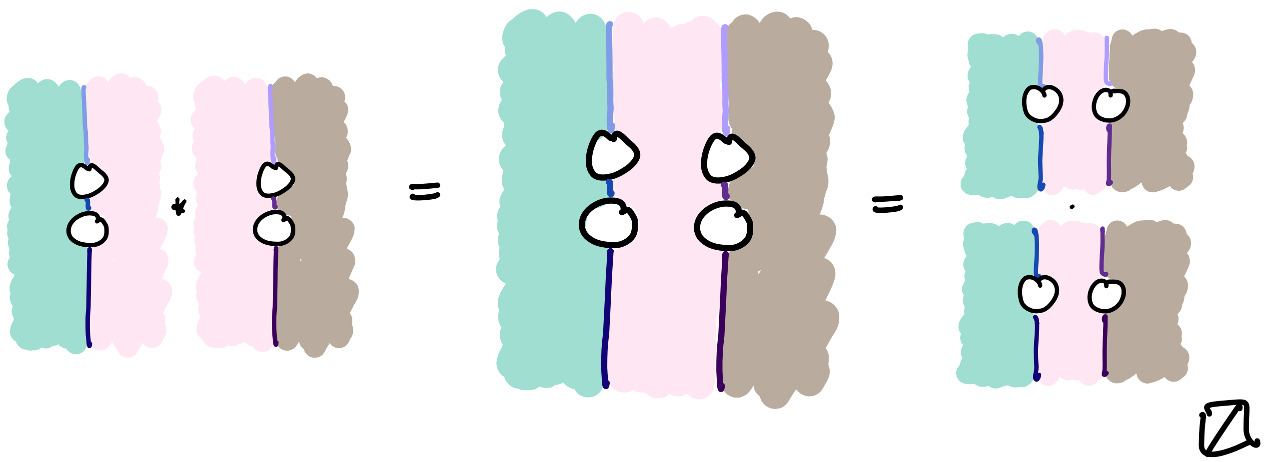


$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{S} bB}\ \gamma \qquad \frac{Bb \xrightarrow{U} cC}{Bb \xrightarrow{V} cC}\ \varepsilon$$

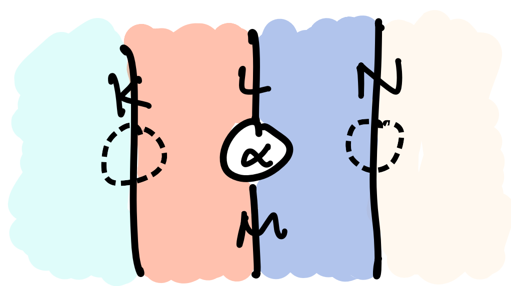
$$\frac{Aa \xrightarrow{R \circ U} cC}{Aa \xrightarrow{S \circ V} cC}\ \gamma * \varepsilon$$

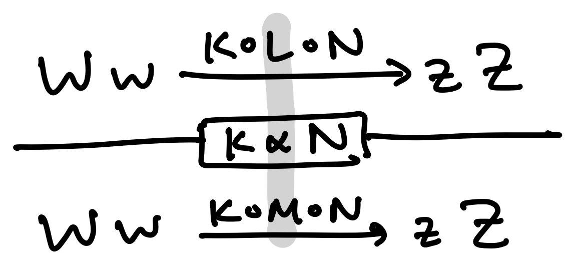
properties [2]

- sequence & parallel composition are associative & unital, and compatible :

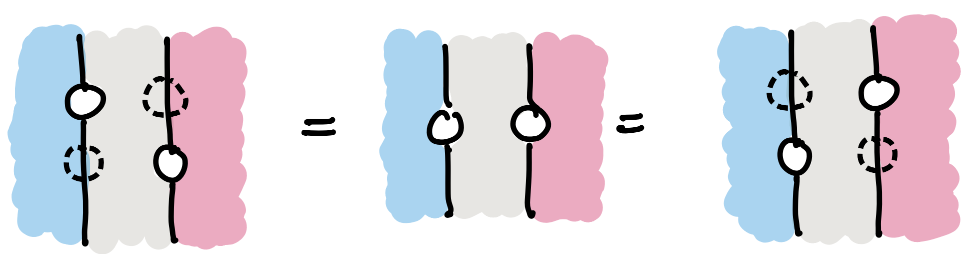


note: identity beads allow for "whiskering"

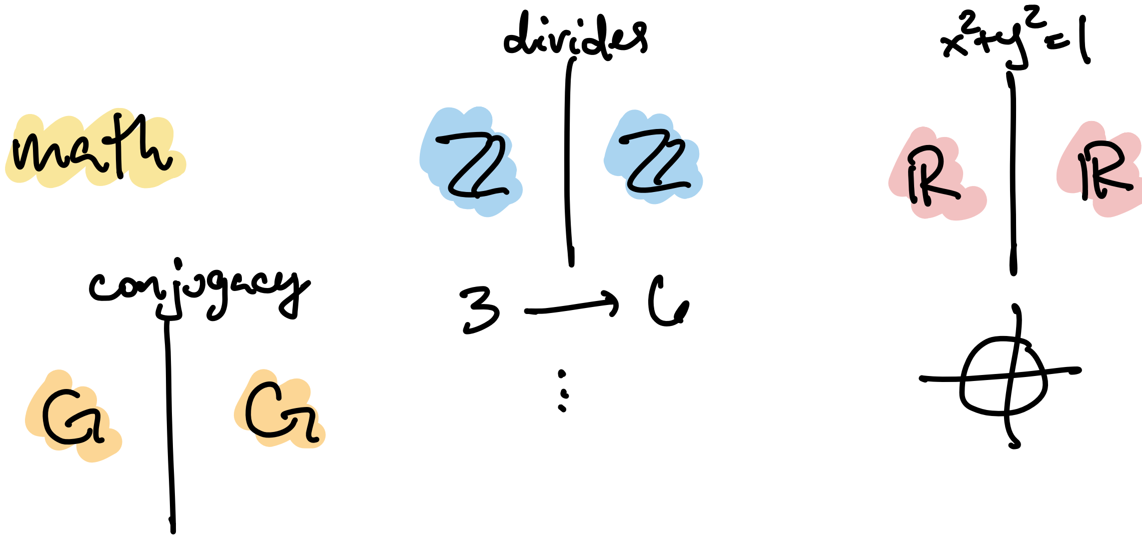




and compatibility implies that parallel beads can slide past each other



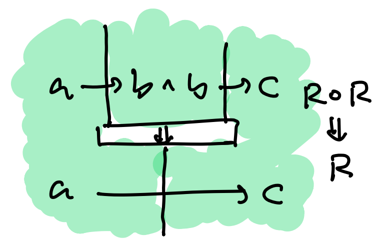
examples



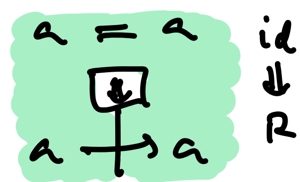
$g C g' = \exists h \in G.\ h g h^{-1} = g'$

How to draw

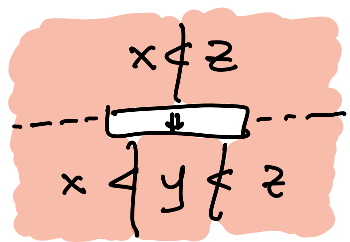
- transitive



- reflexive

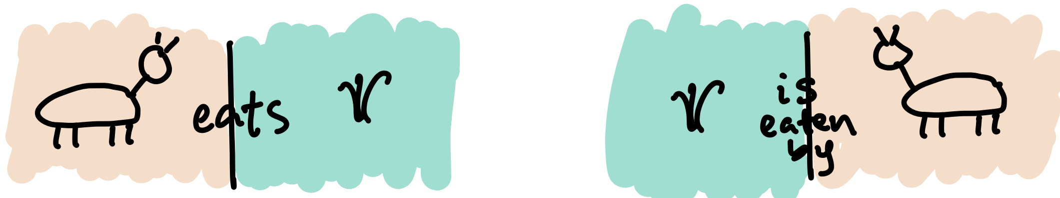


- dense



duality

Every relation has a <u>converse</u>



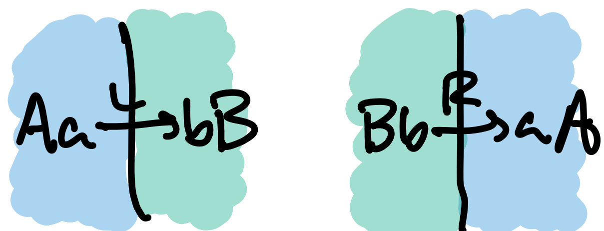
which is not necessarily an <u>inverse</u>.

However, in 2 dimensions a pair of strings can be

$A \rightleftarrows B$

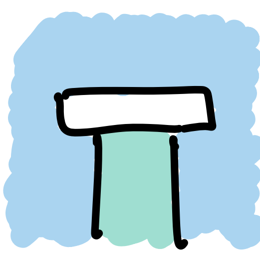
/different/ dual equivalent

An adjunction is a pair



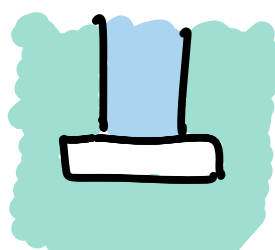
left adjoint right adjoint

with a unit



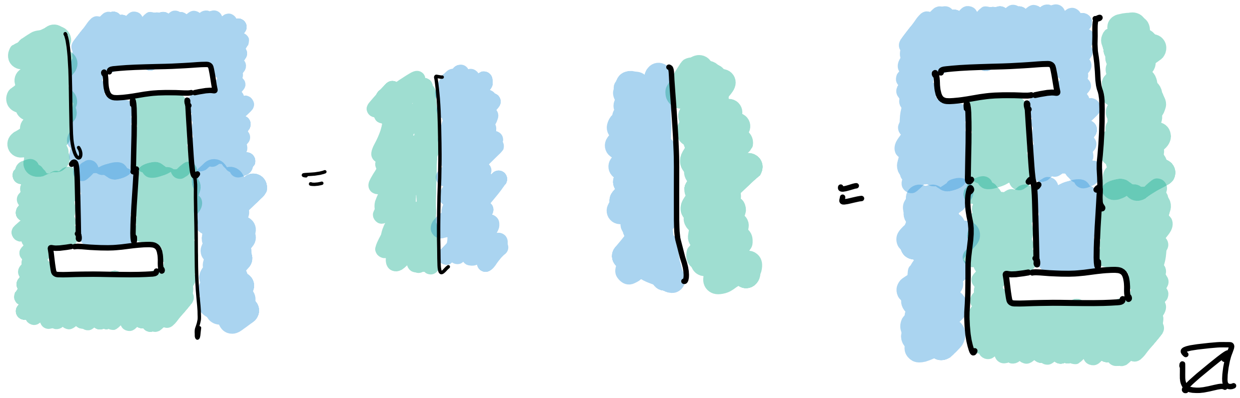
$$\frac{Aa \vdash a'A}{Aa \xrightarrow{L \circ R} a'A}\ (\eta)$$

+ counit



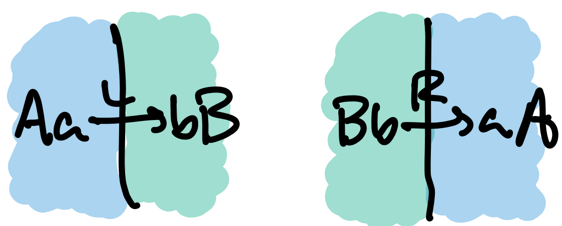
$$\frac{Bb \xrightarrow{R \circ L} b'B}{Bb \vdash b'B}\ (\varepsilon)$$

which cancel along each string:



so each string can "bend + unbend"
— this basic geometry goes a long way.

* what does it mean in Rel?



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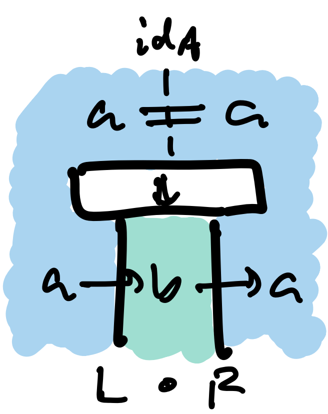
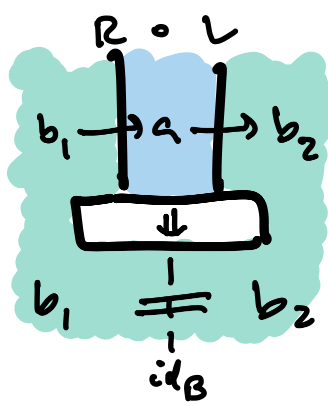


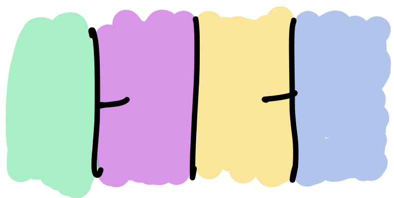
total
"∀ input ∃ output"



deterministic
"output is unique"

so, L is a function! + R is a cofunction.

note:

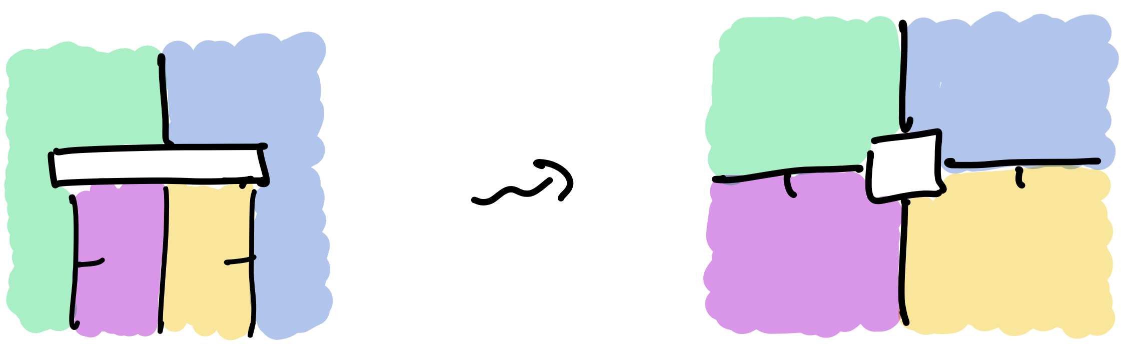


substitution

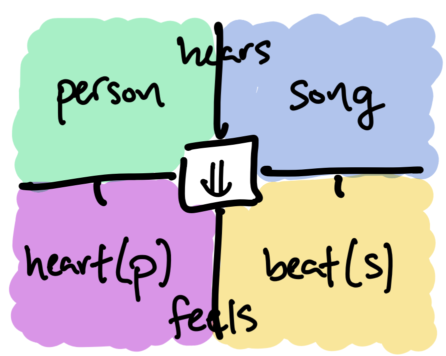
$$= \quad a \xrightarrow{f \circ R \circ g^{-1}} b \;=\; f(a) \xrightarrow{R} g(b)$$

in reality, we want to think about
both functions + relations

so they each deserve
* their own dimension *



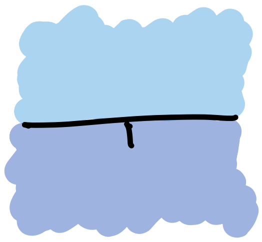
these inferences
are more natural.
for example,



A **double category** is...

like a 2-category, plus

- **vertical morphisms**

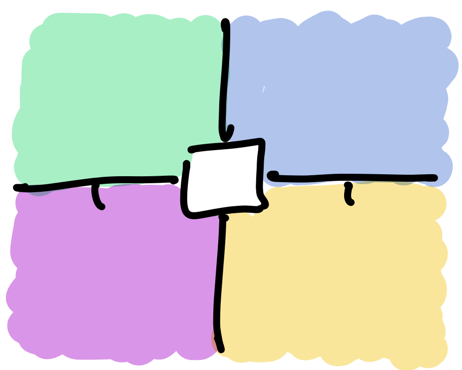


function

term

$$\frac{A\,a \vdash a'\,A}{B\,fa \vdash fa'\,B}\;(f)$$

- vertical composition
- 2-morphisms are squares
  with (horizontal) source/target and vertical s/t.



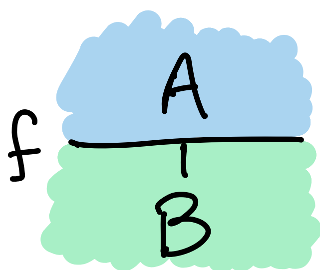
$$\frac{A\,a \xrightarrow{R} b\,B}{X\,f(a) \xrightarrow{S} g(b)\,Y}\;(r)$$

- (composition along both)

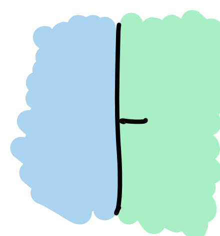
now to formalize that functions can bend into relations :

A fibrant double category ("equipment")
is a double category with:

- for each vertical morphism (term)
  a pair of horizontal morphisms (judgements)

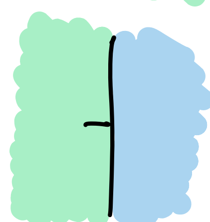


$\rightsquigarrow$



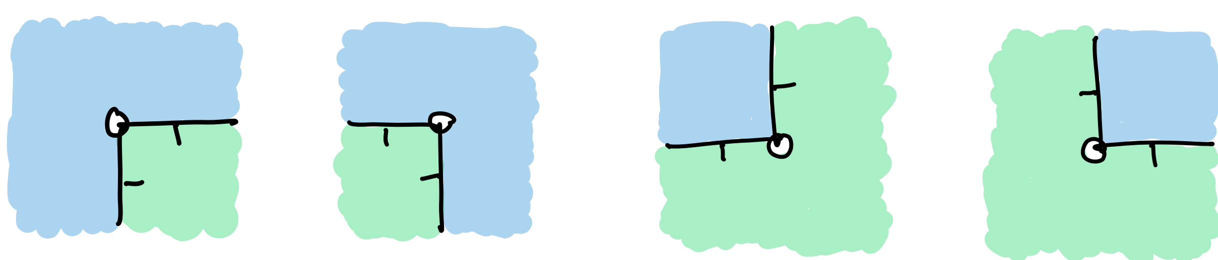
A f(a) ⊢ b B

(the first is "left adjoint" to the second)

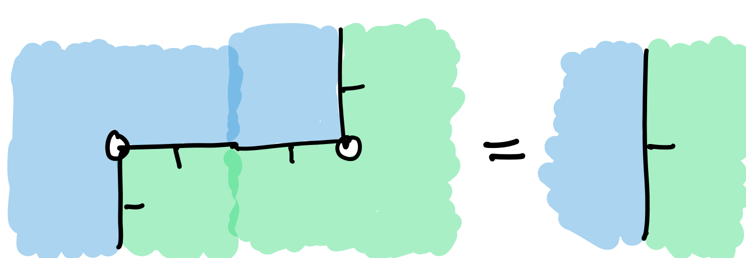


B b ⊢ f(a) A

- equipped with 2-morphisms (what do these inferences mean?)



- such that



etc.

Puzzles

* what can be expressed so far?
  try some favorite concepts/theorems.
* what more structure do we need?

[Rel has all higher-order logic,
when its structure is expounded.
in two lessons, we'll explore quantifiers.]

[* clearly, there's not enough time
to explore everything in depth.
if you're interested, just email me
at cwill041@ucr.edu]

Thanks!