Proof Theory of Partially Normal Skew Monoidal Categories

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Skew monoidal categories

- A *skew monoidal category* (Szlachányi'12) is a category $\mathbb{C}$ together with an object I, a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and nat. transfs. λ, ρ, α with components

$$\begin{aligned} \lambda_A &: \mathsf{I} \otimes A \to A \\ \rho_A &: A \to A \otimes \mathsf{I} \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \end{aligned}$$

such that

(m1)
$$\begin{array}{ccc} & \mathsf{I} \otimes \mathsf{I} & \\ {\scriptstyle \rho_\mathsf{I}} \nearrow & & \searrow {\scriptstyle \lambda_\mathsf{I}} \\ \mathsf{I} & = & \mathsf{I} \end{array}$$

(m2)
$$\begin{array}{ccc} (A \otimes \mathsf{I}) \otimes B & \xrightarrow{\alpha_{A,\mathsf{I},B}} & A \otimes (\mathsf{I} \otimes B) \\ {\scriptstyle \rho_A \otimes B} \uparrow & & \downarrow {\scriptstyle A \otimes \lambda_B} \\ A \otimes B & = & A \otimes B \end{array}$$

(m3)
$$\begin{array}{ccc} (\mathsf{I} \otimes A) \otimes B & \xrightarrow{\alpha_{\mathsf{I},A,B}} & \mathsf{I} \otimes (A \otimes B) \\ {\scriptstyle \lambda_A \otimes B} \searrow & & \swarrow {\scriptstyle \lambda_{A \otimes B}} \\ & A \otimes B & \end{array}$$

(m4)
$$\begin{array}{ccc} (A \otimes B) \otimes \mathsf{I} & \xrightarrow{\alpha_{A,B,\mathsf{I}}} & A \otimes (B \otimes \mathsf{I}) \\ {\scriptstyle \rho_{A \otimes B}} \nwarrow & & \nearrow {\scriptstyle A \otimes \rho_B} \\ & A \otimes B & \end{array}$$

(m5)
$$\begin{array}{ccccc} (A \otimes (B \otimes C)) \otimes D & & \xrightarrow{\alpha_{A,B \otimes C,D}} & & A \otimes ((B \otimes C) \otimes D) \\ {\scriptstyle \alpha_{A,B,C} \otimes D} \uparrow & & & & \downarrow {\scriptstyle A \otimes \alpha_{B,C,D}} \\ ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A \otimes B,C,D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D)) \end{array}$$

Normality conditions, examples

- A skew monoidal category is
  - *left-normal* if λ is invertible,
  - *right-normal* if ρ is invertible,
  - *associative-normal* if α is invertible.
- A monoidal category is a skew monoidal category satisfying all 3 normality conditions.

- Skew monoidal categories, possibly with different degree of normality, appear in the study of relative monads (Altenkirch et al.'15) and quantum categories (Street & Lack'12).
- (Altenkirch et al.'15) Given categories $\mathbb{J}$ and $\mathbb{C}$ and functor $J : \mathbb{J} \to \mathbb{C}$. The functor category $[\mathbb{J}, \mathbb{C}]$ has a skew monoidal structure:

$$\mathsf{I} = J, \qquad F \otimes G = \mathrm{Lan}_J F \cdot G$$

Our contributions

- In previous work (MFPS'18), we presented a sequent calculus for skew monoidal categories.
- The sequent calculus is a presentation of the free skew monoidal category on a set of generating objects.
- It enjoys cut elimination and admits a focused subsystem of canonical derivations.

- In this work, we develop sequent calculi for *partially normal* skew monoidal categories.
- We prove cut elimination and we show that the calculi admit focusing.
- The result is a family of sequent calculi between those of skew monoidal categories and (fully normal) monoidal categories.
- These define 8 weakenings of the $(\mathsf{I}, \otimes)$ fragment of intuitionistic non-commutative linear logic.

The sequent calculus of skew monoidal categories
(MFPS'18)

Skew monoidal sequent calculus

- *Formulae* over a set At of atoms: $A, B ::= X \mid \mathsf{I} \mid A \otimes B$
- *Sequents* are triples $S \mid \Gamma \longrightarrow C$ where
  - S (stoup) is an optional formula,
  - Γ (context) is a list of formulae,
  - C is a single formula.
- *Derivations* are constructed with these inference rules:

$$\frac{A \mid \Gamma \longrightarrow C}{- \mid A, \Gamma \longrightarrow C}\ \mathsf{pass} \qquad \frac{}{A \mid \ \longrightarrow A}\ \mathsf{ax}$$

$$\frac{- \mid \Gamma \longrightarrow C}{\mathsf{I} \mid \Gamma \longrightarrow C}\ \mathsf{IL} \qquad \frac{}{- \mid \ \longrightarrow \mathsf{I}}\ \mathsf{IR}$$

$$\frac{A \mid B, \Gamma \longrightarrow C}{A \otimes B \mid \Gamma \longrightarrow C}\ \otimes\mathsf{L} \qquad \frac{S \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{S \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}$$

- IL, $\otimes$L only apply to the formula of the stoup, if it is not empty. $\otimes$R sends the stoup formula, if present, to the 1st premise.

What makes this work: unitors

- A derivation corresponding to the unitor λ:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \quad \longrightarrow A}\ \text{ax}}{- \mid A \longrightarrow A}\ \text{pass}}{\mathsf{I} \mid A \longrightarrow A}\ \text{IL}}{\mathsf{I} \otimes A \mid \quad \longrightarrow A}\ \otimes\text{L}$$

- There is no derivation corresponding to λ^{-1}:

$$\dfrac{\overset{??}{X \mid \quad \longrightarrow \mathsf{I}} \qquad \overset{??}{- \mid \quad \longrightarrow X}}{X \mid \quad \longrightarrow \mathsf{I} \otimes X}\ \otimes\text{R}$$

- A derivation corresponding to the unitor ρ:

$$\dfrac{\dfrac{}{A \mid \quad \longrightarrow A}\ \text{ax} \qquad \dfrac{}{- \mid \quad \longrightarrow \mathsf{I}}\ \text{IR}}{A \mid \quad \longrightarrow A \otimes \mathsf{I}}\ \otimes\text{R}$$

- There is no derivation corresponding to ρ^{-1}:

$$\dfrac{\overset{??}{X \mid \mathsf{I} \longrightarrow X}}{X \otimes \mathsf{I} \mid \longrightarrow X}\ \otimes\text{L}$$

What makes this work: associator

- A derivation corresponding to the associator α:

$$\dfrac{\dfrac{\dfrac{}{A \mid\ \longrightarrow A}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{B \mid\ \longrightarrow B}\,\text{ax} \qquad \dfrac{\dfrac{}{C \mid\ \longrightarrow C}\,\text{ax}}{- \mid C \longrightarrow C}\,\text{pass}}{B \mid C \longrightarrow B \otimes C}\,\otimes\text{R}}{- \mid B, C \longrightarrow B \otimes C}\,\text{pass}}{\dfrac{A \mid B, C \longrightarrow A \otimes (B \otimes C)}{A \otimes B \mid C \longrightarrow A \otimes (B \otimes C)}\,\otimes\text{L}}\,\otimes\text{R}}{(A \otimes B) \otimes C \mid\ \longrightarrow A \otimes (B \otimes C)}\,\otimes\text{L}$$

- There is no derivation corresponding to α^{-1}:

$$\dfrac{\dfrac{\overset{??}{X \mid Y \otimes Z \longrightarrow X \otimes Y} \qquad \overset{??}{- \mid\ \longrightarrow Z}}{X \mid Y \otimes Z \longrightarrow (X \otimes Y) \otimes Z}\,\otimes\text{R}}{X \otimes (Y \otimes Z) \mid\ \longrightarrow (X \otimes Y) \otimes Z}\,\otimes\text{L} \qquad \dfrac{\dfrac{\overset{??}{X \mid\ \longrightarrow X \otimes Y} \qquad \overset{??}{- \mid Y \otimes Z \longrightarrow Z}}{X \mid Y \otimes Z \longrightarrow (X \otimes Y) \otimes Z}\,\otimes\text{R}}{X \otimes (Y \otimes Z) \mid\ \longrightarrow (X \otimes Y) \otimes Z}\,\otimes\text{L}$$

Equivalence of derivations

(η-conversions)

$$\dfrac{}{\mathsf{I} \mid \ \longrightarrow \mathsf{I}}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{}{- \mid \mathsf{I}}\ \mathsf{IR}}{\mathsf{I} \mid \ \longrightarrow \mathsf{I}}\ \mathsf{IL}$$

$$\dfrac{}{A \otimes B \mid \ \longrightarrow A \otimes B}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\ \mathsf{ax} \qquad \dfrac{\dfrac{}{B \mid \ \longrightarrow B}\ \mathsf{ax}}{- \mid B \longrightarrow B}\ \mathsf{pass}}{A \mid B \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{A \otimes B \mid \ \longrightarrow A \otimes B}\ \otimes\mathsf{L}$$

(commutative conversions)

$$\dfrac{\dfrac{A' \mid \Gamma \longrightarrow A}{- \mid A', \Gamma \longrightarrow A}\ \mathsf{pass} \qquad - \mid \Delta \longrightarrow B}{- \mid A', \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{A' \mid \Gamma \longrightarrow A \qquad - \mid \Delta \longrightarrow B}{A' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{- \mid A', \Gamma, \Delta \longrightarrow A \otimes B}\ \mathsf{pass}$$

$$\dfrac{\dfrac{- \mid \Gamma \longrightarrow A}{\mathsf{I} \mid \Gamma \longrightarrow A}\ \mathsf{IL} \qquad - \mid \Delta \longrightarrow B}{\mathsf{I} \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{- \mid \Gamma \longrightarrow A \qquad - \mid \Delta \longrightarrow B}{- \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{\mathsf{I} \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \mathsf{IL}$$

$$\dfrac{\dfrac{A' \mid B', \Gamma \longrightarrow A}{A' \otimes B' \mid \Gamma \longrightarrow A}\ \otimes\mathsf{L} \qquad - \mid \Delta \longrightarrow B}{A' \otimes B' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{A' \mid B', \Gamma \longrightarrow A \qquad - \mid \Delta \longrightarrow B}{A' \mid B', \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{R}}{A' \otimes B' \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\mathsf{L}$$

Results

- **Theorem:** Two forms of *cut* are admissible:

$$\frac{S \mid \Gamma \longrightarrow A \quad A \mid \Delta \longrightarrow C}{S \mid \Gamma, \Delta \longrightarrow C}\ \text{scut} \qquad \frac{- \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, \Gamma, \Delta_1 \longrightarrow C}\ \text{ccut}$$

- **Theorem:** The sequent calculus, with derivations quotiented by the equivalence relation $\doteq$, is a presentation of the *free* skew monoidal category on At.

- **Theorem:** We have an equivalent *focused* sequent calculus, in which derivations are canonical representative of $\doteq$-equiv. classes. There is a bijection between
  - derivations of $S \mid \Gamma \longrightarrow_{\mathsf{L}} C$ in the focused sequent calculus
  - derivations of $S \mid \Gamma \longrightarrow C$ in the sequent calculus (up to $\doteq$)

The sequent calculus of partially normal skew monoidal categories
(ACT'20)

Left-normal sequent calculus

- Skew monoidal sequent calculus with an extra rule:

$$\frac{- \mid \ \longrightarrow A \quad A' \mid \Delta \longrightarrow B}{A' \mid \Delta \longrightarrow A \otimes B} \otimes \mathsf{R}_2$$

  + some generating equations in $\doteq$.
- A derivation corresponding to λ^{-1}:

$$\frac{\dfrac{}{- \mid \ \longrightarrow \mathsf{I}}\ \mathsf{IR} \quad \dfrac{}{A \mid \ \longrightarrow A}\ \mathsf{ax}}{A \mid \ \longrightarrow \mathsf{I} \otimes A} \otimes \mathsf{R}_2$$

- The rule pass is invertible up to $\doteq$:

$$\frac{- \mid A, \Gamma \longrightarrow C}{A \mid \Gamma \longrightarrow C} \ \mathsf{act}$$

- This implies that the left-normal sequent calculus admits an equivalent stoup-free presentation.

Right-normal sequent calculus

- Skew monoidal sequent calculus with 2 extra rules:

$$\frac{S \mid \Gamma_0, \Gamma_1 \longrightarrow C}{S \mid \Gamma_0, \mathsf{I}, \Gamma_1 \longrightarrow C}\ \mathsf{IC} \qquad \frac{S \mid \Gamma_0, J, J', \Gamma_1 \longrightarrow C}{S \mid \Gamma_0, J \otimes J', \Gamma_1 \longrightarrow C}\ \otimes\mathsf{C}^{\mathrm{c}}$$

  + many generating equations in $\stackrel{\circ}{=}$.
- J and J' stand for closed formulae, i.e. made of I and $\otimes$ only.
- A derivation corresponding to ρ^{-1}:

$$\dfrac{\dfrac{\dfrac{}{A \mid \ \longrightarrow A}\ \mathsf{ax}}{A \mid \mathsf{I} \longrightarrow A}\ \mathsf{IC}}{A \otimes \mathsf{I} \mid \ \longrightarrow A}\ \otimes\mathsf{L}$$

- The rule $\otimes\mathsf{C}^{\mathrm{c}}$ is needed, since it is important to allow deletion in the context of any closed formula, not just I.

$$X \otimes (\mathsf{I} \otimes \mathsf{I}) \xrightarrow{\mathrm{id} \otimes \lambda} X \otimes \mathsf{I} \xrightarrow{\rho^{-1}} X \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid \ \longrightarrow X}\ \mathsf{ax}}{X \mid \mathsf{I} \longrightarrow X}\ \mathsf{IC}}{X \mid \mathsf{I}, \mathsf{I} \longrightarrow X}\ \mathsf{IC}}{X \mid \mathsf{I} \otimes \mathsf{I} \longrightarrow X}\ \otimes\mathsf{C}^{\mathrm{c}}}{X \otimes (\mathsf{I} \otimes \mathsf{I}) \mid \ \longrightarrow X}\ \otimes\mathsf{L}$$

Associative-normal sequent calculus

- Skew monoidal sequent calculus with an extra rule:

$$\frac{S \mid \Gamma_0, A, B, \Gamma_1 \longrightarrow C}{S \mid \Gamma_0, A \otimes B, \Gamma_1 \longrightarrow C}\ \otimes\mathsf{C}$$

  + many generating equations in $\circeq$.

- A derivation corresponding to α^{-1}:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A \mid \ \longrightarrow A}\ \mathsf{ax} \qquad \cfrac{\cfrac{}{B \mid \ \longrightarrow B}\ \mathsf{ax}}{- \mid B \longrightarrow B}\ \mathsf{pass}}{A \mid B \longrightarrow A \otimes B}\ \otimes\mathsf{R} \qquad \cfrac{\cfrac{}{C \mid \ \longrightarrow C}\ \mathsf{ax}}{- \mid C \longrightarrow C}\ \mathsf{pass}}{A \mid B, C \longrightarrow (A \otimes B) \otimes C}\ \otimes\mathsf{R}}{A \mid B \otimes C \longrightarrow (A \otimes B) \otimes C}\ \otimes\mathsf{C}}{A \otimes (B \otimes C) \mid \ \longrightarrow (A \otimes B) \otimes C}\ \otimes\mathsf{L}$$

Results

- **Theorem:** For all these sequent calculi, cut is admissible. Moreover, each calculus admits focusing (inspired by Chauduri & Pfenning'05 in the right-normal case and in assoc.-normal case).

- **Theorem:** Each partially normal sequent calculus, with derivations quotiented by its equivalence relation $\stackrel{\circ}{=}$, is a presentation of the *free* partially normal (with the same degree of partiality) skew monoidal category on At.

Formalization, future work

- Full formalization in the Agda proof assistant:

  `https://github.com/niccoloveltri/skewmoncats-normal`

- We plan to extend our story to:
  - Skew closed categories (Street'13):

    $j_A : \mathsf{I} \to A \multimap A \qquad i_A : \mathsf{I} \multimap A \to A \qquad L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C)$

  - Skew monoidal closed categories ($\otimes$ and $\multimap$)
  - Skew closed prounital categories (no I, but maps may have no source)
  - Braided/Symmetric skew monoidal categories (Bourke & Lack'20)

$$s_{A,B} : (A \otimes B) \otimes C \to (A \otimes C) \otimes B \qquad \frac{S \mid \Gamma, B, A, \Delta \longrightarrow C}{S \mid \Gamma, A, B, \Delta \longrightarrow C}\ \text{exch}$$

Extra: Free skew monoidal category as deductive system

- The free skew monoidal category over a set At can be viewed as a deductive system (following Lambek's tradition).
  We call it the *categorical calculus*.
- Objects are formulae over At: $A, B ::= X \mid \mathsf{I} \mid A \otimes B$
- Maps are equivalence classes of derivations of sequents $A \Longrightarrow C$ where both A, C are single formulae.
- *Derivations* are constructed with these inference rules:

$$\frac{}{A \Longrightarrow A}\ \mathsf{id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}\ \mathsf{comp}$$

$$\frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D}\ \otimes$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A}\ \lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)}\ \alpha$$

Extra: Free skew monoidal category ctd.

- *Equivalence of derivations* is the congruence $\doteq$ induced by the equations

(category laws)	$\mathrm{id} \circ f \doteq f$	$f \doteq f \circ \mathrm{id}$	$(f \circ g) \circ h \doteq f \circ (g \circ h)$
($\otimes$ functorial)	$\mathrm{id} \otimes \mathrm{id} \doteq \mathrm{id}$	$(h \circ f) \otimes (k \circ g) \doteq h \otimes k \circ f \otimes g$	

(λ, ρ, α nat. trans.)

$$\lambda \circ \mathrm{id} \otimes f \doteq f \circ \lambda$$
$$\rho \circ f \doteq f \otimes \mathrm{id} \circ \rho$$
$$\alpha \circ (f \otimes g) \otimes h \doteq f \otimes (g \otimes h) \circ \alpha$$

(m1-m5)

$$\lambda \circ \rho \doteq \mathrm{id} \qquad \mathrm{id} \doteq \mathrm{id} \otimes \lambda \circ \alpha \circ \rho \otimes \mathrm{id}$$
$$\lambda \circ \alpha \doteq \lambda \otimes \mathrm{id} \qquad \alpha \circ \rho \doteq \mathrm{id} \otimes \rho$$
$$\alpha \circ \alpha \doteq \mathrm{id} \otimes \alpha \circ \alpha \circ \alpha \otimes \mathrm{id}$$

Extra: Skew seq. calc. vs. free skew mon. cat.

- Define

$$\llbracket - \langle\!\langle = \mathsf{I} \qquad\qquad \llbracket A \langle\!\langle = A$$
$$A \langle\!\langle A_1, A_2 \ldots, A_n \rrbracket = (\ldots(A \otimes A_1) \otimes A_2) \ldots) \otimes A_n$$

- **Soundness:** For all $f : S \mid \Gamma \longrightarrow C$, we can define $\mathsf{sound}(f) : \llbracket S \langle\!\langle \, \langle\!\langle \Gamma \rrbracket \Longrightarrow C$.
- **Completeness:** For all $f : \llbracket S \langle\!\langle \, \langle\!\langle \Gamma \rrbracket \Longrightarrow C$, we can define $\mathsf{cmplt}(f) : S \mid \Gamma \longrightarrow C$.
(The scut rule interprets composition in the categorical calculus.)
- Also equational soundness and completeness.
- There is a bijection between
    - derivations of $\llbracket S \langle\!\langle \, \langle\!\langle \Gamma \rrbracket \Longrightarrow C$ in the categorical calculus (up to $\doteq$).
    - derivations of $S \mid \Gamma \longrightarrow_{\mathsf{L}} C$ in the focused sequent calculus
    - derivations of $S \mid \Gamma \longrightarrow C$ in the sequent calculus (up to $\stackrel{\circ}{=}$)