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The dialectica monad and its cousins

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Abstract. I give an expositional account of the dialectica construction and some related constructions from the point of view of quantification in fibrations. This allows for concise conceptual formulations of these constructions and explains their universal properties. There are two main points I wish to convey; the first is that the categorical dialectica construction decomposes into two steps, following the quantifier pattern of the original translation; the second is that the categorical embodiment of Skolemization takes the form of a pseudo-distributive law between the pseudo-monads which freely add universal and existential quantification.

1. Introduction

1.1. Background. The original purpose of Gödel's famous dialectica interpretation [11] was to reduce the problem of proving consistency of first-order arithmetic to the consistency of a simply typed system of computable functionals (called *System T*). The key feature of the translation is that it turns formulae of arbitrary quantifier complexity into formulae of the form $\exists\vec{f}\forall\vec{x}.\alpha(\vec{f},\vec{x})$, where α is quantifier-free, by using the principle of *Skolemization*:

$$\frac{\forall u\exists x.\alpha(u,x)}{\exists f\forall u.\alpha(u,fu)}\ \mathrm{Sk}$$

This principle is a form of choice: it allows us to replace a quantifier combination of the form $\forall\exists$ by one of the form $\exists\forall$ by introducing a choice function f. Because this choice function is a higher-type object, the resulting "Skolemized" formula is no longer first order.

Over the years, several authors have tried to capture the dialectica interpretation in categorical terms. The development which is the main focus of this paper is abstract in nature and aims at defining and analysing "dialectica-like" structures without direct reference to arithmetic. The starting point of this line of research is the PhD work of Valeria de Paiva [7], in which the so-called *dialectica categories* were introduced. Let me briefly sketch the construction. One starts with a category C with finite limits and builds a new category $\mathfrak{Dial}(\mathsf{C})$, the objects of which have the form (X,U,α) where α is a subobject of $X\times U$ in C; such an object is thought

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of as the formula $\exists x \forall u.\alpha(x,u)$. A morphism from (X,U,α) to (Y,V,β) consists of a pair (f,f_0), where $f : X \longrightarrow Y$ and $f_0 : X \times V \longrightarrow U$, subject to the condition[1] that $\alpha(x, f_0(x,v)) \vdash \beta(fx,v)$. This definition of morphism, as explained in loc. cit., is motivated by the way the dialectica interpretation acts on implicational formulae.

The main focus in the original work on the dialectica categories was on the categorical structure of $\mathfrak{Dial}(\mathsf{C})$: one can show that these always have a monoidal structure, and that under additional assumptions on C they form a model of intuitionistic linear logic. De Paiva [8] also considers a variant called the *Girard construction*, and describes how the dialectica category $\mathfrak{Dial}(\mathsf{C})$ can be presented as the result of first applying the Girard construction to C, and then forming the coKleisli category for a comonad on this category.

The original construction of the dialectica categories was formulated in terms of the subobject logic of C. However, this can be generalized quite a bit: the case of fibred preorders was explored in detail by Martin Hyland [13], and Bodil Biering's PhD thesis [5] investigates dialectica categories of the form $\mathfrak{Dial}(\mathsf{p})$, where p is an arbitrary cloven fibration.

These constructions raise various questions. What is the precise categorical nature of the dialectica categories? Can they be described in more conceptual terms, for example in terms of universal properties? And in what way does the construction of these categories capture the essential ingredient of Gödel's original translation, namely the principle of Skolemization?

1.2. Goals of the paper. The main aim of this paper is to give an exposition and interpretation of the dialectica construction from a modern categorical perspective, which gives clear and precise answers to the above questions. The dialectica interpretation has somewhat of a reputation of being complicated; however, by giving centre stage to the well-known concepts of monads, simple products and -coproducts I hope that what emerges is a conceptually pleasing and accessible picture.

The second goal of the paper is to provide explicit statements and reasonably detailed proofs of facts which are perhaps folklore among some experts in the field but which have, to the best of my knowledge, never been stated or worked out in the literature. The first statement of this kind is that the dialectica construction is really a two-step construction: $\mathfrak{Dial}(\mathsf{p})$ (where p is a fibration) is obtained by first applying the monad which freely adds simple universal quantification and then applying the monad which freely adds simple existential quantification. The second fact is that the dialectica construction (when performed on the level of general fibrations and under the usual assumption that the base category is cartesian closed) is itself monadic. The main observation in this general categorical setting is that the principle of Skolemization takes the form of a (pseudo-)distributive law between the abovementioned quantification monads.

The paper has been written with a readership familiar with basic category theory in mind. In particular, I assume that the reader is comfortable with adjunctions, monads, categories of algebras for a monad and distributive laws. A standard reference to these topics is the textbook [3]. Since fibrations are omnipresent in the paper I have included a background section on the basic theory of fibred categories, including all the results we need for the rest of the work.

[1] I use logical notation here; this will be made precise in Section 2.3.

At various places we shall be forced to deal with "up-to-isomorphism" versions of monads and related concepts, i.e. with pseudo-monads, pseudo-algebras and pseudo-distributive laws. Because this is not the right place to learn about this material (and because those with an interest in the matter will likely already know the basic theory) I have not included any review material of these concepts; a concise treatment containing all definitions which are used in this paper can be found in [6] (see also the references therein); we also mention [10], which is a survey of the theory of pseudo-distributive laws.

I should stress however that intimate knowledge of these higher-categorical notions is not necessary in order to follow the paper. First of all, I have made an attempt to organize the material in such a way that it becomes clear why and where certain coherence issues arise and why some of the structures under consideration are not strict. The reader who is interested in the general picture but not in the 2-categorical subtleties can simply pretend that the structure is strict and still follow the main developments. Moreover, for most results I have tried to describe the key ideas for the proof without going into too much distracting detail; and I have delegated more complete proofs involving coherence issues and such to separate sections which can be skipped without losing track of the main story.

I should also point out some things which I have not aimed for in this paper. First of all, the main focus in several papers on the dialectica construction and its variants was on the categorical structure of the dialectica categories and the way in which they model linear logic. While this is clearly important and interesting, it is not within the scope of the current exposition, which focuses solely on the construction itself. Similarly, I will not address variants such as the Diller-Nahm interpretation.

Second, I have not attempted to say much on the history of the subject, nor to be complete in the references. Because of the expositional nature of this paper there appear various results which are either standard, a variation on a standard result, or and explicitation of a result which is implicit in the literature. Whilst trying to avoid cluttering the exposition with too many literature references, I have tried to give credit where that is due, and I hope that I will be forgiven for possible omissions.

1.3. Overview. The paper is structured in the following manner. We begin in Section 2 with a quick review of the theory of fibrations. Readers who are already familiar with fibrations can safely skip this section.

Section 3 deals with notions of quantification in fibrations. We first explain in some detail how quantification is handled on the fibrational level and introduce the main kind of quantification we shall study in the rest of the paper, namely simple (co)products. This is all standard material. Next, in Section 3.2, we consider constructions which freely add quantification to a fibration, starting with the easiest such, namely the families construction. This allows us to reformulate these matters in monadic terms and sets us up for a better understanding of subsequent matters. After that (Section 3.3) we investigate one of the main constructions of interest, namely adding simple existential quantification. Because this is one of the ingredients for the dialectica construction we spell out the details of this construction, and show that it naturally comes equipped with all the structure needed to make it a pseudo-monad; a formal proof is given in the Appendix. The dual construction for adding simple universal quantification is briefly described in Section 3.4; this is the second main ingredient for the dialectica construction. Finally, Section 3.5 describes a variation giving rise to a decomposition of the constructions described above into a linear part followed by a coKleisli construction.

Section 4 is the heart of the matter: it first introduces the dialectica categories, and describes them in terms of the monads obtained in the previous section. Then we introduce a (pseudo-)distributive law which explains why the dialectica construction can be equipped with the structure of a pseudo-monad (Section 4.2), and characterize the algebras for the dialectica monad. The proof of this central fact is given in Section 4.3.

We end (Section 5) with a brief discussion of several aspects which could not be included in this paper and with some suggestions for further exploration.

1.4. Acknowledgments. As often happens, the present paper is really the consequence of the author trying to understand someone else's work. In this case, it was Bodil Biering's PhD thesis [5] which led me to formulate matters in the way presented here, and to explore various sidelines and related issues. I thank Phil Scott and Michael Warren for their stimulating comments, remarks and questions; I've benefited greatly from talking to various people, including Bodil Biering, Lars Birkedal, Richard Garner, Claudio Hermida, Martin Hyland, Pino Rosolini and Alex Simpson. To Valeria de Paiva I'm grateful for urging me to put my thoughts on paper. Finally, I am greatly indebted to the anonymous referees whose thoughtful and detailed reports have been very helpful in improving the focus and presentation of the material.

2. Background

In this section I guide the reader through the basic theory of fibred categories. I should stress that the aim here is not to provide a comprehensive account of the theory, but to introduce just enough to keep this paper self-contained and to give the reader sufficient insight into the material to be able to follow the rest of the paper. A more thorough introductory text to the subject is [24]; for a lively programmatic discussion, see [4]; finally, there is the textbook [16].

2.1. Basic theory and examples. Fibred category theory is a convenient setting for modeling mathematical situations in which one collection of objects depends on, or is *indexed*, or *fibred* over another. We begin by giving a concrete and elementary definition, and then turn to some instructive examples. Along the way we introduce notation and terminology.

Definition 2.1. Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a functor.

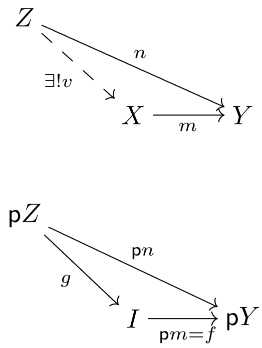
(i) A morphism v in E is *vertical* if $\mathsf{p}(v) = 1$

(ii) For an object I in B, the *fibre of* p *over* I is the subcategory of E consisting of those objects X for which $\mathsf{p}X = I$ and with vertical morphisms between them. We denote this category by $\mathsf{p}^{-1}(I)$, or by E_I when p is understood.

If $\mathsf{p}X = I$ we say that X is an object over I. Similarly, if $\mathsf{p}v = f$, we call v a morphism over f.

Definition 2.2 (Cartesian morphism; fibration).

(i) Let $m : X \longrightarrow Y$ be a morphism over $\mathsf{p}m = f$. We say that m is *cartesian* (over f) if for every other map $n : Z \longrightarrow Y$ and every factorization $\mathsf{p}(n) = fg$ there exists a unique morphism v over g for which $mv = n$. Diagrammatically:



In this case, we call m a *cartesian lifting* of f

(ii) The functor $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is a *fibration* if every map $f : I \longrightarrow \mathsf{p}Y$ has a cartesian lifting with codomain Y. We will usually refer to B as the *base category* and to E as the *total category* of p.

A more abstract way of saying this is: $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is a fibration precisely when each induced functor $\mathsf{p}/Y : \mathsf{E}/Y \longrightarrow \mathsf{B}/\mathsf{p}Y$ has a right adjoint right inverse.

The following properties are readily verified from the definition and will frequently be used:

Lemma 2.3. *Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration.*

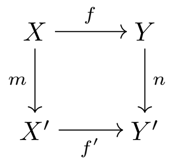
(i) *The composite of two cartesian maps is again cartesian..*
(ii) *If two maps m, m' are both cartesian over f, then there is a unique vertical isomorphism v for which $mv = m'$.*
(iii) *Every morphism in E factors as a vertical map followed by a cartesian map and this factorization is unique up to unique vertical isomorphism.*
(iv) *When two vertical maps v, v' in E satisfy $mv = mv'$ for some cartesian map m, then $v = v'$.*
(v) *Any map in E which is both vertical and cartesian is an isomorphism.*

We now present a couple of standard instructive examples; examples pertaining specifically to logic will be discussed below in Section 2.3.

Examples 2.4.

(1) For any pair B, C of categories consider the projection $\mathsf{B} \times \mathsf{C} \longrightarrow \mathsf{B}$. This is a fibration; maps in $\mathsf{B} \times \mathsf{C}$ of the form $(1, g)$ are vertical, while those of the form $(f, 1)$ are cartesian. Fibrations of this form are sometimes called *constant*, because all fibres are isomorphic (to C).
(2) A fibration p is called *discrete* when the fibre categories are discrete categories, i.e. sets. For such p, the assignment $I \mapsto \mathsf{p}^{-1}(I)$ is the object part of a presheaf on B. The Grothendieck construction (see Section 2.2 below) provides an inverse to this construction, so that discrete fibrations are essentially the same thing as presheaves.

(3) For any category B, let $\mathsf{B}^{\longrightarrow}$ be the arrow category whose objects are maps $f : X \longrightarrow Y$ and whose morphisms are commutative squares



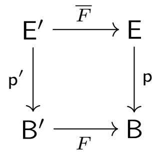
The codomain functor $cod : \mathsf{B}^{\longrightarrow} \longrightarrow \mathsf{B}$ is a fibration (called the *codomain fibration* on B) precisely when B has pullbacks; the cartesian maps are the pullback squares. Later we shall consider subfibrations of the codomain fibration.

The following elementary facts are easily verified.

Lemma 2.5.

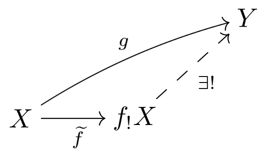
(i) *Fibrations are closed under composition, i.e. when* p *and* q *are fibrations, then so is* pq.

(ii) *Fibrations are preserved by pullback, i.e. when*



is a pullback square in $\mathfrak{Cat}$ *and* p *is a fibration, then so is* p'.

We end this section with a brief mention of *opfibrations*. A functor $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is an opfibration when $\mathsf{p}^{op} : \mathsf{E}^{op} \longrightarrow \mathsf{B}^{op}$ is a fibration. Explicitly, this means that every map $f : \mathsf{p}X \longrightarrow J$ in B has an *opcartesian lifting* $\widetilde{f} : X \longrightarrow f_!X$, which is initial in the sense that for any other map $g : X \longrightarrow Y$ and any factorization $\mathsf{p}(g) = mf$ there is a unique map $f_!X \longrightarrow Y$ over m making



commute.

2.2. Cloven fibrations. Being a fibration is a property of a functor, as opposed to additional structure: cartesian liftings are required to exist, but a fibration does not come equipped with a specific choice of cartesian liftings. The notion of an indexed category, defined below, is structural, and may be regarded as giving a *presentation* of a fibration by explicitly specifying cartesian liftings.

Definition 2.6. A $\mathbb{B}$-*indexed category* is a pseudo-functor $P : \mathsf{B}^{op} \longrightarrow \mathfrak{Cat}$. Explicitly, this gives for each object I of B a category $P(I)$, for each $f : I \longrightarrow J$ a functor $P(f) : P(J) \longrightarrow P(I)$ (called a *reindexing functor*), together with coherence isomorphisms $P(f)P(g) \cong P(gf)$ and $P(1) \cong 1$, subject to coherence axioms.

Moreover, P is called *strict* if these coherence isomorphisms are identities (in which case P can be regarded simply as a functor $P : \mathsf{B}^{op} \longrightarrow \mathfrak{Cat}$).

Any B-indexed category P gives rise to a fibration over B, as follows.

Construction 2.7 (Grothendieck construction). Let $\int_{\mathsf{B}} P$ be the category with objects (I, x) where $x \in Ob(P(I))$. A morphism $(I, x) \longrightarrow (J, j)$ is a pair (f, m) where $f : I \longrightarrow J$ and $m : x \longrightarrow P(f)(y)$. It is easily seen that maps of the form $(f, 1)$ are cartesian, and that morphisms of the form $(1, m)$ are vertical.

The projection $\pi_P : \int_{\mathsf{B}} P \longrightarrow \mathsf{B}$ is a fibration, and by construction it is endowed with a natural choice of cartesian liftings: the lift of $f : I \longrightarrow J = \pi_P(J, x)$ may be taken to be $(f, 1) : (I, P(f)(x)) \longrightarrow (J, x)$. Such fibrations are called *cloven* (and a choice of cartesian liftings is called a *cleavage*). If P is a strict functor, then the cleavage is functorial, meaning that the chosen cartesian liftings compose and that the chosen lifting of identities are again identities. Fibrations with a functorial cleavage are called *split.*

In case a fibration is equipped with a cleavage, we will denote the chosen cartesian lifting of $f : I \longrightarrow \mathsf{p}Y$ by $\overline{f} : f^*Y \longrightarrow Y$, and refer to f^*Y as the *reindexing of* Y *along* f. Similarly, for a vertical morphism $p : Y \longrightarrow Z$, we denote by f^*p the unique mediating map in

$$\begin{array}{ccc} f^*Y & \xrightarrow{\overline{f}} & Y \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ f^*Z & \xrightarrow[\overline{f}]{} & Z. \end{array}$$

We shall be sloppy about one aspect concerning cleavages here: even though a cleavage is additional structure on a fibration, we omit it from the notation, since we shall not be dealing with fibrations equipped with different cleavages at the same time.

For a detailed treatment of how the above construction can be extended to a 2-equivalence between indexed categories and cloven fibrations, see [24, 16].

We mention one construction which is easily understood in terms of indexed categories and which we will use later in the paper, namely that of the *opposite* of a fibration; it should not be confused with the opposite of a functor.

Definition 2.8 (Opposite of a fibration). For a given B-indexed category $P : \mathsf{B}^{op} \longrightarrow \mathfrak{Cat}$ corresponding to a cloven fibration $\int_{\mathsf{B}} P = \mathsf{p}$, composition with the involution $(-)^{op} : \mathfrak{Cat} \longrightarrow \mathfrak{Cat}$, gives a new indexed category $(-)^{op} \circ P : \mathsf{B}^{op} \longrightarrow \mathfrak{Cat}$. Now define the fibration $\mathsf{p}^{op} =_{def} \int_{\mathsf{B}} (-)^{op} P$.

The fibre over I of p^{op} is $\mathsf{p}^{-1}(I)^{op}$, while its cartesian maps are the same as those of p.

2.3. Logical aspects. We now turn to the logical perspective on fibred categories. We first discuss an important example.

Example 2.9 (Subobject fibration). For a set J, we may consider $\mathcal{P}(J)$, the powerset of J; since $\mathcal{P}(J)$ is partially ordered by inclusion, we may regard it as a posetal category. When $f : I \longrightarrow J$ is a function, we get an induced order-preserving function $f^* : \mathcal{P}(J) \longrightarrow \mathcal{P}(I)$, sending $U \subseteq J$ to $f^*U = \{i \in I | f(i) \in J\}$. This defines a functor

$$\mathcal{P} : \mathsf{Set}^{op} \longrightarrow \mathfrak{Cat}$$

and hence a fibration denoted $\mathsf{Sub}_{\mathsf{Set}} \longrightarrow \mathsf{Set}$ called the *subobject fibration.*

Remark 2.10. The above example can be generalized to any category B in which pullbacks of monomorphisms exist: for any object J define $\mathsf{Sub}_{\mathsf{B}}(J)$ to be the partial order of subobjects (equivalence classes of monics) of J, and define reindexing to be given by pulling back subobjects. Then we obtain a subobject fibration $\mathsf{Sub}_{\mathsf{B}} \longrightarrow \mathsf{B}$. (In the case $\mathsf{B} = \mathsf{Set}$ we used the fact that $\mathcal{P}(I) \cong \mathsf{Sub}(I)$ to pick canonical representatives of subobjects.)

Let us think in logical terms about the subobject fibration $\mathsf{Sub}_{\mathsf{Set}} \longrightarrow \mathsf{Set}$. An object A in the fibre over I is a subset of I, and hence can be regarded as a *predicate* on I. Similarly, the inclusion relation on subsets may be thought of as *entailments* between such predicates. Finally, for a function $f : J \longrightarrow I$ and a predicate A on I, we may regard the predicate $f^*A = \{j \in J : f(j) \in A\}$ as the result of making the *substitution* $[f(j)/i]$ in $A(i)$.

We generalize this as follows: given a fibration $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$, we think of the base category B as a category of *types*, or *contexts*. When I is such a type, then the fibre category E_I is thought of as a category of *predicates* with free variables of type I. To stress this viewpoint, we sometimes write $\alpha(i)$ for such an object.

Furthermore, morphisms $\alpha(i) \longrightarrow \beta(i)$ in the fibre E_I are thought of as *proofs* that α implies β. We denote such a proof by $\phi(i) : \alpha(i) \longrightarrow \beta(i)$. Because we may have more than one proof of such an implication, the fibre category is generally not a pre-order.

What about cartesian maps? Well, morphisms in B are thought of as *terms*, or as *context morphisms*; more specifically, a morphism $f : I \longrightarrow J$ may be regarded as a term of type J with free variables from I (in context I). When $\alpha(j)$ is a predicate over J, then the reindexing of $\alpha(j)$ along f is the predicate $\alpha(fi)$ over I obtained by substituting the term f for j in $\alpha(j)$; the cartesian lifting $\overline{f}$ of f then relates $\alpha(j)$ to its substitution instance $\alpha(fi)$.

$$
\begin{array}{ccc c c}
\alpha(fi) & \xrightarrow{\overline{f}} & \alpha(j) & \qquad & \mathsf{E} \\
 & & & & \downarrow{\scriptstyle \mathsf{p}} \\
I & \xrightarrow[f]{} & J & & \mathsf{B}
\end{array}
$$

It is important to keep in mind that this logical notation for fibrations suppresses coherence data. For example, given a predicate $\alpha(j, x)$ in two variables and maps $f : I \longrightarrow J$ and $g : Y \longrightarrow X$, we could first form $\alpha(fi, x)$ and then $\alpha(fi, gy)$, or the other way around. The notation suggests that the two resulting predicates are identical, whereas in general we only have a vertical isomorphism between the two.

Another class of examples which is relevant to the theme of this paper arises by considering fibrations built from the syntax of a theory $\mathbb{T}$ (at least regular). Good concrete examples to keep in mind are Heyting Arithmetic or Gödel's system T.

From such a theory we first build a base category whose objects are the variable contexts of the theory; the morphisms in the base are the (equivalence classes of) provably functional relations of the theory.

Next, we build a fibration over this base by taking the objects in the fibre over a type T to be the formulae of $\mathbb{T}$ with free variable from T. Reindexing is given by substitution. Such a fibration is usually referred to as the *syntactic fibration* associated to the theory $\mathbb{T}$; see [16] for more information.

Remark 2.11. This paper is mainly about quantification in fibrations; however, it seems strange to discuss the logical perspective and not say a word about the interpretation of the propositional connectives. In a nutshell: in order for a fibration to interpret a certain propositional connective, the fibres need to have appropriate categorical structure and this structure needs to be preserved by reindexing. For example, to interpret conjunction the fibres should have cartesian products, and reindexing should preserve these products. Note that in a posetal fibration this just means that the fibres have binary meets preserved by reindexing. Similarly, disjunction is interpreted using fibrewise coproducts, implication using fibred exponentials, and so on.

2.4. 2-categorical aspects. We now describe how fibrations, as well as fibrations over a fixed base B naturally form a 2-category. We will not worry about size issues, although typically we will have in mind that the base is small. After that, we expore some properties of this 2-category which will be used in the rest of the paper, in particular in connection to coherence matters.

We first define the 2-category $\mathfrak{CFib}$ of cloven fibrations over arbitrary base as follows. The objects of $\mathfrak{CFib}$ are cloven fibrations $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$. For objects $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ and $\mathsf{q} : \mathsf{D} \longrightarrow \mathsf{B}'$, a 1-cell is a commutative square

$$\begin{array}{ccc} \mathsf{D} & \xrightarrow{F_0} & \mathsf{E} \\ {\scriptstyle \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}} \\ \mathsf{B}' & \xrightarrow[F]{} & \mathsf{B} \end{array} \tag{2.1}$$

in which F_0 is a *fibred functor*, in the sense that it sends q-cartesian maps to p-cartesian maps. It is *not* required that F_0 preserves the cleavage on the nose.

Finally, given two such 1-cells (F, F_0) and (G, G_0) from q to p, a 2-cell from (F, F_0) to (G, G_0) is a pair of natural transformations $\phi : F \longrightarrow G, \phi_0 : F_0 \longrightarrow G_0$, where ϕ_0 lies over ϕ, in the sense that for each X in D, the component $(\phi_0)_X$ is sent by p to $\phi_{\mathsf{q}X}$.

There is a forgetful 2-functor $\mathfrak{CFib} \longrightarrow \mathfrak{Cat}$ which sends $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ to the base B. When we take the fibre of this forgetful functor at a base category B, we obtain the 2-category $\mathfrak{CFib}(\mathsf{B})$ of cloven fibrations over B, fibred functors and vertical natural transformations.

Definition 2.12 (Fibred functor, fibred natural transformation). Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ and $\mathsf{q} : \mathsf{D} \longrightarrow \mathsf{B}$ be functors.

(i) A functor $F : \mathsf{E} \longrightarrow \mathsf{D}$ over B is called a *fibred functor*[2] if it preserves cartesian maps.

(ii) A natural transformation $\phi : F \Rightarrow F'$ between two functors over B is called *vertical* when all of its components are vertical maps, i.e. when $\mathsf{q}(\phi_X) = 1$ for each object X of E.

To summarize: $\mathfrak{CFib}(\mathsf{B})$ is the 2-category with

- **0-Cells:** cloven fibrations $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$
- **1-Cells:** fibred functors
- **2-Cells:** vertical natural transformations.

[2]Some authors call this a *cartesian* functor, but we will not follow this.

Remark 2.13. The forgetful 2-functor $\mathfrak{CFib}(\mathsf{B}) \longrightarrow \mathfrak{Cat}/\mathsf{B}$ is in fact 2-monadic: the free (split) fibration on a functor $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is the projection $\mathsf{B}/\mathsf{p} \longrightarrow \mathsf{B}$; moreover, "fibration structure" (i.e. a cleavage) on a functor $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is precisely a pseudo-algebra structure for this 2-monad (strict algebras correspond to splittings). See [23] for a detailed treatment of this approach to fibrations.

The following elementary fact will often be used:

Lemma 2.14. *Let F be a functor over B from $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ to $\mathsf{q} : \mathsf{D} \longrightarrow \mathsf{B}$, where p, q are cloven. Then F is a fibred functor if and only if each of the unique mediating maps*

$$\begin{array}{ccc} F(f^*\alpha) & & \\ \downarrow & \searrow & \\ f^*(F\alpha) & \longrightarrow & F\alpha \end{array}$$

is an isomorphism.

This encourages us to extend the logical notation for fibrations to include functors. For example, we write $F\alpha(i, x)$ for the image under the fibred functor F of the predicate $\alpha(i, x)$; then the expression $F\alpha(fj, x)$ is ambiguous, but by the above lemma the two possible interpretations are coherently isomorphic.

Similarly, for a fibred transformation $\tau : F \Rightarrow G$, we note that the square

$$\begin{array}{ccc} F(f^*\alpha) & \xrightarrow{\tau_{f^*\alpha}} & G(f^*\alpha) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ f^*(F\alpha) & \xrightarrow[f^*\tau_\alpha]{} & f^*(G\alpha) \end{array}$$

commutes.

3. Quantification in Fibrations

In this section we first explain various notions of quantification in more detail; then we turn to the process of freely adding existential quantification to a given fibration, showing that this process is monadic in a suitable 2-categorical sense. We do the same for universal quantification, which is then a relatively easy dualization. Finally, we discuss the linear variants of these free constructions.

From now on all fibrations are equipped with a cleavage.

3.1. Sums and products. The starting point for the interpretation of the quantifiers in the fibrational setting is the observation that the quantifiers $\forall, \exists$ may be characterized in terms of adjointness relations. Let me explain this: consider a predicate $\alpha(x, y)$ over $X \times Y$. Existentially quantifying over the variable y results in the predicate $\exists y.\alpha(x, y)$, which now only has free variable x and hence must live in the fibre over X. Thus, this form of quantification should be an operation $\exists_Y : \mathsf{p}^{-1}(X \times Y) \longrightarrow \mathsf{p}^{-1}(X)$. The main observation, due to Lawvere, is now that this operation is *left adjoint* to the reindexing functor $\mathsf{p}^{-1}(X) \longrightarrow \mathsf{p}^{-1}(X \times Y)$ (this functor is also referred to as a *weakening* functor; it introduces a dummy variable). Similarly, universal quantification can be expressed in terms of right adjoints to reindexing functors.

Suppose then that B is a category with (chosen) pullbacks. For an arbitrary map $f : I \longrightarrow J$ in B, we may ask whether the functor $f^* : \mathsf{p}^{-1}(J) \longrightarrow \mathsf{p}^{-1}(I)$ has a left adjoint. If it does, we think of this adjoint as giving *existential quantification along* f and denote it by $\exists_f : \mathsf{p}^{-1}(I) \longrightarrow \mathsf{p}^{-1}(J)$.

However, merely asking that p has left adjoints to reindexing functors is not sufficient: we have to make sure that these left adjoints behave well with respect to substitution. To this end, consider a pullback square in B:

$$\begin{array}{ccc} I & \xrightarrow{f} & J \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[k]{} & W \end{array} \tag{3.1}$$

Given a predicate on J, we may now either first make the substitution $[f(i)/j]$ and then quantify along h, or we first quantify along g and then substitute $[k(v)/w]$. We want those two to agree. Formally, there is a canonical comparison map

$$\kappa : \exists_h f^* \longrightarrow k^* \exists_g$$

obtained as follows: first take the unit $1 \longrightarrow g^*\exists_g$, and apply to it the functor f^* to get a map $f^* \longrightarrow f^*g^*\exists_g$. Then use the canonical isomorphism $h^*k^* \cong f^*g^*$ to get a map $f^* \longrightarrow h^*k^*\exists_g$. Finally, transpose this along the adjunction $\exists_h \dashv h^*$ to get the desired $\exists_h f^* \longrightarrow k^*\exists_g$. We say that p satisfies the *Beck-Chevalley Condition* (for the given pullback square) when this mediating map is an isomorphism.

Definition 3.1. We say that a fibration p over a category B with pullbacks has *existential quantification* when all reindexing functors have left adjoints satisfying the Beck-Chevalley Condition (BCC). Dually, p has *universal quantification*, when reindexing functors have right adjoints satisfying the BCC.

For fibrations with existential quantification, observe that for $f : I \longrightarrow J$ in B and $\alpha \in \mathsf{p}^{-1}(I)$, there is a canonical map

$$\widetilde{f} : \alpha \longrightarrow \exists_f(\alpha)$$

obtained by precomposing the unit $\alpha \longrightarrow f^*\exists_f(\alpha)$ with the cartesian map $\overline{f} : f^*\exists_f(\alpha) \longrightarrow \exists_f(\alpha)$. This map is opcartesian over f, a fact which we will exploit later.

We have already seen examples of fibrations with existential quantification: the archetypical such is the codomain fibration $\mathsf{B}^{\longrightarrow} \longrightarrow \mathsf{B}$. Given a map $f : I \longrightarrow J$ in B, the left adjoint to the pullback functor $f^* : \mathsf{B}/J \longrightarrow \mathsf{B}/I$ is usually denoted by Σ_f, and acts by composition with f. (Moreover, the codomain fibration has universal quantification whenever the base category is locally cartesian closed, see e.g. [16]).

For a different example, consider a complete Heyting algebra (frame, locale) Ω, and consider the assignment $I \mapsto \Omega^I = \mathsf{Set}[I, \Omega]$. Reindexing is given by precomposition. The reindexing functors $f^* : \Omega^J \longrightarrow \Omega^I$ have both adjoints, given by (for $\alpha : I \longrightarrow \Omega$)

$$\exists_f(\alpha)(j) = \bigvee_{f(i)=j} \alpha(i) \qquad \forall_f(\alpha)(j) = \bigwedge_{f(i)=j} \alpha(j)$$

Now that we understand this general form of quantification, it is easy to consider more restricted versions by only asking for adjoints to certain reindexing functors. We will be mostly concerned with reindexing along projections. This is usually referred to as *simple quantification*. Explicitly:

Definition 3.2. A fibration p over a category B with finite products has *simple existential quantification*, or *simple coproducts*, when all weakening functors have left adjoints satisfying the Beck-Chevalley Condition (BCC) for pullback squares of the form

$$\begin{array}{ccc} I \times X & \xrightarrow{\pi_I} & I \\ {\scriptstyle f\times X}\downarrow & & \downarrow{\scriptstyle f} \\ J \times X & \xrightarrow[\pi_J]{} & J. \end{array}$$

Dually, p has *simple universal quantification*, or *simple products*, when weakening functors have right adjoints satisfying the BCC.

Remark 3.3. The idea of selecting a particular class of maps to quantify along (i.e. by specifying a subcategory of $\mathsf{B}^{\rightarrow}$ which is pullback-stable) allows us to define quite general notions of quantification; this has been made more precise via notions such as CT-structure and comprehension categories, see [16].

What does it mean for a fibred functor to preserve coproducts? Consider two fibrations with coproducts and a fibred functor F between them. Given a morphism $f : I \longrightarrow J$ in the base and a predicate α over I, we have a unique vertical comparison map induced by opcartesianness of $\tilde{f}$.

$$\begin{array}{ccc} & & F(\exists_f \alpha) \\ & \overset{F\tilde{f}}{\nearrow} & \uparrow{\scriptstyle \psi_{f,\alpha}} \\ F\alpha & \xrightarrow[\tilde{f}]{} & \exists_f(F\alpha) \end{array}$$

Definition 3.4. A fibred functor F *preserves coproducts* when each of the comparison morphisms $\psi_{f,\alpha}$ is an isomorphism.

Of course, we have a similar definition for preservation of products, and it is now also clear what is meant by a fibred functor preserving simple (co)products. We introduce the following 2-categories: $\mathfrak{CFib}_{\forall}(\mathsf{B})$ is the 2-category with

- **0-Cells:** Cloven fibrations over B with (chosen) simple products
- **1-Cells:** Simple product-preserving fibred functors
- **2-Cells:** Fibred natural transformations

and similarly $\mathfrak{CFib}_{\exists}(\mathsf{B})$ has

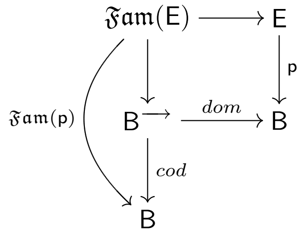
- **0-Cells:** Cloven fibrations over B with (chosen) simple coproducts
- **1-Cells:** Simple coproduct-preserving fibred functors
- **2-Cells:** Fibred natural transformations

We will extend the logical notation for fibrations to include the quantifiers, writing $\exists x.\alpha(i,x)$ for the result of applying $\exists_X$ to $\alpha(i,x)$. Then an expression such as $\exists x.\alpha(fj,x)$ is again ambiguous, but the BCC guarantees that both possible interpretations are coherently isomorphic.

3.2. The families construction. We now turn to the question of how to freely add existential quantification to a given fibration. The answer is easiest for adding all coproducts; we first briefly review this, as it makes the subsequent constructions clear and gives us the opportunity to point out some structural features.

First recall that $cod : \mathsf{B}^{\to} \longrightarrow \mathsf{B}$ (for B with pullbacks) is a fibration with existential quantification along all maps. In fact, it is the free such on the terminal fibration $1 : \mathsf{B} \longrightarrow \mathsf{B}$ on B: given any fibration with coproducts $\mathsf{E} \xrightarrow{\mathsf{p}} \mathsf{B}$ and any fibred functor F from the terminal to p, there is an induced fibred functor from the codomain fibration to p which sends an arrow $f : I \longrightarrow J$ to $\exists_f(FI)$.

We now exploit this by defining, for general p, a new fibration $\mathfrak{Fam}(\mathsf{p})$ as follows:



Here, the square is a pullback. Thus, we label the objects and arrows of the free structure on the terminal by objects of E; explicitly $\mathfrak{Fam}(\mathsf{E})$ is the category with

- **Objects:** pairs $(I \xrightarrow{f} J, \alpha)$ where $\alpha \in \mathsf{p}^{-1}(I)$
- **Arrows:** an arrow from $(I \xrightarrow{f} J, \alpha)$ to $(H \xrightarrow{g} K, \beta)$ consists of a triple (p, q, ϕ), where $gp = qf$ is a commutative square in B and $\phi : \alpha \longrightarrow \beta$ is a map in E over p.

One may think of an object $(f : I \longrightarrow J, \alpha)$ as the predicate $\exists_f \alpha$; alternatively, one regards it as an I-indexed family of objects of E, whence the name of the construction.

Using the universal property of the pullback, this construction is easily seen to be 2-functorial. Moreover, one can construct a unit and a multiplication map. Explicitly, the unit is given by

$$\eta_{\mathsf{p}} : \mathsf{E} \longrightarrow \mathfrak{Fam}(\mathsf{E}); \qquad \alpha \in \mathsf{p}^{-1}(I) \mapsto (I \xrightarrow{1} I, \alpha)$$

and the multiplication is

$$\mu_{\mathsf{p}} : \mathfrak{Fam}^2(\mathsf{E}) \longrightarrow \mathfrak{Fam}(\mathsf{E}); \qquad (I \xrightarrow{f} J, (J \xrightarrow{g} K, \alpha)) \mapsto (I \xrightarrow{gf} K, \alpha).$$

The monad laws are just the composition laws in B.

Theorem 3.5. *The construction* $\mathsf{p} \mapsto \mathfrak{Fam}(\mathsf{p})$ *extends to a strict 2-monad on* $\mathfrak{CFib}(\mathsf{B})$*, which has the KZ property. The 2-category of pseudo-algebras is 2-equivalent to the 2-category of cloven fibrations with chosen coproducts and coproduct-preserving fibred functors.*

Remark 3.6. Let me first comment on the KZ-property: a pseudo-monad is said to be *Kock-Zöberlein* (see [20, 19, 21]) when pseudo-algebra structures on p are left adjoint to the unit at p. (To show that a monad has this property, it suffices to test it for free algebras). This has useful consequences: first, when p admits a pseudo-algebra structure it is essentially unique. Second, any morphism between algebras is automatically a lax algebra morphism in a unique way, and is a pseudo algebra map whenever the unique mediating 2-cell is an isomorphism. And third, any 2-cell between lax algebra morphisms is automatically an algebra 2-cell. The dual notion is called a co-KZ monad, and is characterized by algebra structure maps being right adjoint to the unit.

Now let me sketch why a pseudo-algebra structure on p amounts to having chosen coproducts in p. A pseudo-algebra structure on p is, by the KZ property, a (fibred!) left adjoint to the unit $\eta_{\mathsf{p}} : \mathsf{p} \longrightarrow \mathfrak{Fam}(\mathsf{p})$. Given such left adjoint K we thus have adjunctions for each fibre

$$\mathsf{E}_I \underset{\eta_I}{\overset{K_I}{\rightleftarrows}} \mathfrak{Fam}(\mathsf{E})_I \quad (K_I \dashv \eta_I)$$

where the counit is an isomorphism, and hence η_I is fully faithful. It is straightforward to verify that the fact that K is a fibred functor translates into the BCC for the local adjunctions $K_I \dashv \eta_I$, in the sense that given a map $f : I \longrightarrow J$ the canonical natural transformation $K_I f^* \longrightarrow f^* K_J$ is an isomorphism (this map depends on the adjunction morphisms). We now define existential quantification $\exists_f : \mathsf{E}_I \longrightarrow \mathsf{E}_J$ in p along f to be the composite

$$\mathsf{E}_I \xrightarrow{\eta_I} \mathfrak{Fam}(\mathsf{E})_I \xrightarrow{\exists_f} \mathfrak{Fam}(\mathsf{E})_J \xrightarrow{K_J} \mathsf{E}_J; \qquad \alpha \mapsto K(I \xrightarrow{f} J, \alpha).$$

The adjointness $\exists_f \dashv f^*$ for p is now immediate from the following sequence of bijections:

$$\begin{array}{cl} K_J \exists_f \eta_I \alpha \longrightarrow \beta & \\ \hline \exists_f \eta_I \alpha \longrightarrow \eta_J \beta & \text{by } K_I \dashv \eta_I \\ \hline \eta_I \alpha \longrightarrow f^* \eta_J \beta & \text{by } \exists_f \dashv f^* \text{ for } \mathfrak{Fam}(\mathsf{p}) \\ \hline \eta_I \alpha \longrightarrow \eta_I f^* \beta & \eta \text{ is fibred} \\ \hline \alpha \longrightarrow f^* \beta & \eta_I \text{ is fully faithful} \end{array}$$

The BCC for the K_I easily implies that for the $\exists_f$. Note that the adjunction morphisms for $K \dashv \eta$ determine the adjunction morphisms of $\exists_f \dashv f^*$.

Conversely, when we have left adjoints to reindexing functors then we can use the same definition

$$K_I : \mathfrak{Fam}(\mathsf{E})_I \longrightarrow \mathsf{E}_I; \qquad (J \xrightarrow{f} I, \alpha) \mapsto \exists_f(\alpha)$$

to locally construct a left adjoint to the unit. The BCC now guarantees that these fit together to form a global adjoint $K \dashv \eta$. Details are straightforward.

3.3. The simple fibration, or adding simple coproducts. Next, we investigate adding simple coproducts to a fibration. We follow the pattern from the previous section. First of all, the codomain fibration gets replaced by a smaller fibration called the *simple fibration* over B. This is defined as follows:

Construction 3.7. The category $\mathfrak{Sum}(\mathsf{B})$ has:

Objects: pairs (I, X), where I and X are objects of B.

Morphisms: a map from (I, X) to (J, Y) is a pair $(f, f_0,)$, where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : I \times X \longrightarrow Y$ is a morphism in B.

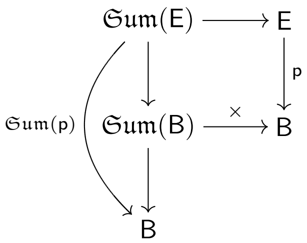
Identities: the identity on (I, X) is the map $(1_I, \pi_X)$.

Composition: given $(f, f_0) : (I, X) \longrightarrow (J, Y)$ and $(g, g_0) : (J, Y) \longrightarrow (K, Z)$, the composite of these maps is defined to be the pair (h, h_0) where

$$h = gf; \qquad h_0 = g_0\langle f\pi_I, f_0\rangle; \qquad h_0(i, x) = g_0(fi, f_0(i, x)).$$

This category is fibred over B via the first projection. The fibre over I is also referred to as the *simple slice* over I. There will be no notational distinction made between $\mathfrak{Sum}(\mathsf{B})$ regarded as a fibration and as category.

We now extend this construction in the canonical way to general fibrations: given $\mathsf{E} \xrightarrow{\mathsf{p}} \mathsf{B}$, consider



where the square is a pullback. The total category $\mathfrak{Sum}(\mathsf{E})$ may be described explicitly as follows:

Construction 3.8. Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration. The category $\mathfrak{Sum}(\mathsf{E})$ has:

Objects: triples (I, X, α), where I and X are objects of the base B, and where $\alpha \in \mathsf{p}^{-1}(I \times X)$ is an object in the fibre over $I \times X$.

Morphisms: a map from (I, X, α) to (J, Y, β) is a triple (f, F, ϕ), where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : I \times X \longrightarrow Y$ is a morphism in B
- $\phi = \phi(i, y) : \alpha(i, x) \longrightarrow \beta(f(i), f_0(i, x))$ is a morphism in the fibre over $I \times X$.

The rest of the structure is straightforward, and it is readily verified that $\mathsf{p} \mapsto \mathfrak{Sum}(\mathsf{p})$ is 2-functorial.

Theorem 3.9. *The 2-functor $\mathsf{p} \mapsto \mathfrak{Sum}(\mathsf{p})$ can be endowed in a canonical way with the structure of a pseudo-monad on $\mathfrak{CFib}(\mathsf{B})$, which is a strict 2-monad when the product structure on B is strict. This pseudo-monad is KZ, and its 2-category of pseudo-algebras is 2-equivalent to the 2-category $\mathfrak{CFib}_\exists$.*

Note first that in case products are strict in B, it is easy to see that we get a 2-monad: for then the collection of first projections is closed under composition, so all of the 2-monad structure is obtained by simply restricting that of the families monad.

Formally, to specify a pseudo-monad structure on $\mathfrak{Sum}$ means to give the following data:

- a pseudo-natural transformation $(\eta, \overline{\eta}) : 1 \longrightarrow \mathfrak{Sum}$
- a pseudo-natural transformation $(\mu, \overline{\mu}) : \mathfrak{Sum}^2 \longrightarrow \mathfrak{Sum}$
- invertible modifications $\lambda : \mu \circ \mathfrak{Sum}\eta \longrightarrow 1, \rho : \mu \circ \eta\mathfrak{Sum} \longrightarrow 1$ and $\theta : \mu \circ \mu\mathfrak{Sum} \longrightarrow \mu \circ \mathfrak{Sum}\mu$.

Moreover, this data is supposed to satisfy certain coherence conditions. Here I will simply give a direct description of this structure; in the Appendix the coherence will be discussed in some detail.

First, the unit for $\mathfrak{Sum}$. At a fibration p, this is the map which acts as follows on an object α and a morphism $\phi : \alpha \longrightarrow \beta$ in E_I:

$$\eta_{\mathsf{p}} : \mathsf{E} \longrightarrow \mathfrak{Sum}(\mathsf{E}); \qquad \alpha \mapsto (I, 1, \alpha'); \qquad \phi \mapsto (1_I, !, \phi')$$

where α' is the result of reindexing α along the unit isomorphism $I \times 1 \longrightarrow I$, and ϕ' is the unique vertical map making

$$\begin{array}{ccc} \alpha' & \xrightarrow{\overline{\pi_I}} & \alpha \\ \big\downarrow{\scriptstyle \phi'} & & \big\downarrow{\scriptstyle \phi} \\ \beta' & \xrightarrow[\overline{\pi_I}]{} & \beta \end{array}$$

commute.

We see right away why we get a complication which was not present in the case of $\mathfrak{Fam}$ (and which also doesn't show up when products are strict): in order to get a well-defined object of $\mathfrak{Sum}(\mathsf{E})$, we need to move to an isomorphic fibre.

As a consequence, the unit is no longer strictly natural, but only pseudo-natural: given $F : \mathsf{p} \longrightarrow \mathsf{q}$, consider the square

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{\eta_{\mathsf{p}}} & \mathfrak{Sum}(\mathsf{E}) \\ {\scriptstyle F}\big\downarrow & \Downarrow \overline{\eta}_F & \big\downarrow{\scriptstyle \mathfrak{Sum}(F)} \\ \mathsf{D} & \xrightarrow[\eta_{\mathsf{q}}]{} & \mathfrak{Sum}(\mathsf{D}) \end{array} \tag{3.2}$$

The mediating vertical natural transformation $\overline{\eta}_F$ arises from the fact that F preserves reindexing up to unique vertical isomorphism: thus the component of $\overline{\eta}_F$ at α over I is the unique vertical isomorphism making

$$\begin{array}{ccc} F(\alpha') & & \\ \big\downarrow & \searrow^{F\overline{\pi_I}} & \\ (F\alpha)' & \xrightarrow[\overline{\pi_I}]{} & F\alpha. \end{array}$$

commute (where $\pi_I : I \times 1 \longrightarrow I$).

The $\overline{\eta}_F$ are the components of a natural transformation

$$\begin{array}{ccc} \mathfrak{CFib}(\mathsf{p}, \mathsf{q}) & \xrightarrow{\mathfrak{Sum}} & \mathfrak{CFib}(\mathfrak{Sum}(\mathsf{p}), \mathfrak{Sum}(\mathsf{q})) \\ \Big\| & \Downarrow \overline{\eta} & \Big\downarrow \\ \mathfrak{CFib}(\mathsf{p}, \mathsf{q}) & \longrightarrow & \mathfrak{CFib}(\mathsf{p}, \mathfrak{Sum}(\mathsf{q})) \end{array}$$

making $(\eta, \overline{\eta})$ a well-defined pseudo-natural transformation $1 \longrightarrow \mathfrak{Sum}$.

A similar story can be told for the multiplication. A typical object of $\mathfrak{Sum}^2(\mathsf{E})$ has the form $(I, X, (I \times X, U, \alpha))$, with $\alpha \in \mathsf{E}_{(I \times X) \times U}$. We may write (I, X, U, α) for such an object. The multiplication map μ_{p} acts as follows:

$$\mu(I, X, U, \alpha) = (I, X \times U, \alpha'),$$

where α' is now the reindexing of α along the associativity isomorphism. As for the unit, we can construct a family of natural transformations $\overline{\mu}$ making $(\mu, \overline{\mu})$ a pseudo-natural transformation $\mathfrak{Sum}^2 \longrightarrow \mathfrak{Sum}$.

The modifications λ, ρ, θ up to which the monad laws hold, are now easily seen to be the canonical isomorphisms

$$\lambda : (I, 1 \times X, \alpha') \longrightarrow (I, X, \alpha)$$

$$\rho : (I, X \times 1, \alpha') \longrightarrow (I, X, \alpha)$$

$$\theta : (I, X \times (U \times V), \alpha') \longrightarrow (I, (X \times U) \times V, \alpha).$$

(In each of these, α' refers to the reindexing of α along the appropriate coherence isomorphism.)

3.4. Adding products. We now dualize the above results. First we give an explicit description:

Construction 3.10. Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a cloven fibration. Construct a category $\mathfrak{Prod}(\mathsf{E})$ as follows:

Objects: triples (I, X, α), where I and X are objects of the base B, and where $\alpha \in \mathsf{p}^{-1}(I \times X)$ is an object in the fibre over $I \times X$.

Morphisms: a map from (I, X, α) to (J, Y, β) is a triple (f, f_0, ϕ), where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : I \times Y \longrightarrow X$ is a morphism in B
- $\phi = \phi(i, y) : \alpha(i, f_0(i, y)) \longrightarrow \beta(f(i), y)$ is a morphism in the fibre over $I \times Y$.

The identity map on (I, X, α) is $(1_I, \pi_X, 1_\alpha)$. The composition of $(f, f_0, \phi) : (I, X, \alpha) \longrightarrow (J, Y, \beta)$ and $(g, g_0, \psi) : (J, Y, \beta) \longrightarrow (K, Z, \gamma)$ is given by (h, h_0, χ) where

$$h = gf : I \longrightarrow K, \qquad h_0(i, z) = f_0(i, g_0(fi, z)) : I \times Z \longrightarrow X$$

and where $\chi(i, z)$ is the composite

$$\alpha(i, h_0(i, z)) = \alpha(i, f_0(i, g_0(fi, z))) \xrightarrow{\phi(i, g_0(fi, z))} \beta(fi, g_0(fi, z)) \xrightarrow{\psi(fi, z)} \gamma(gfi, z).$$

Proposition 3.11. *There is an isomorphism of fibrations*

$$\mathfrak{Prod}(\mathsf{p}) \cong \mathfrak{Sum}(\mathsf{p}^{op})^{op}.$$

This isomorphism is natural in p.

Here one has to recall that p^{op} stands for the *fibrewise opposite* of p as described in Definition 2.8.

Proof. The first statement of the theorem is easily checked by hand: both categories have the same objects. In the fibre over I, a map $(I, X, \alpha) \xrightarrow{(1, f_0, \phi)} (I, Y, \beta)$ of $\mathfrak{Sum}(\mathsf{p}^{op})$ has $f_0 : I \times X \longrightarrow Y$ and $\phi : \beta(i, f_0(i, x)) \longrightarrow \alpha(i, x)$. But this is precisely a map $(I, Y, \beta) \longrightarrow (I, X, \alpha)$ in $\mathfrak{Sum}(\mathsf{p})$.

The second statement is left as a straightforward exercise. □

As a consequence, $\mathfrak{Prod}$ is also a pseudo-monad on $\mathfrak{CFib}(\mathsf{B})$. (This may of course also be verified directly along the same lines as for $\mathfrak{Sum}$.) We state:

Theorem 3.12. *The assignment* $\mathsf{p} \mapsto \mathfrak{Prod}(\mathsf{p})$ *can be endowed in a canonical way with the structure of a pseudo-monad on* $\mathfrak{CFib}(\mathsf{B})$*, which is a strict 2-monad when the product structure on* B *is strict. This pseudo-monad is co-KZ, and its 2-category of pseudo-algebras is 2-equivalent to the 2-category* $\mathfrak{CFib}_\forall$.

Because of this result, we are justified to think of an object (I, X, α) of $\mathfrak{Prod}(\mathsf{p})$ as the predicate $\forall x.\alpha(i, x)$.

3.5. Variations. In this section we make a brief excursion to the linear approximations of the monads described above.

Let us first look at the linear variant of the monad $\mathfrak{Sum}$. Explicitly, it is constructed as follows:

Construction 3.13. Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a cloven fibration. Construct a category $\mathfrak{Sum}_L(\mathsf{p})$ as follows:

Objects: triples (I, X, α), where I and X are objects of the base B, and where $\alpha \in \mathsf{p}^{-1}(I \times X)$ is an object in the fibre over $I \times X$.

Morphisms: a map from (I, X, α) to (J, Y, β) is a triple (f, f_0, ϕ), where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : X \longrightarrow Y$ is a morphism in B
- $\phi = \phi(i, y) : \alpha(u, x) \longrightarrow \beta(f(i), f_0(x))$ is a morphism in the fibre over $I \times Y$.

It is easy to see that this construction is the result of restricting the simple slice over B to the category with the same objects, but only those maps $(I, X) \longrightarrow (J, Y)$ for which the second factor $f_0 : I \times X \longrightarrow Y$ is *independent* of X, i.e. factors as $f_0'\pi_X$ for some $f_0' : X \longrightarrow Y$.

We have the expected result:

Theorem 3.14. *The assignment* $\mathsf{p} \mapsto \mathfrak{Sum}_L(\mathsf{p})$ *carries the structure of a pseudomonad on* $\mathfrak{CFib}(\mathsf{B})$*. When products in* B *are strict, then* $\mathfrak{Sum}_L$ *is a strict 2-monad.*

All structure is inherited from $\mathfrak{Sum}$, and we have an inclusion of pseudo-monads $\mathfrak{Sum}_L \longrightarrow \mathfrak{Sum}$. However, it should be noted that $\mathfrak{Sum}_L$ does not have the KZ-property.

For the following result which further describes how $\mathfrak{Sum}_L$ relates to $\mathfrak{Sum}$, recall that a fibration p has *equality* when it has left adjoints to contraction functors $\delta^* : \mathsf{p}^{-1}(U \times U) \longrightarrow \mathsf{p}^{-1}(U)$ subject to the Beck-Chevalley condition.

Theorem 3.15. *Assume that* p *has equality. Then there is a comonad* $!_\mathsf{p}$ *on* $\mathfrak{Sum}_L(\mathsf{p})$ *such that the coKleisli object for this comonad is* $\mathfrak{Sum}(\mathsf{p})$.

Proof. First note that if p has equality, then so does $\mathfrak{Sum}_L(\mathsf{p})$: this is given by $(I \times I, X, \alpha) \mapsto (I, X, \exists_{\Delta \times X}(\alpha))$.

For the comonad on $\mathfrak{Sum}_L(\mathsf{p})$, we define $!_\mathsf{p}(I, X, \alpha) = (I, I \times X, \hat{\alpha})$, where $\hat{\alpha}(i, i', x) = \alpha(i, x) \wedge (i = i')$. Equivalently, $\hat{\alpha} = \exists_{\Delta \times X}(\alpha)$. It is now easily seen that a map $!_\mathsf{p}(I, X, \alpha) \longrightarrow (J, Y, \beta)$ in $\mathfrak{Sum}_L(\mathsf{p})$ is precisely a map $(I, X, \alpha) \longrightarrow (J, Y, \beta)$ in $\mathfrak{Sum}(\mathsf{p})$. □

What are the (pseudo-)algebras for $\mathfrak{Sum}_L$? These are the fibrations equipped with a notion of quantification (again denoted $\exists$) satisfying the following inference rule:

$$\frac{\alpha(i,x) \longrightarrow \beta(fi, f_0 x)}{\exists x.\alpha(i,x) \longrightarrow \exists y.\beta(fi,y)}$$

There is a sense in which this structure may be thought of as linear quantification; see the comments in the last section.

We now turn to the dual results:

Construction 3.16. The category $\mathfrak{Prod}_L(\mathsf{p})$ has

Objects: triples (I, X, α), where I and X are objects of the base B, and where $\alpha \in \mathsf{p}^{-1}(I \times X)$ is an object in the fibre over $I \times X$.

Morphisms: a map from (I, X, α) to (J, Y, β) is a triple (f, F, ϕ), where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : Y \longrightarrow X$ is a morphism in B
- $\phi = \phi(i, y) : \alpha(i, f_0(y)) \longrightarrow \beta(f(i), y)$ is a morphism in the fibre over $I \times Y$.

Again, this category is fibred over the base B via the first projection. Because all the structure on $\mathfrak{Prod}$ restricts to $\mathfrak{Prod}_L$ (or using the isomorphism $\mathfrak{Prod}_L(\mathsf{p}) \cong (\mathfrak{Sum}_L(\mathsf{p}^{op}))^{op}$), the following is immediate:

Theorem 3.17. *The assignment* $\mathsf{p} \mapsto \mathfrak{Prod}_L(\mathsf{p})$ *carries the structure of a pseudo-monad on the category* $\mathfrak{CFib}(\mathsf{B})$ *of fibrations over* B.

Under the assumption that the base category B is cartesian closed, the inclusion $\mathfrak{Prod}_L(\mathsf{p}) \longrightarrow \mathfrak{Prod}(\mathsf{p})$ is part of a coKleisli adjunction for a comonad ! on $\mathfrak{Prod}(\mathsf{p})$. This comonad is defined by

$$!_{\mathsf{p}}(I, X, \alpha) = (I, X^I, \tilde{\alpha}); \qquad \tilde{\alpha}(i, F) = \alpha(i, Fi),$$

where $\tilde{\alpha}$ is the obvious reindexing of α.

This indeed makes $!_{\mathsf{p}}$ into a fibred functor; moreover, it is indeed a comonad: the counit

$$!_{\mathsf{p}}(I, X, \alpha) = (I, X^I, \tilde{\alpha}) \longrightarrow (I, X, \alpha)$$

is induced by $X \longrightarrow X^I$, the transpose of the projection. The comultiplication

$$!_{\mathsf{p}}(I, X, \alpha) = (I, X^I, \tilde{\alpha}) \longrightarrow (I, (X^I)^I, \tilde{\tilde{\alpha}}) = !_{\mathsf{p}}!_{\mathsf{p}}(I, X, \alpha)$$

is induced by $X^{\pi_1} : X^I \longrightarrow X^{I \times I} \cong (X^I)^I$.

Finally, a map

$$!_{\mathsf{p}}(I, X, \alpha) = (I, X^I, \tilde{\alpha}) \xrightarrow{(f, f_0, \phi)} (J, Y, \beta)$$

in $\mathfrak{Prod}_L(\mathsf{p})$ amounts to $f : I \longrightarrow J, f_0 : Y \longrightarrow X^I, \phi : \tilde{\alpha}(i, f_0 y) \longrightarrow \beta(fi, y)$; by taking the transpose of F this corresponds to a morphism

$$(I, X, \alpha) \xrightarrow{(f, \hat{f}_0, \phi)} (J, Y, \beta)$$

in $\mathfrak{Prod}(\mathsf{p})$, showing that the latter is indeed the coKleisli category for $!_{\mathsf{p}}$.

This gives:

Theorem 3.18. *For each cloven fibration* p *over a cartesian closed base category, there is a coKleisli adjunction*

$$\mathfrak{Prod}(\mathsf{p}) \underset{\longleftarrow}{\overset{!_{\mathsf{p}}}{\underset{\bot}{\longrightarrow}}} \mathfrak{Prod}_L(\mathsf{p})$$

Monad	Pattern	Inference Rule	Structure
$\mathfrak{Sum}_L$	$\begin{array}{ccc} I & & X \\ \downarrow & & \downarrow \\ J & & Y \end{array}$	$\dfrac{\alpha(i,x) \longrightarrow \beta(fi, f_0x)}{\exists x\alpha(i,x) \longrightarrow \exists y\beta(fi,y)}$	Linear simple coproducts
$\mathfrak{Sum}$	$\begin{array}{ccc} I & & X \\ \downarrow & \searrow & \downarrow \\ J & & Y \end{array}$	$\dfrac{\alpha(i,x) \longrightarrow \beta(fi, f_0ix)}{\exists x\alpha(i,x) \longrightarrow \exists y\beta(fi,y)}$	Simple coproducts
$\mathfrak{Prod}_L$	$\begin{array}{ccc} I & & X \\ \downarrow & & \uparrow \\ J & & Y \end{array}$	$\dfrac{\alpha(i,f_0y) \longrightarrow \beta(fi, y)}{\forall x\alpha(i,x) \longrightarrow \forall y\beta(fi,y)}$	Linear simple products
$\mathfrak{Prod}$	$\begin{array}{ccc} I & & X \\ \downarrow & \nearrow & \uparrow \\ J & & Y \end{array}$	$\dfrac{\alpha(i,f_0iy) \longrightarrow \beta(fi, y)}{\forall x\alpha(i,x) \longrightarrow \forall y\beta(fi,y)}$	Simple products

TABLE 1. Quantification monads

Again, the algebras for the monad $\mathfrak{Prod}_L$ may be roughly described as cloven fibrations equipped with a form of linear simple products; in logical terms, these obey the inference rule

$$\frac{\alpha(i, f_0x) \longrightarrow \beta(fi, x)}{\forall x.\alpha(i,x) \longrightarrow \forall y.\beta(fi,y)}$$

The reason for calling this a linear quantifier is that while the ordinary universal quantifier behaves as a generalized conjunction, this linear quantifier behaves more like a tensor product, since one does not necessarily have projections (i.e. the instantiation rule $\forall x.\alpha(x) \vdash \alpha(a)$ is generally not valid).

We summarize the four constructions in Table 3.5.

4. The Dialectica Monad

In this section we consider the dialectica construction and analyse it in terms of the monads considered earlier. We begin by showing directly that the dialectica construction may be decomposed into two steps, following the $\exists\forall$ quantifier pattern suggested by the original interpretation. We then indicate another decomposition via a linear approximation followed by an exponential comonad. After that we come to the main issue, namely the observation that the dialectica construction is itself a (pseudo-)monad, provided the base category is cartesian closed; to this end, we show that the type-theoretic axiom of choice (i.e. Skolemization) in fact takes the form of a pseudo-distributive law of $\mathfrak{Prod}$ over $\mathfrak{Sum}$, thus making the composite into pseudo-monad. This explains the categorical content of Skolemization and the universal property of the dialectica construction.

4.1. The dialectica construction. We now introduce the main construction of interest.

Construction 4.1 (Dialectica construction). Let $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration. Define a category $\mathfrak{Dial}(\mathsf{p})$ as follows:

Objects: quadruples (I, X, U, α), where I, X and U are objects of the base B, and where $\alpha \in \mathsf{p}^{-1}(I \times X \times U)$ is an object in the fibre over $I \times X \times U$.

Morphisms: a map from (I, X, U, α) to (J, Y, V, β) is a quadruple (f, f_0, f_1, ϕ), where

- $f : I \longrightarrow J$ is a morphism in B
- $f_0 : I \times X \longrightarrow Y$ is a morphism in B
- $f_1 : I \times X \times V \longrightarrow U$ is a morphism in B
- $\phi = \phi(i, x, v) : \alpha(i, x, f_1(i, x, v)) \longrightarrow \beta(f(i), f_0(i, x), v)$ is a morphism in the fibre over $I \times X \times V$.

This is again fibred over B via the first projection. Note that the fibre over 1 is, up to equivalence, just the the total category of $\mathfrak{Prod}(\mathsf{p})$; in the original presentation [7] it was the latter category (for the special case where p was the subobject fibration) which was studied. This category can thus be viewed as a category of *dialectica propositions.*

As a first step towards understanding the structure of the fibration $\mathfrak{Dial}(\mathsf{p})$, we observe that the construction can be decomposed into two steps:

Lemma 4.2. *There is an isomorphism of fibrations, natural in* p*:*

$$\mathfrak{Dial}(\mathsf{p}) \cong \mathfrak{Sum}\mathfrak{Prod}(\mathsf{p}).$$

Proof. This is an easy direct verification; an object of $\mathfrak{Sum}\mathfrak{Prod}(\mathsf{p})$ has the form $(I, X, (I \times X, U, \alpha))$, and we may identify this with (I, X, U, α) as usual. A morphism $(I, X, U, \alpha) \longrightarrow (J, Y, V, \beta)$ has the form (f, f_0, F), where $f : I \longrightarrow J$, $f_0 : I \times X \longrightarrow Y$ and

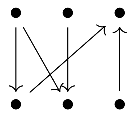
$$F : (I \times X, U, \alpha) \longrightarrow \langle f\pi_I, f_0\rangle^*(J \times Y, V, \beta)$$

in $\mathfrak{Prod}(\mathsf{p})$. Thus, F consists of maps $f_1 : I \times X \times V \longrightarrow U$ and $\phi(i, x, v) : \alpha(i, x, f_1(i, x, v)) \longrightarrow \beta(fi, f_0(i, x), v)$, and hence the tuple (f, f_0, f_1, ϕ) constitutes a map in $\mathfrak{Dial}(\mathsf{p})$. Further details are straightforward and left to the reader. $\square$

Concretely this says that $\mathfrak{Dial}(\mathsf{p})$ arises from p by first adding universal quantification, and then existential quantification. Thus we are justified in thinking of an object (I, X, U, α) as $\exists x \forall u.\alpha(i, x, u)$, and this makes clear how this abstract categorical construction is related to Gödel's original interpretation.

4.2. Skolemization as a distributive law. As already mentioned in the introduction, Skolemization is a syntactic translation which is based on the idea of replacing a quantifier combination of the form $\forall x : X.\exists u : U.\alpha(x, u)$ by one of the form $\exists f : U^X.\forall x : X.\alpha(x, fx)$. Since the existential quantifier ranges over a function type, this construction is only available in the presence of function types. Skolemization is the key ingredient in Gödel's dialectica interpretation, because it tells us how to replace a sequence of alternating quantifiers by a sequence of the form $\exists \vec{f} \forall \vec{x}$.

First, recall from the previous section that the monad $\mathfrak{Prod}$ admits a decomposition into a linear component followed by a coKleisli construction. The same idea works for $\mathfrak{Dial}$: we consider an auxiliary fibration $\mathfrak{Dial}_i(\mathsf{p})$ by taking the same objects as $\mathfrak{Dial}(\mathsf{p})$, but only those morphisms for which the third coordinate does not depend on the second (but it may depend on the first). Thus schematically the morphisms have the form



and we may think of objects (I, X, U, α) of $\mathfrak{Dial}_i(\mathsf{p})$ as predicates

$$\begin{bmatrix} \exists x \\ \forall u \end{bmatrix} \alpha(i, x, u),$$

where the branching quantifier indicates that u and x are independent.

How does this help? Well, if we have an expression with two such quantifiers, the quantifier independence allows us to contract it as follows:

$$\begin{bmatrix} \exists x \\ \forall u \end{bmatrix} \begin{bmatrix} \exists y \\ \forall v \end{bmatrix} \alpha(i, x, u, y, v) \mapsto \begin{bmatrix} \exists xy \\ \forall uv \end{bmatrix} \alpha(i, x, u, y, v).$$

This operation may be recast in terms of fibrations as a fibred functor

$$\mathfrak{Dial}_i^2(p) \longrightarrow \mathfrak{Dial}_i(\mathsf{p}); \qquad (I, X, U, Y, V, \alpha) \mapsto (I, X \times Y, U \times V, \alpha)$$

(the reader can easily verify that this assignment works on the level of morphisms precisely because of the independence conditions) which is the component at p of a multiplication on $\mathfrak{Dial}_i$. One can now work out that $\mathfrak{Dial}_i$ can be equipped with the structure of a pseudo-monad.

Now back to the general case of $\mathfrak{Dial}(\mathsf{p})$. Here, we cannot use this formula for the multiplication directly. Instead, we need to first break the quantifier dependence by using the axiom of choice.

Lemma 4.3. *Let* B *be a cartesian closed category. Then the inclusion of fibrations* $\mathcal{I}_{\mathsf{p}} : \mathfrak{Dial}_i(\mathsf{p}) \longrightarrow \mathfrak{Dial}(\mathsf{p})$ *has a fibred right adjoint.*

PROOF. This is a straightforward calculation along the same lines as for $\mathfrak{Prod}_L$. Explicitly, the right adjoint $\mathcal{R}_{\mathsf{p}}$ sends an object (I, X, U, α) to $(I, X, U^X, \breve{\alpha})$, where $\breve{\alpha}(i, x, f) = \alpha(i, x, fx)$. Then we have a bijective correspondence of maps in $\mathfrak{Dial}_i(\mathsf{p})$ from (J, Y, V, β) to $(I, X, U^X, \breve{\alpha})$ and maps in $\mathfrak{Dial}(\mathsf{p})$ from (J, Y, V, β) to (I, X, U, α). Details are left to the reader. $\Box$

We may also prove (although we shall not need this) that $\mathfrak{Dial}(\mathsf{p})$ is in fact the Kleisli category for the induced monad $!_{\mathsf{p}}$ on $\mathfrak{Dial}_i(\mathsf{p})$. Finally, we note that the condition that B is cartesian closed is not only sufficient but is also necessary for this result. (Just take p to be the identity fibration on B.)

Consider now the following map $\lambda_{\mathsf{p}} : \mathfrak{Prod}\mathfrak{Sum}(\mathsf{p}) \longrightarrow \mathfrak{Sum}\mathfrak{Prod}(\mathsf{p})$:

$$(I, X, U, \alpha) \mapsto (I, U^X, X, \hat{\alpha}), \qquad \hat{\alpha}(i, f, x) = \alpha(i, x, fx).$$

Thus λ_{p} is the composite of the right adjoint $\mathcal{R}_{\mathsf{p}}$ and the twist map.

Theorem 4.4. *When* B *is cartesian closed, the maps* λ_{p} *are the components of a pseudo-natural transformation*

$$\lambda : \mathfrak{Prod}\mathfrak{Sum} \longrightarrow \mathfrak{Sum}\mathfrak{Prod}$$

which underlies a pseudo-distributive law of $\mathfrak{Prod}$ *over* $\mathfrak{Sum}$.

PROOF. A direct verification is long and tedious, but straightforward (the many coherence conditions pose no real difficulty, as the coherence morphisms are all defined using uniqueness properties). In Section 4.3 we give a more conceptual proof using the fact that pseudo-distributive laws of $\mathfrak{Prod}$ over $\mathfrak{Sum}$ correspond to liftings of $\mathfrak{Sum}$ to the category of algebras for $\mathfrak{Prod}$. □

Remark 4.5. Even when $\mathfrak{Sum}$ and $\mathfrak{Prod}$ happen to be strict, this pseudo-distributive law will not be strict, unless (i) the cartesian closed structure on B is strict and (ii) the fibrations involved are split. The reason is that the coherence isomorphisms involved in the distributive law arise as comparison morphisms between various ways of reindexing along structure maps for the cartesian closed structure on the base.

The immediate consequence of Theorem 4.4 is of course that the composite $\mathfrak{Dial}$ can also be equipped with the structure of a pseudo-monad, and that the multiplication is given by

$$\mathfrak{Dial}^2(p) \longrightarrow \mathfrak{Dial}(\mathsf{p}); \quad (I, X, U, Y, V, \alpha) \mapsto (I, X \times Y^U, U \times V, \hat{\alpha})$$

where of course $\hat{\alpha}(i, x, f, u, v) = \alpha(i, x, u, fx, v)$.

Theorem 4.6. *When* B *is cartesian closed, the dialectica construction is (the underlying 2-functor of) a pseudo-monad on* $\mathfrak{CFib}$. *The pseudo-algebras for this monad are the cloven fibrations which have (chosen) simple products and simple coproducts satisfying the distributivity condition*

$$\forall u \exists x.\alpha(i, u, x) \cong \exists f \forall u.\alpha(i, fu, u). \tag{4.1}$$

Indeed, when $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ is a (pseudo-)algebra for $\mathfrak{Dial}$, then the following diagram must commute up to natural isomorphism:

$$\begin{array}{ccccc}
\mathfrak{Prod}\mathfrak{Sum}(\mathsf{E}) & & \xrightarrow{\quad\lambda\quad} & & \mathfrak{Sum}\mathfrak{Prod}(\mathsf{E}) \\
{\scriptstyle \mathfrak{Prod}(\exists)}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathfrak{Sum}(\forall)} \\
\mathfrak{Prod}(\mathsf{E}) & \xrightarrow[\forall]{} & \mathsf{E} & \xleftarrow[\exists]{} & \mathfrak{Sum}(\mathsf{E})
\end{array}$$

Chasing this diagram both ways around for an object (I, X, U, α) gives $\forall u \exists x.\alpha(i, u, x)$ on the one hand, and $\exists f \forall x.\alpha(i, fx, x)$ on the other.

We remark in passing that there is a dual result, which comes for free now: there is also a pseudo-distributive law

$$\kappa : \mathfrak{Sum}\mathfrak{Prod} \longrightarrow \mathfrak{Prod}\mathfrak{Sum}$$

making the composite $\mathfrak{Prod}\mathfrak{Sum}$ into a pseudo-monad whose algebras are fibrations with (chosen) simple products and coproducts satisfying the distributivity condition

$$\exists x \forall u.\alpha(i, x, u) \cong \forall f \exists x.\alpha(i, fx, x). \tag{4.2}$$

4.3. Proof of main result. We now give a proof that $\mathfrak{Dial}$ carries a pseudo-monad structure by verifying that $\mathfrak{Prod}$ distributes over $\mathfrak{Sum}$ (throughout, we of course assume that B is cartesian closed). As is well-known (see e.g. [6]), to give a distributive law of $\mathfrak{Prod}$ over $\mathfrak{Sum}$ is equivalent to giving a lifting of $\mathfrak{Sum}$ to the 2-category of pseudo-algebras for $\mathfrak{Prod}$. In our case, the fact that the monad $\mathfrak{Sum}$

has the KZ property implies that such a lifting is essentially unique. It has been shown[3] that to give this lifting it suffices to prove the following three lemmata:

Lemma 4.7. *When a fibration* $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$ *has (chosen) simple products, then so does* $\mathfrak{Sum}(\mathsf{p})$.

Lemma 4.8. *When* p *has (chosen) simple products, then the unit* $\eta : \mathsf{p} \longrightarrow \mathfrak{Sum}(\mathsf{p})$ *preserves them.*

Lemma 4.9. *When* p *has (chosen) simple products and* $H : \mathsf{p} \longrightarrow \mathfrak{Sum}(\mathsf{q})$ *preserves them, then so does the extension* $H^+ : \mathfrak{Sum}(\mathsf{p}) \longrightarrow \mathfrak{Sum}(\mathsf{q})$.

In the proofs of these lemmas, we shall suppress some coherence data arising from the reindexing of the fibrations involved and the fibred functors between them in order to keep things readable.

Proof of Lemma 4.7. Assume p has chosen simple products, i.e. right adjoints $\forall_{I,X}$ to weakening functors $\pi_I^* : \mathsf{E}_I \longrightarrow \mathsf{E}_{I\times X}$ subject to the BCC. We need to show that $\mathfrak{Sum}(\mathsf{p})$ has the same structure. Let

$$\forall_{I,X} : \mathfrak{Sum}(\mathsf{E})_{I\times X} \longrightarrow \mathfrak{Sum}(\mathsf{E})_I$$

be defined by

$$(I \times X, U, \alpha(i, x, u)) \mapsto (I, U^X, \forall x.\alpha(i, x, fx))$$

where f is of type U^X and where we write $\forall x$ instead of the more cumbersome $\forall_{I,X}$.

This mapping is right adjoint to weakening: a morphism

$$(I, V, \beta(i, v)) \longrightarrow (I, U^X, \forall x.\alpha(i, x, fx))$$

consists of

$$f : I \times V \longrightarrow U^X \qquad \text{and} \qquad \phi(i, v) : \beta(i, v) \longrightarrow \forall x.\alpha(i, x, f(i, v)(x)).$$

Such data is in 1-1 correspondence with maps

$$\hat{f} : (I \times X) \times V \longrightarrow U \qquad \text{and} \qquad \hat{\phi}(i, x, v) : \beta(i, v) \longrightarrow \alpha(i, x, \hat{f}(i, x, v))$$

i.e. with morphisms $\pi^*(I, X, \beta(i, v)) = (I \times X, V, \beta(i, v)) \longrightarrow (I \times X, U, \alpha(i, x, u))$.

For the BCC, consider a morphism $h : I \longrightarrow J$, and an object $(I\times X, U, \alpha(i, x, u))$ over I, and consider the diagram

$$\begin{array}{ccccc}
J \times U^X & \xleftarrow{\ \pi\ } & J \times U^X \times X & \xrightarrow{\ v\ } & J \times X \times U \\
\Big\downarrow {\scriptstyle h\times U^X} & & \Big\downarrow {\scriptstyle h\times U^X\times X} & & \Big\downarrow {\scriptstyle (h\times X)\times U} \\
I \times U^X & \xleftarrow[\ \pi\ ]{} & I \times U^X \times X & \xrightarrow[\ w\ ]{} & I \times X \times U
\end{array}$$

Here, the left hand square is a pullback, and v, w are the obvious maps built from the evaluation and pairing, so the right hand square commutes.

Using the coherence for reindexing we get a canonical isomorphism

$$(h \times U^X \times X)^* v^* \alpha \longrightarrow w^*(h \times X \times U)^* \alpha$$

[3]I am grateful to the referee for pointing me to this result, of which the preorder enriched version can be found in [15].

Applying $\forall_X$ to this isomorphism and precomposing with the canonical isomorphism from the BCC for p gives

$$(h \times U^X)^* \forall_X v^* \alpha \longrightarrow \forall_X w^* (h \times X \times U)^* \alpha.$$

It is now readily verified that this map is the third component of the canonical map

$$h^* \forall_X (I \times X, U, \alpha) \longrightarrow \forall_X (h \times X)^* (I \times X, U, \alpha)$$

so that the BCC holds for the product in $\mathfrak{Sum}(\mathsf{p})$ as we have defined it. $\square$

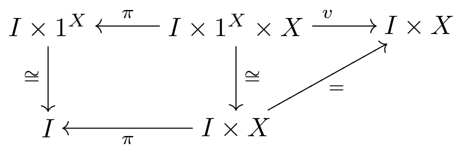
Proof of Lemma 4.8. Consider $\alpha \in \mathsf{E}_{I\times X}$. One the one hand we have

$$\eta \forall_X \alpha = (I, 1, \pi_I^* \forall_X \alpha)$$

and on the other hand we have

$$\forall_X \eta(\alpha) = \forall_X (I \times X, 1, \pi_{I\times X}^* \alpha) = (I, 1^X, \forall_X v^* \pi_{I\times X}^* \alpha)$$

where $v : I \times 1^X \times X \longrightarrow I \times X \times 1$ is again the canonical map. These two objects are canonically isomorphic, since we may use the BCC for p for the pullback in the diagram below.



$\square$

Proof of Lemma 4.9. We are given a product-preserving map $H : \mathsf{p} \longrightarrow \mathfrak{Sum}(\mathsf{q})$; we will write, for $\alpha \in \mathsf{E}_I$,

$$H(\alpha) = (I, H_0[\alpha], H_1[\alpha])$$

First, to say that H is a fibred functor means that for a map $f : J \longrightarrow I$ there are canonical isomorphisms

$$\begin{aligned} I \times H_0[f^*\alpha] &\cong I \times H_0[\alpha] \\ H_1[f^*\alpha] &\cong (f \times 1)^* H_1[\alpha] \end{aligned}$$

To say that H preserves simple products means that there are canonical isomorphisms

$$\begin{aligned} I \times H_0[\forall x.\alpha] &\cong I \times H_0[\alpha]^X \\ H_1[\forall x.\alpha] &\cong \forall x.\widehat{H_1[\alpha]} \end{aligned}$$

where $\widehat{H_1[\alpha]}$ is the result of reindexing $H_1[\alpha]$ along $I \times H_0[\alpha]^X \times X \longrightarrow I \times X \times H_0[\alpha]$.

Next, the extension $H^\dagger$ is defined as $\mu_{\mathsf{q}} \mathfrak{Sum}(H)$. Explicitly,

$$H^\dagger(I, X, \alpha) = (I, X \times H_0[\alpha], H_1[\alpha])$$

where we suppress the reindexing along the associativity isomorphism. To show that $H^\dagger$ preserves products, we consider an object $(I \times X, U, \alpha)$, and compute both

$\forall_X H^\dagger(I \times X, U, \alpha)$ and $H^\dagger \forall_X(I \times X, U, \alpha)$. We have

$$\begin{array}{rcll}
H^\dagger \forall_X(I \times X, U, \alpha) & = & H^\dagger(I, U^X, \forall x.\hat{\alpha}) & \text{by def. of } \forall \text{ in } \mathfrak{Sum} \\
& = & (I, U^X \times H_0[\forall x.\hat{\alpha}], H_1[\forall x.\hat{\alpha}]) & \text{by def. of } H^\dagger \\
& \cong & (I, U^X \times H_0[\hat{\alpha}]^X, \forall x.\widehat{H_1[\hat{\alpha}]}) & H \text{ preserves } \forall \\
& \cong & (I, (U \times H_0[\alpha])^X, \forall x.\widehat{H_1[\alpha]}) & \\
& = & \forall_X(I \times X, U \times H_0[\alpha], H_1[\alpha]) & \text{by def. of } \forall \text{ in } \mathfrak{Sum} \\
& = & \forall_X H^\dagger(I \times X, U, \alpha) & \text{by def. of } H^\dagger
\end{array}$$

The isomorphisms in the middle arise as follows: for the first one, we use the fact that H preserves reindexing and simple products. For the second, it seems as if we've lost a hat in the process, but consider the diagram

$$\begin{array}{ccccc}
I \times (U \times H_0[\alpha])^X & \longleftarrow & I \times (U \times H_0[\alpha])^X \times X & \xrightarrow{v} & I \times X \times U \times H_0[\alpha] \\
\cong \downarrow & & \cong \downarrow & & \uparrow w \\
I \times U^X \times H_0[\hat{\alpha}]^X & \longleftarrow & I \times U^X \times H_0[\hat{\alpha}]^X \times X & \xrightarrow[v']{} & I \times U^X \times X \times H_0[\hat{\alpha}]
\end{array}$$

The object $H_1[\alpha]$ lives over the top right corner; reindexing along w and then along v' gives $H_1[\hat{\alpha}]$ (because H commutes with reindexing) and then $\widehat{H_1[\hat{\alpha}]}$. When we reindex the latter along the canonical isomorphism in the middle, we get something isomorphic to $\widehat{H_1[\alpha]}$. Finally, an application of the BCC for the left hand square gives the desired isomorphism. $\square$

5. Concluding thoughts

We end with a few loose ends, and some possible avenues for further research.

The Chu construction. There is a cousin of the dialectica construction which we haven't mentioned yet, namely the Chu construction (see e.g. [2]); this construction has been compared to the dialectica interpretation in [9] because of certain formal similarities. One can define, for an arbitrary fibration p, a category $\mathsf{Chu}(\mathsf{p})$, by taking the subcategory of $\mathfrak{Sum}_L(\mathsf{p})$ on the same objects, but only on those morphisms (f, f_0, ϕ) for which the third component is an isomorphism. At present, I'm not sure whether this is a fruitful course of action: for one thing, it is not clear whether it gives a nice universal property of the Chu construction along the same lines as for the other constructions. I should point out that a completely different approach to finding a universal property of the Chu construction is given in a paper by Dusko Pavlovic [22].

The case of locally cartesian closed categories. In this paper we have considered the simple (in the sense of simply typed) version of the dialectica construction, mainly because of the fact that this is how the construction appears in the literature. However, in [14] a dependently typed version is suggested. In this case, one works over a locally cartesian closed base category, in which case one can use the families monad and its dual.

Linear quantifiers. We have been brief about the monads for "linear" quantification and in particular about their algebras. This is partly because it requires a bit more work to describe these structures in detail, and also because at this point it is not fully clear what role they should play. One thing which is worth mentioning is that there is a connection with independence-friendly and branching quantifiers; the linear quantifiers introduced here may be viewed as an instance of these. An example of a fibration which has this structure arises from work by Abramsky and Väänänen on the Hodges semantics for independence-friendly logic [1]. However, at present there is no general categorical treatment of branching quantifiers available, and this will be left for future work.

Bicompletion of a fibration. The dialectica construction combines the monads for simple products and coproducts by means of a distributive law between them. However, there are other ways of combining monads than by using distributive laws. Here, we would like to mention that there is another pseudo-monad on $\mathfrak{CFib}(\mathsf{B})$, which takes a fibration p and produces a fibration $\Lambda(\mathsf{p})$, the free bicompletion of p. For this construction, cartesian closure of the base category is not needed. The pseudo-algebras for Λ are the fibrations which have (chosen) simple products and -coproducts. Roughly, $\Lambda(\mathsf{p})$ is constructed by iterating the monads $\mathfrak{Sum}$ and $\mathfrak{Prod}$ and by taking a suitable colimit. Explicitly, an object of $\Lambda(\mathsf{p})$ is a tuple $(I, X_1, X_2, \ldots, X_n, \alpha)$, where α is an object of $\mathsf{p}^{-1}(I \times X_1 \times \cdots \times X_n)$. We think of such an object as the predicate $\exists x_1 \forall x_2 \ldots \forall x_n . \alpha(i, x_1, \ldots, x_n)$. The morphisms in $\Lambda(\mathsf{p})$ are generated by those of the iterates $(\mathfrak{Sum}\mathfrak{Prod})^n(\mathsf{p})$, together with newly added isomorphisms forcing the resulting category to have the desired structure, e.g. $(I, X_1, 1, 1, X_2, \alpha) \cong (I, X_1, X_2, \alpha)$, et cetera. A more detailed description and analysis of this construction is in preparation.

The free bicompletion of an ordinary category was first studied in [17], where it was characterized in terms of its so-called *softness* property.

Appendix A. Coherence for $\mathfrak{Sum}$.

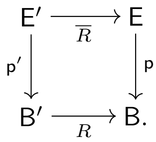
This appendix establishes the coherence for the pseudo-monad structure on $\mathfrak{Sum}$. Let me say first that a direct verification is not difficult; the work is greatly reduced by making use of that the multiplication is left adjoint to the unit. Another approach would be to replace $\mathfrak{Sum}(\mathsf{p})$ by its closure in $\mathfrak{Fam}(\mathsf{p})$; the coherence for the resulting pseudo-monad would then be automatic, and one would only need to show that it is equivalent to $\mathfrak{Sum}$.

However, one may still ask for a more conceptual explanation; part of the motivation here is that there is a sense in which $\mathfrak{Sum}$ is a club[4] (see [18]), i.e. determined by its action on the terminal fibration; moreover, one would hope that coherence for $\mathfrak{Sum}$ is a formal consequence of the coherence at the terminal object. This should also help in dealing with pseudo-monads on $\mathfrak{CFib}(\mathsf{B})$ similar to $\mathfrak{Sum}$ (but perhaps not satisfying the KZ property).

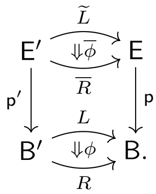
In what follows I heavily use the fact that the forgetful 2-functor $\mathfrak{CFib} \longrightarrow \mathfrak{Cat}$ is a 2-fibration (see [12] for detailed discussion). In particular, we can lift natural transformations from $\mathfrak{Cat}$ to $\mathfrak{CFib}$ as follows: consider a fibration $\mathsf{E} \xrightarrow{p} \mathsf{B}$, functors

[4] As far as I'm aware, no precise definition of pseudo-club exists.

$L, R : \mathsf{B}' \longrightarrow \mathsf{B}$ in the base, and a natural transformation $\phi : L \Rightarrow R$. Now suppose we have a lifting $\overline{R}$ of R, as in

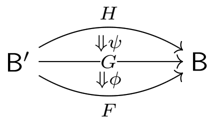


Then we may define a lift of L by letting $\widetilde{L}(\alpha) = \phi_I^*(\alpha)$, where $\mathsf{p}'(\alpha) = I$. Moreover, for each object α over I we get a cartesian morphism $\overline{\phi}_{(I,\alpha)} : \phi_I^*(\alpha) \longrightarrow \alpha$ over ϕ_I. These assemble to give a natural transformation $\overline{\phi} : \widetilde{L} \Rightarrow \overline{R}$ over ϕ, as in



We call $\overline{\phi}$ the *cartesian lifting* of ϕ. Note that by construction all components of $\overline{\phi}$ are cartesian. The assignment $\phi \mapsto \overline{\phi}$ has various other good naturality properties, which we won't spell out.

Instead of just lifting a single natural transformation we can also lift entire pasting diagrams; for example, a diagram



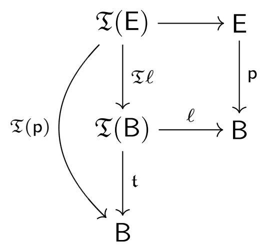
can be lifted by first lifting ϕ, and then lifting ψ. One can also lift the composite $\psi\phi$ in one single step, and the two outcomes differ by a unique vertical transformation, whose component at an object α over I is simply the unique vertical map mediating between $\psi_I^*\phi_I^*(\overline{F}\alpha)$ and $(\phi\psi)_I^*(\overline{F}(\alpha))$. A similar observation can be made for whiskering ϕ on either side. The result of this is, that, up to unique vertical natural isomorphism, the lifting of a pasting diagram is well-defined.

We are now in a position to explain in which sense the coherence for $\mathfrak{Sum}$ comes for free. To this end, I will describe data on the terminal object from which one can construct a pseudo-monad on $\mathfrak{CFib}(\mathsf{B})$.

Datum 1. A (split) fibration $\mathsf{t} : \mathfrak{T}(\mathsf{B}) \longrightarrow \mathsf{B}$.

Datum 2. A functor $\ell : \mathfrak{T}(\mathsf{B}) \longrightarrow \mathsf{B}$ (which is not supposed to be a fibred functor).

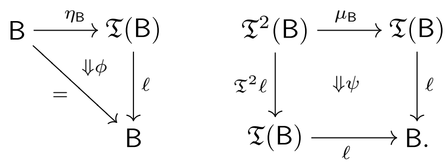
Given these, we may to extend $\mathfrak{T}$ to a 2-functor on $\mathfrak{CFib}$ via pullback:



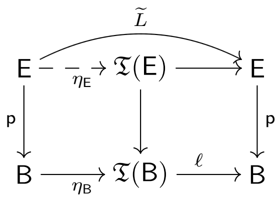
just as we did for $\mathfrak{Sum}$ and $\mathfrak{Fam}$.

Datum 3. Fibred functors $\eta_{\mathsf{B}} : \mathsf{B} \longrightarrow \mathfrak{T}(\mathsf{B})$ and $\mu_{\mathsf{B}} : \mathfrak{T}^2(\mathsf{B}) \longrightarrow \mathfrak{T}(\mathsf{B})$.

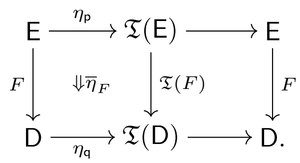
Datum 4. Natural isomorphisms (not fibred)



These allow us, using the lifting properties of natural transformations, to extend η_{B} and μ_{B} to pseudo-natural transformations $(\eta, \overline{\eta}) : 1 \longrightarrow \mathfrak{T}, (\mu, \overline{\mu}) : \mathfrak{T}^2 \longrightarrow \mathfrak{T}$. More explicitly, here is how we get the unit at $\mathsf{p} : \mathsf{E} \longrightarrow \mathsf{B}$. We lift the natural transformation $\phi : \ell\eta_{\mathsf{B}} \longrightarrow 1_{\mathsf{B}}$ to a natural transformation $\overline{\phi}_{\mathsf{E}} : \widetilde{L} \longrightarrow 1_{\mathsf{E}}$. Now factorize this using the fact that the right hand square in the diagram below is a pullback:



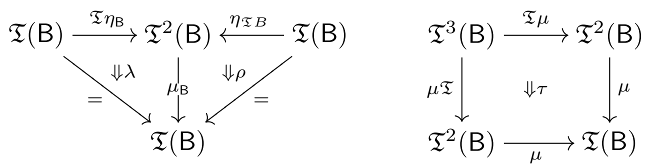
The left hand square is in general not a pullback. We now obtain (using the lifting properties of natural transformations) a natural transformation $\overline{\eta}_F$ as in



making $(\eta, \overline{\eta})$ a pseudo-natural transformation. Completely analoguous is the construction of μ and $\overline{\mu}$ from μ_{B} giving $(\mu, \overline{\mu}) : \mathfrak{T}^2 \longrightarrow \mathfrak{T}$.

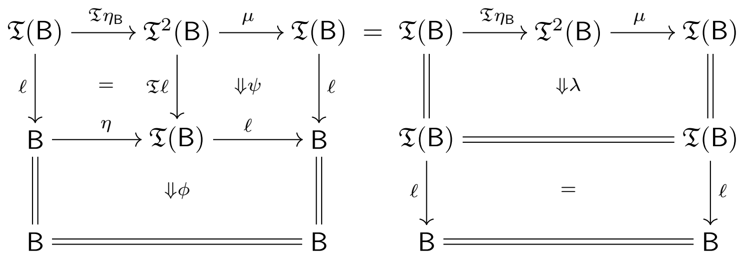
Finally, we specify the data needed to ensure that we can endow $\mathfrak{T}$ with the structure of a pseudo-monad.

Datum 5. Fibred natural isomorphisms

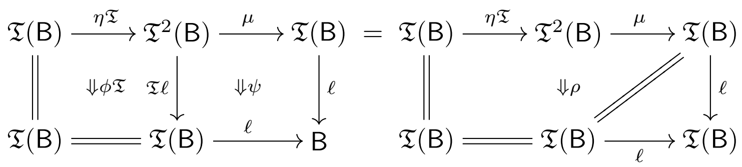


subject to the usual coherence conditions for the modifications up to which the monad laws hold, and in addition three conditions relating λ, ρ and τ to ϕ and ψ:

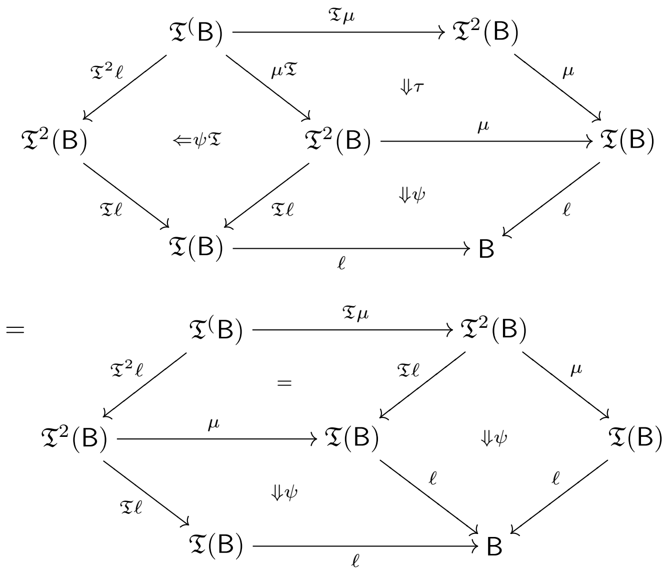
(a)



(b)



(c)



In the case of $\mathfrak{Sum}$, all of the isomorphisms $\lambda, \rho, \tau, \phi, \psi$ are simply coherence isomorphisms for the product structure on B, and the coherence conditions are immediate.

We can now lift λ, ρ and τ. The point is here that we know that λ, ρ and τ fit into pasting diagrams whose composite is equal to a pasting diagram involving ψ, ϕ, and thus that we can find liftings which are of the right type (i.e. have the correct domain and codomain). It now also follows immediately by uniqueness of such liftings that the coherence conditions are inherited from those at B.

This gives the main observation of this section:

Theorem A.1. *The data 1-5 described above, induces a pseudo-monad structure on $\mathfrak{T}$. The underlying pseudo-functor is strict, and when each of $\phi, \psi, \lambda, \rho$ and τ are strict then $(\mathfrak{T}, \eta, \mu)$ is a strict 2-monad.*

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