HEREDITARILY COMPACT SPACES.*

By A. H. Stone.

1. Introduction. By definition, a *hereditarily compact* space, or a "Zariski" or "Noether" space, is a topological space all of whose subspaces are compact.[1] Such spaces have received some attention [9, 10] because they arise in algebraic geometry (in the Zariski topology) and in some other algebraic constructions. Here we study these spaces on their own account. In the applications they are usually T_1 but not T_2; in fact, a T_2 hereditarily compact space is necessarily finite. However, we do not assume any separation axioms except where they are explicitly stated. We begin by giving some alternative characterizations (§ 2), and considering some properties related to some of them (§ 3). In § 4 we associate to every hereditarily compact space a topologically invariant ordinal number, its "type"; this corresponds to the dimension in the application to algebraic geometry. This permits the "construction" of all hereditarily compact spaces (§ 5). In § 6 we discuss the effect of various standard operations on such spaces on their types, and in § 7 we consider the countable hereditarily compact spaces in more detail.

Notation. A space X is *discrete* if each point $p \in X$ has a neighborhood in X consisting of p itself; X is *weakly discrete* if each $p \in X$ has a neighborhood in X consisting of a finite set of points. (For T_1 spaces these notions are equivalent.) An indexed family $\{U_\lambda\}$ of subsets of X is called *finite* if $U_\lambda = \emptyset$ for all but finitely many values of λ.

2. Characterizations. We begin by observing that, in the definition of hereditary compactness, it is not necessary to specify that *all* subspaces are compact, and moreover the kind of compactness considered does not greatly matter. Incidentally we also obtain some further characterizations.

THEOREM 1. *The following statements about an arbitrary space X are equivalent:*

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[1] Throughout this paper, "compact" means "quasicompact" in the sense of Bourbaki; that is, every open covering has a finite subcovering.

(1) *Every subspace of X is compact.*

(1_c) *Every countable subspace of X is compact.*

(1_o) *Every open subspace of X is compact.*

(2), (2_c), (2_o) *Every subspace (or countable, or open subspace) of X is sequentially compact.*

(3), (3_c), (3_o) *Every subspace (or countable, or open subspace) of X is countably compact.*

(4) *X has no weakly discrete infinite subspace.*

(5) *Every strictly decreasing sequence of closed subsets of X is finite.*

(6) *X has a sub-base $\mathfrak{B}$ of open sets such that every strictly increasing sequence of finite unions of members of $\mathfrak{B}$ is finite.*

Remark. The equivalence of (1), (1_o) and (5) is known (see [10] and Exposé 1 (by P. Cartier) of the Séminaire C. Chevalley, vol. 1, 1956-8).

Proof. Because countable compactness is implied by compactness or sequential compactness, it is enough to prove the implications (3_c) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (5), (3_o) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2). All are easy; by way of example we prove (4) $\Rightarrow$ (5). If $C_1 \supset C_2 \supset \cdots$ is an infinite strictly decreasing sequence of closed subsets of X, pick $p_n \in C_n - C_{n+1}$ $(n = 1, 2, \cdots)$; the points p_n are all distinct, so the set $P = \{p_n\}$ is infinite. But each p_n has the neighborhood $P \cap (X - C_{n+1})$ in P, and this consists of the n points $p_1, \cdots, p_n$ only. Thus (4) is contradicted.

Consider now the following modified compactness conditions (all weaker than compactness) on a space X. (The list could easily be extended.)

(A) Every open covering of X has a finite subsystem whose union is dense in X.

(B_1) Every locally finite system of open sets in X is finite.

(B_2) Every locally finite system of disjoint open sets is finite.

(B_3) Every countable covering of X has a finite subsystem whose union is dense in X.

(C_1) Every locally finite open covering of X has a finite subcovering.

(C_2) Every countable locally finite open covering of X has a finite subcovering.

(D_1) Every locally finite open covering of X has a finite dense subsystem.

(D_2) Every countably infinite open covering of X has a proper dense subsystem.

(E_1) Every star-finite open covering of X is finite.

(E_2) Every countably infinite open covering of X by sets each of which meets at most two others has a proper dense subsystem.

(F) Every continuous real-valued function on X is bounded.

Property (B_2) is "feeble compactness" [7]; (B_1) has been called "light compactness" [1]; (F) is "pseudocompactness" [2]. It is easily seen that each of these properties implies the next, and that (B_1), (B_2), (B_3) are equivalent (see [1, 6]), and similarly (C_1) and (C_2), (D_1) and (D_2), and (E_1) and (E_2) are equivalent; it can be shown by examples that there are no other implications between them in general.[2] All except (A) are implied by countable compactness, and are equivalent to it for normal T_1 spaces [2], but not in general. Weaker separation axioms suffice for some other equivalences (for instance, regularity makes (B)—(E) equivalent [3, 4, 5]). But the hereditary forms of all these properties are equivalent, irrespective of separation axioms, as the next theorem shows.

THEOREM 2. *The following statements about an arbitrary space X are equivalent:*

(1) (1_c) *Every subspace (or countable subspace) of X has property* (A).

(2) (2_c) *Every subspace (or countable subspace) of X has property* (F).

(3) *X has no infinite discrete subspace.*

Further, if X is T_1, these statements are equivalent to the statements in Theorem 1.

It follows, of course, that any of the properties (B_1)—(E_2) could replace (A) or (F) here.

Proof. As (A) $\Rightarrow$ (F), it is enough to prove (2_c) $\Rightarrow$ (3) $\Rightarrow$ (1) and that if X is T_1 then (3) implies property (4) of Theorem 1. The first and last of these are trivial; to prove the second, suppose Y is a subspace of X which

[2] Even for T_1 spaces. An example having property (D) but not (C) is given in [1, p. 502] (note, however, that the statement on p. 503 lines 6, 7 is incorrect). It can be modified to give an example satisfying (E) but not (D). The example given at the beginning of § 3 below has property (A) without being countably compact. The usual space of countable ordinals is countably compact, and so satisfies (B), but does not satisfy (A). A suitable union of a sequence of spaces, each of which has no non-constant continuous function, will satisfy (F) but not (E). A T_0 space satisfying (C) but not (B) (from which a T_1 example can be derived by standard technique) is the set of all finite non-empty sets F of positive integers, in which each F has the single neighborhood $U(F)$ = family of non-empty subsets of F. For the properties discussed here (and many others), see [3, 4, 5, 6].

does not have property (A), and let $\{U_\lambda\}$ be an open covering of Y. If no finite subsystem of $\{U_\lambda\}$ is dense in Y, pick $z_1 \in Y$, say $z_1 \in U_{\lambda_1}$; pick $z_2 \in Y - \bar{U}_{\lambda_1}$, say $z_2 \in U_{\lambda_2}$, and generally pick $z_n \in Y - (\bar{U}_{\lambda_1} \cup \cdots \cup \bar{U}_{\lambda_{n-1}})$, say $z_n \in U_{\lambda_n}$. Put $V_n = U_{\lambda_n} \cap Y - (\bar{U}_{\lambda_1} \cup \cdots \cup \bar{U}_{\lambda_{n-1}})$, an open set containing z_n. The sets $V_1, V_2, \cdots$, are disjoint, so $Z = \{z_n\}$ is an infinite discrete subset of Y, contradicting (3).

Remark. In Theorem 2, in contrast to Theorem 1, it is not enough to require that every *open* subspace of X has the relevant properties, even if X is T_1. This is shown by the following example. Let X be the union of two disjoint infinite sets Y, Z; a subset of X is to be open if it is $\emptyset$ or contains all but finitely many points of Z. Then X is a T_1 space and every open subspace of X has property (A), but the closed subspace Y of X is discrete and does not even have property (F).

Further, the T_0 axiom (instead of T_1) would not suffice for the equivalence of the statements in Theorems 1 and 2. For let X be the space of positive integers, with $\emptyset$, X and the sets $\{1, 2, \cdots, n\}$ $(n = 1, 2, \cdots)$ as the only open sets. Every subset of X has property (A), but X is not compact.

3. Irreducibility. A space X is *irreducible* if it is not the union of two proper closed subsets; equivalently, every two non-empty open subsets of X intersect. It is known [10] that a hereditarily compact space is always expressible as the union of a finite number of irreducible sets. Here we amplify this property. We say that a space X is *semi-irreducible* if every family of disjoint (non-empty) open subsets of X is finite. Thus every hereditarily compact space is semi-irreducible; but the converse is false, even for T_1 spaces. (Take, for example, X to be an uncountable set in which the closed sets are X and its countable subsets; X is T_1 and irreducible but not even countably compact.) We note the following easily verified properties:

(1) If $A \subset X$, A is irreducible, or semi-irreducible, if and only if $\bar{A}$ has the corresponding property.

(2) If X is irreducible, or semi-irreducible, then so is every open subset of X.

(3) If X is semi-irreducible and non-empty, then X contains a non-empty maximal open irreducible subspace, and also a non-empty maximal irreducible subspace (which must be closed, from (1)).

(4) X is hereditarily irreducible if and only if the open sets of X are linearly ordered by inclusion; if X is T_1, it is hereditarily irreducible if and only if it has at most one point.

THEOREM 3. *The following statements about an arbitrary space X are equivalent:*

(i) *X is semi-irreducible.*

(ii) *There is a finite system of disjoint open irreducible subspaces $U_1, \cdots, U_n$ of X such that $\bigcup \bar{U}_i = X$.*

(iii) *X is the union of a finite number of disjoint irreducible subspaces, each the difference between two closed sets.*

(iv) *X is the union of a finite number of closed irreducible subspaces.*

(v) *X is the union of a finite number of semi-irreducible subspaces.*

(vi) *There is an integer N such that X does not contain more than N disjoint non-empty open sets.*

(vii) *X has only finitely many regular open sets.*[3]

Proof. (i) $\Rightarrow$ (ii) By Zorn's lemma there is a maximal system $\mathcal{U}$ of disjoint open irreducible subsets of X; from (i), this system is finite, say $\mathcal{U} = \{U_1, \cdots, U_n\}$. Let $V = X - \bigcup \bar{U}_i$; from (2) and (3) above, if $V \neq \emptyset$, V contains a non-empty open irreducible subset U_{n+1}, contradicting the maximality of $\mathcal{U}$. Hence $V = \emptyset$ and $X = \bigcup \bar{U}_i$.

(ii) $\Rightarrow$ (iii) Put $Y_i = \bar{U}_i - \bigcup \{\bar{U}_j \mid j < i\}$ $(1 \leqq i \leqq n)$; then $U_i \subset Y_i \subset \bar{U}_i$, so $\bar{Y}_i = \bar{U}_i$ and Y_i is irreducible by (1) above. Since $X = \bigcup Y_i$, (iii) follows.

(iii) $\Rightarrow$ (iv) If $X = \bigcup Y_i$, where Y_i is irreducible, then $X = \bigcup \bar{Y}_i$, where $\bar{Y}_i$ is irreducible.

(iv) $\Rightarrow$ (v) trivially, because every irreducible space is semi-irreducible.

(v) $\Rightarrow$ (vi) Say $X = X_1 \cup \cdots \cup X_n$, where each X_i is semi-irreducible. Because (i) implies (iii), each X_i is the union of a finite number of irreducible sets, so we may write $X = Y_1 \cup \cdots \cup Y_N$, where each Y_i is irreducible. Suppose that $U_1, \cdots, U_{N+1}$ are disjoint non-empty open subsets of X. Each U_i meets some $Y_{j(i)}$, and we must have $j(i_1) = j(i_2)$ for two distinct integers i_1, i_2 (between 1 and $N+1$). Thus we may assume that both U_1 and U_2 meet Y_1; but this contradicts the irreducibility of Y

The implication (vi) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (vii) Let V be any regular open set in X; we show V is one of the 2^n interiors of unions of the sets $\bar{U}_i$ in (ii). We may suppose V meets $U_1, \cdots, U_r$ and is disjoint from $U_{r+1}, \cdots, U_n$ (where $0 \leqq r \leqq n$). Then, if $i \leqq r$, the closure $\mathrm{Cl}(V \cap U_i)$ of $V \cap U_i$ in X must be $\bar{U}_i$; for, as U_i is irreducible, the non-empty open set $V \cap U_i$ is dense in U_i. Hence

[3] A set G is "regular open" if and only if $G = \mathrm{Int}(\bar{G})$.

$$V = \operatorname{Int}(\bar{V}) = \operatorname{Int}(\bigcup \operatorname{Cl}(V \cap U_i)) = \operatorname{Int}(\bar{U}_1 \cup \cdots \cup \bar{U}_r).$$

(vii) $\Rightarrow$ (i) If X has an infinite family of disjoint (non-empty) open sets $G_1, G_2, \cdots$, the sets $X - \bar{G}_1, X - \bar{G}_2, \cdots$ provide infinitely many distinct regular open sets.

COROLLARY 1. *If X is semi-irreducible (a fortiori if X is hereditarily compact), X has only a finite number of components.*

For an irreducible space is connected.

COROLLARY 2. *If X is regular, X is semi-irreducible if and only if X has only finitely many open sets.*[4]

For, in a regular space, every open set is a union of regular open sets.

Remarks. (a) From (iv) of Theorem 3, we can write any semi-irreducible space X as $X_1 \cup \cdots \cup X_n$, where each X_i is closed and irreducible, and where no X_i is contained in any other. It is easy to see that the sets $X_1, \cdots, X_n$ are then uniquely determined, apart from their order. (Cf. [10] for the hereditarily compact case.)

(b) A connected semi-irreducible T_1 space need not be irreducible.

THEOREM 4. *For any Hausdorff space X, the following assertions are equivalent:*

(I) *X is hereditarily compact.*
(II) *X is semi-irreducible.*
(III) *X is finite.*

For trivially (I) implies (II) and (III) implies (I); that (II) implies (III) follows from Theorem 3(iv) since an irreducible Hausdorff space can have at most one point.

THEOREM 5. *The following statements about an arbitrary space X are equivalent to those in Theorem 2, and thus to the hereditary compactness of X if X is T_1:*

(1) *Every subspace of X is semi-irreducible.*
(1_c) *Every countable subspace of X is semi-irreducible.*

For if every subspace of X has property (A) (§ 2), it is clearly semi-

[4] "Regular" means that each point has a basis of closed neighborhoods; the T_1 axiom is not assumed. In the hereditarily compact case, Corollary 2 is due to Nollet [9].

irreducible. Conversely, if every countable subspace of X is semi-irreducible, X can contain no infinite discrete subspace.

Remark. The analogous statement (1_0)—that every *open* subspace of X is semi-irreducible—would not be equivalent to the statements in Theorem 5 in general, being equivalent to the semi-irreducibility of X.

4. The type of a hereditarily compact space. Let X be a hereditarily compact space, fixed for the moment. We assign, to each closed subspace of X, an ordinal number, its "type," as follows. The empty set (exceptionally) has type -1. When all the closed subsets of X of types $< \alpha$ have been dealt with, and if X has other closed subsets, then (by Theorem 1(5)) X has *minimal* closed subsets not of type $< \alpha$; each of these is said to be *irreducibly of type* α. The finite unions of sets irreducibly of type $\leqq \alpha$ are said to be of type $\leqq \alpha$, and a (closed) set of type $\leqq \alpha$ which is not of type $< \alpha$ is of type α. Ultimately all closed subsets of X (including X) are assigned types. The sets which are irreducibly of type 0 are precisely the non-empty *trivial* closed subsets of X; if X is T_1 they are the 1-point sets.[5] (Further examples will be given later.) The following two properties follow at once from the definitions.

(1) If $Y_1, \cdots, Y_n$ are closed subsets of X of types $\leqq \alpha$, then $Y_1 \cup \cdots \cup Y_n$ is of type $\leqq \alpha$.

(2) If Y is closed in X and is irreducibly of type α, then every closed proper subset Z of Y is of type $< \alpha$.

In the following statements it is to be understood that Y is a closed subset of X—a restriction which will later be removed.

(3) If Y is of type α, then every closed subset Z of Y is of type $\leqq \alpha$.

For $Y = Y_1 \cup \cdots \cup Y_n$, where Y_i is irreducibly of type $\alpha_i \leqq \alpha$. Then $Y_i \cap Z$ is of type $\leqq \alpha_i$, by (2), so Z is of type $\leqq \alpha$, by (1).

(4) *Y is of type α if and only if $Y = F_1 \cup \cdots \cup F_n$, where F_i is closed and irreducibly of type α_i and* $\max(\alpha_1, \cdots, \alpha_n) = \alpha$.

If Y is expressible in this form, Y has type $\leqq \alpha$ by (1); but if Y is of type $< \alpha$, then each $\alpha_i < \alpha$ by (3), and therefore $\max(\alpha_1, \cdots, \alpha_n) < \alpha$, which is impossible. Conversely, if Y is of type α, the definition shows that $Y = F_1 \cup \cdots \cup F_n$, where F_i is closed and irreducibly of type $\leqq \alpha$, say of

[5] A space Y is "trivial" if its only closed subsets are $\emptyset$ and Y.

type α_i. Let $\beta = \max(\alpha_1, \cdots, \alpha_n)$; thus $\beta \leqq \alpha$. But (1) shows that Y has type $\leqq \beta$; hence $\beta = \alpha$.

(5) *Y is irreducibly of type α, if and only if Y is irreducible and of type α.*

If Y is irreducible and of type α, we express Y as in (4) with n as small as possible. Because Y is irreducible, $n = 1$, and then $Y = F_1$, irreducibly of type $\alpha_1 = \alpha$. Conversely, if Y is irreducibly of type α, suppose $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are proper closed subsets of Y; by (2), Y_1 and Y_2 have types $< \alpha$, and (1) gives a contradiction.

(6) *If Y is of type α, and $\beta < \alpha$, then Y has a closed subset Z of type β.*[6]

As Y is a closed subset of X which is not of type $< \beta$, Y contains a minimal closed subset Z (of X) with this property; and Z is irreducibly of type β, by definition.

(7) *The type of a closed subset Y of X does not depend on the containing space X.*

It is enough to show that if Y has type α in X, Y has type α when the containing space is Y. We prove by transfinite induction on β that if a closed subset Z of Y has type β in X, it has type β in Y, and conversely. For $\beta = -1$ this is clear. Assume it true for all $\beta < \gamma$, where $\gamma \leqq \alpha$. If Z is *irreducibly* of type γ in X, then Z is a minimal closed subset of X which is not of type $< \gamma$ in X; in view of the induction hypothesis, it is also a minimal closed subset of Y which is not of type $< \gamma$ in Y, and so it is (irreducibly) of type γ in Y. If Z is of type γ in X but not necessarily irreducible, if follows from (4) and the preceding that Z is of type γ in Y. The converse is established by substantially the same argument.

We can thus speak of the *type* of a hereditarily compact space Y, independent of the containing space X; it is, of course, a topological invariant of Y. It follows from (7) that, in propositions (2)-(6), the hypothesis that Y is closed in X can be omitted; these propositions apply to arbitrary hereditarily compact spaces Y.

(8) *There exist hereditarily compact T_1 spaces of type α, for every ordinal α.*

To construct one such space X, let A denote the section of ordinals $< \alpha$, beginning with -1 (that is, we count -1 as an ordinal), let I denote any

[6] It follows that, if α is finite, Y has a family of non-empty closed proper irreducible subsets, well-ordered under inclusion and of ordinal α. This need not be true when α is infinite.

infinite set, and put $X = A \times I$. The closed sets in X are defined to be those of the form $(B \times I) \cup F$, where B is an arbitrary section of A (or A itself) and F is an arbitrary finite set. It is easily verified (using Theorem 1(5)) that X is a hereditarily compact T_1 space; and a straightforward transfinite induction on α shows that X is irreducibly of type α.

The hereditarily compact T_1 spaces of type 0 are finite unions of 1-point spaces—that is, they are the finite (non-empty) T_1 spaces. The hereditarily compact T_1 spaces X irreducibly of type 1 are those of the following form: X is an infinite set and its closed subsets are just X and its finite subsets. The hereditarily compact spaces of finite type n are those of "dimension n" in the sense: $n+1$ is the greatest length of any strictly decreasing sequence of irreducible closed non-empty subsets. This agrees with the usual dimension for algebraic varieties in the Zariski topology. For $n > 1$, and still more for infinite types, there are surprisingly many of them; we return to this in § 7. In the next section we show how all hereditarily compact T_1 spaces of type α can be "constructed" if we know enough about those of types $< \alpha$.

5. Dual direct systems. Let $\{F_\lambda, f_\lambda{}^\mu\}$ be a direct system of spaces F_λ (the suffixes λ running over a directed set Λ) and maps $f_\lambda{}^\mu\colon F_\lambda \rightarrow F_\mu$ $(\lambda < \mu)$ subject to the usual rule $f_\mu{}^\nu f_\lambda{}^\mu = f_\lambda{}^\nu$ for $\lambda < \mu < \nu$. We assume further that the maps $f_\lambda{}^\mu$ are *closed*. Let S be the limit space; thus a point of S is an equivalence class $\{x\}$ of representatives $x = \{x_\lambda\}$, where $x_\lambda \in F_\lambda$ for $\lambda > \lambda(x)$ and $f_\lambda{}^\mu(x_\lambda) = x_\mu$ for $\mu > \lambda > \lambda(x)$; two representatives $\{x_\lambda\}$ and $\{y_\lambda\}$ are equivalent if and only if $x_\lambda = y_\lambda$ for $\lambda > \lambda(x,y)$. We give S, *not* the usual direct limit topology, but one which (roughly speaking) uses closed sets instead of open sets. Let f_λ be the usual mapping of F_λ in S, defined as follows: given $x_\lambda \in F_\lambda$ write $x_\mu = f_\lambda{}^\mu(x_\lambda)$ for $\mu > \lambda$, and put

$$f_\lambda(x_\lambda) = \{\{x_\mu \mid \mu > \lambda\}\} \in S.$$

The closed sets of S are to be the intersections of sets of the form $f_\lambda(K_\lambda)$, where K_λ is closed in F_λ; that is, the sets $S - f_\lambda(K_\lambda)$ form a basis of open sets. It is easily verified that, if $\lambda < \nu$, $f_\nu f_\lambda{}^\nu = f_\lambda$, and thence that, if $\nu > \lambda, \mu$, $f_\lambda(K_\lambda) \cup f_\mu(K_\mu) = f_\nu(f_\lambda{}^\nu(K_\lambda) \cup f_\mu{}^\nu(K_\mu)) = f_\nu(K_\nu)$, where K_ν is closed in F if K_λ, K_μ are closed in F_λ, F_μ. Hence this does define a topology on S, the coarsest in which each f_λ is closed. (In general, the mappings f_λ will not be continuous, even if each $f_\lambda{}^\mu$ is continuous.) We call S, with this topology, the "dual direct limit space" of the system $\{F_\lambda, f_\lambda{}^\mu\}$. Clearly S is a T_1 space if each F_λ is T_1.

We are particularly concerned with the case in which each $f_\lambda{}^\mu$ is 1-1 and continuous (and thus a homeomorphism into); we then call $\{F_\lambda, f_\lambda{}^\mu\}$ an *imbedding system.* In this case the closed proper subsets of S are simply the sets $f_\lambda(K_\lambda)$, where K_λ is closed in $F_\lambda(\lambda \in \Lambda)$, and each f_λ is a homeomorphism into.

THEOREM 6. *The dual direct limit space S of an imbedding system $\{F_\lambda, f_\lambda{}^\mu\}$ of hereditarily compact spaces, each of type $< \alpha$, is hereditarily compact and of type $\leqq \alpha$; it is irreducible providing no $f_\lambda{}^\mu$ is onto, and T_1 if each F_λ is. Conversely, every irreducible hereditarily compact T_1 space of type $\alpha > 0$ is homeomorphic to the dual direct limit space of an imbedding system $\{F_\lambda, f_\lambda{}^\mu\}$, where each F_λ is hereditarily compact, T_1, and of type $< \alpha$, and no $f_\lambda{}^\mu$ is onto.*

In a sense, this theorem determines all hereditarily compact T_1 spaces by transfinite induction over the type; for any such space is a finite union of closed irreducible subsets of no greater type (4(4) and 4(5)).

Proof. Suppose each F_λ is hereditarily compact and of type $< \alpha$. If there could be an infinite strictly decreasing sequence of closed proper subsets $f_{\lambda_n}(K_{\lambda_n})$ of S $(n = 1, 2, \cdots)$, where K_{λ_n} is closed in F_{λ_n}, the sets $f_{\lambda_1}{}^{-1}(f_{\lambda_n}(K_{\lambda_n}))$ would form a strictly decreasing sequence of closed subsets of F_{λ_1}, which is impossible (Theorem 1(5)). Hence S is hereditarily compact. Each proper closed subset of S, being homeomorphic to a closed subspace of some F_λ, is of type $< \alpha$ (by 4(3) and 4(7)); hence S is of type $\leqq \alpha$. If S is reducible, it is the union of two sets of the form $f_\lambda(K_\lambda)$, $f_\mu(K_\mu)$. Take $\nu > \lambda, \mu$: it follows that $f_\nu(F_\nu) = S$, and thence (because the mappings are 1-1) that $f_\nu{}^\rho$ is onto for all $\rho > \nu$. The T_1 property is obvious.

(Conversely, if the direct limit space S of an imbedding system $\{F_\lambda, f_\lambda{}^\mu\}$ is hereditarily compact and of type $\leqq \alpha$, or is T_1, then the same is true of each F_λ; for F_λ is homeomorphic to a closed subspace of S.)

If X is irreducible, T_1, and hereditarily compact of type $\alpha > 0$, let $\{F_\lambda\}$ be the family of its closed proper subsets, ordered by (proper) inclusion; as $F_\lambda \cup F_\mu$ is also a closed *proper* subset, the family is directed.[7] Let $f_\lambda{}^\mu$ be the inclusion map ("identity") for $F_\lambda \to F_\mu$. This defines an imbedding system; let S be its dual direct limit space. It is easily verified that S is homeomorphic to X (the T_1 axiom guarantees that the obvious map of S in X is onto), and that the other properties asserted hold good.

Remark. In the first part of Theorem 6, to ensure the hereditary com-

[7] The T_1 axiom is used here to produce a closed proper subset of X properly containing F_λ and F_μ when $\lambda = \mu$.

pactness of the direct limit S of a direct system of hereditarily compact spaces, we have assumed that each $f_\lambda{}^\mu$ is closed, continuous and 1-1. None of these assumptions can be omitted; nor can the usual (instead of the dual) direct limit topology be used.

6. Standard operations and types.

LEMMA 1. *If every proper closed subset Z of a space X is hereditarily compact and of type $< \alpha$, then X is hereditarily compact and of type $\leqq \alpha$.*

That X is hereditarily compact follows from Theorem 1(5); the rest follows from the way in which types were defined.

THEOREM 7. *If Y is any subspace of a hereditarily compact space X of type α, then Y is hereditarily compact and of type $\leqq \alpha$.*

This is proved by transfinite induction on α. We may assume that the theorem is true for all smaller types, and also (since we may replace X by $\bar{Y}$, in view of 4(3)) that $X = \bar{Y}$. Suppose first that Y is irreducible; then X is also irreducible (3(1)). Any proper relatively closed subset of Y is of the form $Y \cap Z$, where Z is a closed proper subset of X; say Z has type β. Then $\beta < \alpha$ because X is irreducible; the hypothesis of induction then gives that the type of $Y \cap Z$ is $\leqq \beta < \alpha$, and by Lemma 1 the type of Y is $\leqq \alpha$. Finally, if Y is not irreducible, we have $Y = Y_1 \cup \cdots \cup Y_n$, where each Y_i is (relatively) closed and irreducible and of type α_i say. By the result just established, $\alpha_i \leqq \alpha$ $(i = 1, \cdots, n)$; thus the type of $Y = \max(\alpha_1, \cdots, \alpha_n) \leqq \alpha$.

THEOREM 8. *The union X of a finite number of hereditarily compact spaces $Y_1, \cdots, Y_n$ is hereditarily compact; and if Y_i is of type α_i and X of type α, then*

$$\max(\alpha_1, \cdots, \alpha_n) \leqq \alpha \leqq 1 + \alpha \leqq (\Sigma)(1 + \alpha_i).$$

Here $(\Sigma)\alpha_i$ denotes the "natural" sum of the ordinals $\alpha_1, \cdots, \alpha_n$; that is, we express each α_i in the form $\omega_0{}^{\xi_1}k_{i1} + \omega_0{}^{\xi_2}k_{i2} + \cdots + \omega_0{}^{\xi_m}k_{im}$, where the ordinals ξ_j satisfy $\xi_1 > \xi_2 > \cdots > \xi_m = 0$, and $k_{i1}, \cdots, k_{im}$ are positive integers or 0, and define

$$(\Sigma)\alpha_i = \omega_0{}^{\xi_1} \sum k_{i1} + \cdots + \omega_0{}^{\xi_m} \sum k_{im}.$$

(See [11, pp. 363, 364].) When $\alpha_1, \cdots, \alpha_n$ are all finite, this coincides with their ordinary sum.

Note that $1 + \alpha = \alpha + 1$ if α is finite, but $1 + \alpha = \alpha$ otherwise.

Proof. That X is hereditarily compact is obvious, and that $\alpha \geqq \max(\alpha_1, \cdots, \alpha_n)$ follows from Theorem 7. To prove the remaining inequality, we use transfinite induction over the ordered n-ples $(\alpha_1, \cdots, \alpha_n)$ of ordinal numbers (each $\leqq$ some large enough α^*), ordered lexicographically (n being fixed); this is a well-ordered family. It is convenient to count -1 as an ordinal here. Thus the induction starts with each $\alpha_i = -1$; each Y_i is empty, so X is empty and of type $\alpha = -1$ as required. Now suppose that the assertion is true for all $(\alpha_1', \cdots, \alpha_n') < (\alpha_1, \cdots, \alpha_n)$. We first assume that each Y_i is *irreducibly* of type α_i. If Z is any proper closed subset of $X = \bigcup Y_i$, then $Y_i \cap Z$ is for at least one i a proper closed subset of Y_i; hence if $Y_i \cap Z$ has type β_i, we have $\beta_i \leqq \alpha_i$ $(1 \leqq i \leqq n)$, and $\beta_j < \alpha_j$ for at least one value of j. Thus $(\beta_1, \cdots, \beta_n) < (\alpha_1, \cdots, \alpha_n)$, and it follows from the induction hypothesis that the type β of Z satisfies

$$1 + \beta \leqq (\Sigma)(1 + \beta_i) < (\Sigma)(1 + \alpha_i).$$

Hence, by Lemma 1, the type α of X satisfies $1 + \alpha \leqq (\Sigma)(1 + \alpha_i)$.

In the general case, let $Y_i = \bigcup \{Y_{ij} \mid j = 1, 2, \cdots, m(i)\}$, where Y_{ij} is relatively closed and irreducible, and for each of the $m(1)m(2)\cdots m(n)$ choices λ of suffixes, put $Z_\lambda = \bigcap_i \bar{Y}_{i\lambda(i)}$. Then $Z_\lambda \cap Y_i \subset Y_{i\lambda(i)}$, so we have $Z_\lambda \subset \bigcup \{Y_{i\lambda(i)} \mid i = 1, 2, \cdots, n\}$. By Theorem 7 and the case already dealt with, the type γ_λ of Z_λ satisfies

$$(1 + \gamma_\lambda) \leqq (\Sigma)(1 + \text{type of } Y_{i\lambda(i)}) \leqq (\Sigma)(1 + \alpha_i).$$

But $X = \bigcup Z_\lambda$, a finite union of *closed* sets; hence the type α of X satisfies $\alpha = \max(\gamma_\lambda)$, and the desired relation $(1 + \alpha) \leqq (\Sigma)(1 + \alpha_i)$ follows.

Remark. The inequalities in (2) are "best possible," even for T_1 spaces, as can be seen by taking X to be the example constructed to prove 4(8).

THEOREM 9. *The product X of a finite number of hereditarily compact spaces $Y_1, \cdots, Y_n$ is hereditarily compact; and if Y_i is of type α_i and no Y_i is empty, then the type of X is $(\Sigma)\alpha_i$ $(1 \leqq i \leqq n)$.*

It will suffice to prove this when $n = 2$, as then the general result follows by induction over n. As in the proof of Theorem 8 we use transfinite induction over the ordered pairs (α_1, α_2) in lexicographic ordering, and may assume the theorem for products of spaces of types β_1 and β_2 whenever $(\beta_1, \beta_2) < (\alpha_1, \alpha_2)$. Again, as in Theorem 8, we can easily reduce the proof to the case in which Y_1 and Y_2 are irreducible. It readily follows that X is irreducible too. Let Z be any proper closed subset of X; then $X - Z$ contains a set of the form

$U_1 \times U_2$, where U_1, U_2 are non-empty open subsets of Y_1, Y_2. Then $Y_1 - U_1$ and $Y_2 - U_2$ are of types (say) β_1 and β_2, where $\beta_1 < \alpha_1$ and $\beta_2 < \alpha_2$. Hence, by the induction hypothesis, $(Y_1 - U_1) \times Y_2$ and $Y_1 \times (Y_2 - U_2)$ are hereditarily compact and of types $\beta_1 (+) \alpha_2$, $\alpha_1 (+) \beta_2$. As they are closed in X, Theorem 8 and 4(1) show that their union T is hereditarily compact and of type $\max(\beta_1 (+) \alpha_2, \alpha_1 (+) \beta_2) < \alpha_1 (+) \alpha_2$. But $Z \subset T$, so the same is true of Z; and from Lemma 1 it follows that X is hereditarily compact and of type $\leqq \alpha_1 (+) \alpha_2$. To obtain the reverse inequality, suppose (say) $\alpha_2 \neq 0$. For every ordinal $\gamma_2 < \alpha_2$, Y_2 contains a closed proper subset of type γ_2; applying the induction hypothesis again shows that X contains a closed proper subset of type $\alpha_1 (+) \gamma_2$. Thus the type of X is greater than $\alpha_1 (+) \gamma_2$ for every $\gamma_2 < \alpha_2$, and so is $\geqq \alpha_1 (+) \alpha_2$.

If $\alpha_1 = \alpha_2 = 0$ (i.e., to start the induction), Y_1 and Y_2 are trivial spaces; consequently X is trivial too, and so is hereditarily compact and of type 0.

Remark. A product of infinitely many non-trivial spaces is *never* hereditarily compact. For it contains a subspace homeomorphic to $\prod Y_n$ $(n = 1, 2, \cdots)$, where Y_n consists of two points a_n, b_n and (b_n) is open in Y. But this contains the infinite discrete subset $(b_1, a_2, \cdots, a_n, \cdots)$, $(a_1, b_2, a_3, \cdots)$, etc.

THEOREM 10. *Let f be a continuous mapping of a hereditarily compact space X of type α. Then $f(X)$ is hereditarily compact and of type less than $\omega_0^{\alpha+1}$.*

Let $Y = f(X)$. Each $Z \subset Y$ is compact, being a continuous image of the compact set $f^{-1}(Z) \subset X$. To prove the remainder of the assertion, suppose first that X is irreducible. We show by transfinite induction over α that Y has type $\leqq \omega_0^{\alpha}$, if $\alpha \geqq 0$. When $\alpha = 0$, X is trivial; hence Y is trivial, so its type $= 0 < \omega_0^{0}$. In general, if Z is any closed proper subset of Y, which we may assume to be non-empty, let $f^{-1}(Z) = S_1 \cup \cdots \cup S_n$, where each S_i is a non-empty closed irreducible subset of X, necessarily proper. Let S_i have type β_i, and put $\beta = \max(\beta_1, \cdots, \beta_n)$; thus $\beta < \alpha$, because X is irreducible. By the hypothesis of induction, the type of $f(S_i)$ is $\leqq \omega_0^{\beta}$; and by Theorem 8 the type of $Z = \bigcup f(S_i)$ is $\leqq \omega_0^{\beta} n < \omega_0^{\beta+1} \leqq \omega_0^{\alpha}$. Hence, by Lemma 1, Y has type $\leqq \omega_0^{\alpha}$.

In the general case, we have $X = X_1 \cup \cdots \cup X_m$, where X_j is irreducible of type α_j, and $\alpha = \max(\alpha_1, \cdots, \alpha_m)$ (4(4) and 4(5)). By Theorem 8 and the foregoing, the type of Y is $\leqq \omega_0^{\alpha} m < \omega_0^{\alpha+1}$.

Remark. The bound for the type of $f(X)$ here is sharp, even for T_0

spaces, and even if f is 1-1. But if α is finite and $f(X)$ is T_1, its type is $< \omega_0{}^\alpha$, which is now "best possible"; for infinite α, Theorem 10 is sharp even for T_1 spaces and 1-1 mappings. However, if f is *closed* and continuous, it is easily seen that the type of $f(X)$ does not exceed the type of X.

7. Countable spaces. The simplest hereditarily compact spaces are those which have at most countably many closed (or open) sets. Concerning these we have:

THEOREM 11. *Let X be a hereditarily compact space. Then:*

(1) *The family of open subsets of X is countable if and only if X has a countable base.*

(2) *If X is T_0 and has a countable base, then X is countable.*

(3) *If X is T_1, X has a countable base if and only if it satisfies the first axiom of countability.*

Proof. (1) Let $B_1, B_2, \cdots$ be a countable base of open sets. We show that the open sets coincide with the *finite* unions of the sets B_i—which evidently form a countable family. In fact, if U is open, U is a union of sets B_i, and being compact is covered by a finite number of them. "Only if" is trivial.

(2) By (1), X has at most $\aleph_0$ distinct closed sets; but the sets $\bar{x}$ $(x \in X)$ are all distinct.

(3) Assuming X is "first countable," we first show that X is countable, using transfinite induction over the type α of X. We may clearly assume that X is irreducible and not empty. Pick $p \in X$ and let $U_1, U_2, \cdots$ be a basis of open neighborhoods of p; thus $\bigcap U_n = (p)$. Put $F_n = X - U_n$; then F_n has type $< \alpha$, so by the induction hypothesis F_n is countable. Hence X is countable.[8] If the points of X are enumerated as $q_1, q_2, \cdots$, and if $V_{n1}, V_{n2}, \cdots$ is a basis of open neighborhoods of q_n, the sets V_{nm} evidently form a countable basis for X.

The converse implication is trivial.

Remark. There are hereditarily compact T_0 spaces which satisfy the first axiom of countability and have arbitrarily large cardinal, and there are hereditarily compact T_1 spaces (of type 1) which are separable but have arbitrarily large cardinal.

[8] This argument applies, more generally, if instead of assuming that X is T_1 and first countable, we assume that each point of X is a G_δ in X.

One might expect that, conversely, a countable T_1 hereditarily compact space has to satisfy the first axiom of countability, at least at one point, especially in view of a theorem of S. Mrówka [8] asserting that a compact T_2 space with fewer than $2^{\aleph_1}$ points must satisfy the first axiom of countability at some point. But this is not the case, as the following example shows.

Example 1. *There exists a countable hereditarily compact T_1 space of type 2, having c closed subsets, and not having a countable base of neighborhoods at any point.*

The example requires the following lemma, which is due to Sierpinski (cf. [11, p. 77]).

LEMMA 2. *Let S be a set with $\aleph_0$ elements. There is a family of c distinct infinite subsets A_x of S, every two of which intersect in at most a finite set.*

We may take S to be the set of rational numbers, and for each real number x take A_x to be a sequence of rational numbers converging to x.[9]

Now topologize S by taking its closed sets to be: S, and all sets of the form $E \cup \bigcup A_{x_i}$ $(i = 1, 2, \cdots, n)$, where n is a non-negative integer and E is finite. S is easily seen to be irreducible, hereditarily compact and T_1. The closed sets of type 0 are the non-empty sets E; the irreducible closed sets of type 1 are the sets A_x; and thus S is of type 2. We may assume that, given $p \in S$, there are uncountably many sets A_x which do not contain p; for the set of points p for which this is not true must be finite, and we simply omit them from S. If $V_1, V_2, \cdots$ is a countable basis of open neighborhoods of p, we have $V_m = S - (E_m \cup \bigcup \{A_x \mid x \in F_m\})$, where E_m, F_m are finite. As $\bigcup F_m$ is countable, there exists a suffix $y \notin F_m$ for which $p \notin A_y$, and $S - A_y$ is a neighborhhod of p. It must contain some V_m and then $A_y \subset E_m \cup \bigcup \{A_x \mid x \in F_m\}$. But each $A_y \cap A_x$ $(x \in F_m)$ is finite, so A_y is finite, giving a contradiction as required.

A *countable* hereditarily compact space X can evidently have at most c closed subsets; its type must therefore have cardinal $\leqq c$ (from 4(6)). Further, as there are at most 2^c ways of selecting the c sets which are to be closed in X, there can be at most 2^c nonhomeomorphic countable hereditarily compact (or, indeed, countable) spaces. We show now that these trivial estimates are in fact "best possible," even for T_1 spaces. That there can be as many as c closed sets has been shown by Example 1.

[9] This simple proof of Lemma 2 is also due to Sierpinski.

Example 2. There exists, for each ordinal λ of cardinal $\leqq c$, a countable hereditarily compact T_1 space of type λ.

We use transfinite induction over λ. Using the sets S, A_x of Lemma 2, and noting that the number of ordinals $\beta < \lambda$ is at most c, we assign to each $\beta < \lambda$ one or more spaces A_x and topologise them as (countable) hereditarily compact T_1 spaces of type β. Now define a topology on S by taking the closed sets to be: S, and all sets of the form $\bigcup F_{x_i}$ $(i = 1, 2, \cdots, n)$, where F_{x_i} is closed in A_{x_i}. One easily verifies that this does give a topology on S, in which S is T_1 and hereditarily compact, and that the subspace topology it induces on each A_x coincides with the topology originally assigned to A_x. Hence S is irreducibly of type λ.

THEOREM 12. *There are 2^c nonhomeomorphic countable hereditarily compact T_1 spaces.*

Let Ω denote the smallest ordinal of cardinal c; let P be the set of ordinals less than Ω, Q the set of non-limit ordinals in P, and R any subset of Q. Thus there are 2^c distinct sets R, and for each of them there are c elements in $P - R$. We construct for each R a corresponding space as follows. Again we use Lemma 2. Let $x \leftrightarrow \alpha(x)$ be a 1-1 correspondence between the set of suffixes x and the set $P - R$. From Example 2, we can give each set A_x a hereditarily compact T_1 topology, irreducibly of type $\alpha(x)$. As in Example 2, we can extend these topologies to a hereditarily compact T_1 topology on S. Now the sets A_x will be precisely the maximal proper irreducible closed subsets of S. Hence the topology of S determines the family of types of the sets A_x, and hence determines R. That is, different sets R give nonhomeomorphic spaces S, and the theorem follows.

It would be interesting to know how many nonhomeomorphic countable hereditarily compact T_1 spaces have a given type α. By a slight modification of the above argument one can show that this number is at least $2^{|\alpha|}$ if $\aleph_0 \leqq |\alpha| \leqq c$, where $|\alpha|$ denotes the cardinal of α.

It would also be interesting to have corresponding estimates for hereditarily compact T_1 spaces of larger cardinals. The above methods can of course be extended, but do not suffice to settle the questions in general; the difficulty is that the analogue of Lemma 2 is false for "most" cardinals (see [12]).

THE UNIVERSITY,
MANCHESTER, ENGLAND.

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