Sketches, Cones, Commas.

The “*Algebraic Databases*” paper says that “database schemas are formalized as sketches of various sorts.” (§1, ‘Introduction’)

It adds that “EA sketches = finite limits + coproducts”.

The nLab defines a “sketch” as “a small category T equipped with a set L of cones and a set C of cocones.

I want to reduce a definition which contains many aspects into a *single* most characterizing feature, from which the rest can be inferred.

I have not yet done this with “category”.

I have been exploring the idea that “a category of objects” is just like a “set of objects”, except you make explicit *some* relationship the objects have to each other.

But that class of objects could have different kinds of relationships between the objects.

Would an ideal representation of that class of things depict *all* the **kinds** of relationships they have to each other?

For example, the collection of all sets has at least 2 choices for its morphisms: functions, denoted $\mathbf{Set}$; and relations, denoted $\mathbf{Rel}$.

How many different categories exist on the “collection of all sets”?

This is just as relevant for real-world things like bicycles, biological species, etc.

It is a frequent impulse of mine to be able to talk about “all” of some thing. This is perfectly possible in the language of sets. What categories do, is enable more discovery, because they take implicit structure and relationships between the things (which is already there, implicitly), and make it explicit. It is like those external relations encode way, way more insightful and useful information that you can work with to discover more things, about that class of things.

I wonder how that idea might be expressed categorically.

I know the exponential object is a way of expressing “all possible functions” between two things.

I wonder if we could adapt that idea to where we do not mean all possible *instances of one class of relations*, but all possible *classes* of relations.

I am told that a category with multiple classes of arrows is the same as distinct categories with the same collection of objects.

Then what I am referring to must be the collection of all categories X in $\mathbf{Cat}$, for whom $Ob(X) = Ob(C)$, for a collection of objects of interest, $Ob(C)$.

Thus, my intuition is there being an exponential object in some category C, which represents all possible “types of morphisms” for some category D.

What category would C be?

We need to distinguish between two choices of arrows which are different instances of the same *kind*, and two choices which are of a fundamentally different *kind*.

For example, if the morphisms are functions on sets, we could make an arbitrary restriction on the functions, say, only functions with a codomain of 2 elements.

This is clearly not a conceptual leap from one kind of relationship to another, it is just a subcollection of the same kind of relationship.

If my category consists of “people”, my morphisms could be “is friends with”, or it could be “is siblings to”.

These are morphisms that are not (immediately clearly) the “same kind”.

In order to have an exponential object collect all distinct “kinds” of morphisms, I need a way to collapse *all possible choices of morphisms* (of the same kind or different kind) into “equivalence classes” of the same “kind”.

My guess is, you could show mathematically how one collection of morphisms embeds fully into a different one.

This would probably be a functor.

I think the idea of a “limit object” is sort of how it “contains” a bunch of other objects.

(I have been thinking that the limit object of the Cartesian product of sets would be an infinite product.)

So, maybe I am looking for the following:

In a category of categories, any category is defined by its morphisms.

Functors can easily show which categories perfectly embed into each other, as sub-categories.

My hope is that a chain of functorial embeddings implies a limit object, which is a category which “contains” a slew of other categories.

I want to collapse the category of (small) categories into the category of all *limit* categories.

This represents “fundamentally distinct kinds of morphisms” way better.

I then want to associate any collection of objects with an exponential object in that category of limit categories.

This object represents “all fundamental kinds of relations those objects can have to each other.”

I did not even get to the definition of “cone”, but I need to jump on a bus and will continue then.