# Relative differential categories, differential clones and Fermat theories

Jean-Baptiste Vienney

A *differential category* [2] is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ enriched over commutative monoids, together with a monad $S$ on $\mathcal{C}$ and a differentiation operator $\mathrm{d}\colon SA \to SA \otimes A$ satisfying standard axioms. Relative monads are a generalization of monads where the underlying functor is not necessarily an endofunctor [1]. Our main contribution in this talk is the introduction of the notion of a *relative differential category* where the monad is replaced by a relative monad. Every differential category is a relative differential category but the converse is false. An example is $\mathsf{Vec}_k$ with its usual tensor product and with $S(n) = k[x_1, \ldots, x_n]$. Another one is $\mathsf{Vec}_{\mathbb{R}}$ with its usual tensor product and with $S(n) = C^\infty(\mathbb{R}^n, \mathbb{R})$. This framework allows one to work directly with polynomials and smooth functions, instead of using coordinate-free constructions such as the symmetric algebra.

These two examples can be generalized using clones [4]. A rig clone $\mathcal{O}$ is given by a commutative rig $\mathcal{O}(n)$ for every $n \geq 0$ together with projection and composition operations satisfying familiar identities. We introduce the notion of a *differential clone* as a rig clone together with partial derivative operations $\partial_i\colon \mathcal{O}(n) \to \mathcal{O}(n)$. Differential clones are equivalent to the differential theories of [2]. For every differential clone, the symmetric monoidal category $(\mathrm{Mod}_{\mathcal{O}(0)}, \otimes, \mathcal{O}(0))$ is a relative differential category. In particular, we recover our two previous examples of relative differential categories by choosing $\mathcal{O}(n) = k[x_1, \ldots, x_n]$ or $\mathcal{O}(n) = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

A Fermat theory [3] is a ring clone where differentiation is axiomatized in a geometric way. We show that every Fermat theory is a differential clone. The differential clones $\mathcal{O}(n) = k[x_1, \ldots, x_n]$ and $\mathcal{O}(n) = C^\infty(\mathbb{R}^n, \mathbb{R})$ are Fermat theories. However, not every differential clone is a Fermat theory and we provide a counterexample inspired from differential Galois theory [5]. Finally, given any topological field $k$ whose topology is nontrivial and nondiscrete, we define smooth functions from $k^n$ to $k$ following [6]. We show that $\mathcal{O}(n) = C^\infty(k^n, k)$ gives a Fermat theory. This provides the first connection between smooth functions on topological fields and differential categories.


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