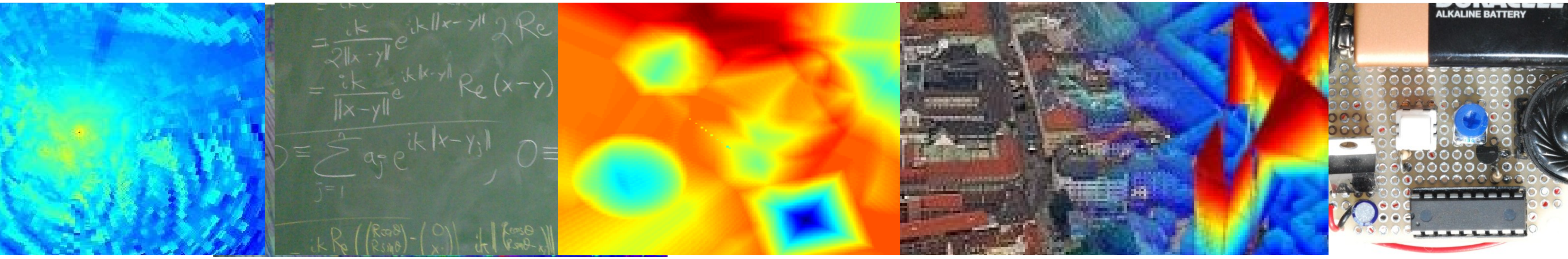
Assignments to sheaves *of* pseudometric spaces



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Acknowledgements

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Key ideas

- Sheaves of **pseudometric spaces** rather than of sets
- **Motivate** *filtrations of partial covers* as generalizing *consistency filtrations* of sheaf assignments
- **Explore** filtrations of partial covers as interesting mathematical objects in their own right with a dual categorial/topological nature:
  - The consistency filtration is a covariant functor
  - The consistency filtration is a continuous function

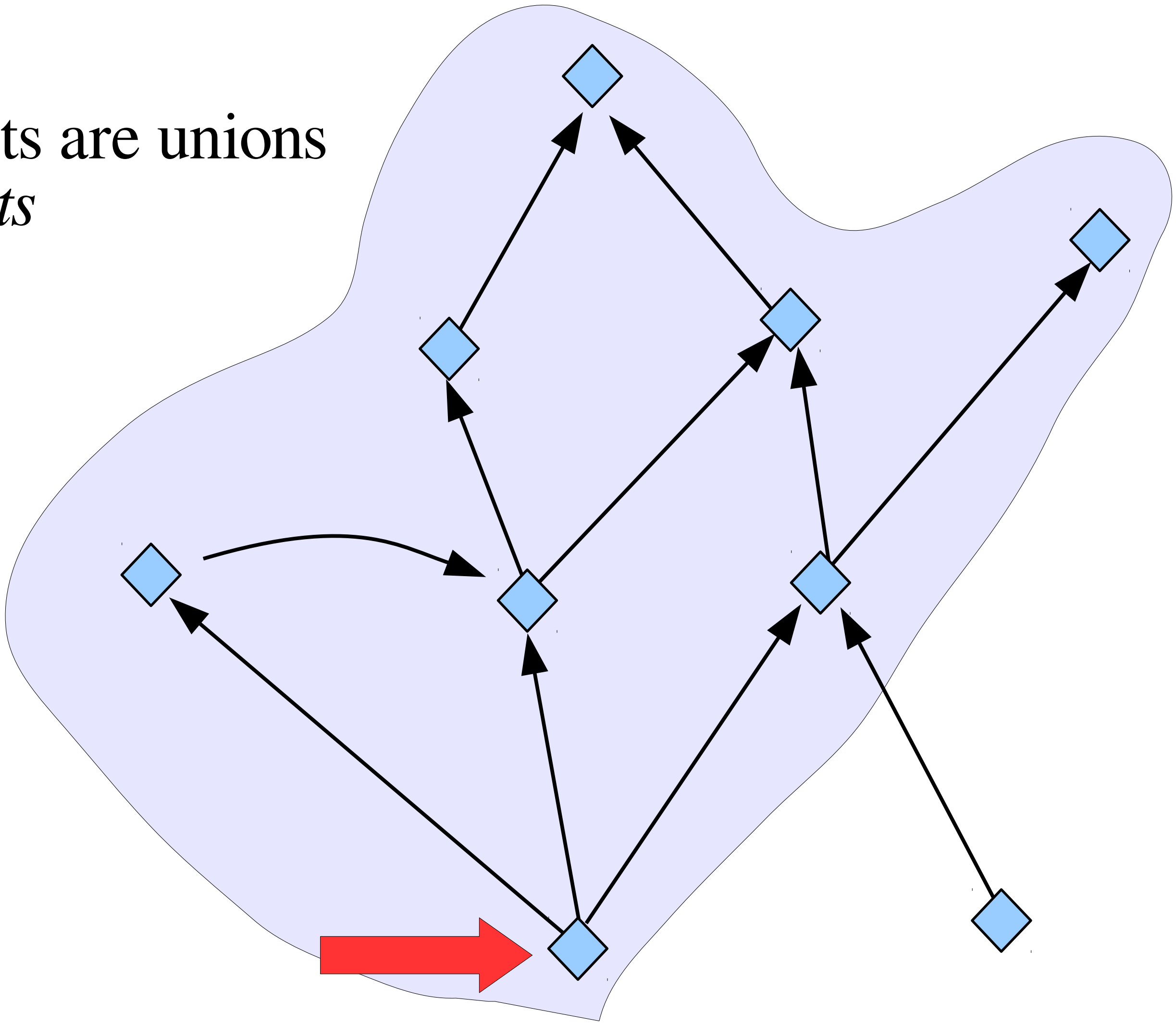
Big caveat: Only finite spaces are under consideration!

Context

- Assemble stochastic models of data locally into a global topological picture
  - *Persistent homology* is sensitive to outliers
  - Statistical tools are less sensitive to outliers, but cannot handle (much) global topological structure
  - *Sheaves* can be built to mediate between these two extremes... this is what I have tried to do for the past decade or so
- The output is the *consistency filtration* of a sheaf *assignment*

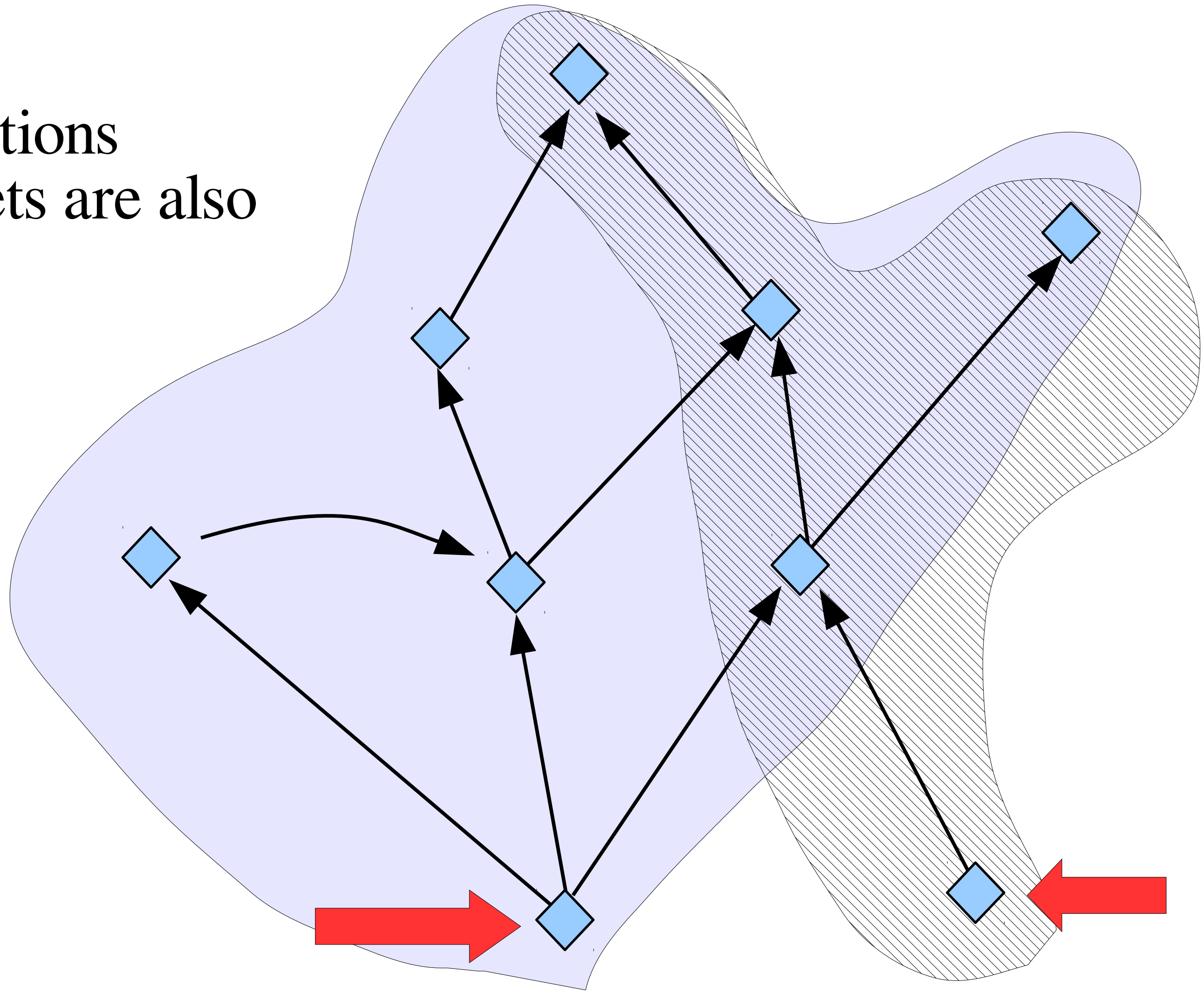
Topologizing a partial order

Open sets are unions of *up-sets*



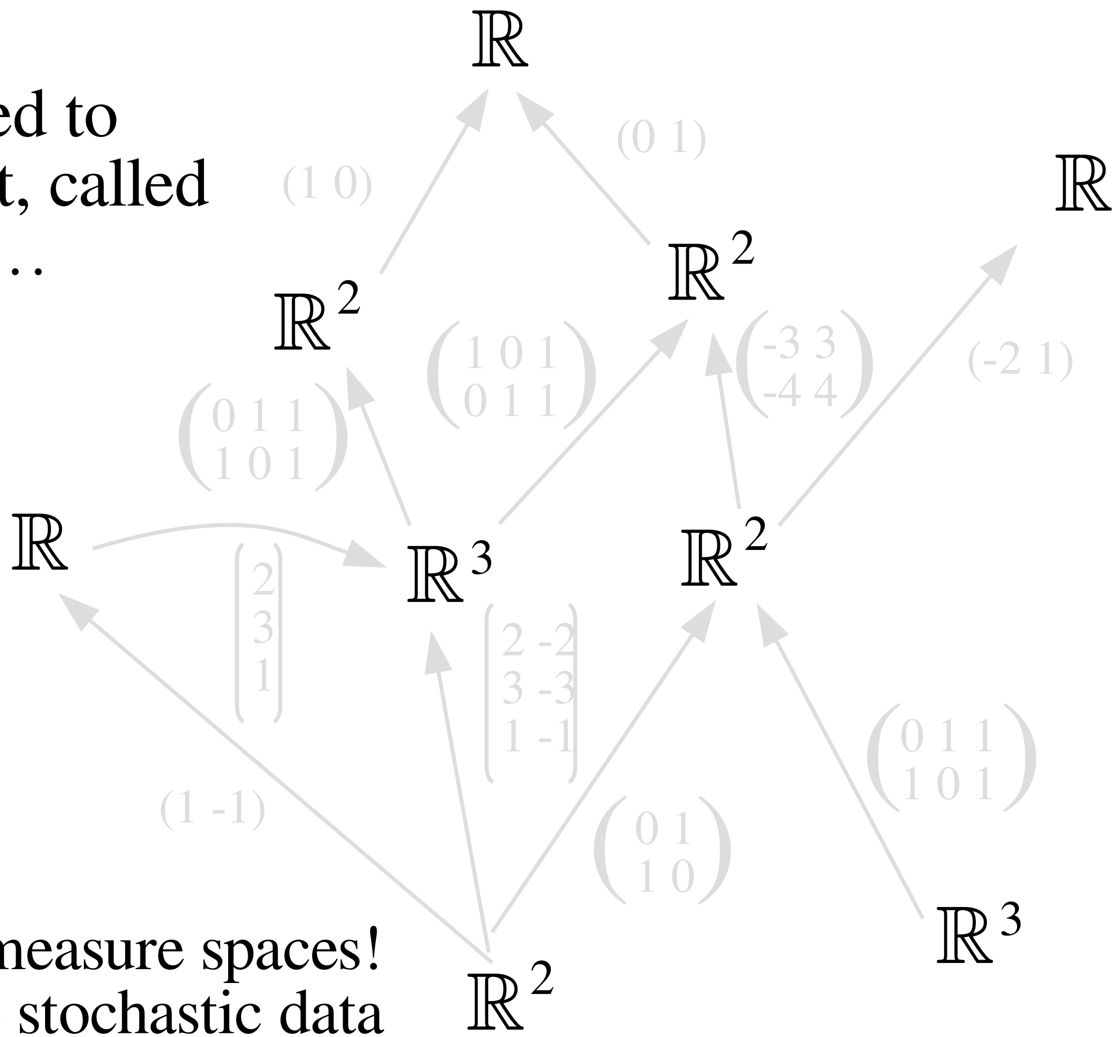
Topologizing a partial order

Intersections
of up-sets are also
up-sets



A *sheaf* on a poset is...

A set assigned to each element, called a *stalk*, and …

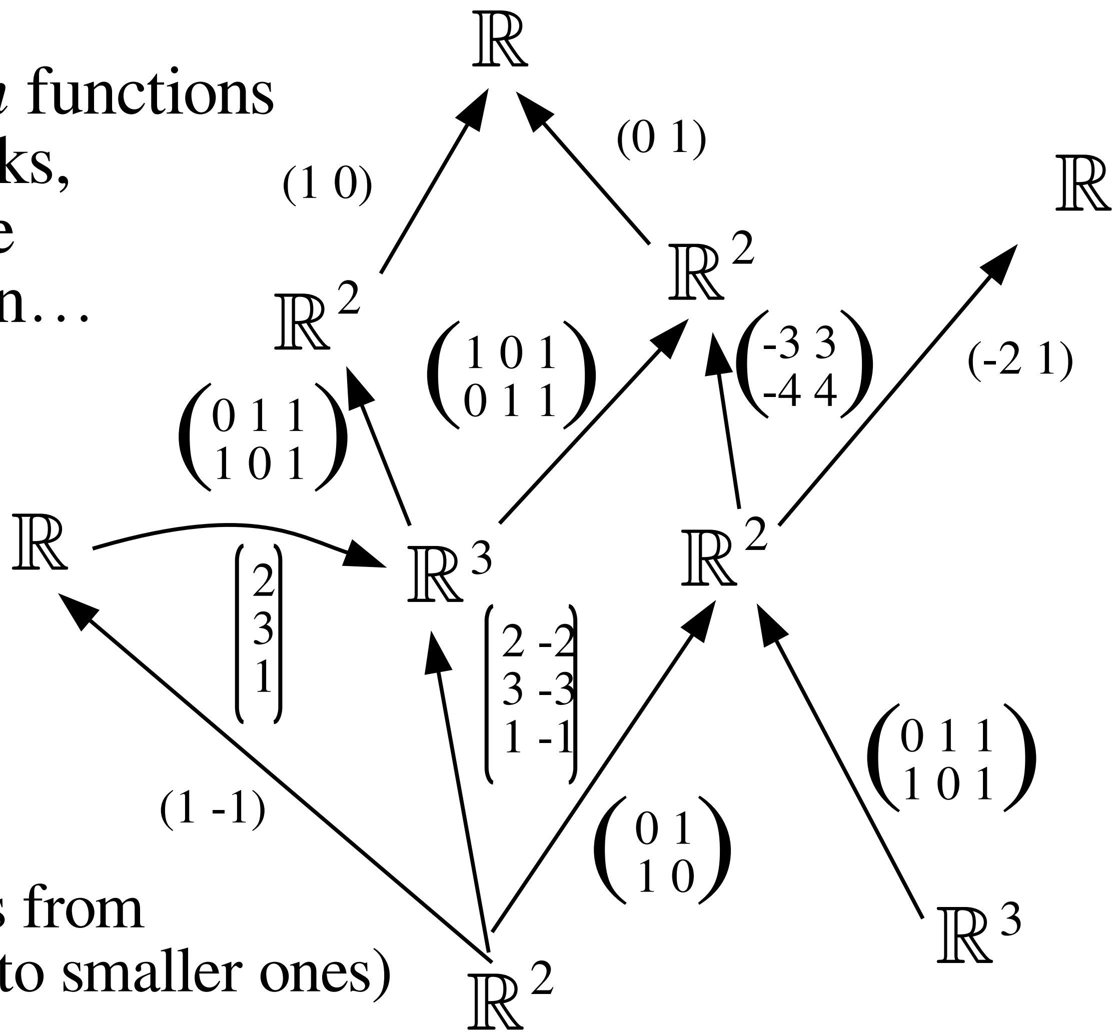


Stalks can be measure spaces!
We can handle stochastic data

This is a *sheaf* **of** vector spaces **on** a partial order

A *sheaf* on a poset is...

… *restriction* functions between stalks, following the order relation…



("Restriction" because it goes from bigger up-sets to smaller ones)

This is a *sheaf* **of** vector spaces **on** a partial order

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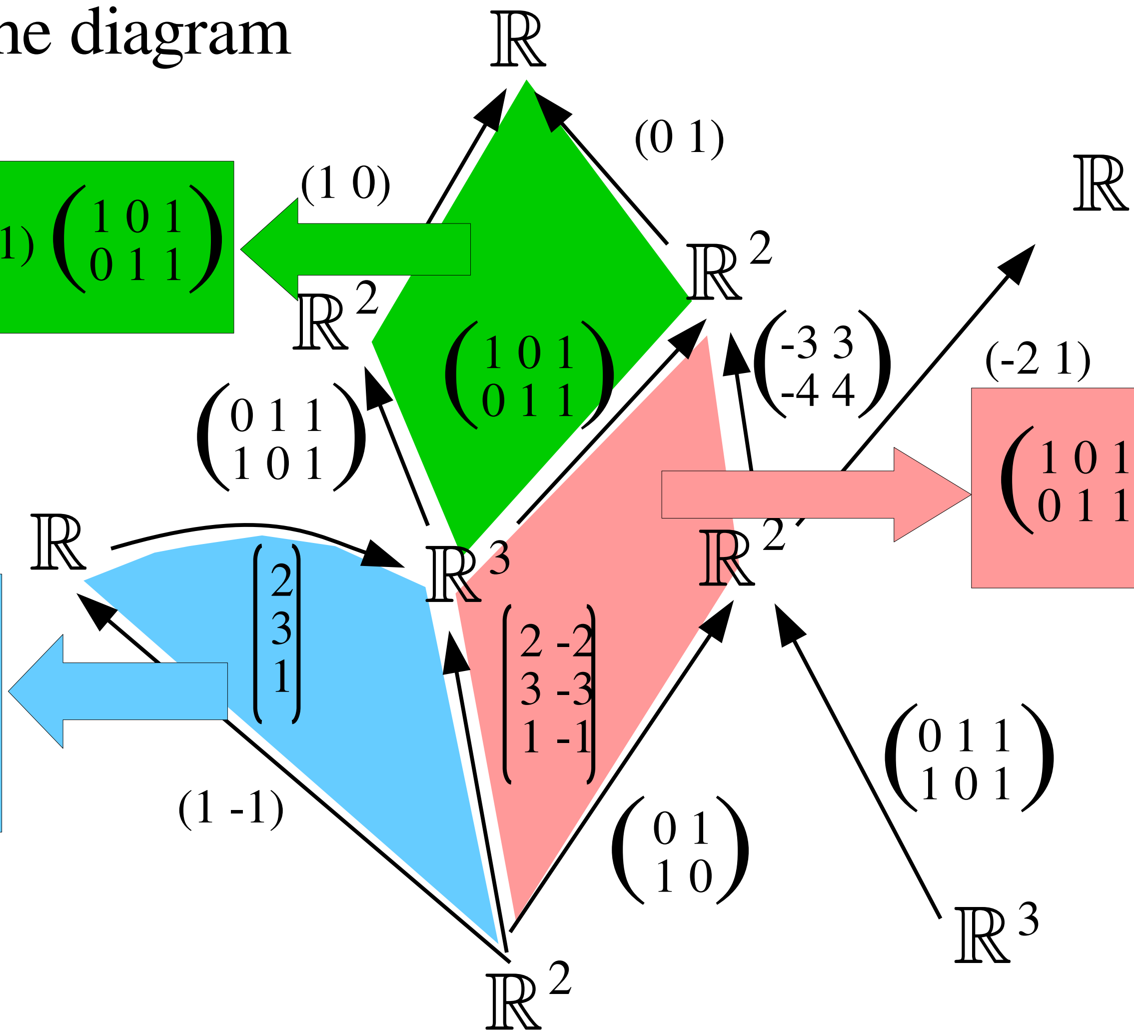
A *sheaf* on a poset is...

… so that the diagram commutes!

$$(1\ 0)\begin{pmatrix}0&1&1\\1&0&1\end{pmatrix} = (0\ 1)\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$$



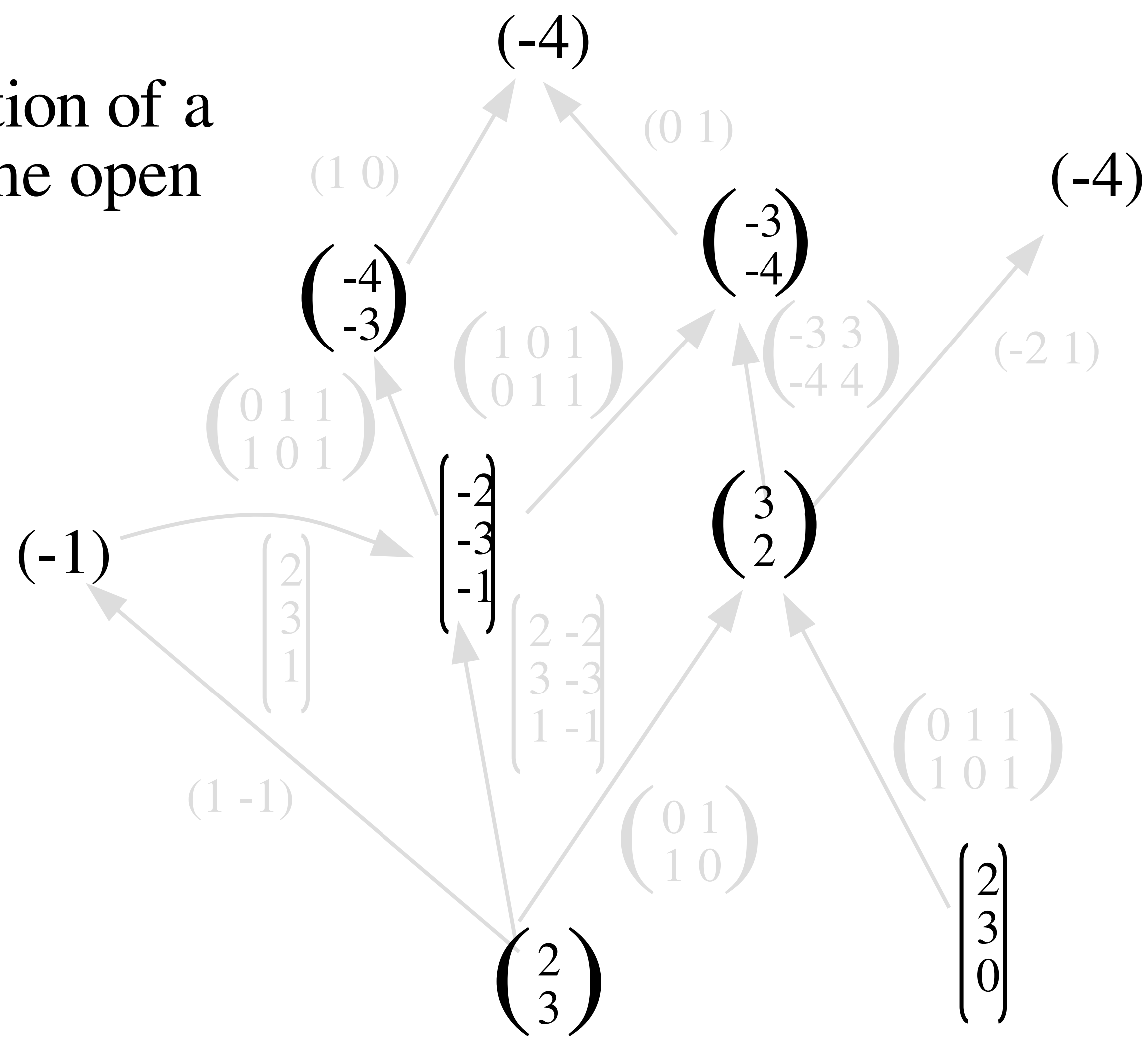
$$\begin{pmatrix}2\\3\\1\end{pmatrix}(1\ -1) = \begin{pmatrix}2&-2\\3&-3\\1&-1\end{pmatrix}$$

$$\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}2&-2\\3&-3\\1&-1\end{pmatrix} = \begin{pmatrix}-3&3\\-4&4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

This is a *sheaf* **of** vector spaces **on** a partial order

An *assignment* is...

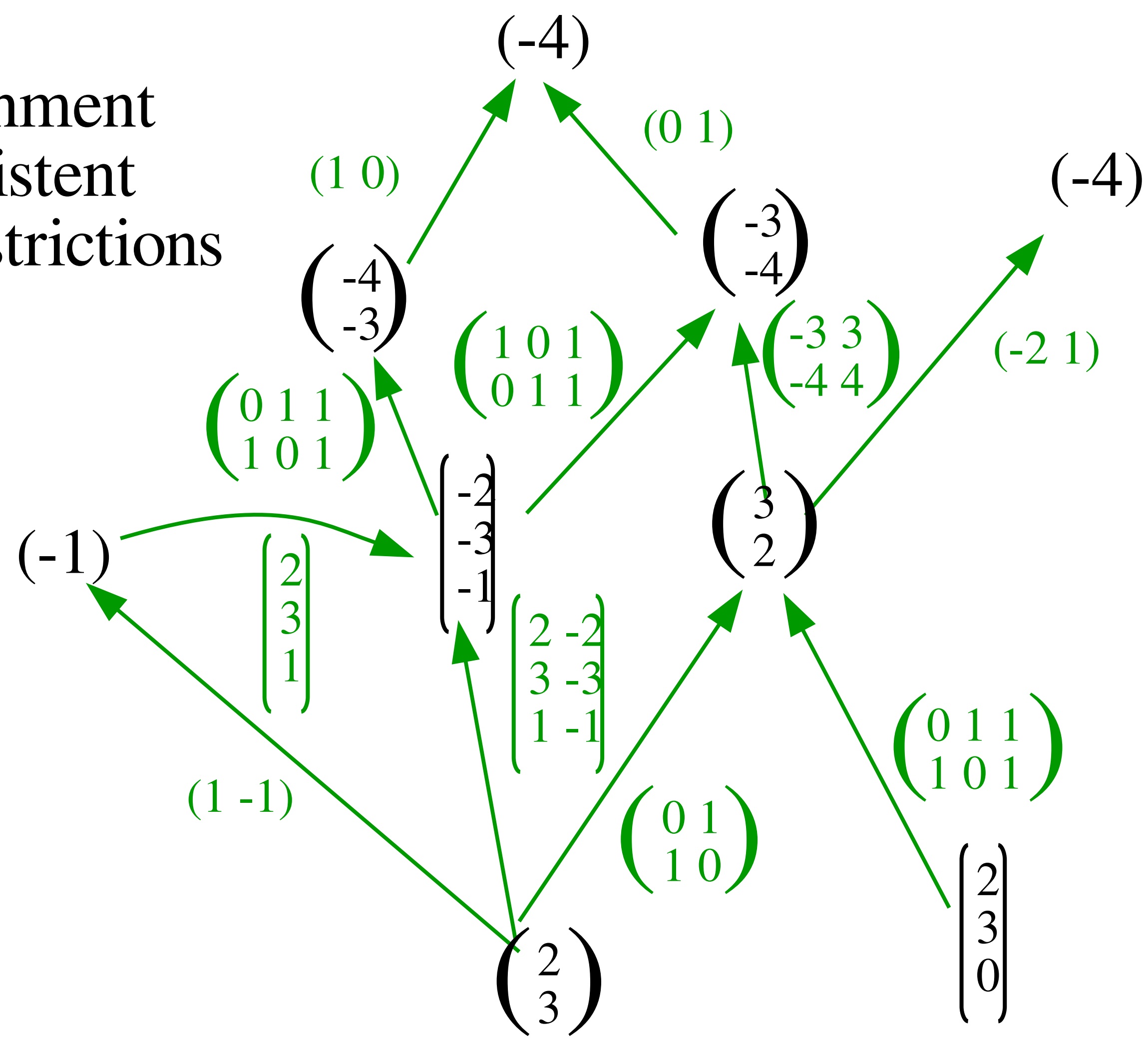
… the selection of a value on some open sets



The term *serration* is more common, but perhaps more opaque.

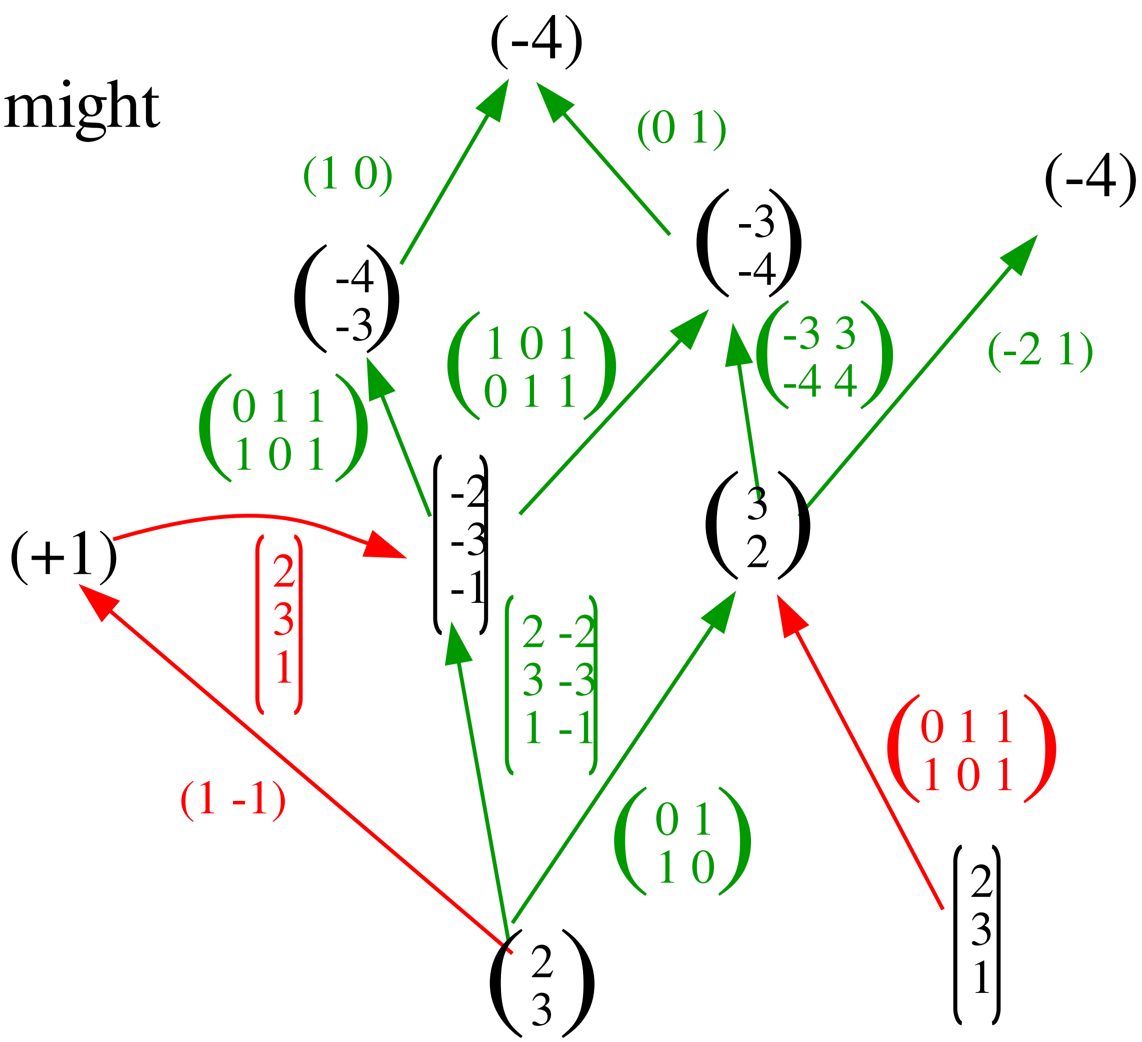
A *global section* is...

… an assignment that is consistent with the restrictions



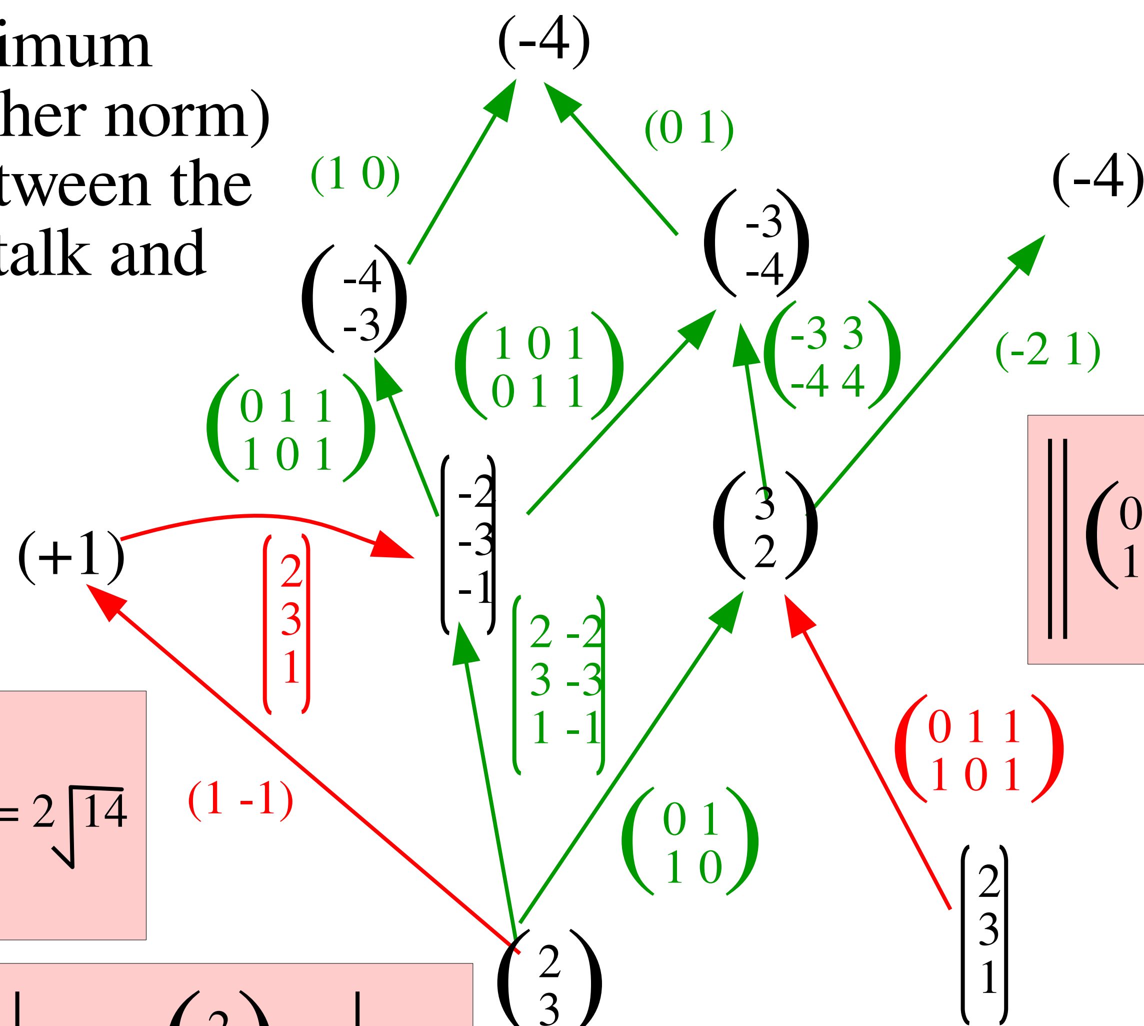
Some assignments aren't consistent

… but they might be partially consistent



Consistency radius is…

… the maximum (or some other norm) distance between the value in a stalk and the values propagated along the restrictions



$$\left\| \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = \sqrt{2}$$

$$\left\| \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} (+1) - \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} \right\| = 2\sqrt{14}$$

$$\left| (1\ -1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 1 \right| = 2$$

$$\text{MAX} \geq 2\sqrt{14}$$

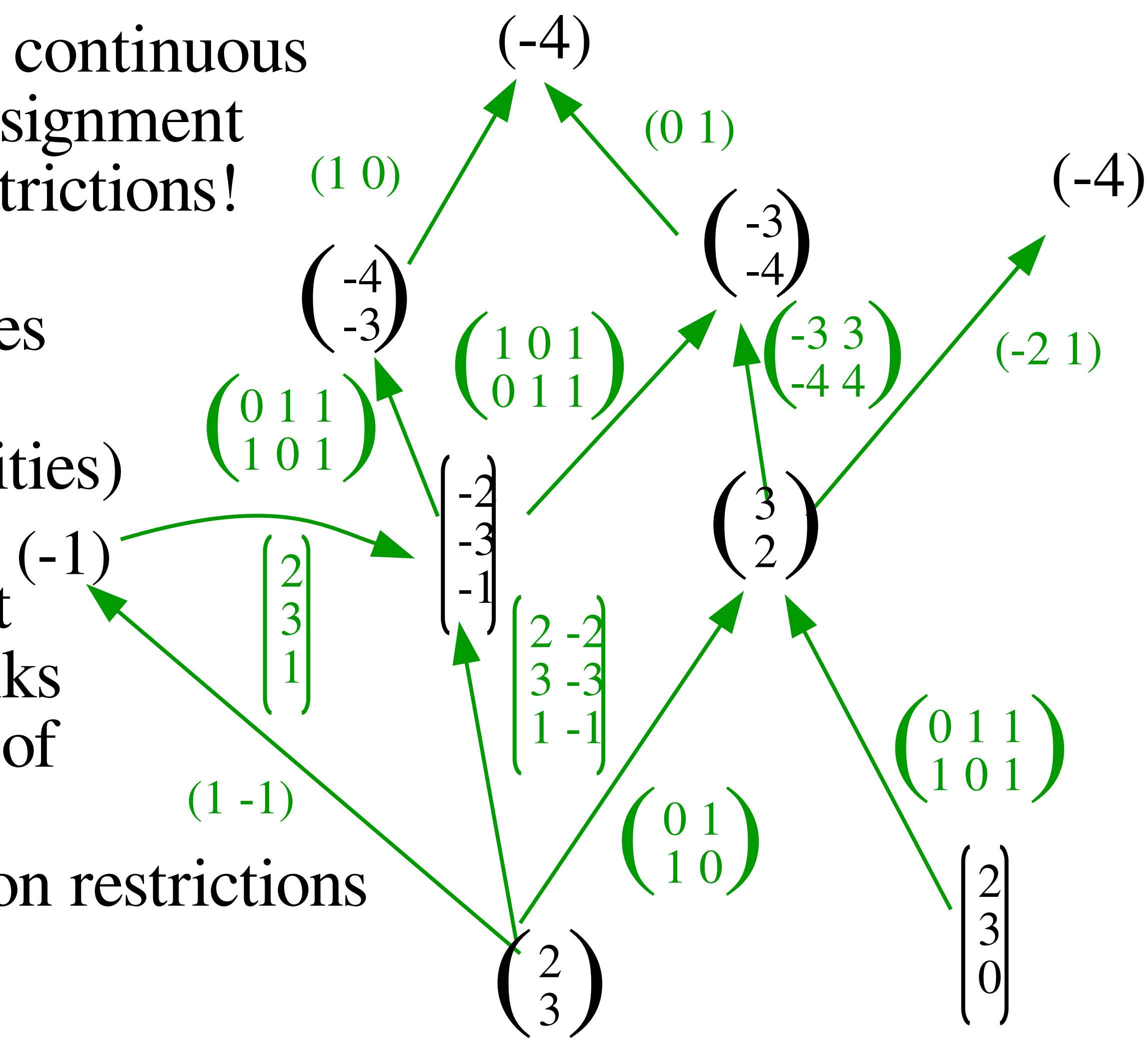
Note: lots more restrictions to check!

Consistency radius is continuous

Theorem: It's continuous both in the assignment and in the restrictions!

(Proof involves establishing some inequalities)

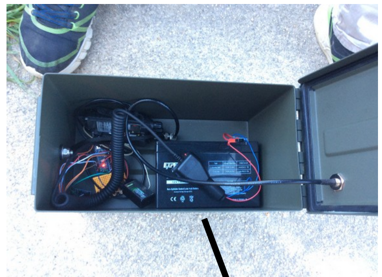
Use a product metric on stalks and topology of uniform convergence on restrictions

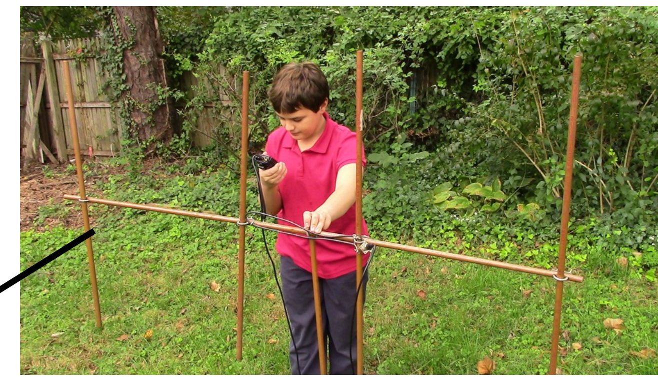


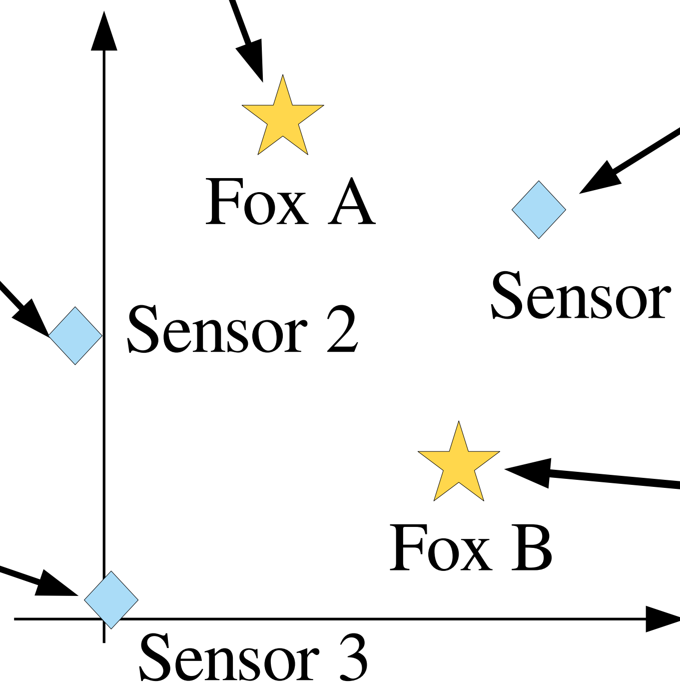
Amateur radio foxhunting

Typical sensors:

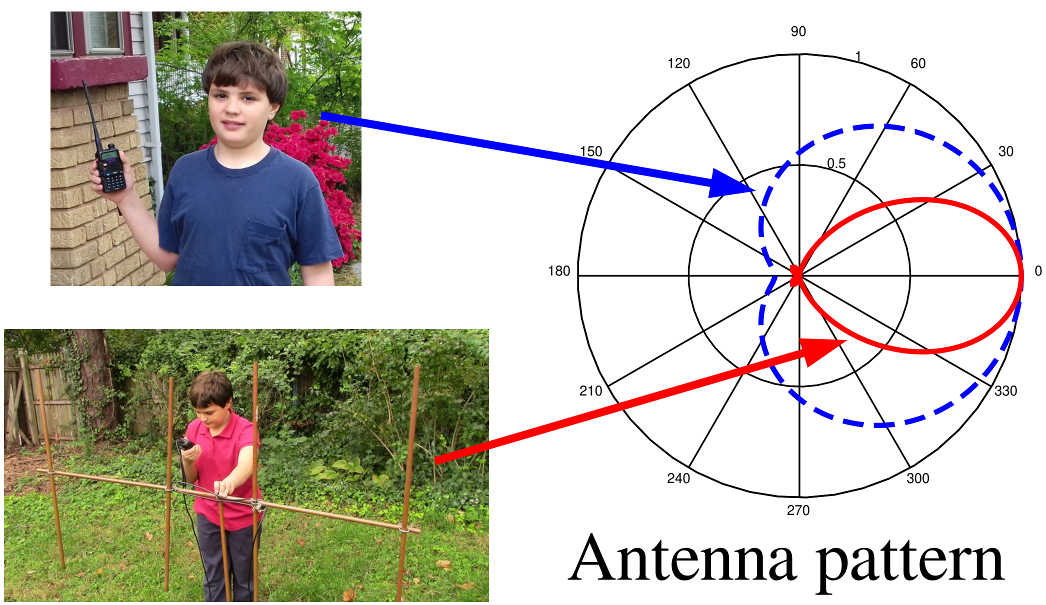
- Bearing to Fox
- Fox signal strength
- GPS location





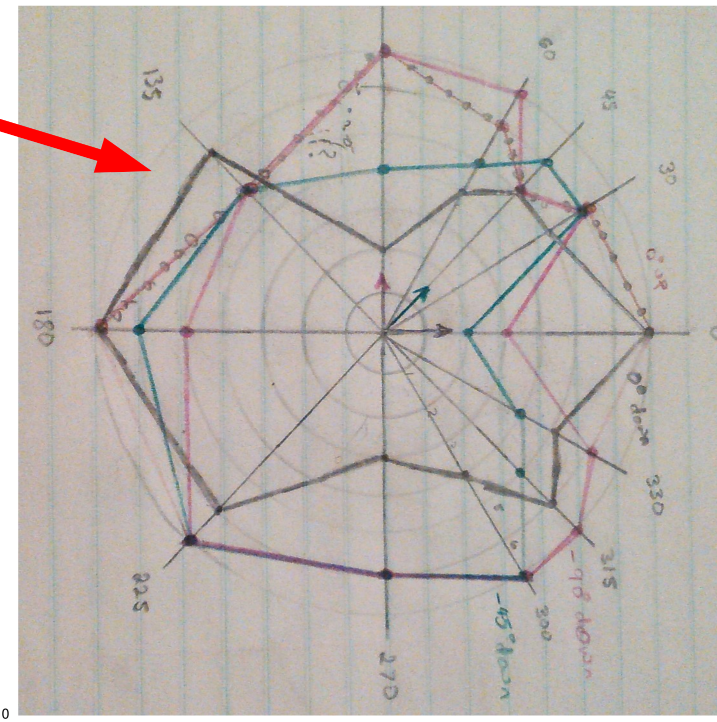


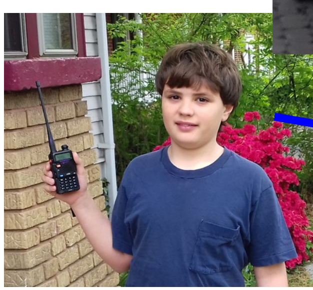
Bearing sensors

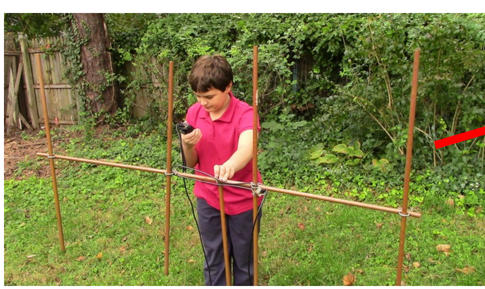


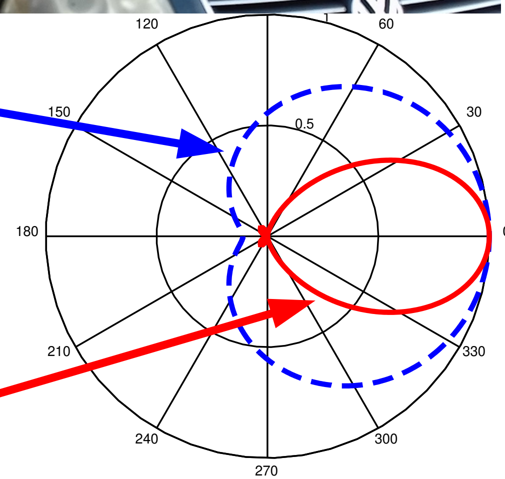
Antenna pattern

Bearing sensors… reality…



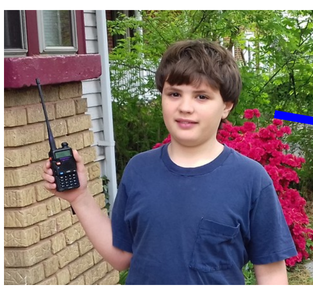


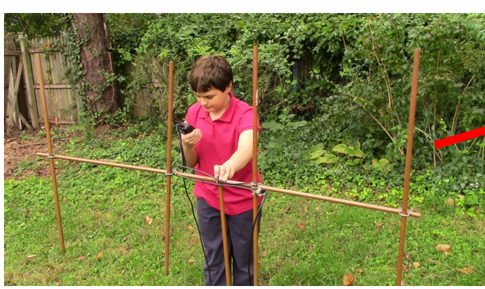


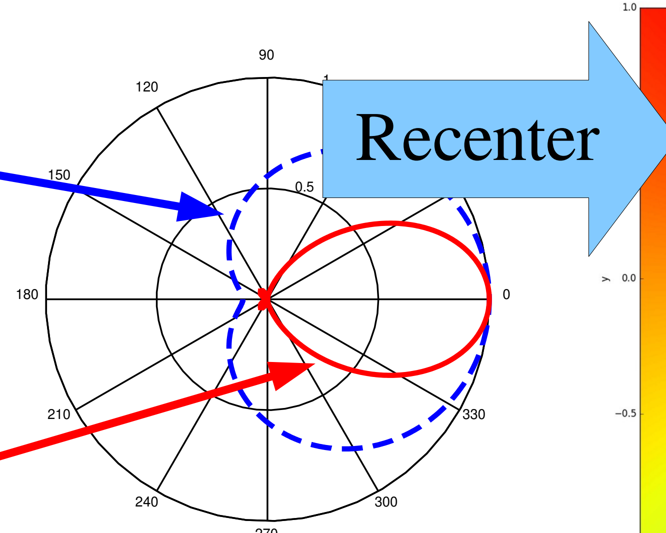


Antenna pattern

Bearing observations

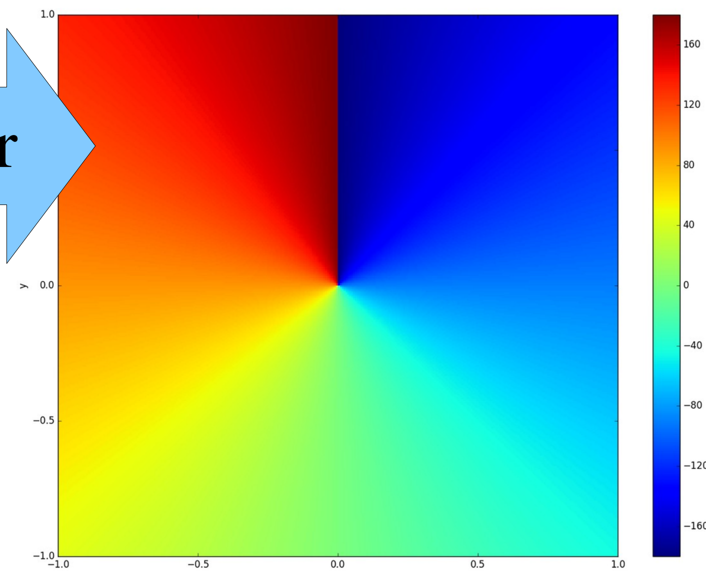






Antenna pattern

Bearing as a function of sensor position



Bearing sheaf

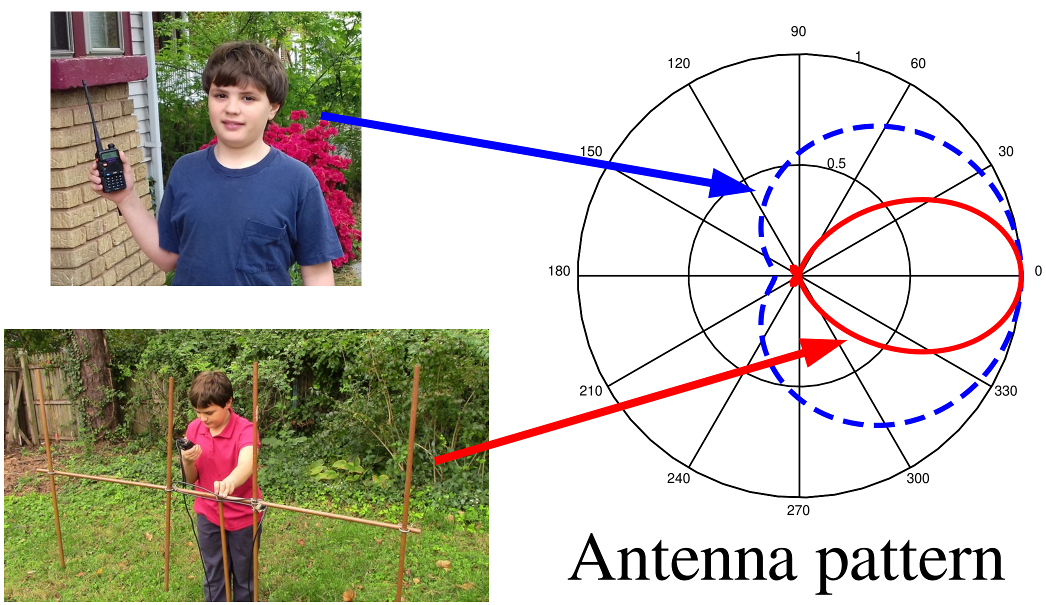
Fox position — **Sensor position, Bearing**

$$\mathbb{R}^2 \xleftarrow{\ \mathrm{pr}_1\ } \mathbb{R}^2\times\mathbb{R}^2 \xrightarrow{\ (\mathrm{pr}_2,\, M_{bearing})\ } \mathbb{R}^2\times S^1$$

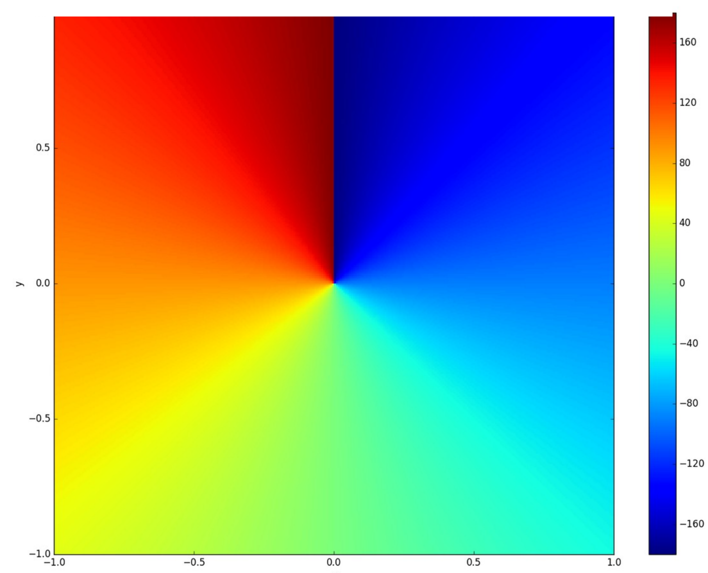
True observations

Virtual observations

Fox position, Sensor position



Antenna pattern



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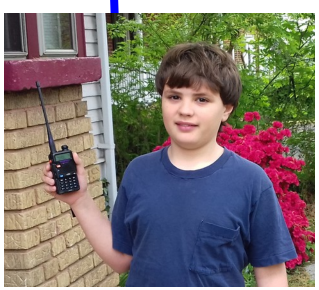
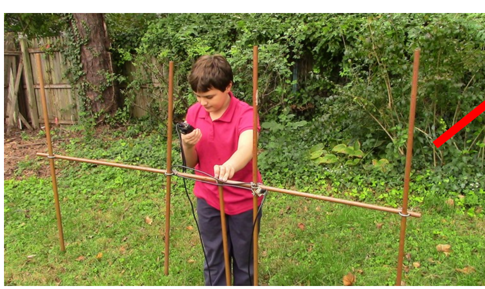
Bearing sheaf (two sensors)

Sensor 1 position, Bearing — **Fox position** — **Sensor 2 position, Bearing**

$\mathbb{R}^2 \times S^1$ $\xleftarrow{(pr_2, M_{bearing})}$ $\mathbb{R}^2 \times \mathbb{R}^2$ $\xrightarrow{pr_1}$ $\mathbb{R}^2$ $\xleftarrow{pr_1}$ $\mathbb{R}^2 \times \mathbb{R}^2$ $\xrightarrow{(pr_2, M_{bearing})}$ $\mathbb{R}^2 \times S^1$

Fox position, Sensor 1 position — Fox position, Sensor 2 position

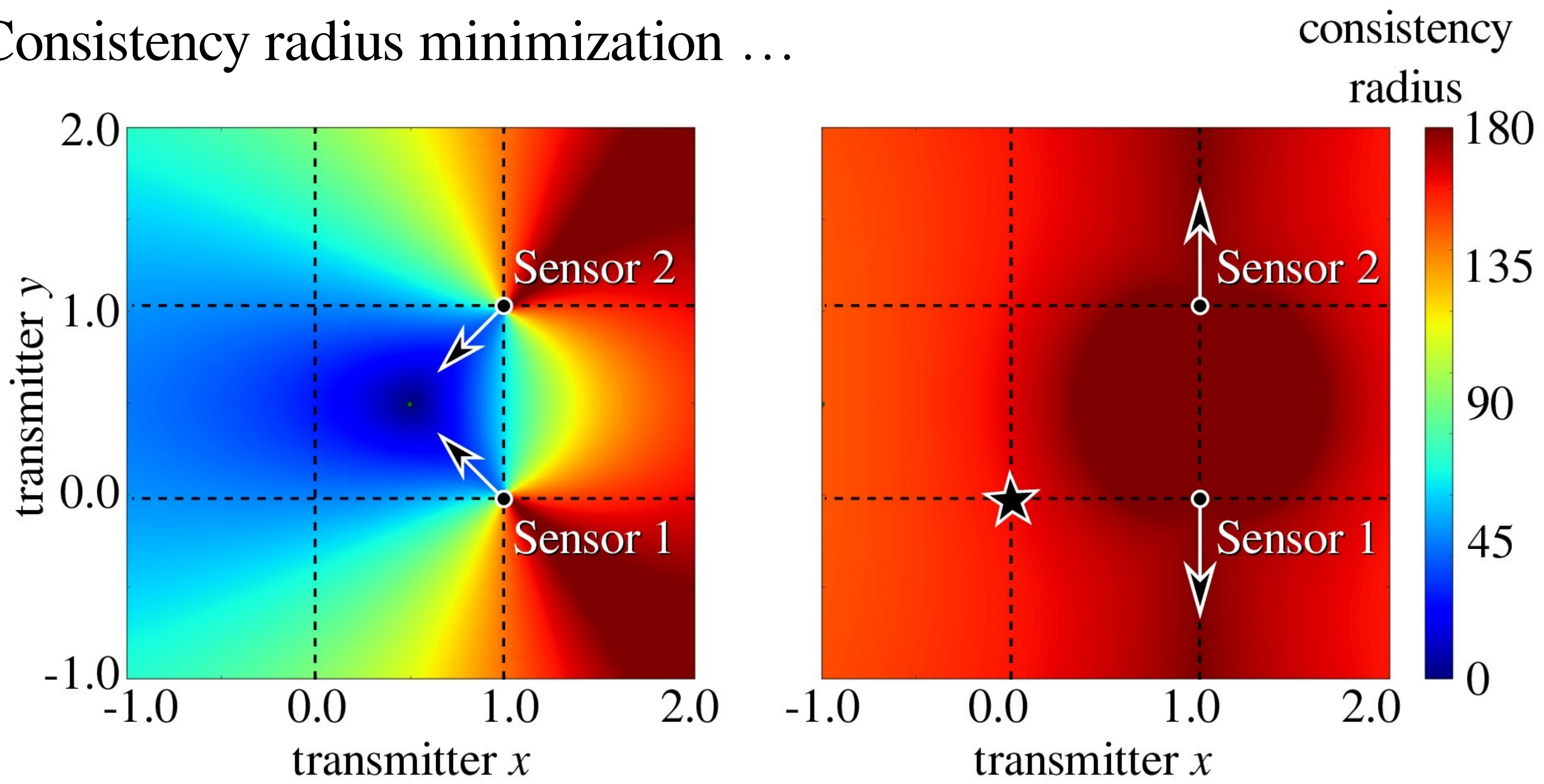




Global sections of this sheaf correspond to two bearings whose sight lines intersect at the fox transmitter

Consistency of proposed fox locations

Consistency radius minimization …

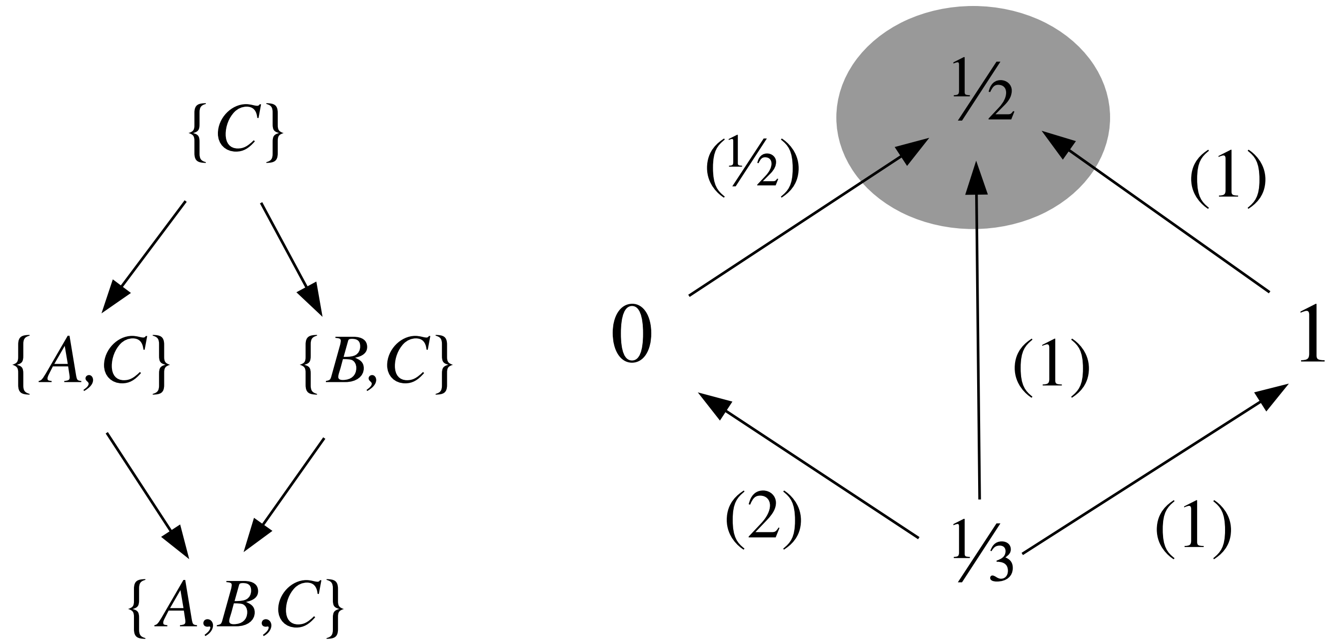


… converges to a likely fox location

… does not converge!

Local consistency radius

Consistency radius of this open set = 0

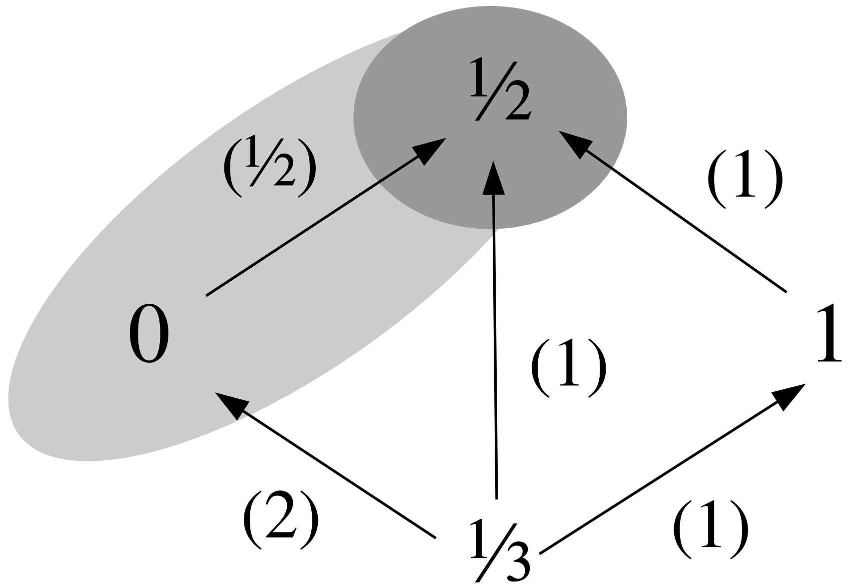


<u>Lemma</u>: Consistency radius on an open set U is computed by only considering open sets $V_1 \subseteq V_2 \subseteq U$

Local consistency radius

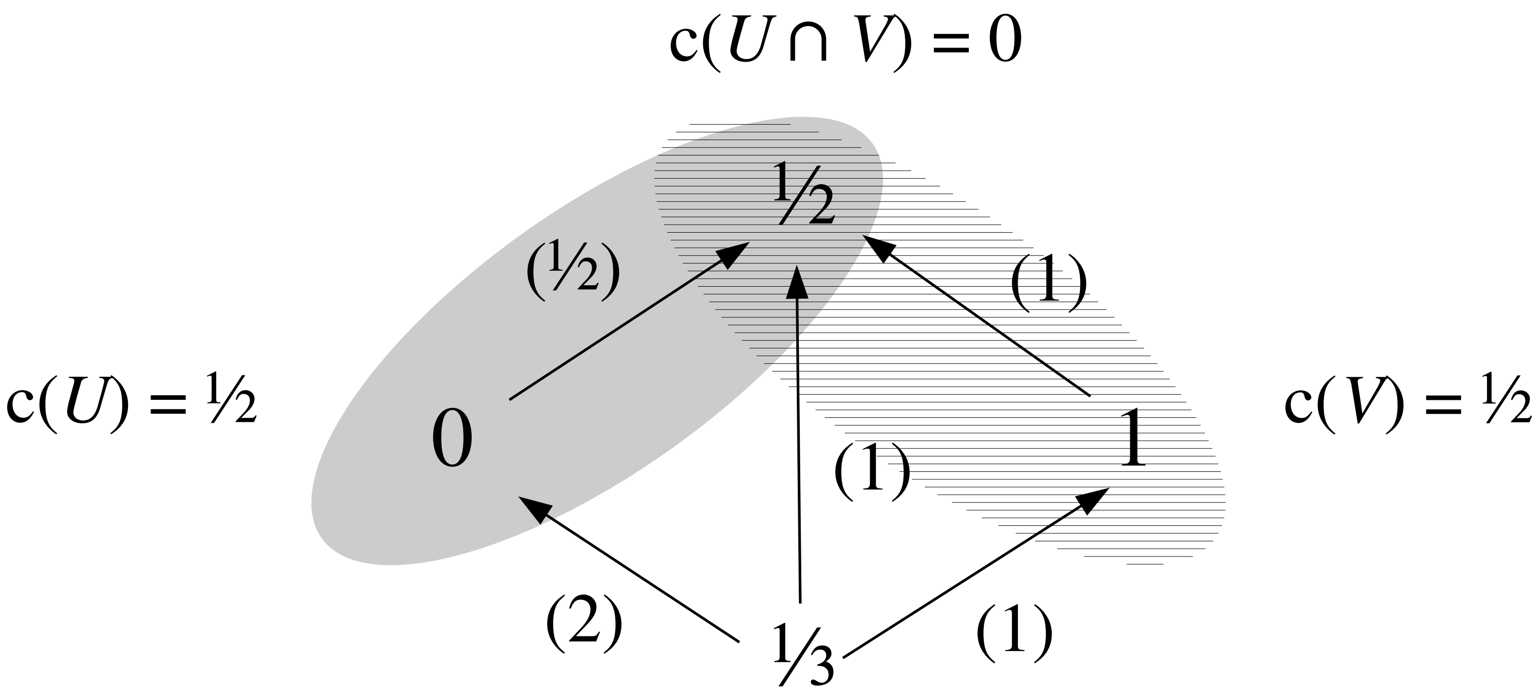
Consistency radius of this open set = 0

c(*U*) = ½



Lemma: Consistency radius does not decrease as its support grows: if $U \subseteq V$ then $c(U) \leq c(V)$.

Local consistency radius

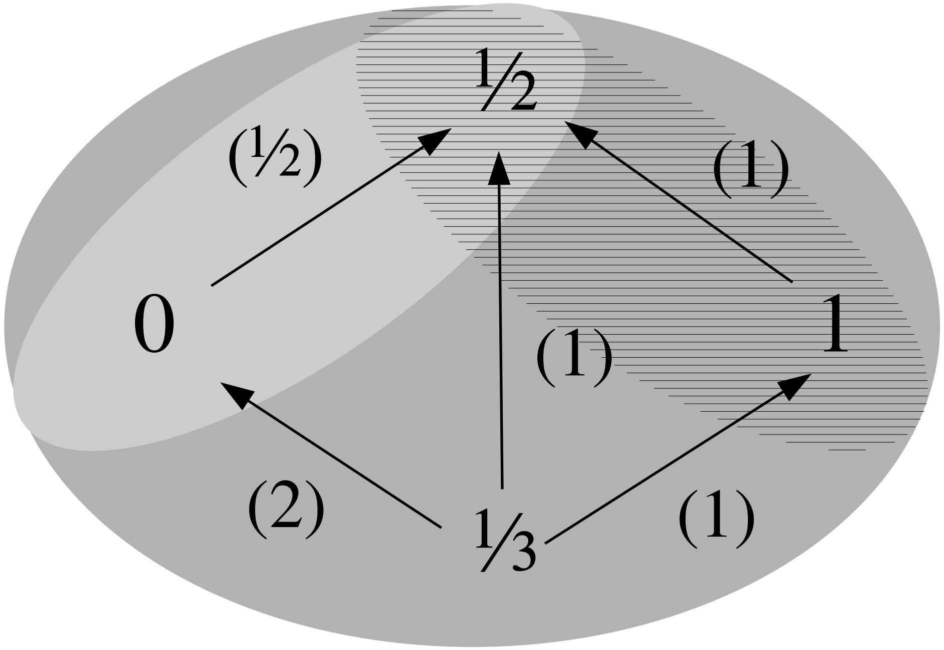


Lemma: Consistency radius does not decrease as its support grows: if $U \subseteq V$ then $c(U) \leq c(V)$.

Consistency radius is not a measure

$c(U \cap V) = 0$

$c(U) = ½$



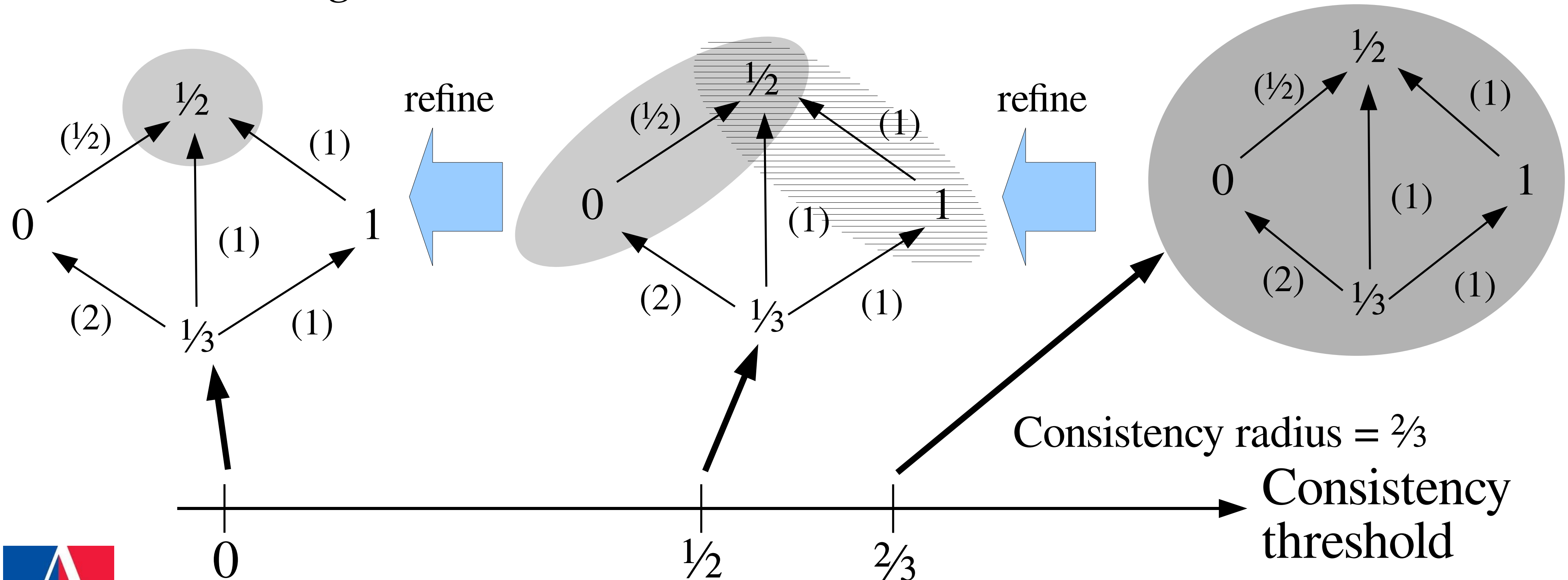
$c(V) = ½$

$c(U \cup V) = ⅔ \neq c(U) + c(V) - c(U \cap V)$

(Consistency radius yields an *inner measure* after some work)

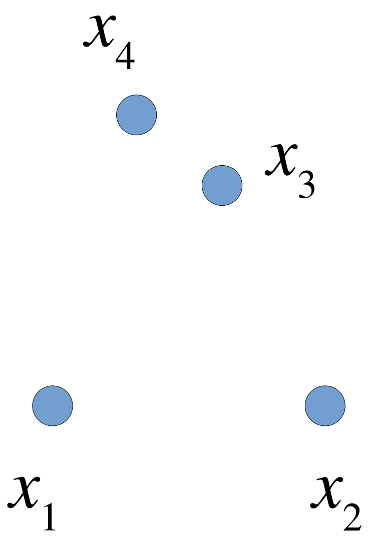
The *consistency filtration*

- … assigns the set of open sets (open cover) with consistency less than a given threshold
- Lemma: consistency filtration **is itself a sheaf** of collections of open sets on $(\mathbb{R},\leq)$. Restrictions in this sheaf are *cover coarsenings*.



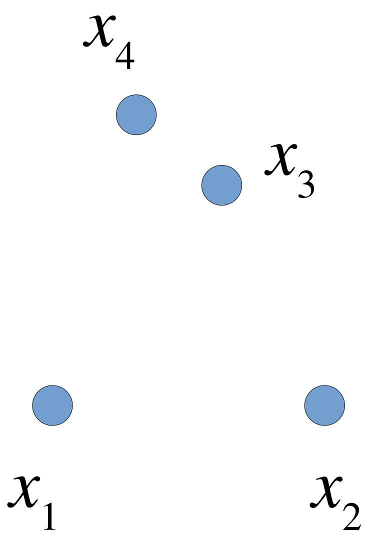
Persistent Čech cohomology

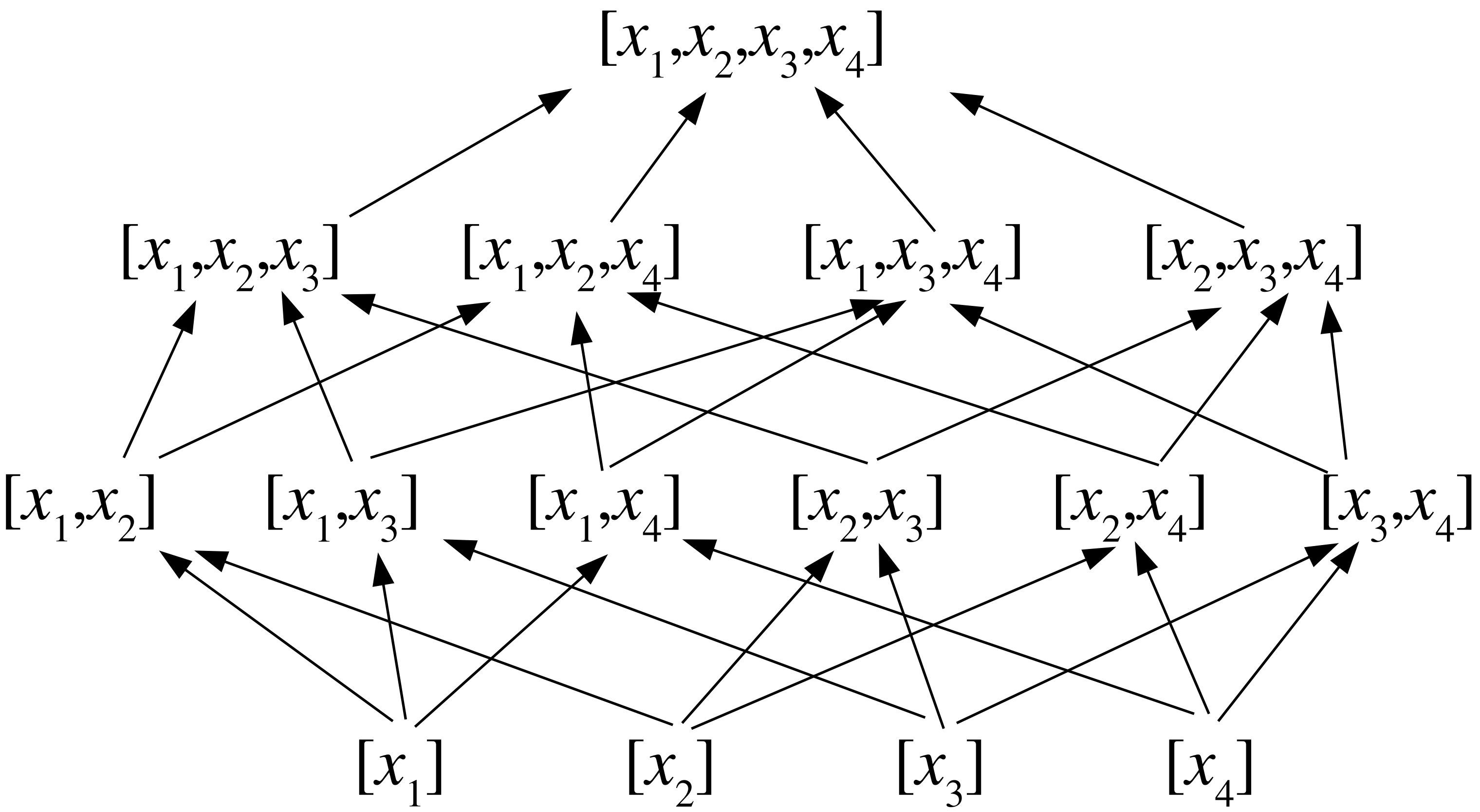
- Consider a point cloud



Persistent Čech cohomology

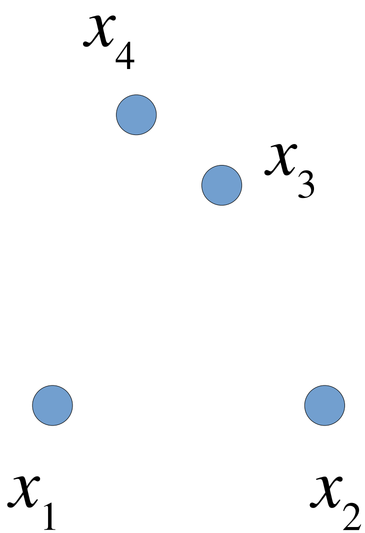
- Build the Alexandrov topology on the complete simplex with the points as vertices...

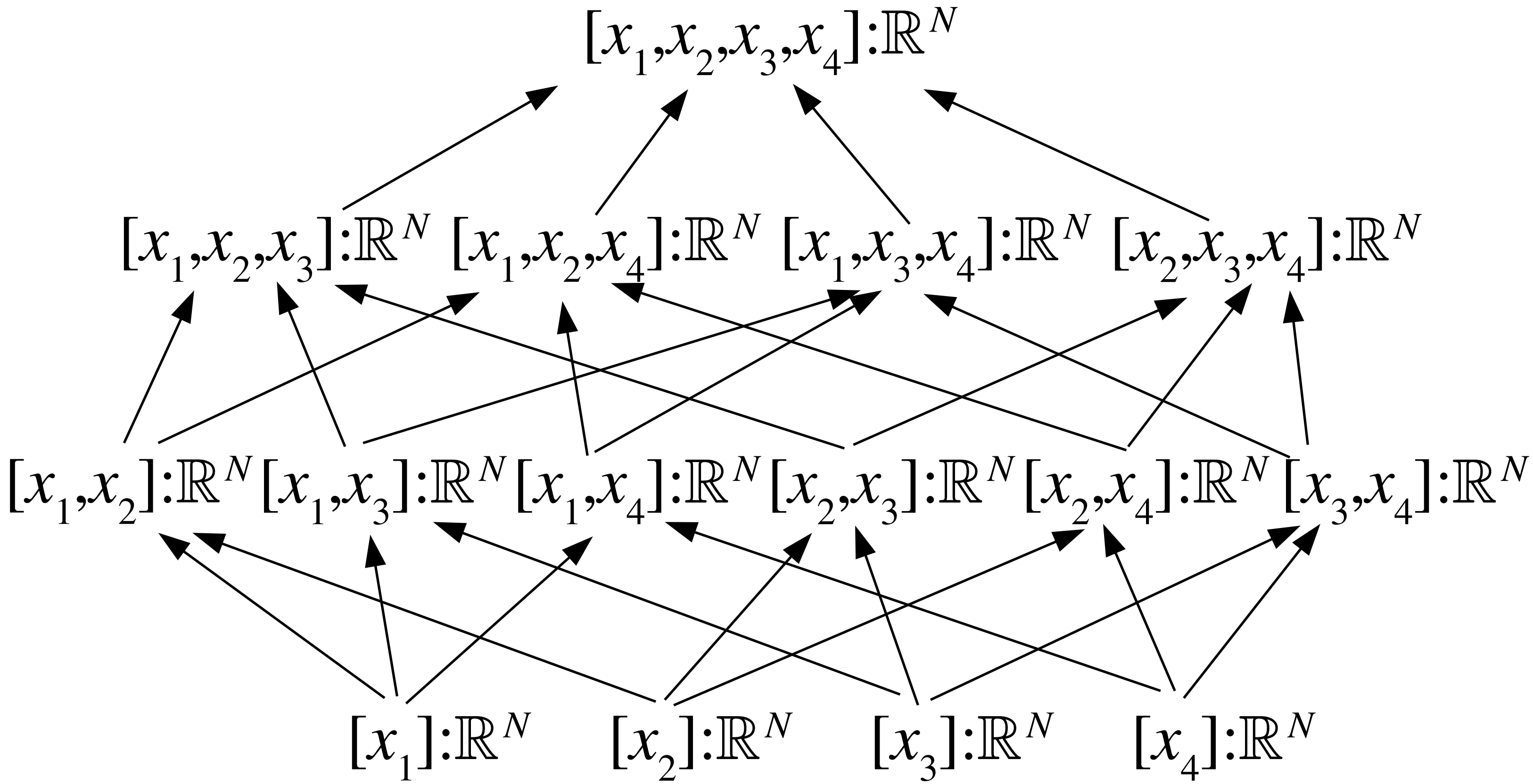




Persistent Čech cohomology

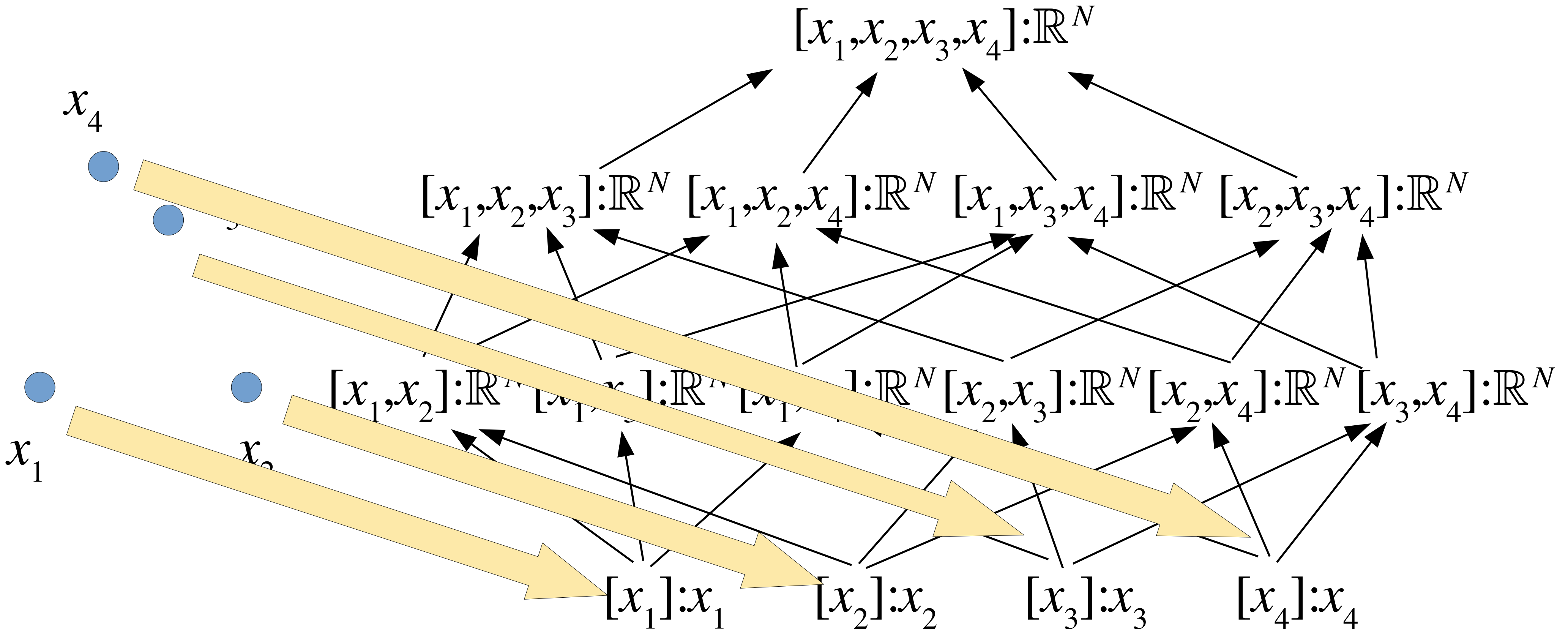
- ...Build the constant sheaf on that. (Restrictions are identity maps.)





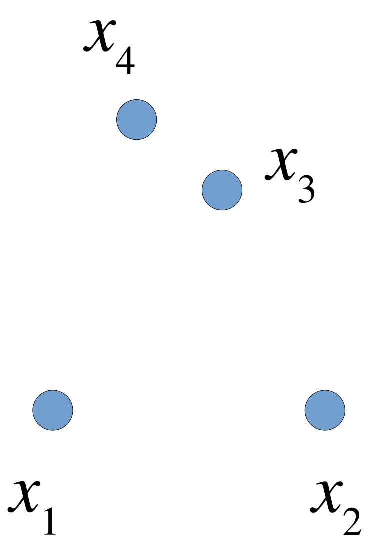
Persistent Čech cohomology

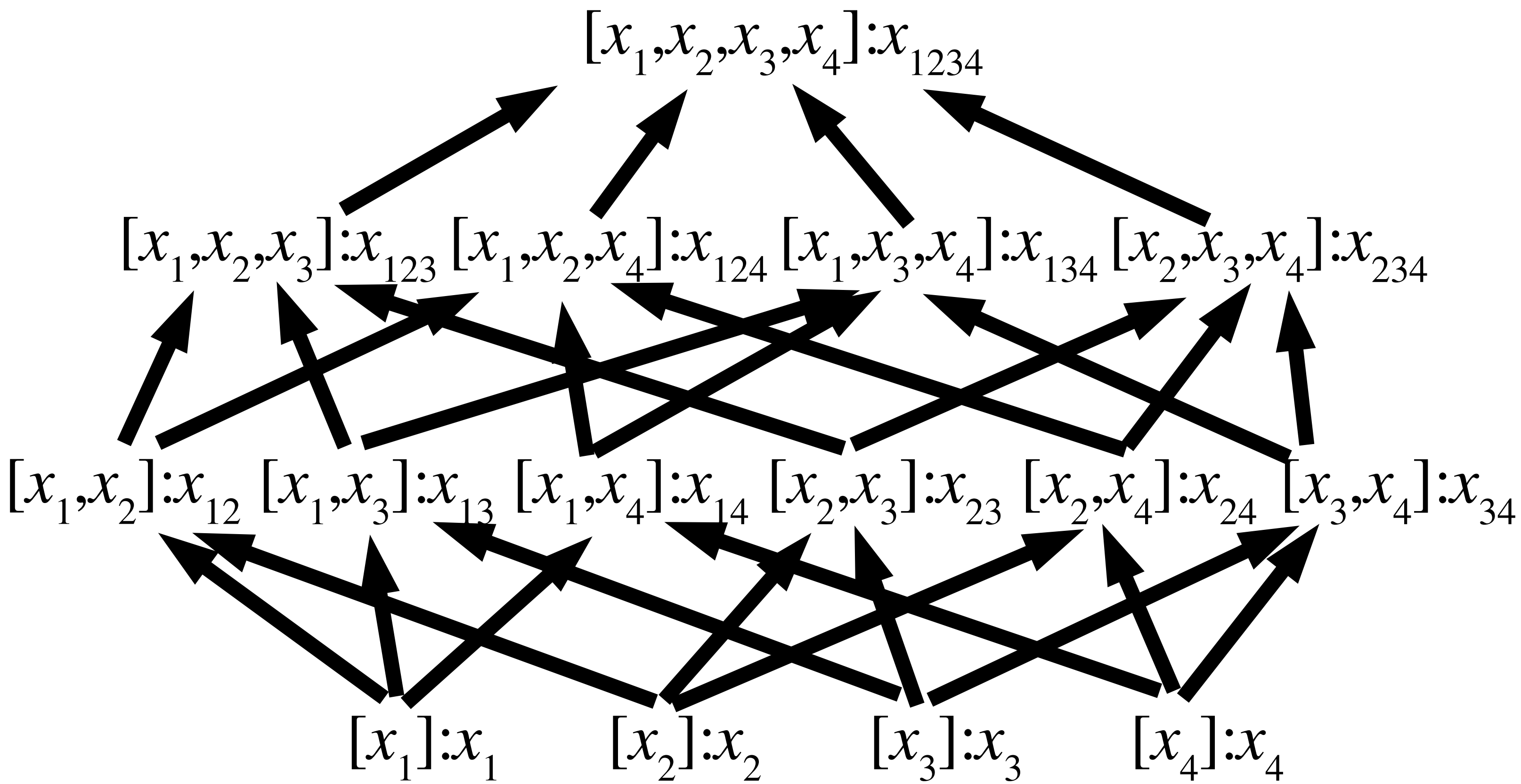
- The coordinates of the points form an assignment to the vertices (lowest level in the poset)



Persistent Čech cohomology

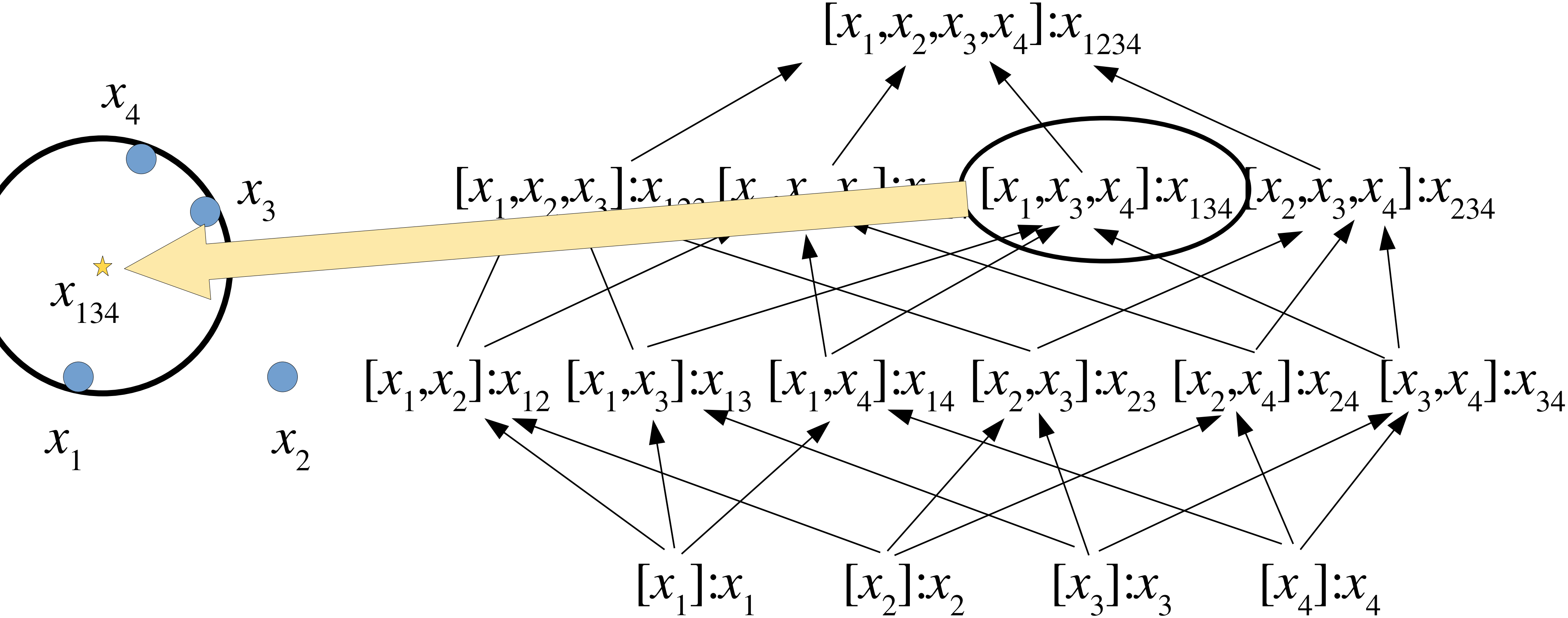
- Find the global assignment with minimal consistency radius





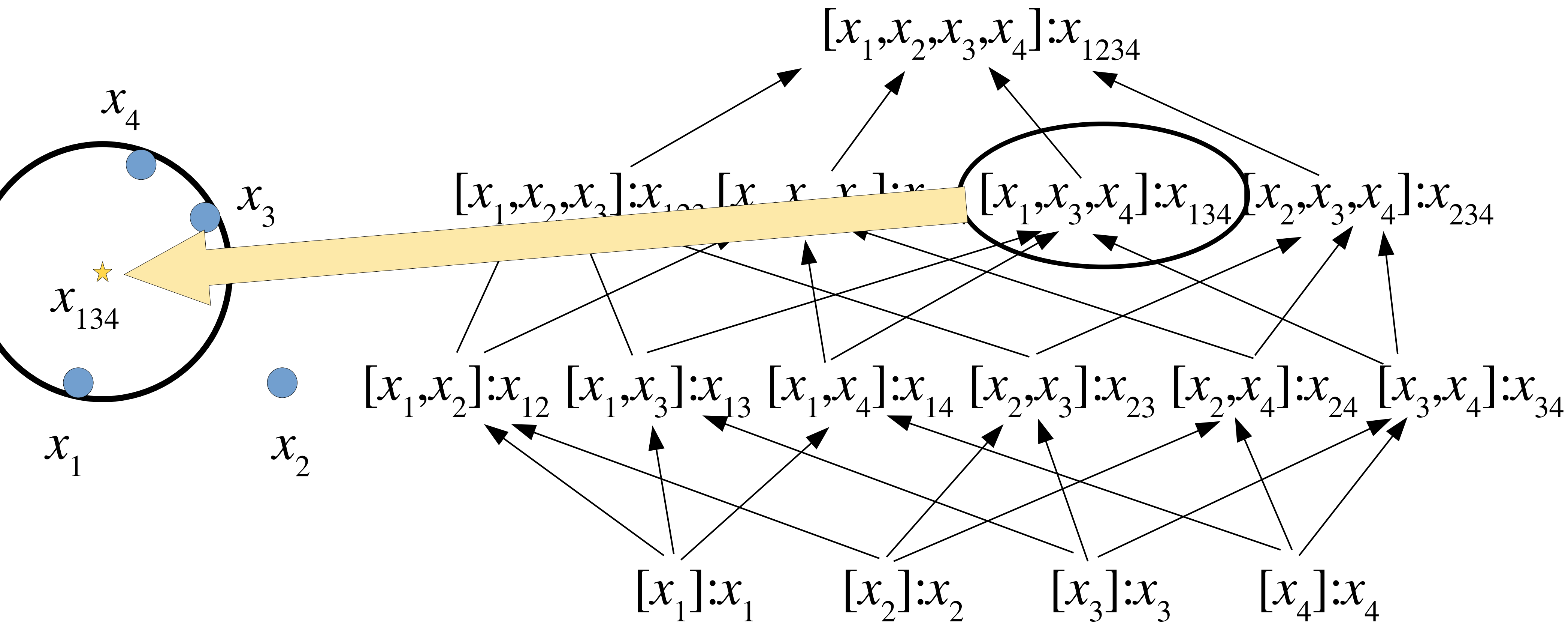
Persistent Čech cohomology

- Each value in the assignment turns out to be the circumcenter of each subset of points



Persistent Čech cohomology

- Theorem: The consistency filtration is isomorphic to the one in "usual" persistent Čech cohomology



Filtrations of partial covers

Covers of topological spaces

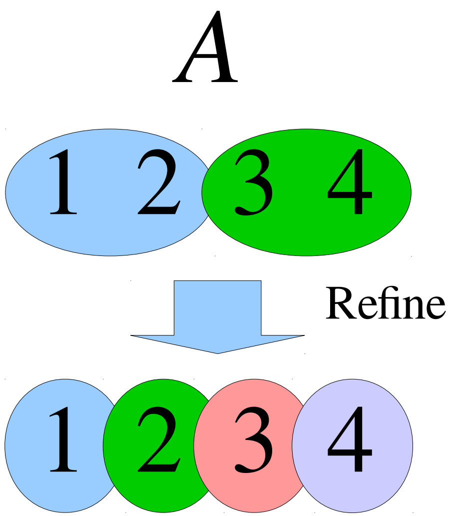
- Classic tool: Čech cohomology
  - Coarse
  - Usually blind to the cover; only sees the underlying space
- Cover measures (Purvine, Pogel, Joslyn, 2017)
  - How fine is a cover?
  - How overlappy is a cover?

Cover measures

- Theorem: (Birkhoff) The set of covers ordered by refinement has an explicit rank function
  - The rank of a given cover is the number of sets in its downset as an antichain of the Boolean lattice
  - This counts the number of sets of consistent faces there are
- Conclusion: An assignment whose maximal cover has a higher rank is more self-consistent

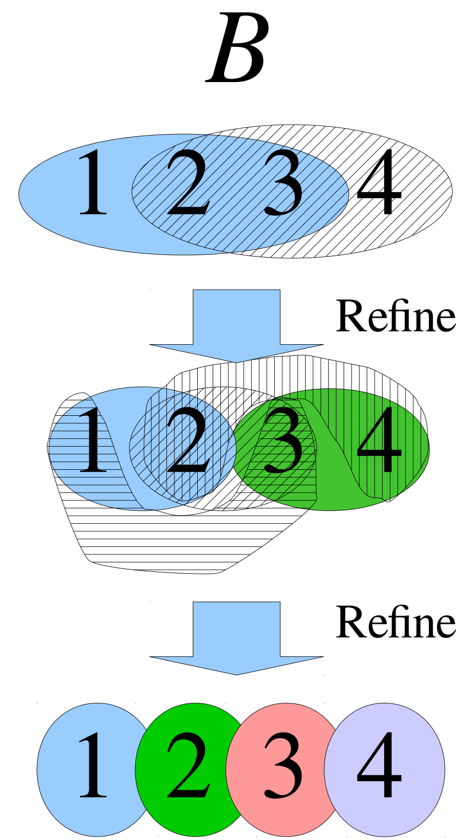
Cover measures

- Consider the following two covers of {1,2,3,4}



Total = 6 sets

Since $6 < 11$,
cover *B* is coarser

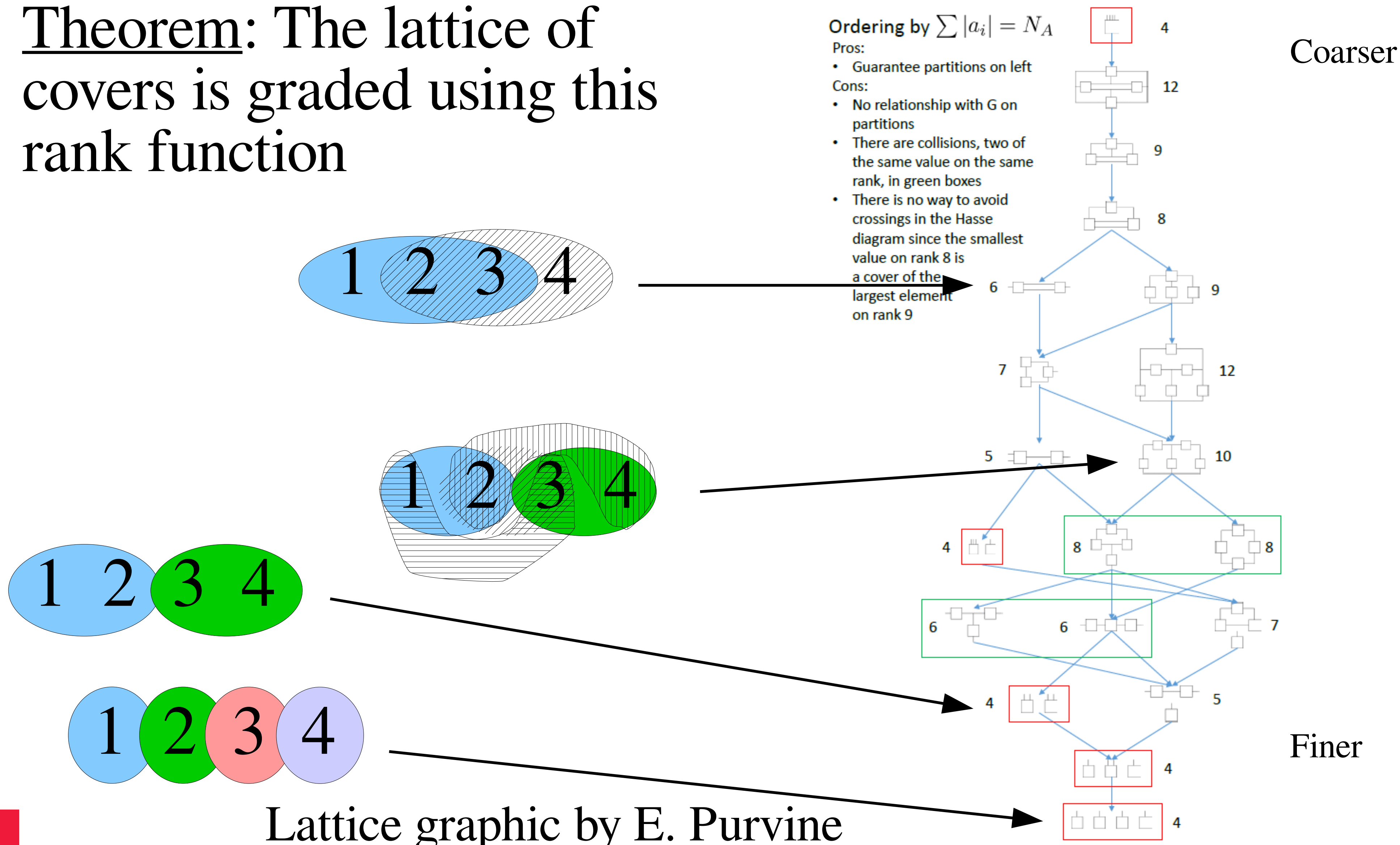


Total = 11 sets

Computing the number of sets in its downset
as an antichain of the Boolean lattice

The lattice of covers

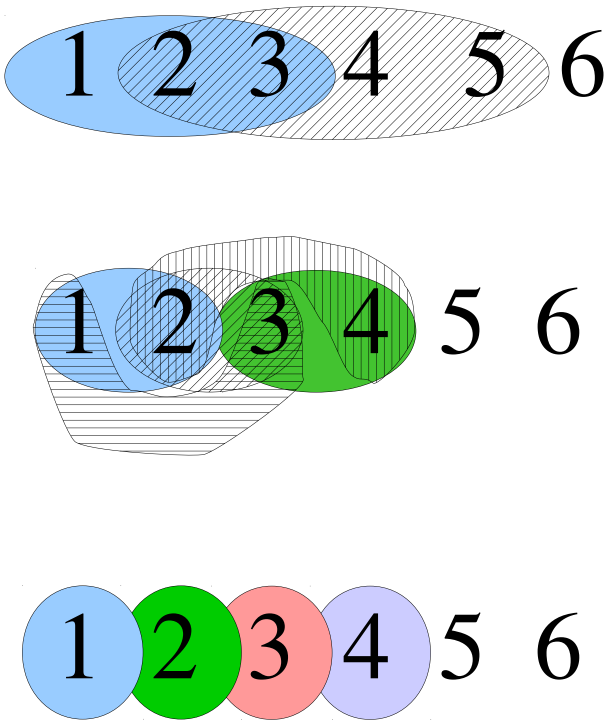
- Theorem: The lattice of covers is graded using this rank function



Lattice graphic by E. Purvine

Defining **PartCovers** : partial covers

- Start with a fixed topological space
- Objects: Collections of open sets
- No requirement of coverage

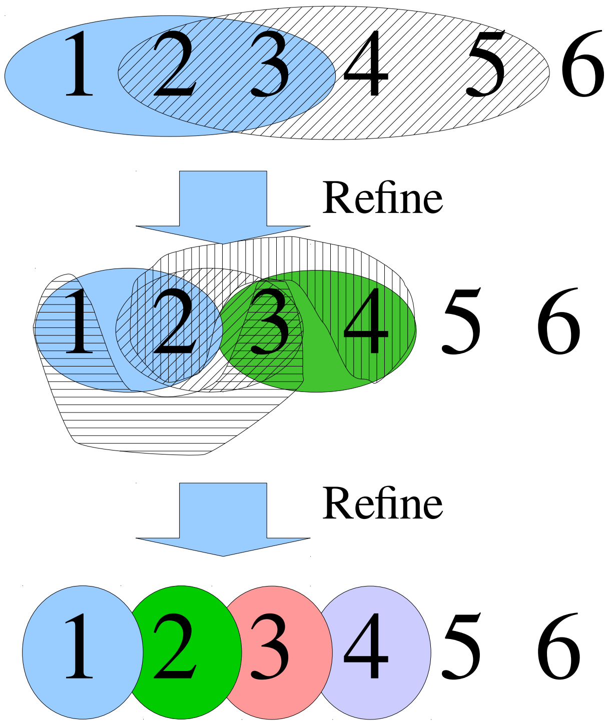


Defining **PartCovers** : partial covers

- Morphisms are r*efinements* of covers:

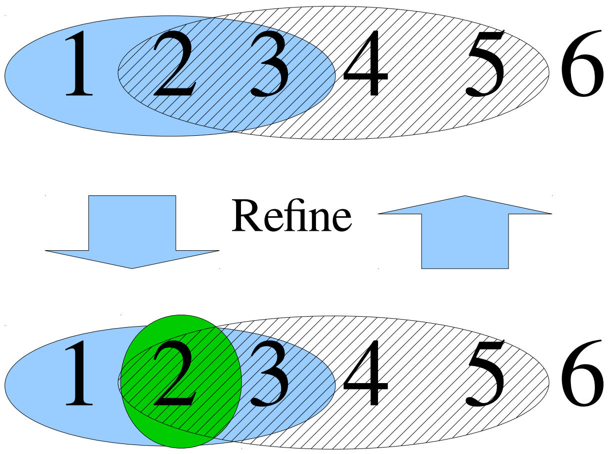
  If $\mathscr{U}$ and $\mathscr{V}$ are partial covers, $\mathscr{V}$ *refines* $\mathscr{U}$ if for all V in $\mathscr{V}$ there is a U in $\mathscr{U}$, with $V \subseteq U$.

- <u>Convention</u>: $\mathscr{U} \rightarrow \mathscr{V}$



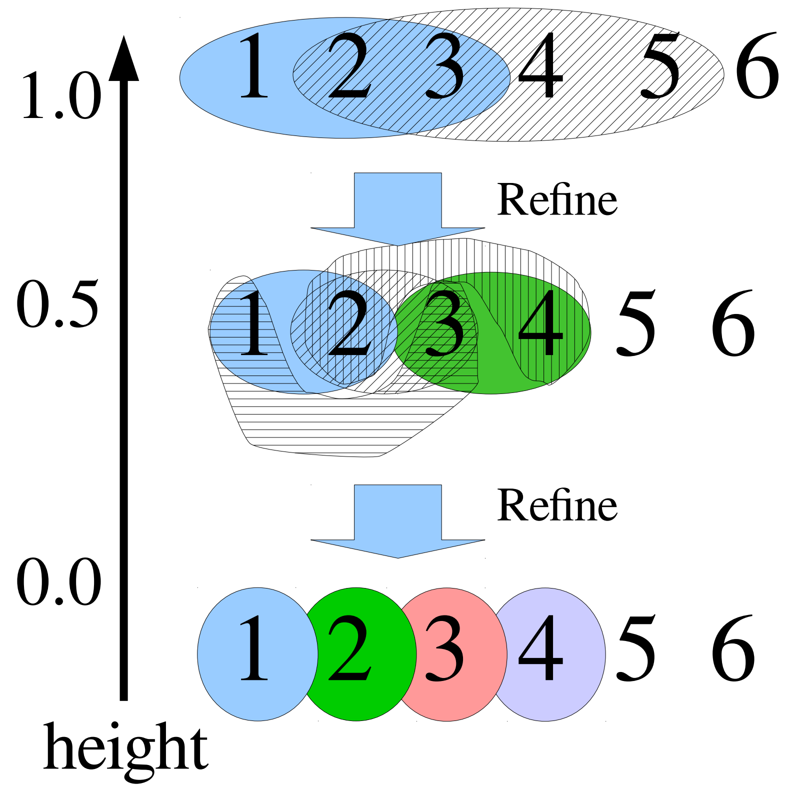
Irredundancy

- *Irredundant cover* has no cover elements contained in others
- Minimal representatives of **PartCovers** isomorphism classes
  - Minimal according to **inclusion**, not refinement
- Lemma: Every finite partial cover is **PartCovers**-isomorphic to a unique irredundant one



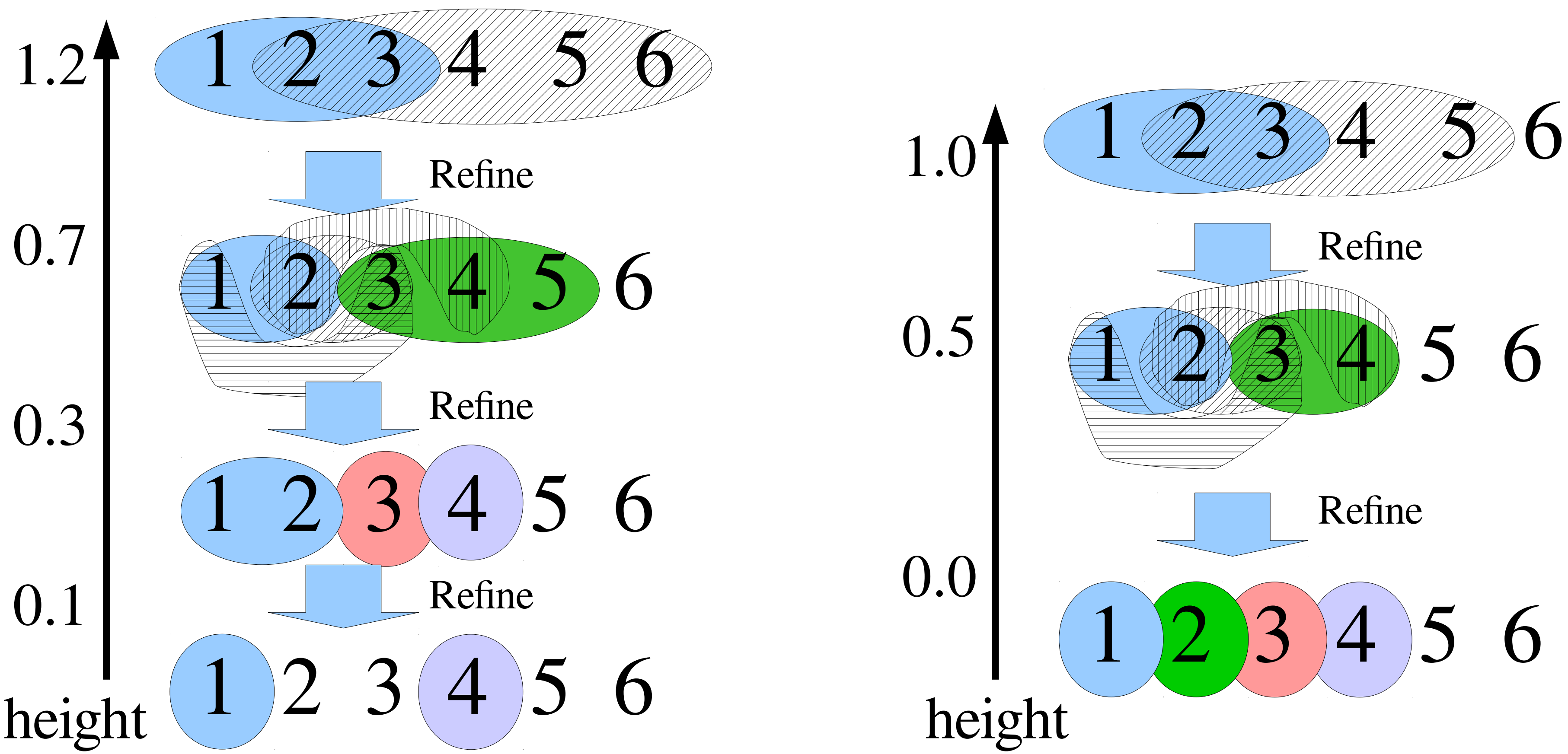
CoarseFilt : Filtrations in **PartCovers**

- Objects are chains of morphisms in **PartCovers** with a monotonic height function
  - Height increases as cover coarsens
  - Could be the cover lattice rank, but need not be



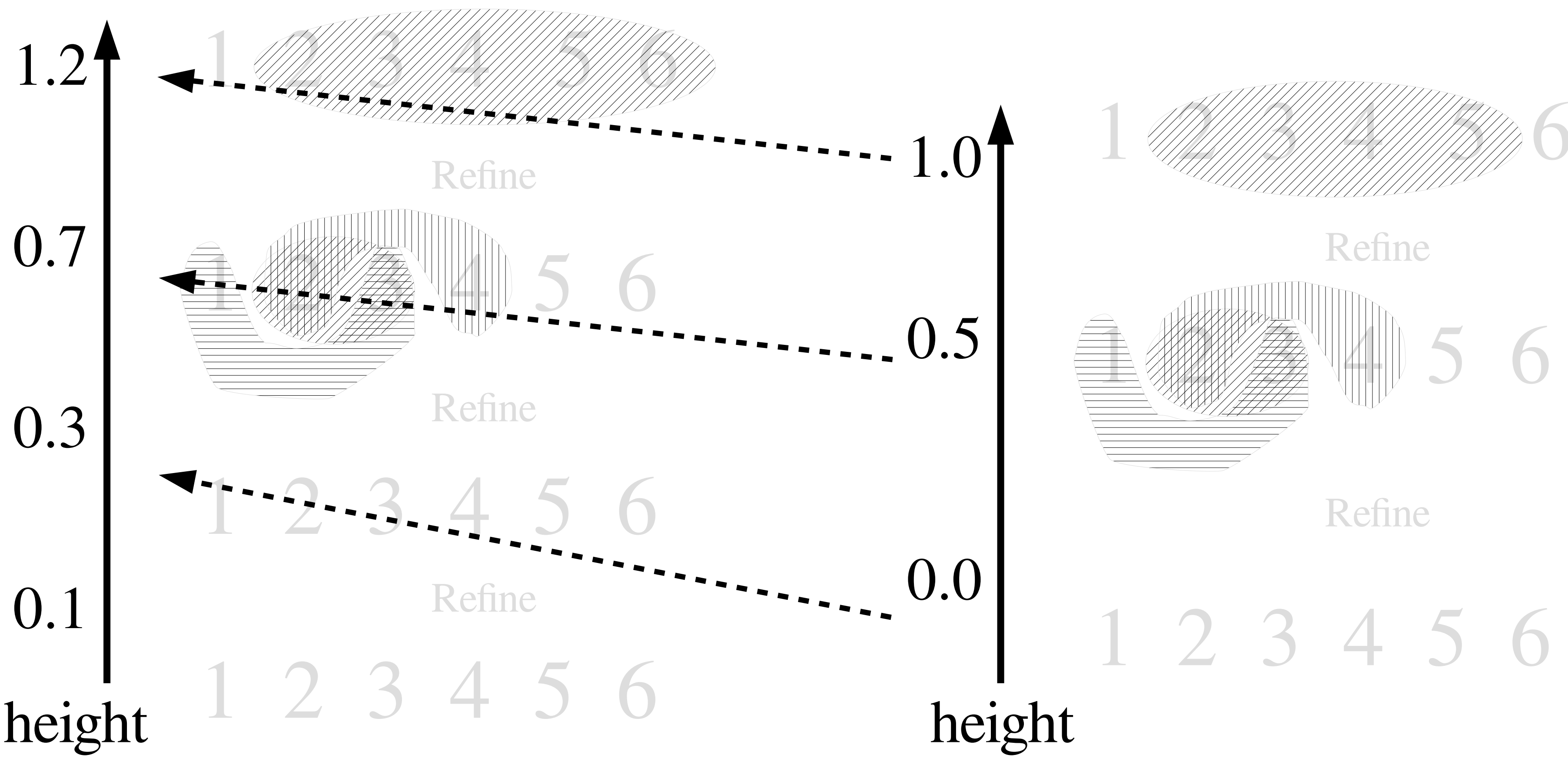
CoarseFilt : Filtrations in **PartCovers**

- Morphisms are commutative ladders of refinements with a monotonic mapping $\phi : \mathbb{R} \rightarrow \mathbb{R}$ of height functions



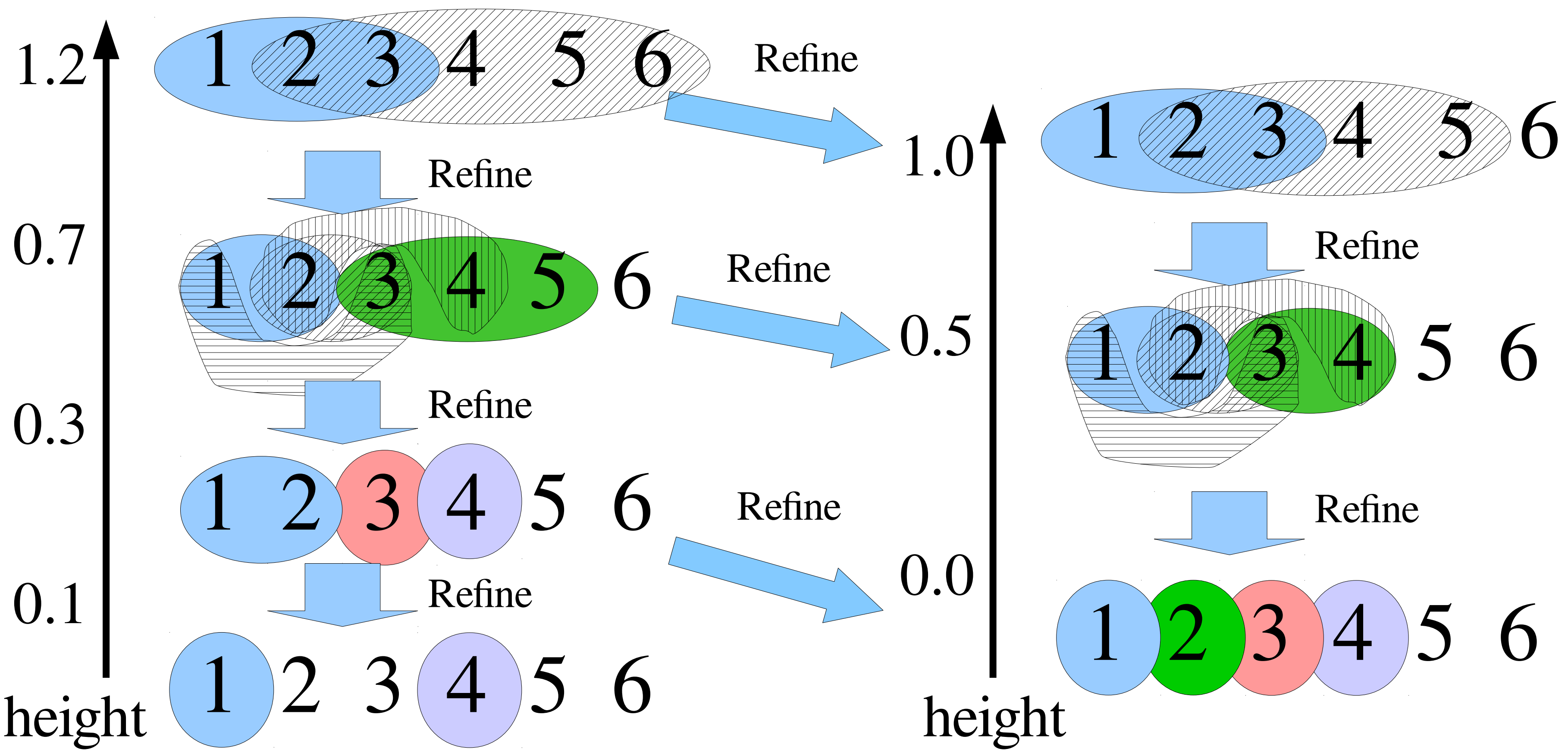
CoarseFilt : Filtrations in **PartCovers**

- Morphisms are commutative ladders of refinements with a **monotonic mapping** $\phi : \mathbb{R} \to \mathbb{R}$ of height functions



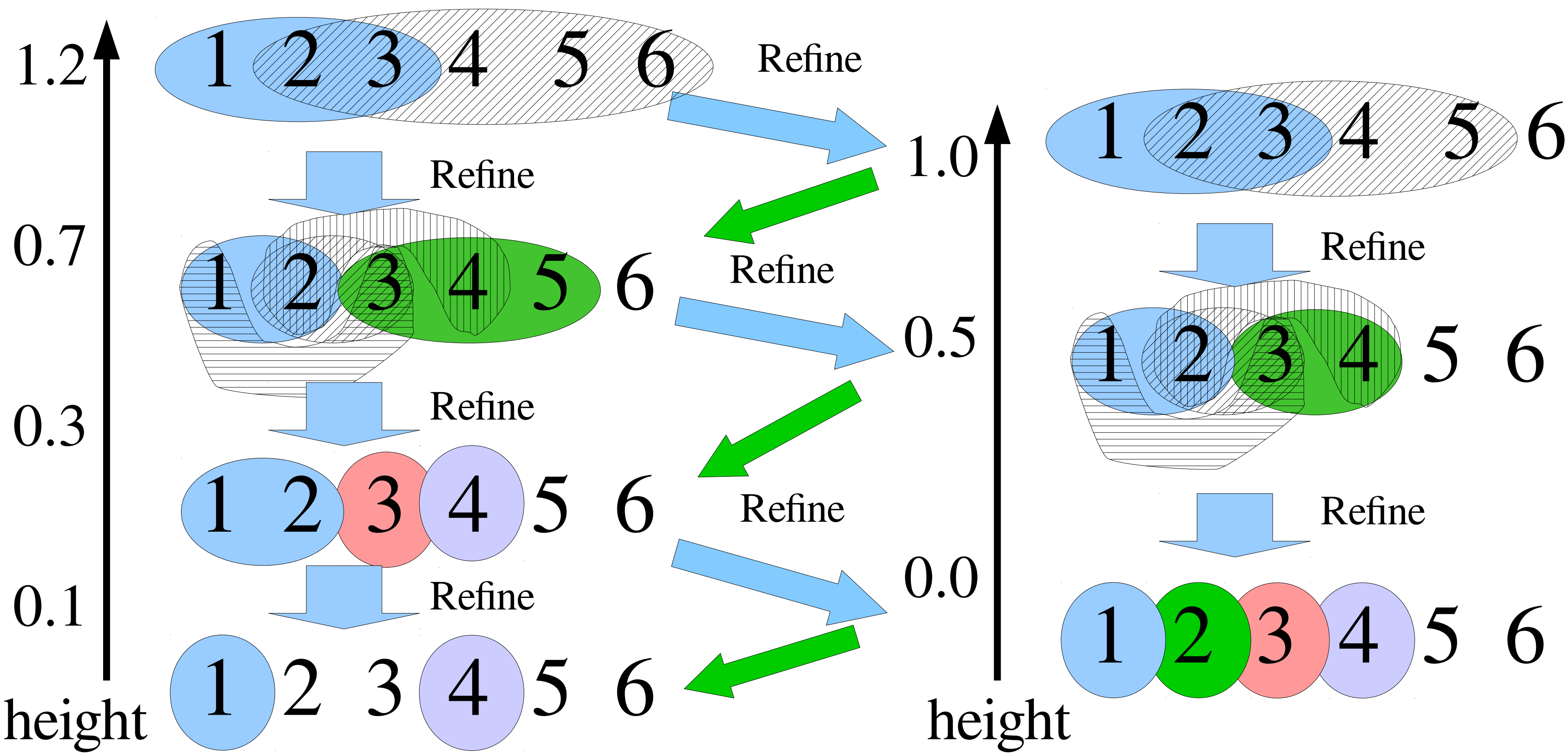
CoarseFilt : Filtrations in **PartCovers**

- Morphisms are **commutative ladders of refinements** with a monotonic mapping $\phi : \mathbb{R} \to \mathbb{R}$ of height functions



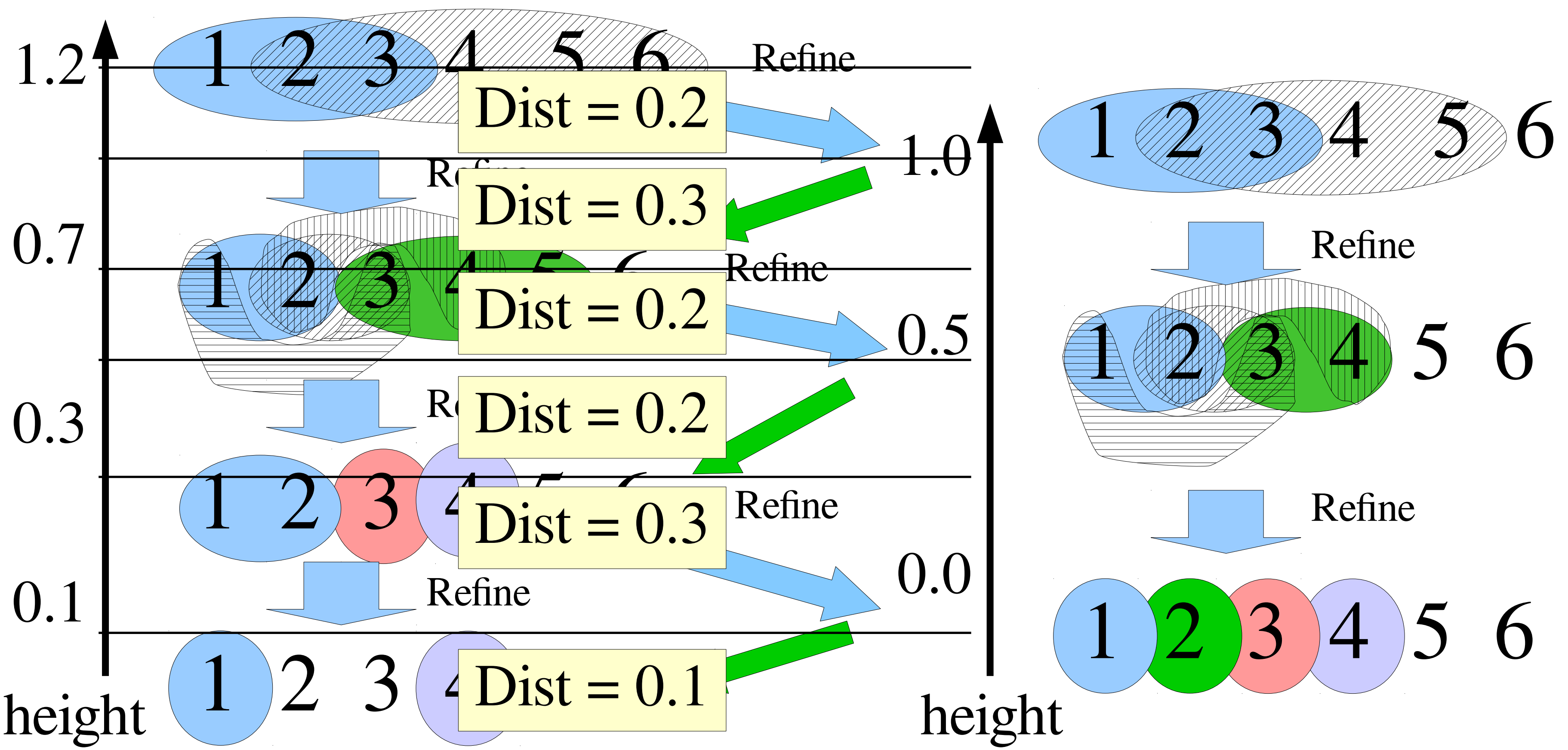
Interleavings in **CoarseFilt**

- Pair of morphisms between two objects



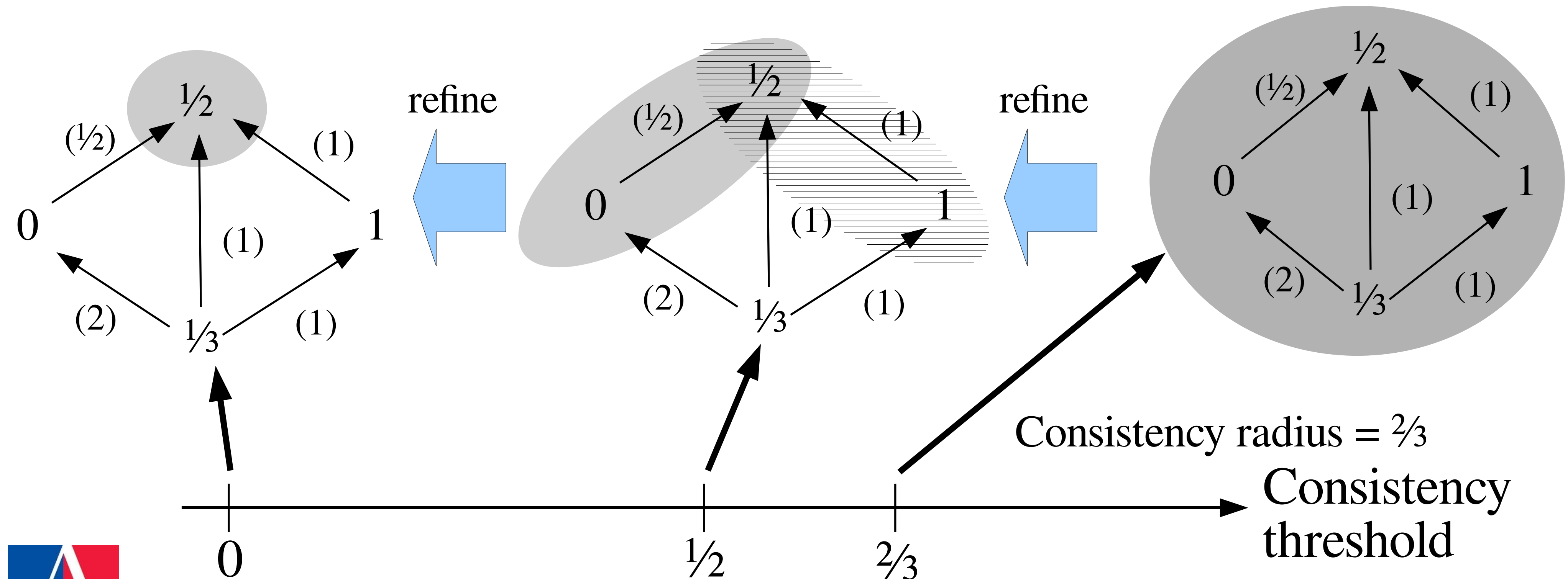
Interleavings in **CoarseFilt**

- Measure the maximum displacement of the heights, minimize over all interleavings = *interleaving distance*



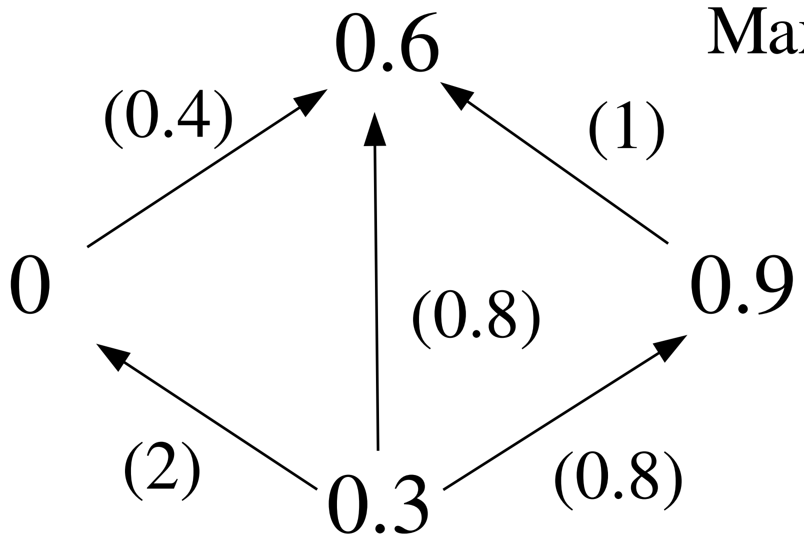
Consistency filtration **stability**

- Theorem: Consistency filtration is continuous under the **CoarseFilt** interleaving distance
- Thus the persistent Čech cohomology of the consistency filtration is **robust** to perturbations

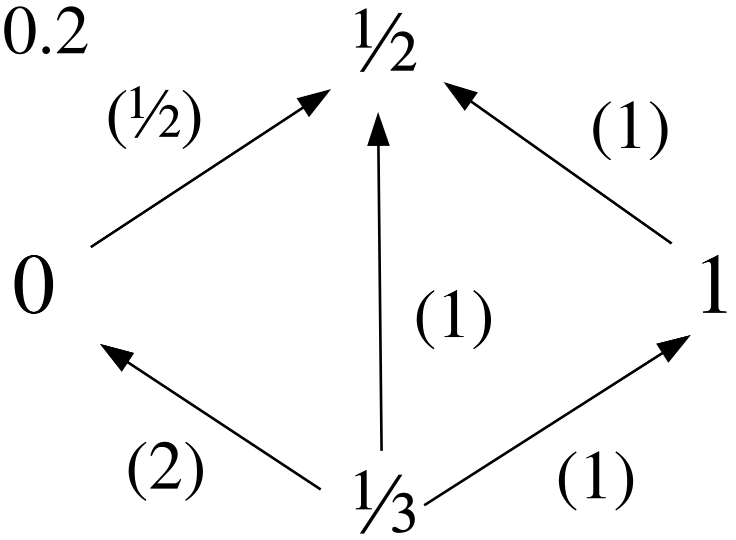


A small perturbation …

- Perturbations allowed in both assignment **and** sheaf (subject to it staying a sheaf!)

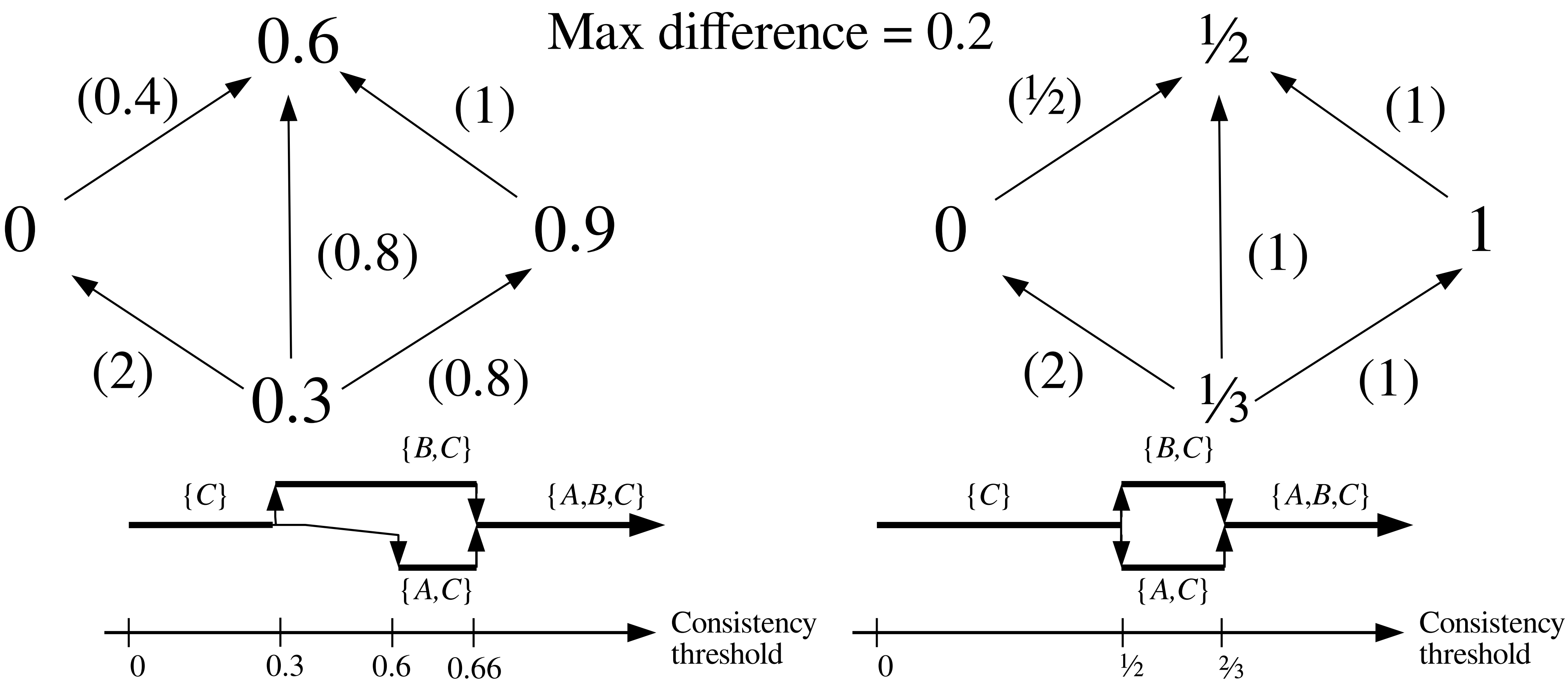


Max difference = 0.2

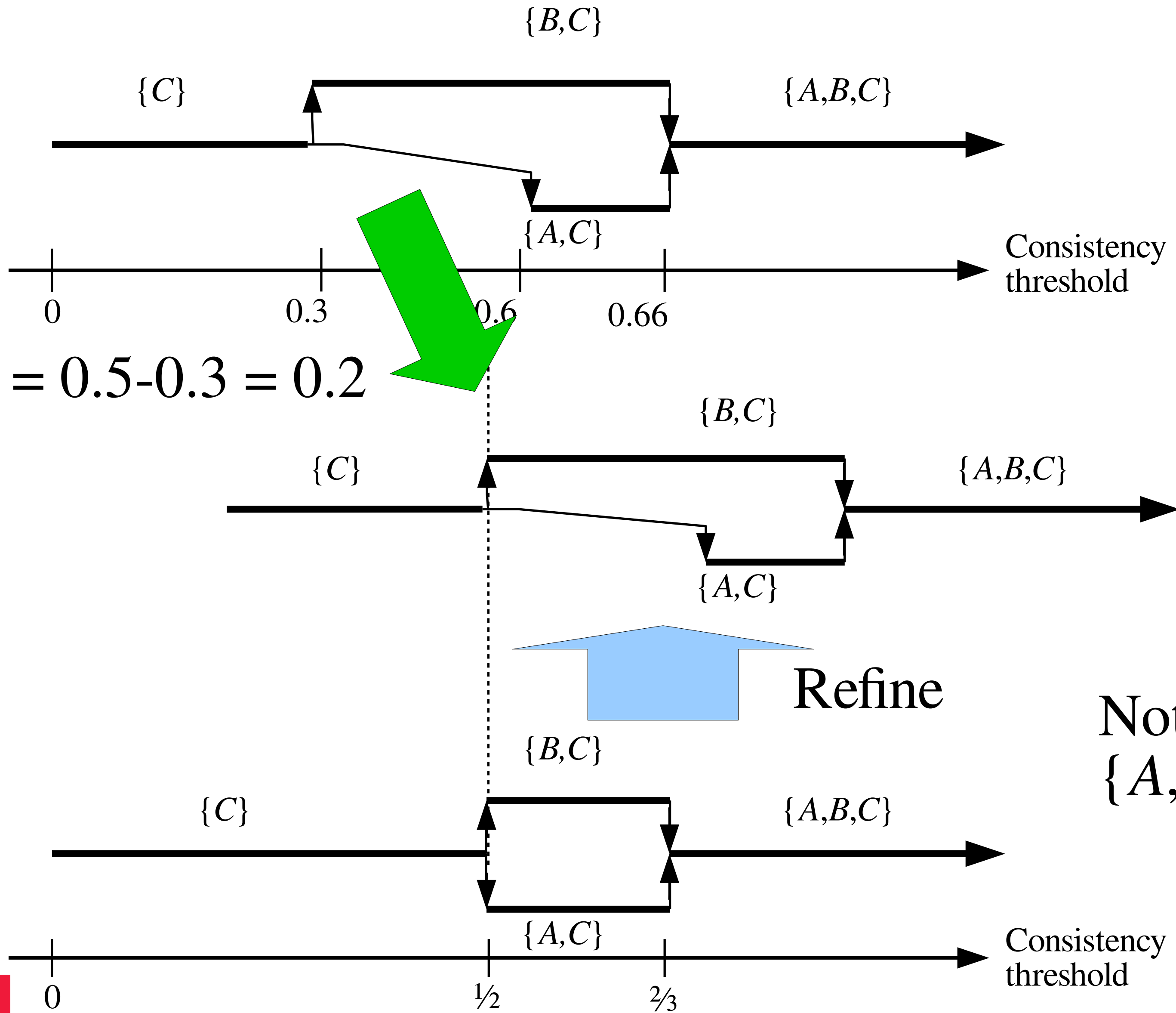


A small perturbation …

- Compute consistency filtrations...



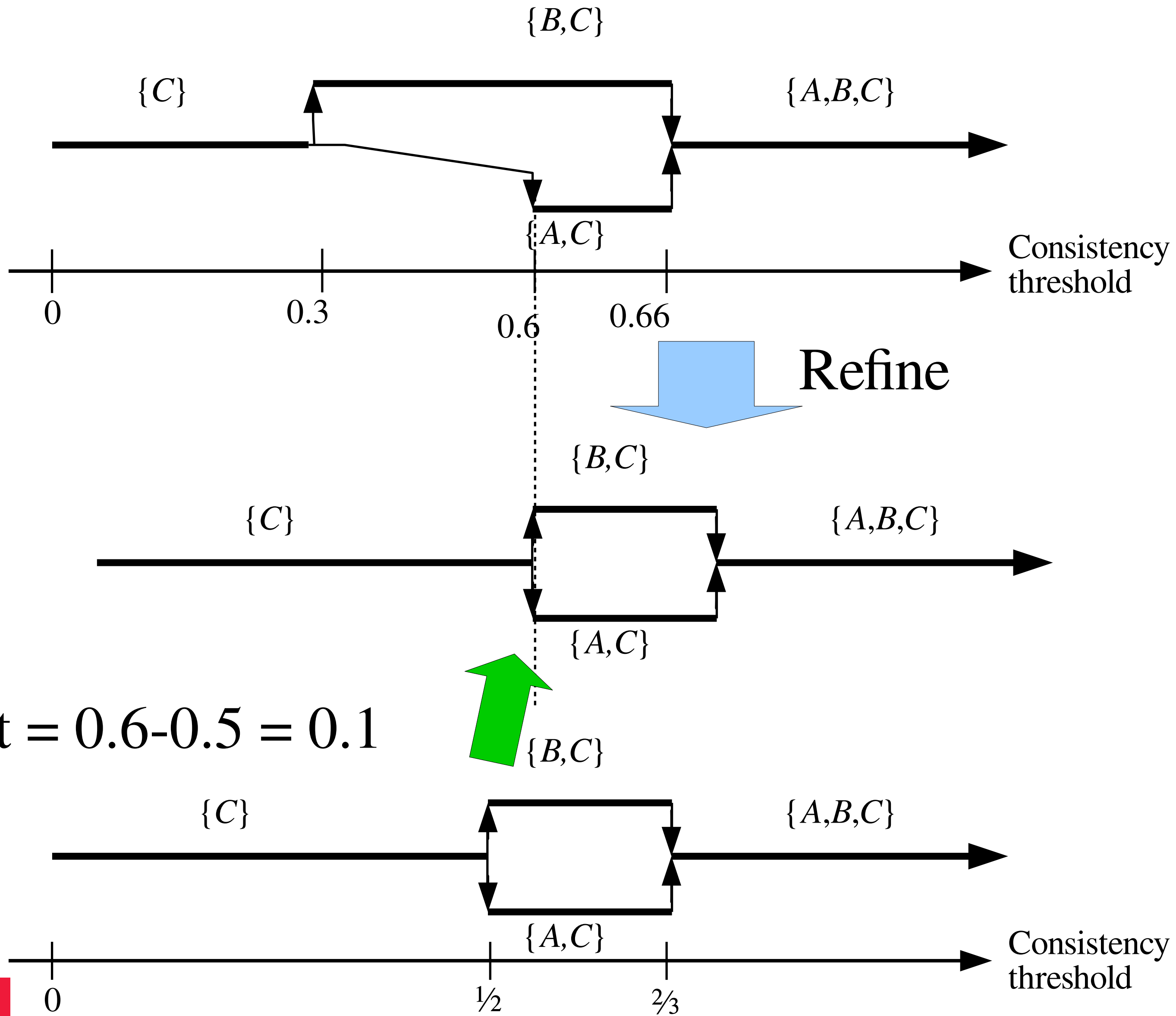
… bounds interleaving distance



Shift = 0.5-0.3 = 0.2

Note that $\{A,C\} \subseteq \{A,B,C\}$ etc.

… bounds interleaving distance



Max shift = 0.2, This is bounded above by constant times the perturbation (0.2 in this case)

Shift = 0.6-0.5 = 0.1

Functoriality of consistency filtrations

Consistency and functoriality

- Category of sheaves **and** assignments: **ShvFPA**

Sheaves on **F**inite spaces with **P**seudometrics paired with **A**ssignments (a bit of a mouthful. Sorry!)

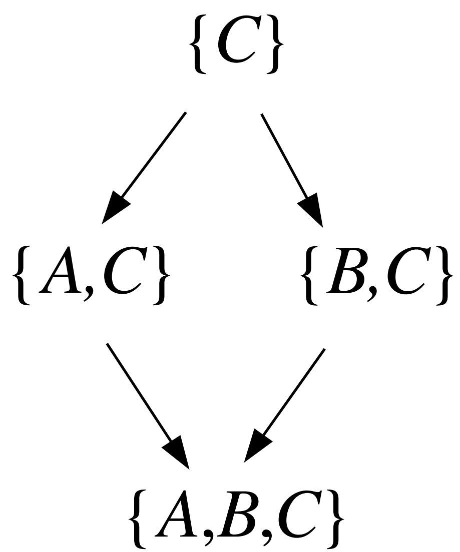
- Suppose $\mathscr{S}$ is a sheaf on X and $\mathscr{R}$ is a sheaf on Y
- Assignments a and b
- A *morphism* m: $(\mathscr{S},a) \to (\mathscr{R},b)$ consists both of
  - A *base space map* on the base spaces $f : X \to Y$ and
  - Component maps $m_U : \mathscr{S}(f^{-1}(U)) \to \mathscr{R}(U)$ for each open set U in Y

  such that $m_U(a(f^{-1}(U))) = b(U)$ for each open U in Y

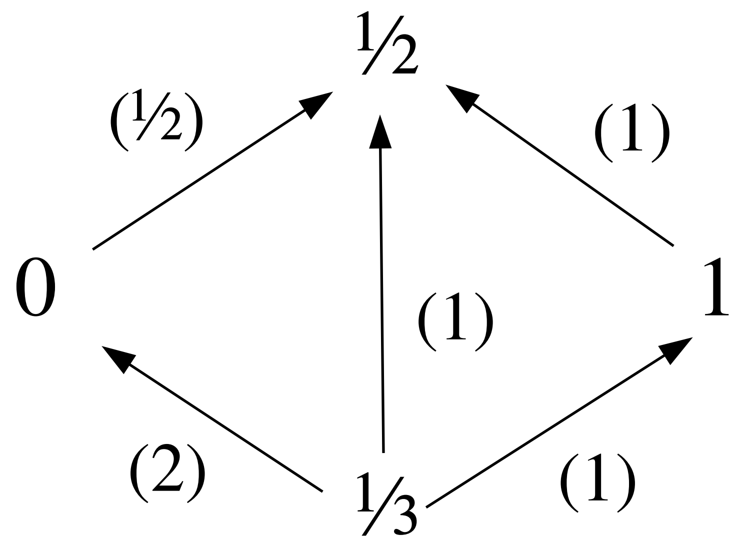
- Theorem: Consistency filtration is a covariant functor **ShvFPA→CoarseFilt**

Defining **Con** : consistency functions

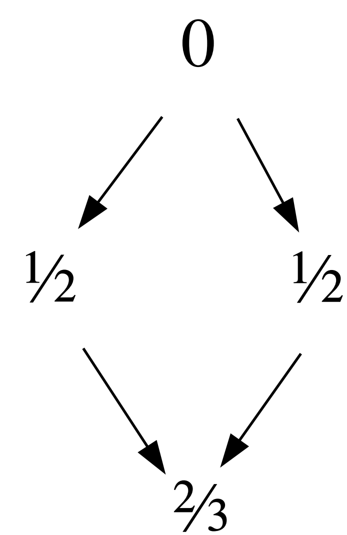
- Objects: order preserving functions $\text{Open}(X) \to \mathbb{R}^+$
- Example: local consistency radius



Open sets



Assignment

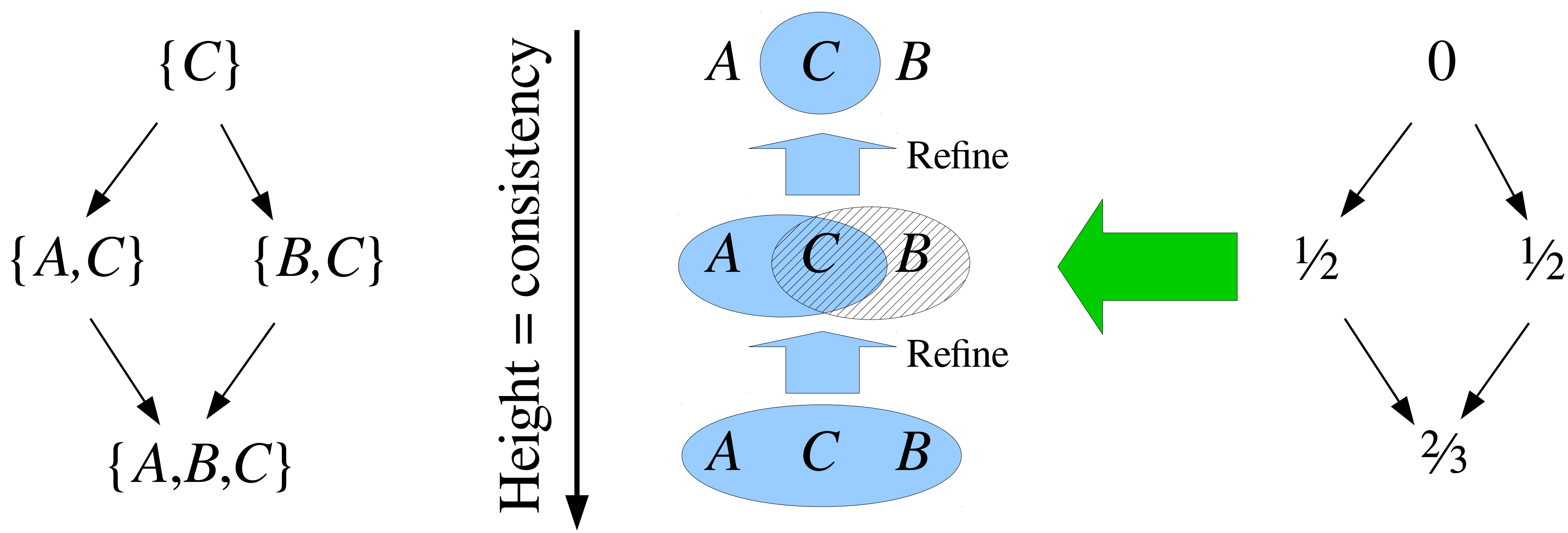


Local consistency radius
Object in **Con**

Defining **Con** : consistency functions

Ideally, we want…

- Consistency radius is a functor **ShvFPA** → **Con**
- A functor **Con** → **CoarseFilt** acting by thresholding



Open sets

Consistency filtration
Object in **CoarseFilt**

Local consistency radius
Object in **Con**

Defining **Con** : consistency functions

Ideally, we want…

- Consistency radius is a functor **ShvFPA** → **Con**
- A functor **Con** → **CoarseFilt** acting by thresholding

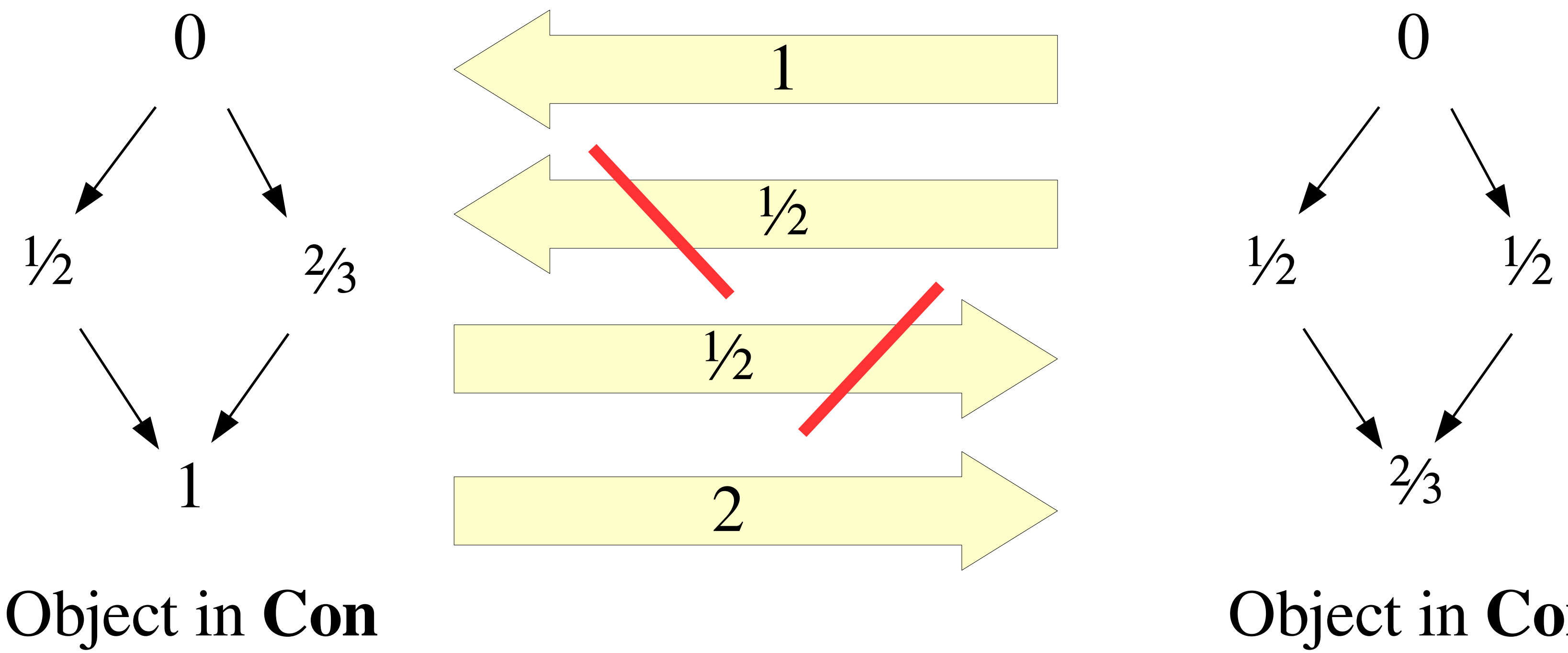
To get this, the morphisms of **Con** are a little strange

A morphism K: $m \to n$ of **Con** is a nonnegative real K so that $m(U) \leq K\, n(U)$ for all open U.

Composition works by multiplication!

Defining **Con** : consistency functions

A morphism K: $m \rightarrow n$ of **Con** is a nonnegative real K so that $m(U) \leq K\, n(U)$ for all open U.



These objects are not **Con**-isomorphic!

Con and **CoarseFilt**

Theorem: **Con** is equivalent to a subcategory of **CoarseFilt** by way of two functors:

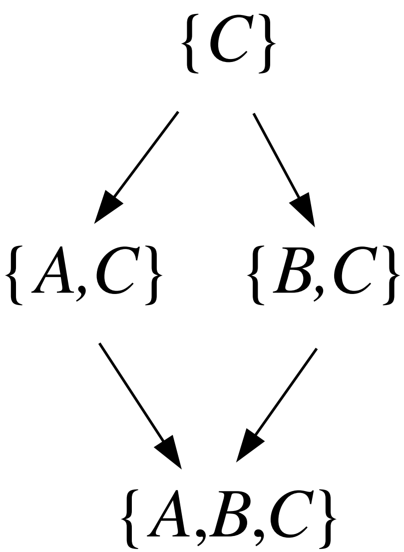
- A faithful functor **Con** → **CoarseFilt**
- A non-faithful functor **CoarseFilt** → **Con**

such that **Con** → **CoarseFilt** → **Con** is the identity functor.

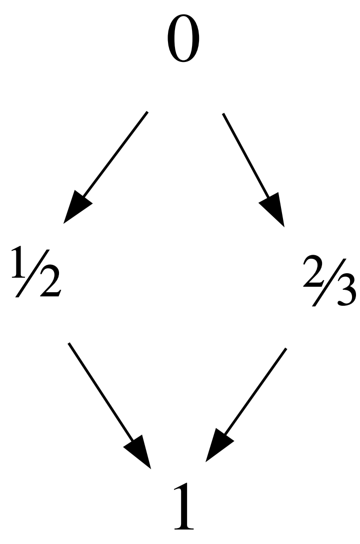
Interpretation: May be able to summarize filtrations of partial covers using consistency functions, but this is lossy!

Con → CoarseFilt

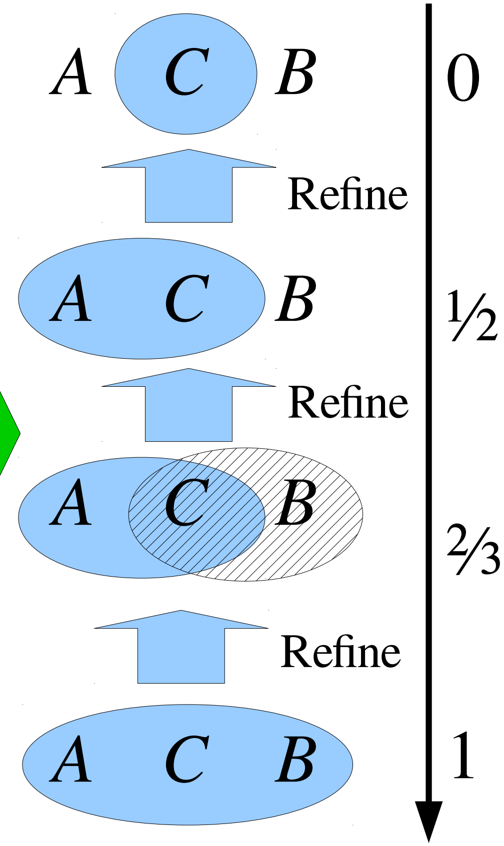
- Motivation: generalization of consistency filtration
- Idea: thresholding!



Open sets



Object in **Con**

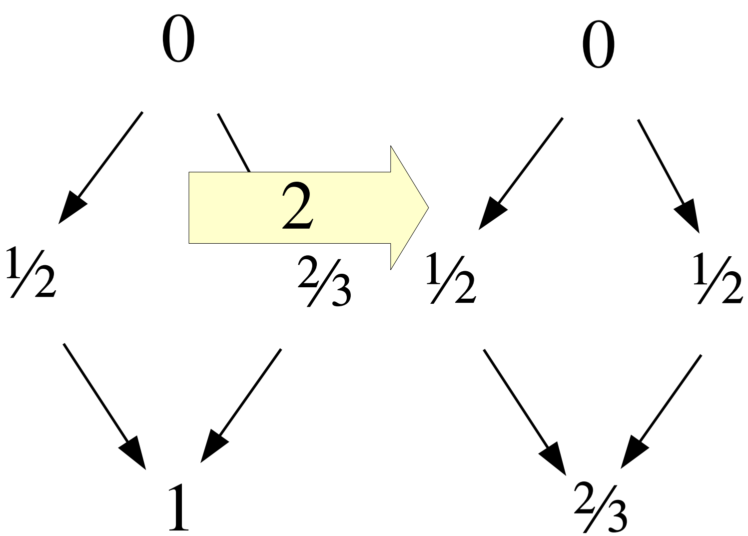


Object in **CoarseFilt**

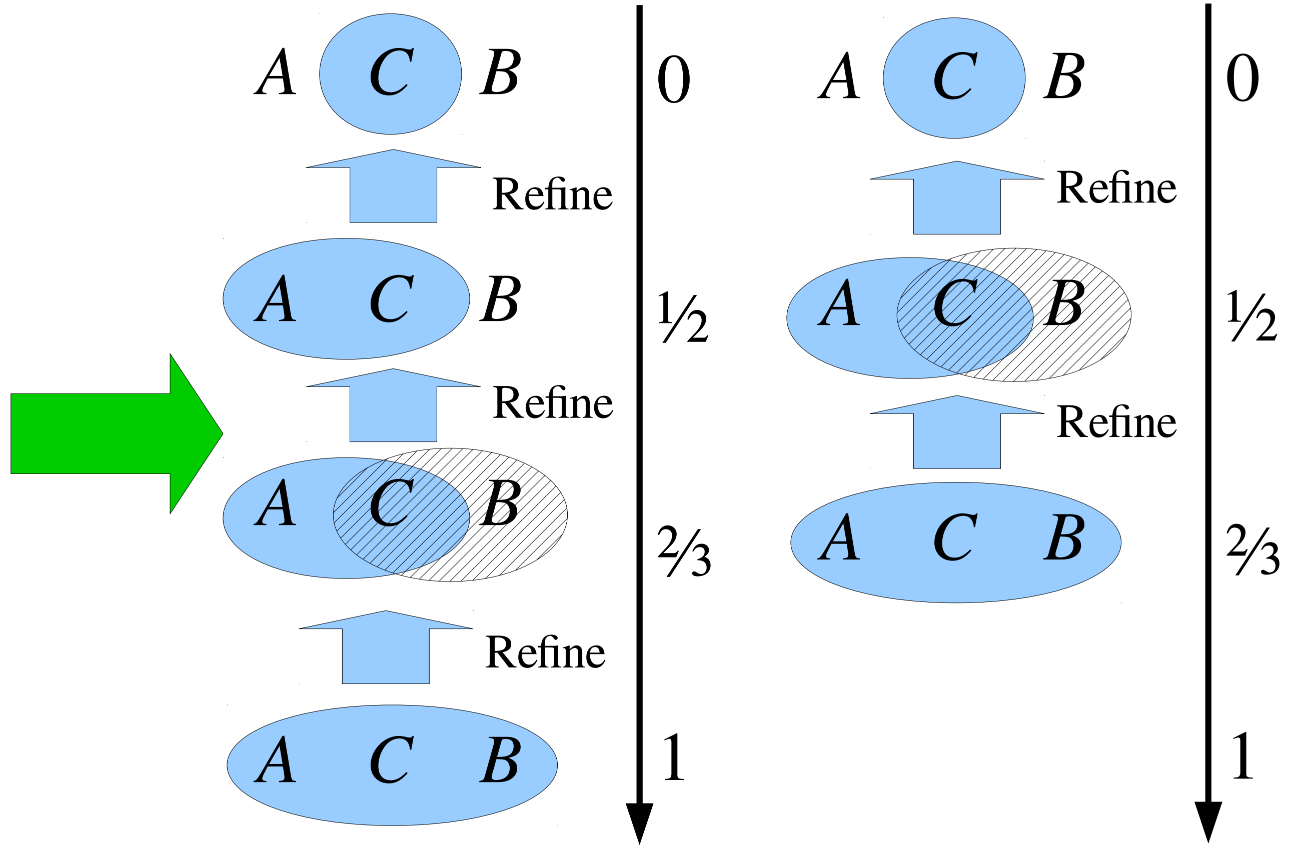
Showing an irredundant representative for this object

Con → CoarseFilt

- Morphisms in **Con** transform to linear rescalings of the heights in **CoarseFilt** … monotonicity does the rest



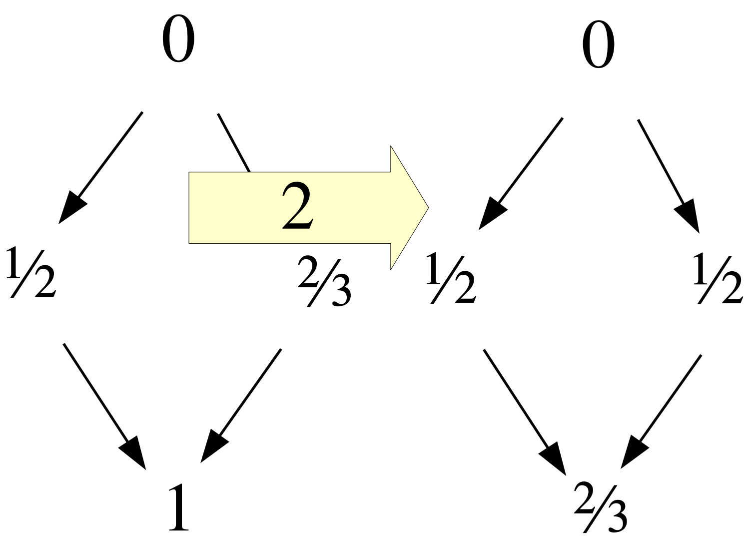
Morphism in **Con**



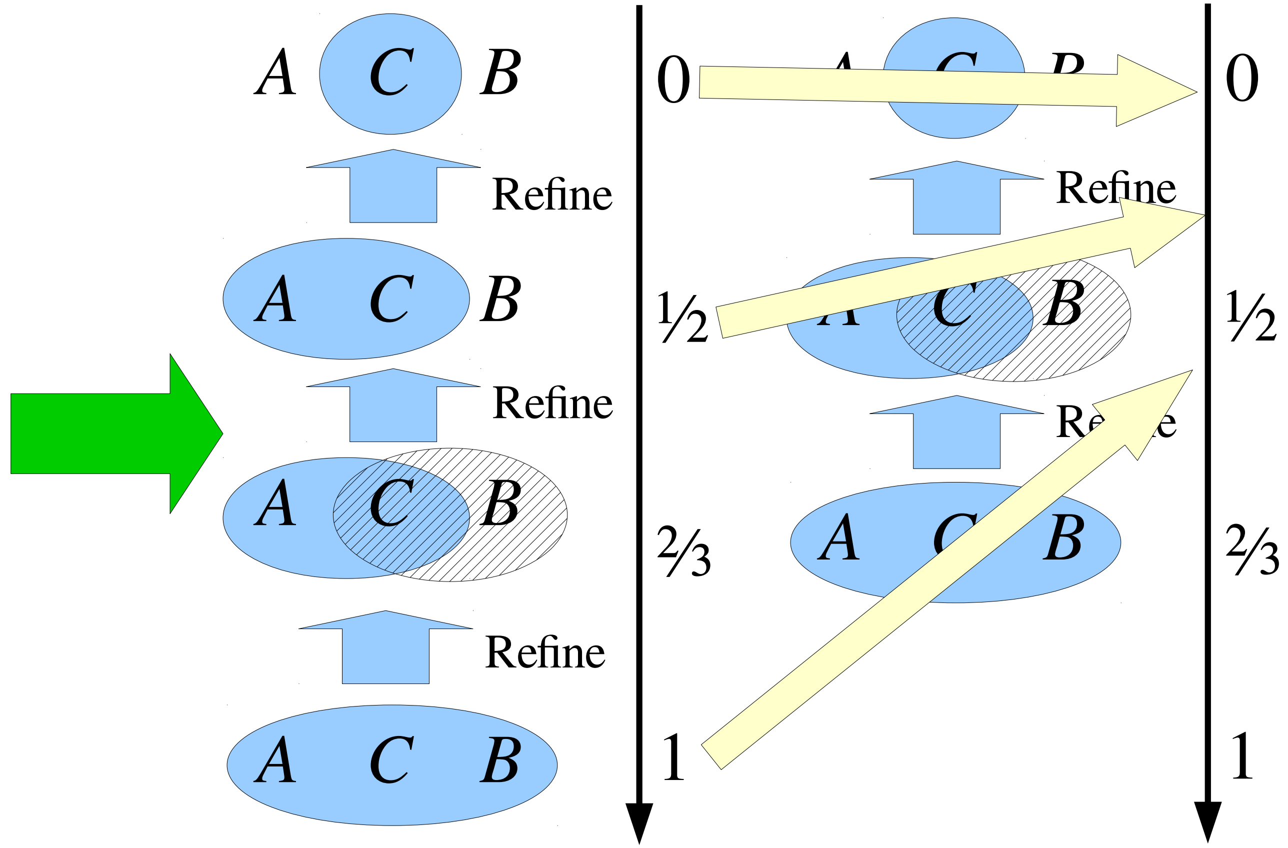
Morphism in **CoarseFilt**

Con → CoarseFilt

- Morphisms in **Con** transform to linear rescalings of the heights in **CoarseFilt** … monotonicity does the rest



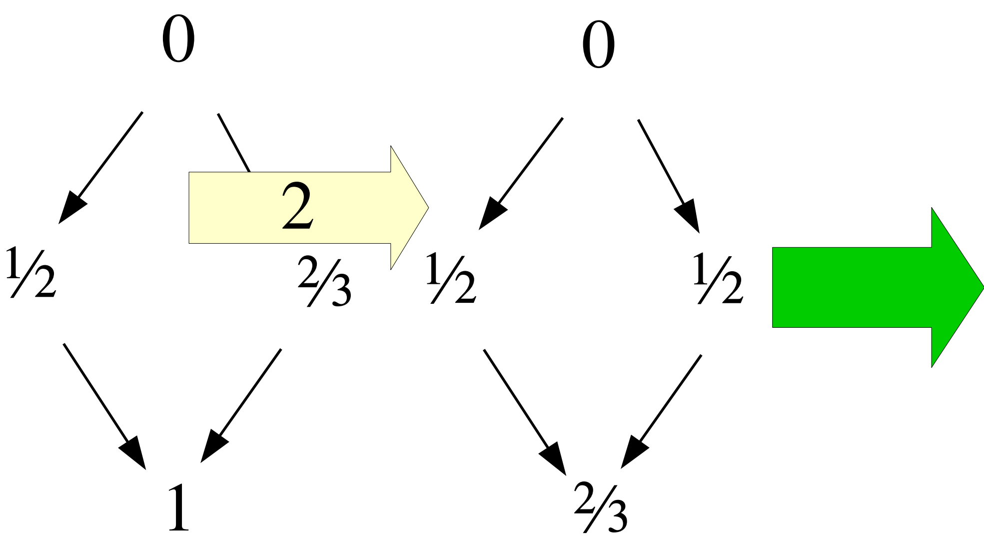
Morphism in **Con**



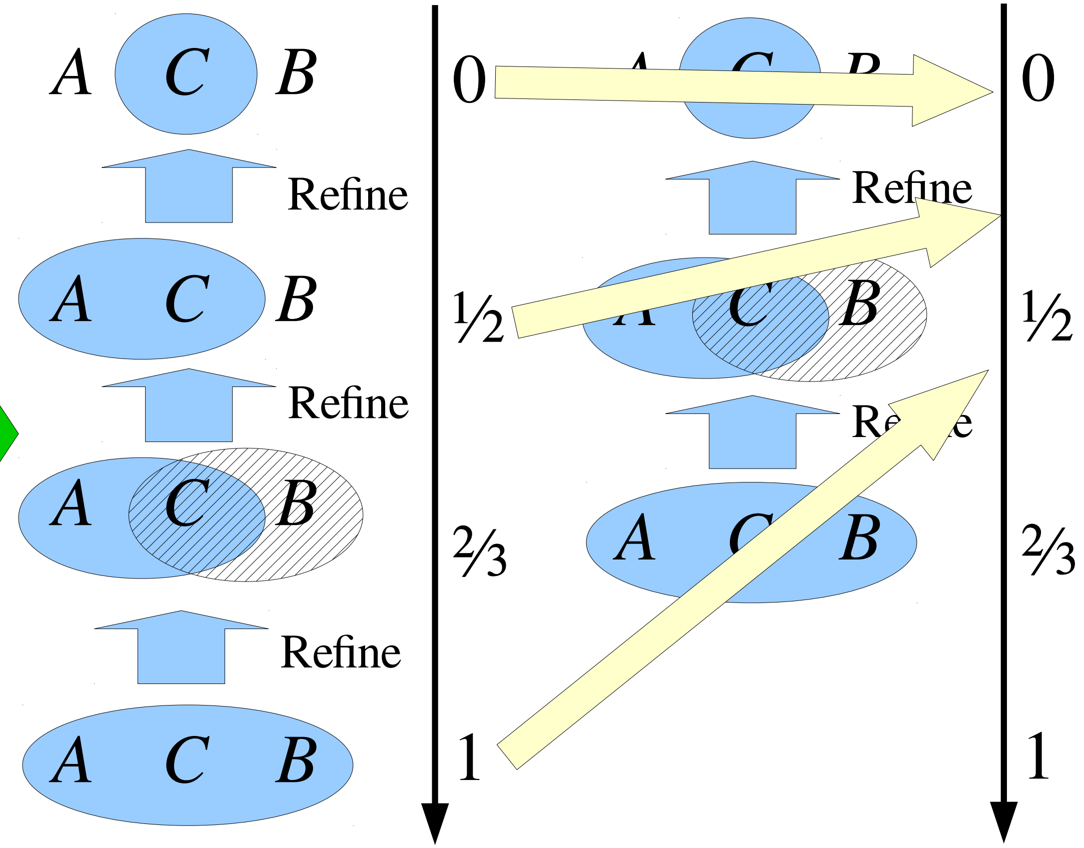
Morphism in **CoarseFilt**

Con → CoarseFilt

- Morphisms in **Con** transform to linear rescalings of the heights in **CoarseFilt** … monotonicity does the rest: **Faithful!**



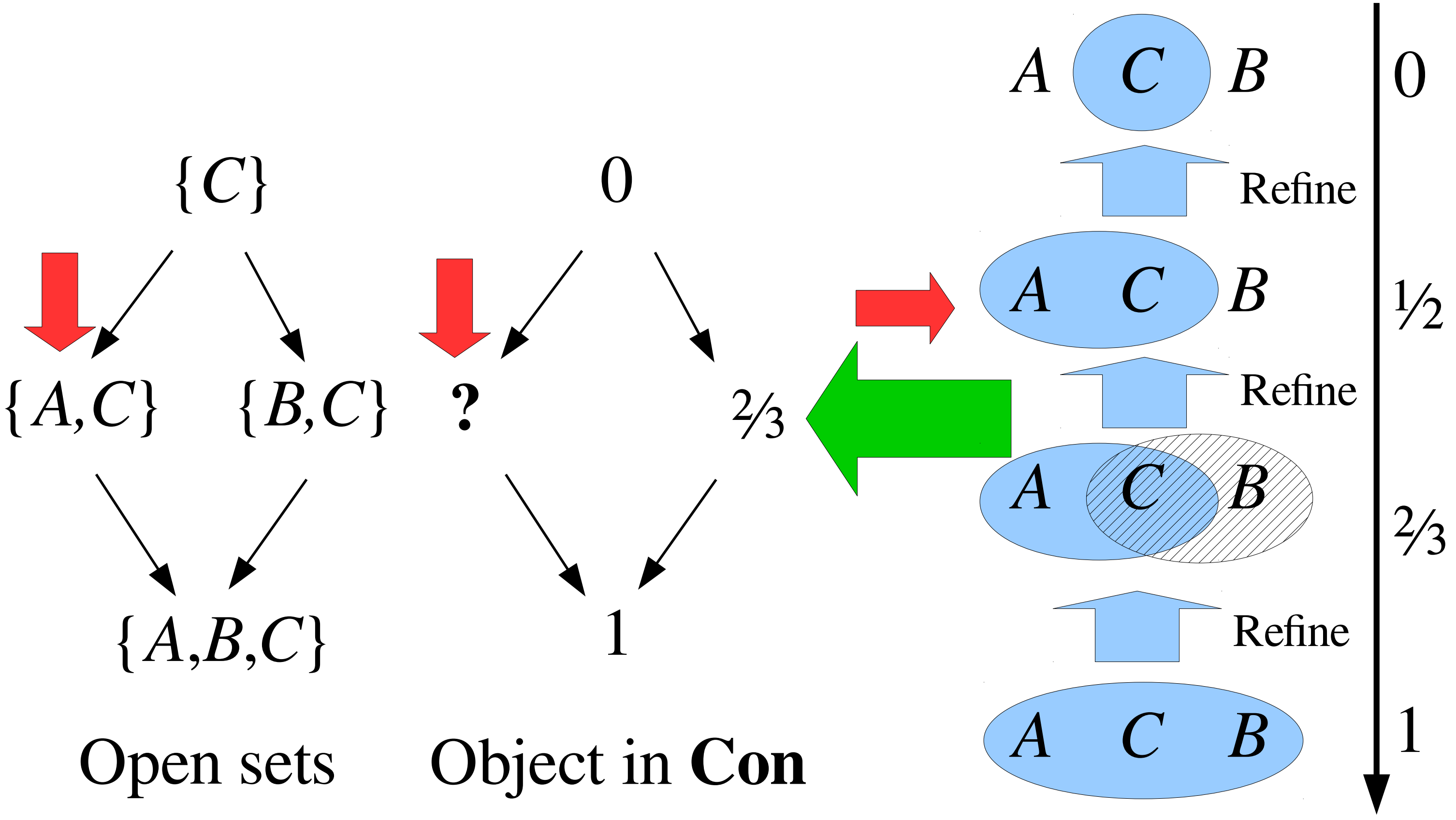
Morphism in **Con**



Morphism in **CoarseFilt**

CoarseFilt → Con

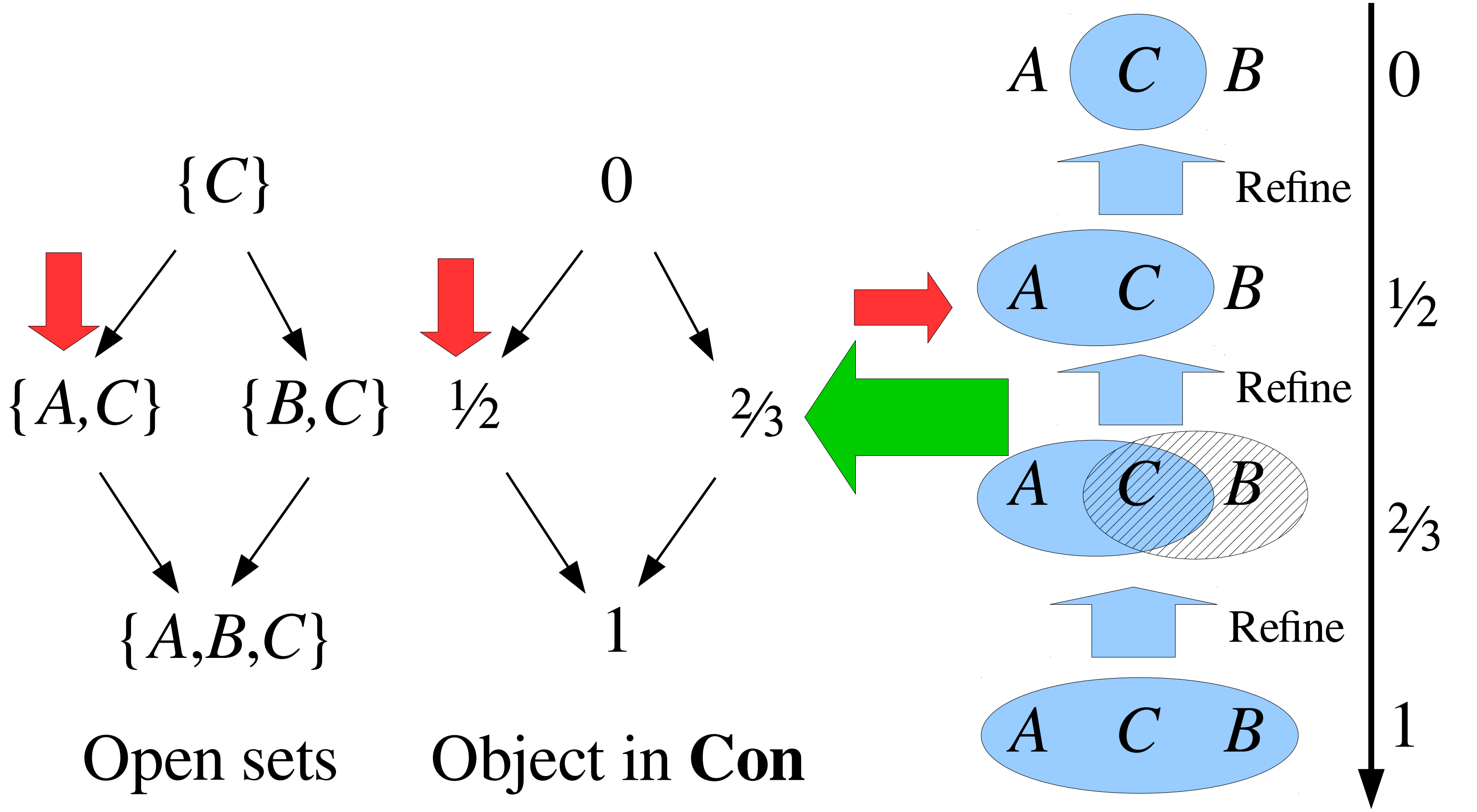
- At first, this seems easy. Just look up the threshold for each open set



Open sets   Object in **Con**   Object in **CoarseFilt**

CoarseFilt → Con

- At first, this seems easy. Just look up the threshold for each open set

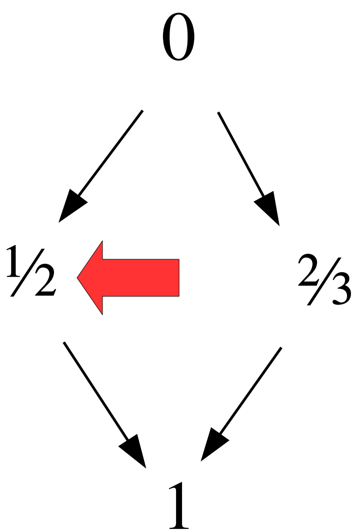


Open sets   Object in **Con**

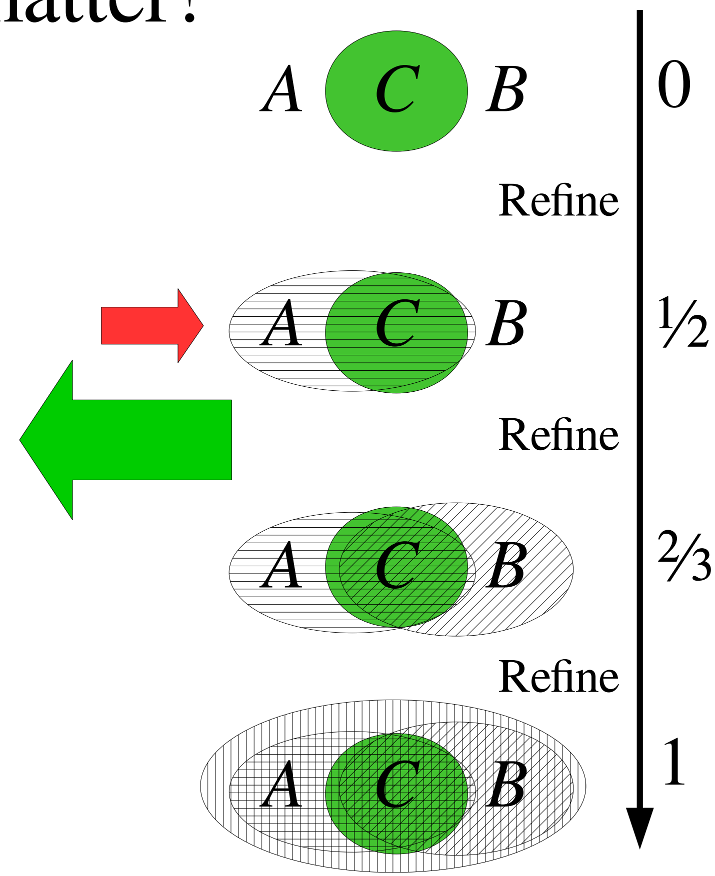
Object in **CoarseFilt**

CoarseFilt → Con

- But what if the cover is not irredundant?
- This does not matter!



Object in **Con**



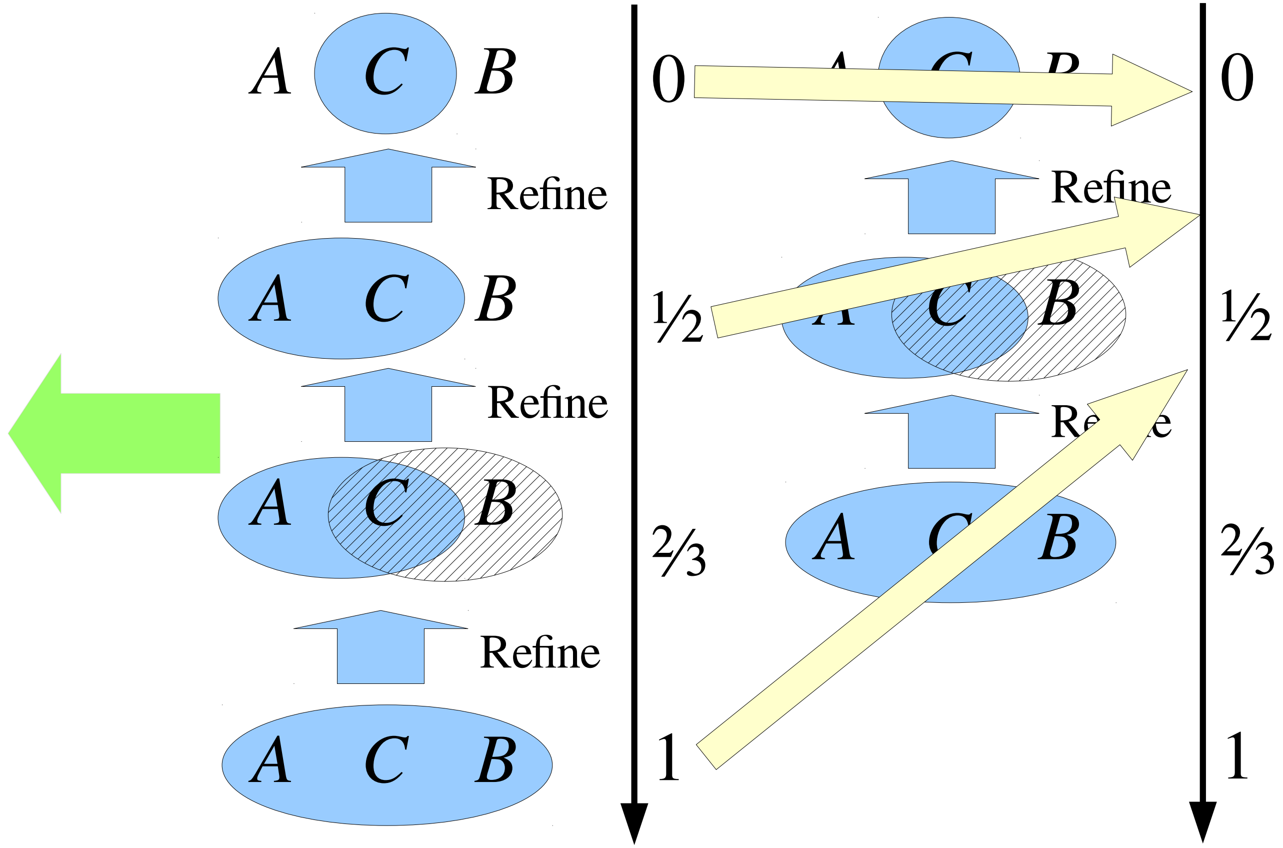
Object in **CoarseFilt**

Fix: take the smallest threshold where the open set is contained in a cover element

CoarseFilt → Con

- Recall: **CoarseFilt** morphisms are given by height rescaling functions ϕ, which may not be linear

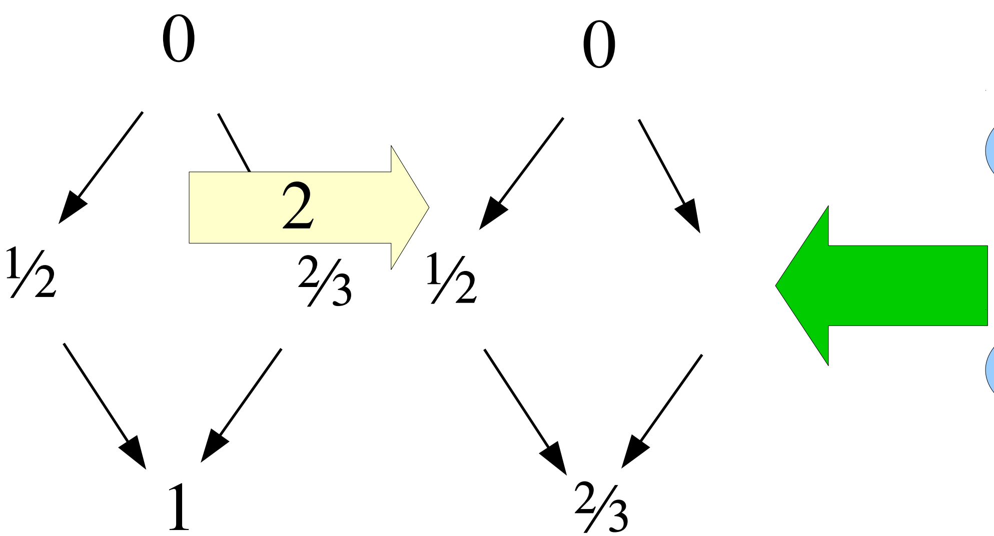
Morphism in **Con**



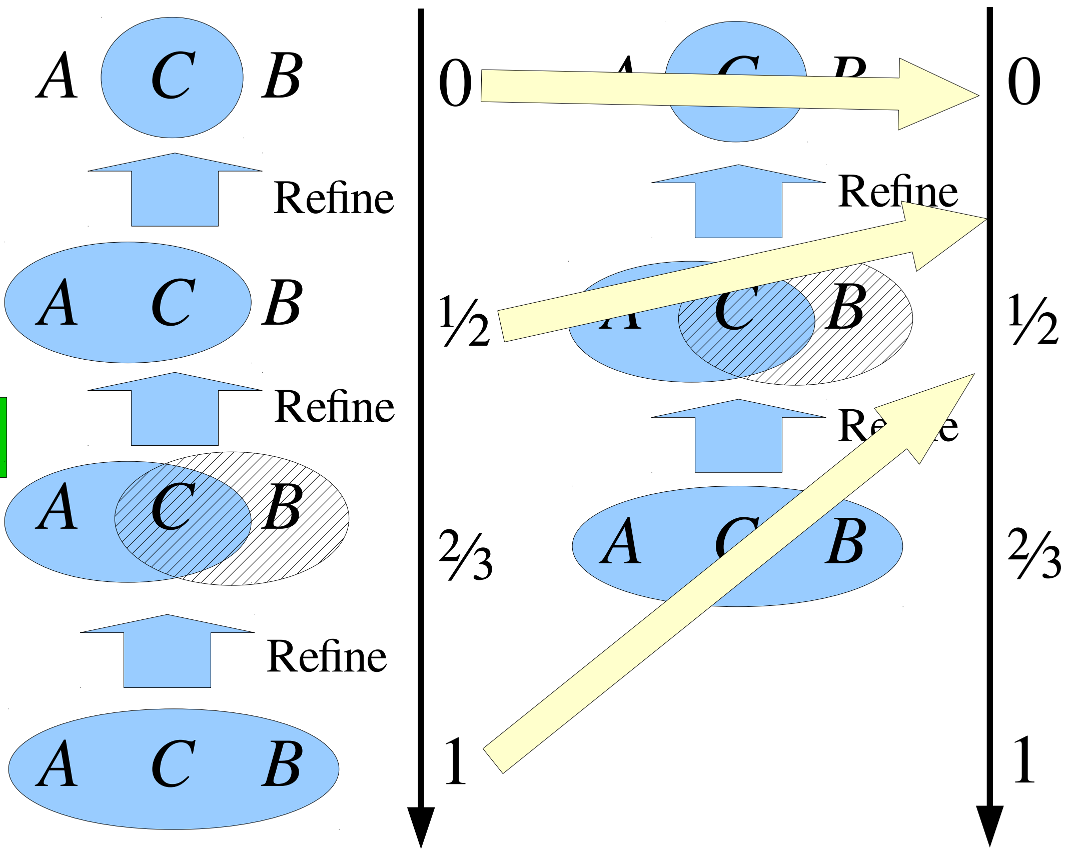
Morphism in **CoarseFilt**

CoarseFilt → Con

- Morphism in **Con** is given by $K = \max \frac{t}{\phi(t)}$
- **Not** faithful!



Morphism in **Con**



Morphism in **CoarseFilt**

Wrapping up...

- Assignments to sheaves can be studied using both topology and category theory…
- The main tools are the consistency radius and consistency filtration
- Although consistency radius isn't functorial, there are fancier invariants that are!
- Open question: Can we relate structure of local consistency of a sheaf assignment to the structure of functions on the base?

To learn more...

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Main reference for this talk: "Assignments to sheaves of pseudometric spaces," *Compositionality*, 2:2, 2020.

Software: https://github.com/kb1dds/pysheaf