Perugia Notes

Lunedì Nov. 13, 1972
17:15 - 19:00

Programma Provvisorio del Corso di
"Teoria delle Categorie Sopra un Topos di Base"
Prof. F. W. Lawvere

Il Corso si compone di due parti. La prima, di carattere più elementare, sarà svolta nei primi due o tre mesi. La seconda, più avanzata, successivamente.

1. Insiemi astratti e applicazioni; elementi, sottoinsiemi, valori di verità e funzioni caratteristiche; prodotti, esponenziazione, strutture algebriche e analitiche; coegualizzatori e prodotti fibrati; l'assioma della scelta. Altri esempi di *Topos*: $S/X, S^{\downarrow}, S^{\circlearrowright}$, permutazioni a orbite finite. Immagini e $\exists$.

   Categorie piccole/S. Numeri naturali e categorie libere; gruppoidi, insiemi ordinati e matrici. Fibrazioni discrete = moduli = prefasci. $S^{C^{op}}/X \cong S^{(C/X)^{op}}$. Lemma di Cayley-Dedekind-Yoneda. Presentazione canonica.

   Categorie grandi/S e la nozione di famiglie piccole di oggetti. Categorie S-basate complete. Prefasci $(C, X) \cong$ Funtori aggiunti $(S^{C^{op}}, X)$. Bimoduli, scissione di idempotenti e "completezza alla Cauchy". Esattezza a sinistra e categorie cofiltrate. Lex $(C^{op}, X) \cong \mathrm{Top}(X, S^C)^{op}$. Categorie localmente a presentazione finita. Proiettivi regolari, categorie esatte, algebra universale, Teorema d'immersione di Joyal-Gödel-Kripke-Freyd-Mitchell-Barr.

2. Topos sopra un topos di base S. Teoremi di struttura di Freyd e Giraud. Operatori modali di Grothendieck, cotriple esatte a sinistra. Topos Booleani/S. Indipendenza dell'ipotesi del continuo. Bicategorie bichiuse/S. Logica intuizionistica. Gruppi algebrici, fasci localmente liberi, spazi metrici.

Theory of Categories over a Base Topos

Introduction

From the mathematical developments of recent years has become clear that the theory of categories is a most effective means for summing up the essential aspects of all the traditional areas of modern mathematics (geometry, analysis, algebra and logic) in order to provide a basis in viewpoint, and general results for further developing these subjects. (Naturally, the detailed development of the just stated general line is still somewhat uneven, having been carried out more fully in some fields, less fully in others.) At the same time it has also become clear that a (for the applications most effective) further sharpening of this categorical summing up process consists in a sort of generalization of category theory within itself, namely, that the role of the category of sets in the formulation of the general results of category theory can be split into two and generalized, into an arbitrary base topos and an arbitrary closed category of values for hom. The second, sharpened, program is even less completely carried out at the present time, but has already demonstrated its effectiveness, at least in each aspect separately, through the work of Grothendieck, Giraud, et al on algebraic geometry over a topos and through the work of Eilenberg-Kelly-Day, et al on closed categories. One of the aims of the *second* part of this course will be to attempt to move forward the sharpened program by consciously combining these two aspects for the

first time (see however the book of Rivano: Catégories tanakiennes, Springer Lecture Notes series). The aim of the *first* part of the course is to give an introduction to the elementary basic aspects of category theory which "every young mathematician should know", such as pullbacks, epimorphisms, exactness, objects with an algebraic structure, adjoint functors, structure of presheaf categories ... presented in such a manner, however, that it can be understood in two ways, in an elementary way
a) in which the base topos is taken as the category of abstract sets and mappings but, because the formulation is (we hope) given a most optimal form, also in a second way
b) in which "base" is taken to be any category satisfying the very general axioms of a topos.

We hasten to add, however, that the theory of categories over a general base topos will in certain special cases be used to illuminate the study of the abstract set case both from above and from below. By "below" we refer to so-called "foundational" questions, for example, the question whether the continuum hypothesis holds for abstract sets and mappings themselves; part of the "above" aspect is analogous to the situation in which algebraic geometry over an arbitrary commutative ring may be used to illuminate the number theory of the abstract integers in a special case where the base ring is taken to be, say, an algebraic number field. We emphasize that even the elementary aim of the first part of the course is by no means completely carried out and therefore stands to benefit immensely by original work on the part of interested students.

The notion of a base topos is closely related to the notion of fiber bundle, a recent exposition of which (in the language of topos in fact) is to be found in Giraud's book (Grundlehren, Band 179), and which presumably had its historical origin in the notion of a family of varieties parameterized by a base variety. The fact that, even with the point of view we are trying to develop in this course, "the category of abstract sets still plays a central role" is to be compared with fiber bundles over a base with one point; more profoundly, this fact may be compared with the primacy (both historically and conceptually) of constant quantities within the theory of variable quantities, in spite of the fact that variable quantities provide a more accurate description of reality. In this connection, we may consider the following quotation from Engels: (Anti-Dühring, part I philosophy, chapter 12, dialectics, quantity and quality).

> In its operations with variable quantities mathematics itself enters the field of dialectics, and it is significant that it was a dialectical philosopher, Descartes, who introduced this advance. The relation between the mathematics of variable and the mathematics of constant quantities is in general the same as the relation of dialectical to metaphysical thought. But this does not prevent the

> great mass of mathematicians from recognizing dialectics only in the sphere of mathematics, and a good many of them from continuing to work in the old, limited, metaphysical way with methods that were obtained dialectically.

The primary subject matter of mathematics is still the variation of quantity in time and space, but that also this primacy is partly of the nature of a "first approximation", is reflected in the increasing importance of structures (not only of quantities) in the mathematics of the last 100 years. For example, a first approximation to a theory of a material situation involving three apples might be simply the number (constant quantity) 3. The idea of an abstract set of three elements is a somewhat more accurate theory. If one of the apples happens to be distinguished, for example, by being rotten, we may consider the simple *structure* of an abstract set with a distinguished element, a theoretical refinement which the quantity three does not really admit; the unique non-trivial auto-morphism of this structure is a theoretical operation which again the quantity itself does not admit. This simple example indicates that at least in some cases the idea of a structure is a refinement of the idea of a quantity – but still constant. But it is variable structures which are in a general way the subject matter of the theory of categories over a base topos. Of course, most of our examples of base topos can be interpreted within *mathematics over* the base topos of abstract sets, but this does not trivialize our aim any more than the continuous is annihilated by the discrete or variable quantities "reduced" to constant quantities through the "construction" of the real numbers within the *higher order theory* of the natural numbers. That the abstraction from structure to quantity (obvious in the above simple example) is present and is significant also among *variable* structures and *variable* quantities, is already exemplified by a flourishing branch of mathematics, K-theory. A general treatment of this latter phenomenon, however, cannot presently be promised for this course.

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Lesson 1

1. The topos of abstract sets and mappings and some other basic examples

By an abstract set A we understand a collection of "elements" each of which has no internal property or structure. Thus the only external property which A by itself has, is the "number" of these elements; however, A itself has an internal structure which a "number" does not have, namely the equality and inequality of pairs of elements; but A in itself has no other internal structure except that just mentioned. The collection of all abstract sets (which by the foregoing is not an abstract set - why?) admits of a very rich internal structure through the concept of mapping. By a mapping f we understand the following: There is an abstract set A called the domain of f and an abstract set B (which may or may not be the same as A) called the codomain of f and further there is a correspondence which to every element x of A associates exactly one element xf of B; the element xf of the codomain of f is often called the *value* of f at the element x of the domain of f, and it is also frequently written alternatively as $f(x)$. The statement that f is a mapping whose domain is A and whose codomain is B is often abbreviated by the diagram

$$A \xrightarrow{f} B$$

As a trivial, but important, example there is for each abstract set A the identity mapping of A satisfying the two conditions

$$A \xrightarrow{1_A} A$$

$$x1_A = x \text{ for every element } x \text{ of } A$$

(alternatively $1_A(x) = x$).
We consider that two mappings of abstract sets are equal if they have the same domain, the same codomain and to each element of the domain the same value.

Given any two mappings f and g with the property that the codomain of f is the same abstract set as the domain of g, there is always a third mapping called their *composition* denoted by fg (alternatively $g \circ f$) determined by the following properties:

the domain of fg is the domain of f
the codomain of fg is the codomain of g
$x(fg) = (xf)g$ for every element x of the domain of fg
(alternatively $(g \circ f)(x) = g(f(x))$.

The statement that a mapping h is the composition f followed by g is often abbreviated by introducing names A, B, C for the abstract sets involved and stating that the following diagram is *commutative*:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

The notions of mapping (of abstract sets) and their composition constitutes a correct idea which is a basis for summing up the mathematics of the past 100 years. The question "what are the abstract sets good for?" can be answered as follows: They can be used for parameterizing (or indexing) families of objects of arbitrary categories, and then both the internal structure of the objects and the external relations between the objects can be theoretically analyzed by means of the mappings between the parameterizing sets; particularly in mathematics, they can be used in order to parameterize families of objects of mathematical categories.

One example of a category of mathematical objects is the ("discrete") category of the elements of a given abstract set B; naturally, by a family in this category, parameterized by A is intended simply a mapping from A to B.

A second example of a mathematical category is the category of abstract sets; experience has shown that a family of abstract sets with parameterizing set A may always be interpreted as a mapping p from some set S to A, the x-th object of the family being the set of all those elements s of S for which

$p(s) = x$. Moreover, the abstract sets can be furnished with structure (again with help of the mappings), and families of such structures may in turn be parameterized by abstract sets, etc.

In our study of the compositions of mappings between abstract sets we will make use of the axiomatic method which should be understood in the following sense: In the course of the experience of working with mathematics, mathematicians have discovered many true facts about and many possible constructions on abstract sets and mappings. We can isolate a small number of these facts and constructions which are on the one hand simple to grasp and remember, and on the other hand powerful enough to logically entail all or nearly all of the further facts and constructions in which we are interested. Taking advantage of this situation, we can formalize these central facts and constructions and call them axioms.

It may be noted that many of the "constructions" formalized through axioms have simply the effect to make sure that there are enough sets and mappings to permit the parameterization of some important types of objects.

The first axiom which we state is the fact that the composition of mappings constitutes a category. This means that the unit and associativity laws hold, i.e. that any diagrams of the form

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "f"] & \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

are commutative and that in any diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "l"] \arrow[dl, "g"'] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

if the two triangles are commutative, then the square is commutative in the sense that $fl = hk$. The reader should convince himself (or herself) that this first axiom is indeed a true fact about the composition of mappings of abstract sets.

Since 1945 it has been found that the dialectics of "categories" in the above defined sense applies to a very great number of situations (not only to mappings of abstract sets) and indeed is a further powerful tool for summing up the essential features of all mathematics. Therefore, in formulating further axioms for sets and mappings, we must both fight against this generality and take advantage of it in the following sense: On the one hand we must give axioms which will clearly distinguish the category of sets and mappings from all the other categories, in the sense that any category satisfying all our axioms must be nearly indistinguishable from the category of sets and mappings. On the other hand, around 1960 it was discovered that there is an important class of categories, the topos, which, while definitely different from the category of sets, are "similar" to it in somewhat the same sense that algebraic number fields and rings of smooth functions on a space are "similar" to the integers in that they are all commutative rings. Because this constitutes a further essential advance in the process of summing up the basic features of mathematics and, in particular, because it serves our aim of considering constant structures as a special case of variable structures, our choice of the further axioms will therefore be strongly determined by the view that the category of sets and mappings is a particular topos. Briefly, in terms which will presently be explained, the further axioms are to the effect that (for a general topos) the notions of element, ordered pair, subset, and mapping, are all "representable" within the category, and further (for the sets and mappings in particular) that the axiom of choice holds, that there are only two truth values, and that the iteration of endomappings is representable (in fact, the last condition ("arithmetic") holds in most topos, unlike the choice axiom).

Before discussing in detail the further axioms, we will in fact first give a few of the simplest examples of topos which indeed can be constructed within the framework of sets and mappings. The examples are really general methods for passing from given base topos $\mathcal{S}$ to more complicated topos "defined over $\mathcal{S}$" applied to the case where $\mathcal{S}$ is the category of sets and mappings.

Given an object B of $\mathcal{S}$, we construct another category $\mathcal{S}/B$ as follows. An *object* of $\mathcal{S}/B$ is any mapping $A \xrightarrow{f} B$ with any object A of $\mathcal{S}$ as domain, a *morphism* of $\mathcal{S}/B$ is any commutative triangle

$$\begin{tikzcd} A' \arrow[rr, "a"] \arrow[dr, "f'"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

of mappings in $\mathcal{S}$ (always with the given B, but with any A', a, A). The *domain* of the above triangle is f', and the *codomain* is f, and thus we may say that a is a morphism from f to f' in $\mathcal{S}/B$.
The *composition* of two morphisms

$$\begin{tikzcd} A'' \arrow[rr, "a'"] \arrow[dr, "f''"'] & & A \arrow[dl, "f'"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[rr, "a"] \arrow[dr, "f'"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

of $\mathcal{S}/B$ is just

$$\begin{tikzcd} A'' \arrow[rr, "a'a=aoa'"] \arrow[dr, "f''"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

which can easily be proved to be again a morphism in $\mathcal{S}/B$ by using the associative law for composition in $\mathcal{S}$ as follows:

$$(a'a)f = a'(af) = a'f' = f''$$

In particular, the *identity morphism* of the object $A \xrightarrow{f} B$ of $\mathcal{S}/B$ is the commutative triangle

$$\begin{tikzcd} A \arrow[rr, "1_A"] \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

of mappings in $\mathcal{S}$. Now to give a category we must not only give an interpretation of the terms "object, morphism, domain, codomain, identity, and composition", but we must also give a verification that the associative law and the left and right unit laws are valid for the given interpretation; such verification is trivial in the case of $\mathcal{S}/B$ because these laws are true in $\mathcal{S}$. (Actually, certain more primitive laws must also be verified, namely that the composition of two morphisms is defined iff the codomain of the first is the same object as the domain of the second and that in that case only one composition is defined, that the domain and codomain of the composition are correct, that every object has exactly one identity morphism, and that the domain and codomain of an identity are correct; of course, all this is also true of $\mathcal{S}/B$ because it is true for $\mathcal{S}$).

Our two interpretations of a mapping with codomain B give rise to two interpretations of the category $\mathcal{S}/B$. When we think of a mapping into B as a family of elements of B parameterized by its domain A, we often speak of it as "an element of B defined over A", and of a morphism in $\mathcal{S}/B$ as a "restriction of the domain of definition for elements of B" (in the sense inverse to the morphism); actually, this terminology is abstracted from another example (to be discussed later [ ]) of a topos $\mathcal{S}$ in which the object B may be an algebraic space and in which among the objects A will be the affine spaces corresponding to possible rings of definition, with morphisms $A' \to A$ corresponding in an inverse sense to ring homomorphisms. For the present, notice that for given $A' \to A$, *not* every element f' of B defined over A' results from restricting an element of B defined over A, and that two elements f_1, f_2 of B defined over A may restrict to the same element of B defined over A'.

On the other hand, when we think of mappings into B as families of sets parameterized by B we often, by abuse of notation, use the letter A alone to denote a pair A, f where $A \xrightarrow{f} B$, and call a morphism

$$\begin{tikzcd} A' \arrow[rr, "a"] \arrow[dr, "f'"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

of $\mathcal{S}/B$ simply a morphism $A' \to A$ "over B" the f' and f being understood. Such a morphism may be thought of as a morphism of families (of sets) parameterized by B, and further may actually be interpreted as a family of mappings indexed by B as follows:
For each element b of B (now again in the primitive sense), denote A by A_b the b-th set in the family f, namely A_b is the set of all those elements x of A for which $xf = b$; similarly, denote by A'_b the b-th set in the family f'. Then the morphism a from f' to f in $\mathcal{S}/B$ gives rise to a family

$$A'_b \xrightarrow{a_b} A_b$$

of mappings of sets, for b ranging over elements of B, uniquely determined by the commutativity of the diagrams

$$\begin{tikzcd} A'_b \arrow[r, "a_b"] \arrow[d, tail] & A_b \arrow[d, tail] \\ A' \arrow[r, "a"'] & A \end{tikzcd}$$

where the vertical arrows denote "inclusions". This is possible because if an element x' of A' belongs to A'_b then $x'a$ belongs to A_bx (i.e., if $x'f' = b$, then since a was a morphism in $\mathcal{S}/B$ from f' to f we have $(x'a)f = x'(af) = x'f' = b$) and hence for x' belonging to A'_b we can define $a_b(x') = a(x')$ considered as an element of A_b and a_b will be a mapping $A'_b \to A_b$ for each b in B. Naturally, the constructions, calculations, and notions (such as "inclusion", "belongs to") just used will be given a precise meaning in the framework of our axioms for the category of sets (and more generally for any topos).

We remark here that both the "replacement axiom scheme" of Fraenkel and the claim that $\mathcal{S}$ is a "Grothendieck universe" have the sole purpose to guarantee that "all" families of sets and families of mappings parameterized by a set B are actually accounted for by objects and morphisms in $\mathcal{S}/B$; but in normal mathematical practice all the families of sets and mappings which arise can in fact be interpreted naturally in $\mathcal{S}/B$ without invoking such principles. We will later [ ] apply the construction of $\mathcal{S}/B$ from $\mathcal{S}$ and B

not only to the category of sets, but to any topos $\mathcal{S}$ and also to any category in $\mathcal{S}$ ("small category") and even to any category over $\mathcal{S}$ ("large category/$\mathcal{S}$"). However, it will be a theorem that $\mathcal{S}/B$ is again a topos if $\mathcal{S}$ is a topos.

Another important topos is $\mathcal{S}^{\underline{2}}$, the category of morphisms of S. Its objects are arbitrary mappings and its morphisms are arbitrary commutative squares

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r, "b"'] & B \end{tikzcd}$$

of mappings, where we consider that the *pair* a, b determines a morphism of $\mathcal{S}^{\underline{2}}$ from f' to f provided $af = f'b$. Composition is defined by erasing f' from a diagram of the form

$$\begin{tikzcd} A'' \arrow[r, "a'"] \arrow[d, "f''"'] & A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B'' \arrow[r, "b'"'] & B' \arrow[r, "b"'] & B \end{tikzcd}$$

We leave it to the reader to show that if the arrows satisfy the laws of a category, then so do the commutative squares. Notice that if we restrict consideration to those morphisms in $\mathcal{S}^{\underline{2}}$ for which $b' = b = 1_B$ for a given B we are back to $\mathcal{S}/B$, so that $\mathcal{S}^{\underline{2}}$ is a natural "gluing together" of all the categories $\mathcal{S}/B$ for B in $\mathcal{S}$, whose morphisms take account of the fact that also B may vary (along mappings). The process

$$\mathcal{S}^{\underline{2}} \xrightarrow{\sum} \mathcal{S}$$

of assigning to each f its domain and to each square its upper arrow, that is, the process of "forgetting" the b's (and the f's) is clearly seen to preserve the notions of domain, codomain and composition as defined in $\mathcal{S}^{\underline{2}}$, respectively $\mathcal{S}$; it is thus an example of a *functor*. In the interpretation of mappings as families of sets, $\sum$ assigns to each family $\{A_b\}, b \in B$ its (disjoint) sum $A = \sum\limits_{b\in A} A_b$ and the fact that $\sum$ applies to morphisms as well as to objects, means that if we are given two families of sets and also a morphism (of $\mathcal{S}^{\underline{2}}$ or possibly of a particular $\mathcal{S}/B$) between them, then there is a natural induced mapping between their sums, and further these induced mappings behave coherently with respect to composition of morphisms. Though the process of remembering the domain and forgetting the codomain is a functor $\mathcal{C}^{\underline{2}} \to \mathcal{C}$ for any category $\mathcal{C}$, our interpretation of it as a summation process is only sensible for a topos.

* There is actually a second natural method for gluing together all the categories $\mathcal{S}/B$ for B ranging in $\mathcal{S}$ to give a single category which we might call

inv($\mathcal{S}$). The objects are again arbitrary mappings (or parameterized families of sets). If f is a family of sets parameterized by B and if $B' \xrightarrow{b} B$ is any mapping, then there is an induced family $b^*(f)$ parameterized by B', whose y'-th set is the $b(y')$-th set of f, for any element y' of B'. Under the abuse of notation whereby A/B stands for an "understood" mapping $A \xrightarrow{f} B$, we also abuse $b^*(f)$ by calling it $b^*(A)/B'$. Here

$$\begin{aligned} b^*(A) = \{< x, y' > \quad &| \quad x \in A, y' \in B'\} \\ &| \quad f(x) = b(y')\} \end{aligned}$$

is the set of all pairs satisfying the stated equations; the obvious mapping $b^*(A) \to B'$ obtained by forgetting x may be safely "understood" (all this will be made more formal and precise by our axioms - what we have just constructed is called a "pullback"). This said, we may define a morphism from

$$\begin{array}{ccc} A & & A' \\ {\scriptstyle f}\downarrow & \text{to} & \downarrow{\scriptstyle f'} \\ B & & B' \end{array}$$

in the category *inv* ($\mathcal{S}$) to be any pair of mappings b, a' for which $B' \xrightarrow{b} B$ and for which $b^*(A) \xrightarrow{a'} A'$ with the triangle

$$\begin{tikzcd} b^*(A) \arrow[rr, "a'"] \arrow[dr] & & A' \arrow[dl, "f'"] \\ & B' & \end{tikzcd}$$

required to be *commutative*. The reader may define for himself/herself the composition in *inv*($\mathcal{S}$); as soon as you have understood the concept of pullback, you may show that $\mathcal{S}$ may be replaced by any category having pullbacks and verify that *inv*($\mathcal{S}$) is then again a category. Notice that the mappings b of the parameters go in the sense *inverse* to the morphisms in *inv*($\mathcal{S}$) of which they are the first components. By the *product*

$$\prod_{b \in B} A_b$$

of a family of abstract sets, we mean the "set" of all choice mappings s which to each parameter b in B assign an element s_b of A_b; when a family is interpreted as a mapping $A \xrightarrow{f} B$, then a choice mapping for the family f is clearly just any mapping $B \xrightarrow{s} A$ for which $sf = 1_B$ (alternatively

$fos = 1_B$) and we will see presently that the "set" of all these can naturally be parameterized biuniquely by an abstract set $\prod_B(A)$. The reader may consider how a morphism $f \to f'$ of families *in the sense of inv*$(\mathcal{S})$ gives rise in a natural way to a mapping

$$\prod_B(A) \to \prod_{B'}(A')$$

(consider first the case where $B = B', b = 1_B$, and then the more general case with $B' \xrightarrow{b} B$ arbitrary) and that composition in *inv*$(\mathcal{S})$ gives rise to composition of induced mappings so that $\prod$ is also a functor. Later [ ] we will see that the notion of $\prod$ has a unique sense in any topos. *

The topos $\mathcal{S}^{\underline{2}}$ has also a simple interpretation which will be very suggestive when dealing with more complicated topos. Namely, in contrast to the objects of $\mathcal{S}$ which are "constant" sets, we may think of an object of $\mathcal{S}^{\underline{2}}$ as a "variable" set with two stages α, β of development and with a definite connection between the two stages in that every element at stage α corresponds to exactly one element at stage β. Thus an element at stage β may also "be" a single element at stage α, but another element at stage β may not exist at all at stage α, while the typical element at stage β will be split into two or more elements at stage α. If $S = [A \xrightarrow{f} B]$ is a set developing or varying in this manner (we could also write $S_\alpha = A, S_\beta = B$) and if $S' = [A' \xrightarrow{f'} B']$ is another one, then by a mapping $S' \xrightarrow{m} S$ we would naturally understand a morphism in the category $\mathcal{S}^{\underline{2}}$, which can thus be understood as mappings $m_\alpha = a, m_\beta = b$ at each stage which respect or compare coherently the details of the two developments in the sense that $m_\beta f = f' m_\beta$. This equation states that m is the simplest type of *natural transformation*; in a more complicated situation there will be many stages α and/or many forms f of connection between any two stages (in the sense that in the above example f and f' are both just two different instances of the *one* form for $\mathcal{S}^{\underline{2}}$), and *naturality* will simply be expressed by many equations, but all of the same type as the one equation here.

The two interpretations of a mapping in $\mathcal{S}$ (a family of elements parameterized by the domain versus a family of sets parameterized by the codomain) are closely related to two basic constructions of small categories. We have said that an abstract set has no internal structure (except equality and inequality of elements). Of course any real object (and even some mathematical objects) has an infinite complexity of internal structure; fortunately we are usually able through experiment and study to find out the most important structure of an object, in the sense that the most important structure of the object influences or largely determines all the other structures of the object and that it is mainly responsible for the workings of the object. Thus the notion of abstract set represents the (nearly) ultimate limit of the mathematical method, which consists of taking the main structure by itself as a first approximation to a theory of the object, i.e. mentally operating (hopefully temporally and hopefully to advantage) as though all further structure of the object simply did not exist.

Similarly, that two objects A, B can enter into some concrete external relation is possible only because A, B separately have *internal* structures (as well as because A, B together are part of a larger whole); in particular, to specify a *particular* mapping between two sets we have to make use of some structure that the sets have. The success of abstract sets is due to the fact that in mathematics the internal structure of sets may be considered to arise by reflection from the external relations specified by mappings. In general a given relation between two objects will be reflected as (additional) internal structure of each of the two objects and in particular if we are given a mapping

$$A \xrightarrow{f} B$$

between abstract sets, there will be by reflection a definite structure $\mathcal{A}_f$ in A and a definite structure $\mathcal{B}_f$ in B, as follows: In fact, both of the structures $\mathcal{A}_f$ and $\mathcal{B}_f$ have the particular form of *categories* (many basic structures are categories, contrary to the idea that "arbitrary" forms of structure may be essential). For any two elements x, x' in A, say that $x \to x'$ in $\mathcal{A}_f$ iff $f(x) = f(x')$ in B. Then we clearly have

$$x \to x \text{ in } \mathcal{A}_f \text{ for any } x \text{ in } A$$

$$x \to x' \text{ and } x' \to x'' \text{ in } \mathcal{A}_f \text{ implies } x \to x'' \text{ in } \mathcal{A}_f$$
$$\text{for any } x, x', x'' \text{ in } \mathcal{A}_f$$

so that $\mathcal{A}_f$ is a *category.* Notice that there is no need to give names to particular arrows in $\mathcal{A}_f$, for if there is a morphism $x \to x'$ in $\mathcal{A}_f$, there is only one; such a category is sometimes called an *order* or even a *trivial* category. Actually, in many contexts categories of the form $\mathcal{A}_f$ are far from being "trivial", though actually they are very special even among orders, since we moreover have

$$x \to x' \text{ in } \mathcal{A}_f \text{ iff } x' \to x \text{ in } \mathcal{A}_f$$

which technically means that $\mathcal{A}_f$ is even a trivial "groupoid" or in more traditional terminology an equivalence relation. If we are given nothing except that $A \xrightarrow{f} B$ is a mapping of arbitrary abstract sets, then $\mathcal{A}_f$ is the main structure or the only structure in A which can be derived from the situation. The same is true of the much richer structure $\mathcal{B}_f$ in B which we now will explain; a family of abstract sets is a richer context than a family of abstract elements. If y, y' are any two elements of B, say that a morphism $y \xrightarrow{t} y'$ in $\mathcal{B}_f$ is any mapping $A_y \xrightarrow{t} A_{y'}$ between the corresponding two abstract sets $A_y, A_{y'}$ in the family which f represents; we already know how to compose mappings in general, and we simply transfer this from $\mathcal{S}$ to $\mathcal{B}_f$ to make the latter a category. $\mathcal{B}_f$ is intuitively a "small" category, and indeed on the basis of our axioms for $\mathcal{S}$ this fact will be given objective form by the theorem that $\mathcal{B}_f$ (as well as $\mathcal{A}_f$, but *unlike*, for example, $\mathcal{S}$ itself) can actually be represented by a category $\mathcal{B}_f$ in $\mathcal{S}$. The category $\mathcal{B}_f$ is usually *not* merely an ordering of B; in fact, it will be so *iff* the given f is an *injection* or *monomorphism* in the sense that

$$f(x) = f(x') \text{ in } B \text{ implies } x = x' \text{ in } A \text{ for any } x, x' \text{ in } A$$

(Thus $\mathcal{B}_f$ reduces to an ordering in B iff $\mathcal{A}_f$ reduces to equality in A.) There are however two ways of deriving from $\mathcal{B}_f$ some orderings $\mathcal{C}_f$ and $\mathcal{I}_f$ in B (these two procedures of "degrading" a category to an ordering can in fact be applied to *any* category). For $\mathcal{C}_f$ we say that $y \to y'$ in $\mathcal{C}_f$ iff $\exists t[y \xrightarrow{t} y'$ in $\mathcal{B}_f$ and t is a *monomorphism*]. To verify that $\mathcal{C}_f$ is a category, one has to first verify that the composition of two monomorphisms is again a monomorphism. In $\mathcal{I}_f$, we say that

$$y \to y' \text{ iff } \exists t[y \xrightarrow{t} y' \text{ in } \mathcal{B}_f]$$

this is a category, since $\mathcal{B}_f$ is. The category $\mathcal{I}_f$ contains information about the *image* of f since in $\mathcal{B}_f$ the existence of maps is related to the emptiness or non-emptiness of the sets $A_y, A_{y'}$. Here we say for any element y of B that

$$y \text{ is the image of } f \text{ iff } \exists x[f(x) = y]$$

To see that the information in the category $\mathcal{I}_f$ is equivalent to specifying which subset of B is the image of f, notice that

$$y \text{ is in the image of } f \text{ iff } A_y \text{ is not empty}$$

(while for a set 0 to be empty it is necessary and sufficient that for any set S, there is exactly one mapping $0 \to S$). Then on some reflection, it is clear that there are just three cases in which we have $y \to y'$ in $\mathcal{I}_f$: either y, y' are both in the image of f, or both are not in the image of f, or y is not in the image while y' is. Neglecting for a moment the third case, we see that $\mathcal{I}_f$ contains an equivalence relation on B with two "equivalence classes"; taking the third case again into account, the equivalence classes are "ordered" in that the complement of the image is inferior to the image.

2. Subsets, images, and choice

This is a good place to explain more exactly what we mean by a *subset* of an abstract set B. In view of the abstractness of our sets, a subset cannot be simply another set I having some *property* relative to B and must in fact be a set I *together with* a specified mapping j detailing *how* I is included in B; this idea, *unlike* that of the traditional set theory, is entirely in accord with mathematical practice in which we say, for example, that the integers are a subset of the rational numbers and the real numbers are a subset of the complex numbers, etc. However, such an "inclusion mapping" j cannot be arbitrary, for if it is not a monomorphism, then by the above discussion, a set I would receive objectively an additional *internal* structure by merely making it a subset of B, and this is *not* a feature of the correct mathematical idea of subset; also, if $I \xrightarrow{j} B$ is not a monomorphism, then some distinct elements of I cannot be distinguished in B, which is also not a feature of subsets. On the other hand, *any* mapping which *is* a monomorphism, involves no more nor less information than a certain subset of its codomain, namely its image. Therefore, we define

j is a *subset* of B iff j is a monomorphism with codomain B.

The discussion of "subsets" can be partly generalized to an arbitrary category, and more completely to an arbitrary topos, if we make the following

Definition: *A morphism* $I \xrightarrow{j} B$ *is a* **monomorphism** *iff for any object* X *and any two morphisms* $X \xrightarrow{t_1} I$, $X \xrightarrow{t_2} I$

$$t_1 j = t_2 j \Rightarrow t_1 = t_2$$

Now in the case of the category $\mathcal{S}$ of abstract sets an element s of a set S is entirely equivalent to a mapping $1 \xrightarrow{s} S$ from a one-element set, so by taking $X = 1$ it is clear that any monomorphism is injective on elements, since under the identification of elements with such mappings the special compositions

$$1 \longrightarrow S \longrightarrow T$$

$$\longrightarrow$$

are entirely equivalent with evaluations of mappings. Conversely, noting that the category of sets has the very special property

$$X \overset{t_1}{\underset{t_2}{\rightrightarrows}} S \text{ and } t_1 \neq t_2 \Rightarrow \exists 1 \xrightarrow{x} X[xt_1 \neq xt_2]$$

a simple calculation shows that any mapping in the category of sets which is injective for elements is in fact a monomorphism in the sense of our general definition. This said, consider two subsets j_1, j_2 of a set B. We say that

$$j_1 \subseteq j_2 \text{ iff } \exists t \qquad \begin{tikzcd} I_1 \arrow[rr, "t"] \arrow[dr, "j_1"'] & & I_2 \arrow[dl, "j_2"] \\ & B & \end{tikzcd}$$

where the diagram is commutative. Such a mapping t may be intuitively thought of as a "proof" that j_1 is included in j_2; since j_2 is a monomorphism, there can be of course *at most one* such "proof". As a special case we may have $I_1 = 1$ (note that *any* mapping with domain a one element set is necessarily a monomorphism), in which case we speak of an element of B being a *member* of a subset, i.e.

$$y \in j \text{ iff } \exists \overline{y} \qquad \begin{tikzcd} 1 \arrow[rr, "\overline{y}"] \arrow[dr, "y"'] & & I \arrow[dl, "j"] \\ & B & \end{tikzcd}$$

That is, if $j \in B$ (short for $y \in 1_B$), then $y \in j$ iff there is an element $\overline{y}$ of I (= domain of j) to which y corresponds under the subset-inclusion j; there is at most one such $\overline{y}$. An easy calculation shows that

$$j_1 \subseteq j_2 \Rightarrow \forall y \in B[y \in j_1 \Rightarrow y \in j_2]$$

and the converse of this is true in the category of sets (it will also be true in a topos *provided* we generalize to consider (in the $\forall$) elements which are not necessarily "global" or "eternal" in the sense of being defined over 1). Thus, if we have both $j_1 \subseteq j_2$ and $j_2 \subseteq j_1$, then the two subsets are *equivalent* though they may not be absolutely equal. In fact, we then have two "proofs" $I_1 \xrightarrow{t} I_2 \quad I_2 \xrightarrow{u} I_1$ and because of the two commutative triangles together with the fact that j_1, j_2 are monomorphisms (a "cancellation" property) we can conclude that

$$tu = 1_{I_1} \text{ and } ut = 1_{I_2}$$

which we express by saying that t (respectively u) is an *isomorphism* with inverse u (respectively t). Isomorphic objects in a category are mathematically indistinguishable, and in particular in the category of sets we have (by Cantor's definition!) that two sets are isomorphic (in the sense that there exists an isomorphism between them) iff they have the same number of elements. However, a subset is not simply a set, it is an inclusion mapping; thus (recalling the commutativity of the triangles) two equivalent subsets of B not only have the same number of elements, but are moreover included into B in the same way. Since inclusion obviously is reflexive and transitive, the notion of subsets of B and their inclusions determines a category $\mathcal{P}(B)$ (actually an ordering); one of the fundamental principles of modern higher mathematics is that this category is small enough to be represented (modulo equivalence) by a category in $\mathcal{S}$, and in fact this will be true in any topos. Note that $\mathcal{P}(B)$ has a smallest object $0 \to B$ as well as a greatest object 1_B.

Now that we have accounted for the notions of element, value, subset, and inclusion entirely in terms of the theory of composition of mappings, we can do the same for the notion of the image of an arbitrary mapping

$$A \xrightarrow{f} B$$

The image of f is certainly a subset $I \xrightarrow{j} B$ of B, and it has the property that

$$\forall x \in A[xf \in j]$$

The domain, codomain, and all the values xp (or $p(x)$) of a mapping $A \xrightarrow{p} I$ are then uniquely determined (since j is a monomorphism) by the property that the following triangle should be commutative

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & I \arrow[d, "j"] \\ & B \end{tikzcd} \qquad pj = f \qquad (\text{or } j \circ p = f)$$

and thus by our notion of mapping in $\mathcal{S}$ we may say that such a p exists. But for given f, there may be many such subsets; for example, if we take $j = 1_B$ then there always exists such a p, namely $p = f$, yet 1_B is not the image of f if there are any elements of B which are not values of f. The further property which characterizes the image of f among all the subsets of B is that it is the *smallest* subset with the property that all values of f are members of it. That is, if we write $f \in j$ to mean that all elements in the family f are members of j (equivalently $f \in j$ iff $\exists \overline{f}[f = \overline{f}j]$, then the subset $im(f)$ is characterized by the two conditions

$$f \in im(f)$$

$$\forall j[j \subseteq B \text{ and } f \in j \Rightarrow im(f) \subseteq j]$$

Here by "characterize" we mean "characterize up to equivalence"; in fact, any two images of f (i.e. any two subsets $im(f), im'(f)$ (each satisfying the above two conditions) will be isomorphic in a unique way which respects not only the inclusions, but also respects the associated mappings p, p' in the factorization of f. That is, we have

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & \cdot \arrow[ur, "im(f)"'] & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p'"'] & & B \\ & \cdot \arrow[ur, "im'(f)"'] & \end{tikzcd}$$

and taking $j = im'(f)$ in the universal property of $im(f)$ we get a "proof" i that $im(f) \subseteq im'(f)$ so that the triangle on the right in

$$\begin{tikzcd} & \cdot \arrow[dr, "im(f)"] \arrow[dd, "i"] & \\ A \arrow[ur, "p"] \arrow[dr, "p'"'] & & B \\ & \cdot \arrow[ur, "im'(f)"'] & \end{tikzcd}$$

is commutative. But then

$$(pi)im'(f) = p\ im(f) = f = p'im'(f)$$

and since $im'(f)$ is a monomorphism, we may "cancel" to obtain

$$pi = p'$$

Notice that in our notation, the membership of elements $y \in j$ is a special case of inclusion $j' \subseteq j$ which in turn is a special case of "membership" $f \in j$ of generalized elements. In fact, the first of these subsumptions was common in early functional analysis and the last notation is, as we already mentioned, quite natural in modern algebraic geometry. Thus again our theory reflects better the actual mathematical practice than does traditional set theory in which membership and inclusion are rigidly opposed and cannot be transformed into each other without complicated notation (this is not to say that the distinction between x and $\{x\}$ is not a correct idea; it arises also in our theory, but in a more appropriate place).

The characterizing property of images may be summed up in the following way, which *any* theory would have to account for. To any subset φ of B, we have associated a monomorphism $j(\phi)$ with codomain B (in fact we have identified ϕ with j). To any mapping f with codomain B we have associated a subset $im(f)$ of B. Further, these two processes are *adjoint* in the sense that

$$\frac{f \to j(\phi)}{im(f) \to \phi}$$

where the horizontal line means that morphisms in $\mathcal{S}/B$ and in $\mathcal{P}(B)$ with the indicated domains and codomains can be biuniquely transformed into each other. Now another idea of "subsets" is that they are *propositional functions*, i.e. mappings

$$B \xrightarrow{\phi} \Omega$$

where Ω is the set of *truth values*; in the case of abstract sets Ω has two elements, and one of them is distinguished as "true". Then

$$j(\phi) = \{y | \phi(y) = true\}$$

usually abbreviated to $\{y|\phi(y)\}$ or even $\{B|\phi\}$; it is clear that to any j there is also a unique corresponding φ, whose values are determined by

$$\phi(y) = true \text{ iff } y \in j$$

Formulas of the predicate calculus of logic correspond (when interpreted) to propositional functions, hence to subsets. The notion of image is closely connected with the logical operation of *existential quantification*, since

$$y \in im(f) \text{ iff } \exists x[f(x) = y]$$

In fact, the *adjointness* mentioned above is essentially the mathematical interpretation of the fundamental rule of proof for existential quantification. We will return to this point later [ ].

Recall that we associated to an arbitrary mapping $A \xrightarrow{f} B$ a category $\mathcal{A}_f$ with objects = the elements of A and a category J_f with objects = elements of B. We saw previously how the image of f, considered as a subset of B, can be described in terms of J_f. Now we may consider the image factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & I \arrow[ur, "j"'] & \end{tikzcd}$$

as a splitting of f into two, the second of which is a monomorphism. Instead of the minimality condition on $j = im(f)$ just discussed, we may characterize the image factorization alternatively by the fact that p is an "epimorphism"; further, epimorphisms with domain A may be described by means of the categories of the type $\mathcal{A}_f$, as *partitions* of A. Before defining epimorphisms, let us consider the notion of surjective mappings, which will turn out to be equivalent in the case of the category of abstract sets. A mapping $A \xrightarrow{f} B$ is called *surjective* (for elements) iff

$$\forall y \in B \exists x \in A[xf = y]$$

i.e. iff $im(f)$ is equivalent to 1_B. Then it is clear that in the *image* factorization of an arbitrary mapping f

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & I \arrow[ur, "j"'] & \end{tikzcd}$$

the mapping p is surjective; moreover, this fact characterizes the image since for any larger subset j' of B through which f can be factored, the factoring mapping p' will not be surjective. For any factorization of f into a mapping f' followed by a monomorphism, we have

$$x_1 f = x_2 f \text{ iff } x_1 f' = x_2 f' \text{ for any } x_1, x_2 \in A$$

and hence

$$\mathcal{A}_f \cong \mathcal{A}_{f'}$$

In particular, for the minimal such f', namely for $f' = p$ where p is the surjective part of the image factorization of f, we have

$$\mathcal{A}_f \cong \mathcal{A}_p$$

Now, not from A nor even from the category $\mathcal{A}_f$ can we expect to reconstruct f, but we can, up to "equivalence of partitions", reconstruct p. This can be seen as follows: if we consider f as a family of sets parameterized by B, then some of the sets A_y in the family may be empty, in fact precisely for those $y \in im(f)$; thus if we restrict the family f to those parameters $y \in im(f)$ (there will be such y iff $im(f) \neq 0_B$ i.e. iff $A \not\cong 0$), then all the A_y will be non empty. But this restricted family is clearly equivalent to the family $\{Ai\}_{i \in I}$ parameterized by I determined by the mapping p. Moreover, two elements x_1, x_2 belong to the same set in the family, iff $x_1 f = x_2 f$ i.e. iff $x_1 \to x_2$ in the equivalence relation $\mathcal{A}_f$. If we say that in that case x_1, x_2 belong to the same cell of the partition of A determined by $\mathcal{A}_f$, then the set I may be constructed by "abstracting" - each cell is given exactly one "name" and different cells are given different names and I is exactly an abstract set which can name or parameterize the cells of the partition biuniquely. Then the mapping p simply assigns to each element x of A the name of the cell to which x belongs.

In any category we say that

$$[A \xrightarrow{f} B \text{ is an epimorphism}] \text{ iff } \forall Y \forall g_1, g_2 \quad B \overset{g_1}{\underset{g_2}{\rightrightarrows}} Y$$

$$[fg_1 = fg_2 \Rightarrow g_1 = g_2]$$

Thus the notion of epimorphism is *dual* to that of monomorphism, where the primitive idea of "dualizing" in category theory consists simply of reversing all the arrows. In spite of this duality, the actual meaning of epimorphisms in a particular category is typically rather deeper than that of monomorphisms. We can establish in general certain simple facts which are dual to general facts about monomorphisms such as

$$f \text{ epi}, g \text{ epi} \;\Rightarrow fg \text{ epi}$$

$$fg \text{ epi} \;\Rightarrow g \text{ epi}$$

and define an *ordering* by "refinement" among those epimorphisms with given domain A (dual to the ordering by inclusion of monos). In certain "good" categories (including all the topos) the epimorphisms with domain A are equivalent to equivalence relations in A (but e.g. the category of commutative rings and homomorphisms is *not* good in this sense). In particular, we have the

Proposition 1.1: *In the category $\mathcal{S}$ of abstract sets, a mapping $A \xrightarrow{f} B$ is an epimorphism iff it is surjective.*

Proof: Suppose that f is surjective and that $fg_1 = fg_2$. Suppose in order to reach an absurdity that $g_1 \neq g_2$. Then by the fact already mentioned that there are enough global elements to distinguish mappings (or more primitively by our notion of equality of mappings) there is $1 \xrightarrow{y} B$ such that $yg_1 \neq yg_2$. Then since f is surjective $\exists x[xf = y]$ is true in $\mathcal{S}$. Since $\exists$ in $\mathcal{S}$ really means that something exists, there is $1 \xrightarrow{x} A$ with $xf = y$. Then

$$(xf)g_1 = yg_1 \neq yg_2 = (xf)g_2$$

so by associativity

$$x(fg_1) \neq x(fg_2)$$

but this contradicts the fact that $fg_1 = fg_2$. Hence f is an epimorphism. Conversely, if f is not surjective, we must construct a suitable pair of mappings g_1, g_2 in order to establish that f is not an epimorphism, and here the two-element set Ω with the distinguished element "true" will again be useful. Let

$$B \xrightarrow{\phi} \Omega$$

be the "characteristic function" of the image of f, i.e.

$$y\phi = \text{ true iff } \exists x[xf = y]$$

and let

$$B \xrightarrow{true_B} \Omega$$

be the mapping which is *constant* with value true, i.e.

$$y \text{ true}_B = \text{ true for every element } y \text{ of } B$$

Since f is not surjective, there is an element y_o of B which is not in the image of f, so

$$y_o\phi = \text{ false.}$$

Thus

$$y_o\phi \neq y_o \text{ true}_B \text{ and hence}$$

$$\phi \neq \text{true}_B$$

at the same time we do have

$$f\phi = f\text{true}_B$$

since $x(f\phi)) = (xf)\phi = $ true for all $x \in A$. This means that the pair $g_1 = \phi$, $g_2 = \text{true}_B$ is a counterexample to the (left) cancellation law for f, i.e. that f is not an epimorphism. ∎

In fact, an epimorphism in the category of sets has a property stronger than "surjectivity for the elements defined over $1''$, namely the property "surjectivity for elements defined over an arbitrary T''. Suppose $A \xrightarrow{f} B$ is epic (i.e. surjective) and suppose $T \xrightarrow{b} B$ is any mapping with codomain B. Then for any element t of T, $y = tb$ is an element of B, hence f being surjective, there are elements x of A which are mapped by f to tb; choose such an x and call it a_t. Then

$$a_{tf} = tb$$

If we choose one such $a_t \in A$ for each $t \in T$, then we can define a mapping $T \xrightarrow{a} A$ by

$$ta = a_t$$

$$(\text{or } a(t) = a_t)$$

Then we have

$$t(af) = (ta)f = tb$$

for all elements t of T and hence

$$\begin{tikzcd} & T \arrow[dl, dashed, "a"'] \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad af = b$$

The fact that such an arbitrary choice process $t \rightsquigarrow a_t$ produces an actual mapping in $\mathcal{S}$ (in contrast to the situation in most toposes or categories) results from the abstractness and constancy of the sets, so that their morphisms (mappings) are not required to preserve anything and there is no development going on inside T and A which could obstruct the existence of such morphisms. Considering the special case $T = B, b = 1_B$, we obtain (for abstract sets) the *Axiom of Choice*

$$f \text{ epic } \Rightarrow \exists s[sf = 1_B]$$

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, dashed, bend right, "s"'] \end{tikzcd}$$

Such a choice mapping is also called a *section* of f and the axiom of choice may be formulated: "every epimorphism has a section". Remembering our introduction of infinite products $\prod_B A$, the axiom of choice says that for any family $\{A_y\}_{y\in B}$ of sets, if every set A_y has at least one element, then the product $\prod_{y\in B} A_y$ of the family has at least one element. Actually, the axiom of choice is true in certain toposes different from the category of sets, for example in any topos of the form $\mathcal{S}/\Theta$, where Θ is any parameter set. In fact, $\mathcal{S}/\Theta$ distinguishes itself from $\mathcal{S}$ mainly because there are many truth values and because 1 (i.e. the family constantly 1) is not sufficient to catch all elements. But although there are many "stages" θ, there are no "connections" between them, which permits the axiom of choice (and the law of the excluded middle) to remain valid. In particular, the reader should be able to repeat our discussion of $\mathcal{A}_f, \mathcal{B}_f$, image factorization, etc. for a morphism f of $\mathcal{S}/\Theta$ without difficulty by simply "doing things for each parameter θ independently".

Exercise 1.1 Prove the previous proposition (that in $\mathcal{S}$, any f is surjective iff epic) by using the axiom of choice instead of characteristic functions.

Even for a topos as simple as $\mathcal{S}^{\underline{2}}$ the axiom of choice and the law of the excluded middle are no longer true, but the construction of $\mathcal{A}_f, \mathcal{B}_f$ and the image factorization (appropriately reformulated) do remain valid. This will be discussed in more detail later [ ], but let us consider now briefly one suggestive aspect.

If we denote the "stages" by 1, 0 then a typical object A of $\mathcal{S}^{\underline{2}}$ is a mapping $A_1 \to A_0$, and if A, B are two objects of $\mathcal{S}^{\underline{2}}$, then a morphism $A \xrightarrow{f} B$ in $\mathcal{S}^{\underline{2}}$ is really a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d] & B_1 \arrow[d] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd}$$

of mappings. The equivalence relation $\mathcal{A}_f$ in A induced by f again has two stages; at stage 1 the equivalence relation induced on A_1 by f_1 (in $\mathcal{S}$) and at stage 0, the equivalence relation in A_0 induced by f_0. Due to the commutativity of the square, equivalent pairs at stage 1 are mapped by $A_1 \to A_0$ into equivalent pairs at stage 0, and thus $\mathcal{A}_f$ having a specific connection between its stages is also "in" $\mathcal{S}^{\underline{2}}$. But note that two elements of A_1 may be disequivalent at stage 1, even if they are equivalent at stage 0. Thus the internal development, simple though it is, of an object in $\mathcal{S}^{\underline{2}}$ can destroy the rigid opposition of $\mathcal{S}$ between equivalence and disequivalence modulo f, transforming one into the other. The same observation is also valid for the opposition between belonging and not belonging to the image of f; typically, there will be elements of B which do not belong to the image of f at stage l, but which do belong at stage 0. Also, the image factorization in $\mathcal{S}^{\underline{2}}$ proceeds "stage by stage"; however, this will no longer be true in the more typical topos of "sheaves" because there the "continuity" will force the stages to be related in a much more intrinsic manner, and that will condition the meaning of "epimorphism".

In order to break the rigid opposition between A and B which has characterized this discussion for several pages, note that while $\mathcal{A}_f$ is the most general equivalence relation in A, there is also a natural equivalence relation which is part of $\mathcal{I}_f$. However, $\mathcal{I}_f$ is very far from being the most general equivalence relation in B. In that vein, the reader can show (for sets and mappings), that if ϕ denotes the characteristic function $B \xrightarrow{\phi} \Omega$ of the image $I \xrightarrow{j} B$ of a mapping $A \xrightarrow{f} B$, where $A \neq 0$, then ϕ is epic iff f is not epic. We have already stated the two properties which distinguish the category of abstract sets from other toposes, namely the axiom of choice and the existence of sufficiently many morphisms with domain 1 (the reader can verify that also the latter fails in $\mathcal{S}^{\underline{2}}$, which is closely connected with the existence of *three* "subsets" of 1 in $\mathcal{S}^{\underline{2}}$, in contrast with the obvious *two* subsets of 1 in $\mathcal{S}$). The topos axioms in themselves (not yet given) will, however, be quite powerful even for sets. First we want to briefly examine four further simple examples, one big category and three of its subcategories.

The big category is a very important example $\mathcal{S}^{\circlearrowleft}$, the category whose objects are pairs A, t where t is an arbitrary endomorphism

$$A \circlearrowleft^{t}$$

of an arbitrary set A, and whose morphisms are arbitrary mappings f which satisfy the equation

$$tf = ft'$$

$$A \circlearrowleft^{t} \xrightarrow{\ f\ } A' \circlearrowleft^{t'}$$

where A, t and A', t' are arbitrary given objects. In such objects there is an internal dynamic (much more complicated than that in $\mathcal{S}^{\underline{2}}$) based on the internal structure which is essentially the opposition of an element x' with one of the form xt; for then we have $x't$ versus xtt, or versus x etc. In general, $x't^n$ may be equal or unequal with xt^m for any two natural numbers n, m, and for any two elements x', x of A (of course, if $x't^n = xt^m$ then also $x't^{n+k} = xt^{m+k}$); in $\mathcal{S}^{\circlearrowleft}$, in particular, one may have or not have $x = xt$. In fact, the category $\mathcal{S}^{\circlearrowleft}$ may be considered as the basis for the existence of the natural numbers as *one set* in $\mathcal{S}$ as we will see later [ ]. The other three examples are subcategories of $\mathcal{S}^{\circlearrowleft}$ which are also toposes, but which are quite different from $\mathcal{S}^{\circlearrowleft}$ in their particularity. That is, we may restrict attention to those objects A, t of $\mathcal{S}^{\circlearrowleft}$ for which t is an *isomorphism* (hence an automorphism, or permutation, of A)

(1) $\exists s[ts = st = 1_A]$

But we could also consider the still smaller category of permutations of *finite orbit*

(2) $\forall x \in A \exists n[xt^n = x]$

Finally, we could consider the subcategory of $\mathcal{S}^{\circlearrowleft}$, which is however not a subcategory of the permutations, consisting of objects satisfying

(3) $\forall x \in A \exists n[xt^{n+1} = xt^n]$

that is the topos of *eventually stationary* endomorphisms of sets.

Lesson 2

1. Generating families

Let $\mathcal{C}$ be any category and let $\mathcal{G}$ be any class of objects in $\mathcal{C}$. (Later [ ] we will be interested in the case where $\mathcal{G}$ can be parameterized by a set, but this plays no role in the formal consideration here.)

Definition: *A morphism* $A \xrightarrow{f} B$ *in* $\mathcal{C}$ *is said to be a* $\mathcal{G}$-**injective** *iff*

$$\forall G \in \mathcal{G}\ \forall x_1, x_2 [G \underset{x_2}{\overset{x_1}{\rightrightarrows}} A \text{ and } x_1 f = x_2 f \Rightarrow x_1 = x_2]$$

In particular, f is called a monomorphism if it is $\mathcal{C}$-injective, i.e. if $\forall G \in \mathcal{G}$ may be replaced by $\forall G$ in the above condition. Say that $\mathcal{G}$ **weakly generates** *$\mathcal{C}$ iff*

$$\forall A, B\ \forall f_1, f_2\ A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B$$

$$[\forall G \in \mathcal{G}\ \forall G \xrightarrow{x} A[x f_1 = x f_2]] \Rightarrow f_1 = f_2$$

or equivalently

$$f_1 \neq f_2 \Rightarrow \exists G \in \mathcal{G}\ \exists G \xrightarrow{x} A[x f_1 \neq x f_2]$$

Proposition 2.1: *Any monomorphism in $\mathcal{C}$ is $\mathcal{G}$-injective. If $\mathcal{G}$ weakly generates $\mathcal{C}$, then any $\mathcal{G}$-injective morphism is a monomorphism of $\mathcal{C}$.*

Definition: *A morphism f is an* **equalizer** *of g_1, g_2 iff the following two conditions are satisfied*

$$fg_1 = fg_2$$
$$\forall y\,[yg_1 = yg_2 \Rightarrow \exists! x[xf = y]]$$

$$\begin{tikzcd} T \arrow[d, dashed, "x"'] \arrow[dr, "y"] & & \\ A \arrow[r, "f"'] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

Here for any condition or property $\phi(x)$

$$\exists\,!\,x\phi(x) \text{ means } \exists x\phi(x) \text{ and } (!\,x)\phi(x)$$

where in turn the "uniqueness" operator is an abbreviation as follows

$$(!x)\phi(x) \text{ means } \forall x_1, x_2\,[\phi(x_1) \text{ and } \phi(x_2) \Rightarrow x_1 = x_2]$$

One may consider that an equalizer of g_1, g_2 is a "best morphism which equalizes g_1, g_2" in the sense that if y also equalizes them (i.e. $yg_1 = yg_2$) then y can be *uniquely* expressed in terms of the equalizer f . Further, we say that f is a *regular monomorphism* iff there exists at least one pair g_1, g_2 of which f is an equalizer.

Proposition 2.2: *Every regular monomorphism is a monomorphism.*
The *proof* depends mainly on the *uniqueness* condition in the definition of equalizer. We will see that in any topos $\mathcal{C}$, every monomorphism is regular. On the other hand, the inclusion $\mathbf{N} \longrightarrow \mathbf{Z}$ is a *non* regular monomorphism in the category $\mathcal{C}$ of commutative monoids; similarly, for most localizations in the category $\mathcal{C}$ of commutative rings.

Definition: *A morphism f is a* **retract** *iff there exists a "retraction" for it, i.e. iff*

$$\exists g[fg = 1A] \qquad \begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, dashed, bend right, "g"'] \end{tikzcd}$$

Proposition 2.3: *Every retract is a* regular *monomorphism.*

Proof: In fact, a retract $A \xrightarrow{f} B$ is an equalizer of the pair $gf, 1_B$ for any chosen retraction g.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "gf"] \arrow[r, shift right, "1_B"'] & B \end{tikzcd}$$

Definition: *f is an* **isomorphism** *in $\mathcal{C}$ iff there exists g in $\mathcal{C}$ for which $fg = 1_A$ and $gf = 1_B$*

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, bend right, dashed, "g"'] \end{tikzcd}$$

clearly every isomorphism is a retract; and any identity morphism is an isomorphism. Moreover, the inverse g of an isomorphism f is uniquely determined by f and is also an isomorphism; we write $g = f^{-1}$. Then if $B \xrightarrow{f_1} C$ is another isomorphism, then ff_1 is also an isomorphism, and in fact $(ff_1)^{-1} = f_1^{-1}f^{-1}$.

Exercise 2.1 If A, B are two objects of a category $\mathcal{C}$ and if we are given an *isomorphism* $A \xrightarrow{f} B$ in $\mathcal{C}$ then there exists a *functor* $F : \mathcal{C} \to \mathcal{C}$ such that

$$\begin{aligned} &F(A) = B \\ &F(B) = A \\ &F(f) = f^{-1} \\ &F(X) = X \quad X \neq A, B \end{aligned}$$

and such that F has a *functor* F^{-1} inverse to it on both sides. This implies that if $\phi(x)$ is any formula in the internal (or external) language of $\mathcal{C}$ whose only free variable is a variable X ranging over objects of $\mathcal{C}$, then we can prove that

$$A \cong B \Rightarrow [\phi(A) \iff \phi(B)]$$

where $A \cong B$ means that there exists an isomorphism $A \xrightarrow{f} B$. The converse implication also holds if we consider *all* “interesting” formulas $\phi(x)$, where interesting means “preserved under any *equivalence* of categories”; the definition of “equivalence of categories” can be found in any book, but a syntactical characterization of “interesting formulas” in no book.*

Definition: *If B is an object in a category, a* **subobject** *of B means any monomorphism with domain B. If $\mathcal{G}$ is a given class of objects, if j is a subobject of B, and if $G \xrightarrow{y} B$ with $G \in \mathcal{G}$, then we say “y is a member of j” iff*

$$y \in j \textit{ iff } \exists \overline{y}[\overline{y}j = y]$$

Note that such $\overline{y}$ is unique if it exists. If j', j are two subobjects of B, we say

$$j' \subseteq j \textit{ iff } \exists i[ij = j']$$

Observation: We clearly have

$$j' \subseteq j \Rightarrow \forall y[y \in j' \Rightarrow y \in j]$$

and further the converse is easy in case $\mathcal{G}$ is the class of all objects of $\mathcal{C}$. For a less trivial converse, consider the following concepts

Definition: *A morphism* $A \xrightarrow{f} B$ *is* $\mathcal{G}$-**surjective** *iff for every* $G \in \mathcal{G}$

$$\forall y \exists x[xf = y] \qquad \begin{tikzcd} & G \arrow[dl, dashed, "x"'] \arrow[dr, "y"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

Say that $\mathcal{G}$ **strongly generates** $\mathcal{C}$ *iff every morphism which is both* $\mathcal{G}$-*surjective and a monomorphism is an isomorphism.*

Exercise 2.2 Suppose that $\mathcal{G}$ strongly generates $\mathcal{C}$ and that pullbacks of monomorphisms exist in $\mathcal{C}$. Let j', j be two subobjects of an object B. Then

$$\forall y(\text{defined over } \mathcal{G})[y \in j' \Rightarrow y \in j] \Rightarrow j' \subseteq j$$

Exercise 2.3 *Lemma: For any subobject* j *of* B *we have* $j \subseteq 1_B$. *But* $1_B \subseteq j$ *iff* j *is an isomorphism.*

Exercise 2.4 Suppose that any two morphisms $\cdot \rightrightarrows \cdot$ in $\mathcal{C}$ have an equalizer in $\mathcal{C}$, and let $\mathcal{G}$ be any class of objects which strongly generates $\mathcal{C}$. Then $\mathcal{G}$ also weakly generates $\mathcal{C}$.

Exercise 2.5 Suppose f is an equalizer of g_1, g_2 in $\mathcal{C}$. Then

$$y \in f \iff yg_1 = yg_2$$

i.e. "the equalizer of a pair of morphisms is precisely the subobject (of their domain) whose members are those elements on which the two have equal values". This even characterizes the equalizer up to equivalence ($\subseteq$ and $\supseteq$) of subobjects at least if we take $\mathcal{G}$ = all of $\mathcal{C}$, or more generally if $\mathcal{G}$ strongly generates $\mathcal{C}$.

Definition: *A* **terminal** *(or final) object of a category* $\mathcal{C}$ *is an object 1 such that*

$$\forall A \exists ! t[A \xrightarrow{t} 1]$$

Exercise 2.6 Any two terminal objects of $\mathcal{C}$ are isomorphic. Say that $U \subseteq 1$ iff the unique morphism $U \to 1$ is a *monomorphism.* If $\mathcal{C}$ has a terminal object, then for any object $U, U \subseteq 1$ iff *every* morphism with domain U is a monomorphism.

Exercise 2.7 Suppose $\mathcal{G}$ is the class of all objects of $\mathcal{C}$ which are subobjects of 1 (this has a unique sense; see the preceding exercise). Then for any subobject j of any object B, and for any element of B defined over an object of $\mathcal{G}$, we have

$$y \in j \Longleftrightarrow y \subseteq j$$

For this notation see Banach, for example. For the category of sets and mappings, or even for any topos of the special kind which we call a "Boolean-valued model of set theory" (and indeed also for any of the classical sheaf topos over a topological space), the subobjects of 1 strongly generate.

Exercise 2.8 In the topos $\mathcal{S}^{\underline{2}}$, there are exactly three subobjects of 1, and they strongly generate $\mathcal{S}^{\underline{2}}$. In the topos $\mathcal{S}^{\circlearrowright}$ there are two subobjects of 1; they do *not* weakly generate.

All the general notions and propositions above can be *dualized*: i.e. by syntactically reversing all arrows, compositions, and diagrams, or by semantically considering the dual category $\mathcal{C}^{op}$. Thus the dual of "monomorphism" is "epimorphism", the dual of "equalizer" is "coequalizer", and the dual of "regular monomorphism" is "regular epimorphism", while the dual of "subobject" is (sometimes) "quotient object". But the dual of "isomorphism" is again "isomorphism", while the dual of "$\mathcal{G}$-injective" is (usually) *not* "$\mathcal{G}$-surjective"!

Exercise 2.9 Ω = two element set co-generates the category $\mathcal{S}$.

Proposition 2.4: *If $\mathcal{G}$ weakly generates $\mathcal{C}$, then every $\mathcal{G}$-surjective morphism f is an epimorphism.*

Proof: Suppose $fg_1 = fg_2$ but $g_1 \neq g_2$. Then there exists $G \in \mathcal{G}$ and a y with domain G such that $yg_1 \neq yg_2$ since $\mathcal{G}$ weakly generates. But then since f is $\mathcal{G}$-surjective, there exists x with $xf = y$. Then

$$yg_1 = (xf)g_1 = x(fg_1) = x(fg_2) = (xf)g_2 = yg_2$$

an inconsistency with our assumption $g_1 \neq g_2$. Thus f is an epimorphism since g_1, g_2 were arbitrary. ∎

Definition: *A class $\mathcal{G}$ of objects of $\mathcal{C}$ is called a* **regular projective generator** *iff the $\mathcal{G}$-surjective morphisms are precisely the regular epimorphisms. For example, $\mathcal{G} = \{\mathbf{Z}\}$ is a regular projective generator for the category of groups, and $\mathcal{G} = \{\mathbf{Z}[t]\}$ is a regular projective generator for the category of rings with unit element. Most topos do not have a regular projective generator.*

Definition: *A category $\mathcal{C}$ is said to admit* **regular image factorizations** *iff every morphism f can be factored*

$$f = pi$$

where p is a regular epimorphism and i is a monomorphism.

Proposition 2.5: *If $f = pi$ is a regular image factorization as above, and also if $f = f'j$ where j is a monomorphism, then $i \subseteq j$; i.e. if $xf \in j$ for all x, then imreg $(f) \subseteq j$.*

Proof: Let t_1, t_2 be a pair of morphisms with p as coequalizer. Then

$$(t_1 f')j = t_1(f'j) = t_1 f = t_1(pi) = (t_1 p)i = (t_2 p)i = t_2(pi) = t_2 f = t_2 f'j$$

and therefore

$$t_1 f' = t_2 f'$$

since j is a monomorphism. But then by the universal property of coequalizers, f' can be expressed in terms of p as

$$f' = pq$$

for some (unique) morphism q. Now we want to use q to prove that $i \subseteq j$, i.e. we must show that we also have $i = qj$. But at least we have

$$pi = f = f'j = (pq)j = p(qj)$$

and hence the desired result, since p is an epimorphism. ∎

$$\begin{tikzcd}
{} \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & \\
{} \arrow[r, "f'"] \arrow[d, two heads, "p"'] & {} \arrow[d, tail, "j"] \\
{} \arrow[r, tail, "i"'] \arrow[ur, dashed] & {}
\end{tikzcd}$$

Lemma: *In any category $\mathcal{C}$, any morphism which is both a monomorphism and also a* regular *epimorphism is in fact an isomorphism. A pair of morphisms which has an equalizer are equal iff any equalizer is an isomorphism, and dually two morphisms are equal iff their coequalizer is an isomorphism, provided their coequalizer exists.*

Proposition 2.6: *Any regular projective generator is a strongly generating class.*

Proof: If f is a monomorphism, suppose further that f is $\mathcal{G}$-surjective. Then since $\mathcal{G}$ is a regular projective generator, f is also a regular epimorphism, hence by the lemma an isomorphism. ∎

Definition: *Suppose $A \xrightarrow{f} B, B' \xrightarrow{b} B$ are given morphisms in a category $\mathcal{C}$. Then a morphism $A' \xrightarrow{f'} B'$ is said to be a* **pullback of f by** *b iff* *we can equip it with a further morphism* $A' \xrightarrow{a} A$ *such that the resulting square has the following two properties, saying essentially that* f', a *is a best completion of* f, b *to a* **commutative** *square* $af = f'b$

$\forall T \forall y' \forall x \; [xf = y'b \Rightarrow \exists! x'[x = x'a \;\&\; y' = x'f']]$

$$\begin{tikzcd}
T \arrow[dr, dashed, "x_1"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y'"'] & & \\
& A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\
& B' \arrow[r, "b"'] & B
\end{tikzcd}$$

Otherwise said, the "elements" of A' defined over any T "are" (i.e. are by f', a in a canonically biunique correspondence with) precisely the pairs $x' = < y', x >$ such that y' is an element of B', and x an element of A (defined over the same T) with the pair satisfying the condition that $y'b = xf$ in B.

Proposition 2.7: *The pullback* f' *of a* monomorphism f *by any morphism* b *is again a monomorphism.*

Proof: Suppose x', x'' are two "elements" of A' having the same value under f'. Then they certainly have the same value under $f'b$, and hence since the square is commutative, we have by associativity that

$$(x'a)f = (x''a)f$$

Now since f is a monomorphism, we can define a *single* element x of A by

$$x'a = x = x''a$$

Similarly, we can directly from the supposition define a single y' by

$$x'f' = y' = x''f'$$

But then both x', x'' satisfy the same pair of commutative triangles, so by the uniqueness condition in the definition of pullback we have $x' = x''$. Hence f' is a monomorphism. ∎

Remark: The pullback f' by b of a monomorphism f will be called the *inverse image* of f by b, since

$$y' \in f' \Longleftrightarrow y'b \in f$$

for any generalized element y' of B'. We may write $b^*(f) = f'$ without ambiguity, since the morphism a is then unique. If f is not a monomorphism, then a will not be unique, but by abuse of notation we may still write $f' = b^*(f)$ with a given a "understood".

Definition: *If j_1, j_2 are both subobjects of B, then a third subobject of B, denoted by $j_1 \cap j_2$, can be constructed in either of two equivalent ways: the pullback of j_1 by j_2 is composed with j_2; or the two subscripts can be switched.*

Exercise 2.10 In any category admitting pullbacks, we have

$$y \in j_1 \cap j_2 \iff y \in j_1 \ \& \ y \in j_2$$

for any two subobjects of an object B and any $T \xrightarrow{y} B$.
In particular

$$j \subseteq j_1 \cap j_2 \iff j \subseteq j_1 \ \& \ j \subseteq j_2$$

for any third subobject j of B. Thus in any category with pullbacks, both the "lattice-theoretic" and the "set-theoretic" notions of intersection are special cases of each other. Further, if $B' \xrightarrow{b} B$ is *any* morphism, then

$$b^*(j_1 \cap j_2) = b^*(j_1) \cap b^*(j_2)$$

$$b^*(1_B) = 1_B$$

up to equivalence of subobjects of B'.

Proposition 2.8: *Let $\mathcal{C}$ be any category with pullbacks, and let $\mathcal{G}$ be any class of objects. Then along any morphism b, the pullback of any $\mathcal{G}$-surjective morphism f is again a $\mathcal{G}$-surjective morphism f'.*

Proof: Suppose $G \xrightarrow{y'} B'$, where B' is the codomain of f (= domain of b). We must find x' with $x'f' = y'$. But $x'b$ is an "element" of the codomain of the $\mathcal{G}$-surjective f. Thus by assumption, there exists x with $xf = x'b$. But then since f' is a pullback of f by b, there is x as required. ∎

Definition: *A* **regular category** *$\mathcal{C}$ is one having pullbacks, regular image factorizations, and the "exactness property": any pullback of any regular epimorphism is a regular epimorphism.*

Corollary: *If $\mathcal{C}$ has pullbacks and regular image factorizations and if there exists a subclass $\mathcal{G}$ of the objects of $\mathcal{C}$ which is a regular projective generator, then $\mathcal{C}$ is regular.*

Remark: The conditions of the corollary are satisfied for any "algebraic" category such as groups, rings, R-modules, Lie algebras, lattices, etc. Another way in which $\mathcal{C}$ may satisfy the corollary is if $\mathcal{C}$ satisfies the "axiom of choice" that every epimorphism is a retraction (= has a section); this is the case for the category of sets and for any Boolean-valued model of set theory. However, most topos do *not* admit a regular projective generator $\mathcal{G}$; on the other hand, we will prove [ ] the theorem that every topos is a regular category. There is also a seemingly different type of regular category which arises directly from the syntax of any logical theory. Another virtue of regular categories is that a generalization due to Lubkin and Barr of the Freyd-Mitchell embedding theorem for abelian categories, as well as a generalization due to Joyal of the Gödel-Henkin-Kripke completeness theorems of logic, can be proved for any small regular category.

Proposition 2.9: *In any regular category, pullback commutes with image.*

Proof: Let b, f be any two morphisms with domain B and let $f = pi$ be a factorization of f into a regular epimorphism p followed by a monomorphism i. Let f' be a pullback of f by b with structural projection a. We want to show that $i' = b^*(i)$ is the image of f'. Let b_0 be the morphism with $i'b = b_0 i$. First, note that $f'b = af = a(pi) = (ap)i$, so since i' is the pullback of i, there is a (unique) p' with $f' = p'i'$ and $p'b_0 = i$. We must show that p' is a regular epimorphism

$$\begin{tikzcd}
 & {} \arrow[rr, "a"] \arrow[dl, "p'"'] \arrow[dd, dashed, "f'"] & & {} \arrow[dl, "p"'] \arrow[dd, "f"] \\
{} \arrow[rr, dashed, "b_0"] \arrow[dr, "i'"'] & & {} \arrow[dr, "i"'] & \\
 & {} \arrow[rr] & & {}
\end{tikzcd}$$

Rather than explicitly constructing a pair for which p is a coequalizer, we will show that the square

$$\begin{tikzcd}
{} \arrow[r, "a"] \arrow[d, "p'"'] & {} \arrow[d, "p"] \\
{} \arrow[r, "b_0"'] & {}
\end{tikzcd}$$

is a pullback and apply regularity. First, we show that the square commutes. This is clear since i is a monomorphism. Now suppose y'', x is any pair of morphisms for which $y''b_0 = xp$. Then $(y''i')b = y''(i'b) = y''(b_0 i) = (y''b_0)i = (xp)i = x(pi) = xf$. Hence there is a unique $x' =< y''i', x >$ for which

$$y''i' = x'f'$$

$$x = x'a$$

We want to show that x' also works for the p-square. But since $f' = p'i'$ and i' is a monomorphism, the first of the above equations yields

$$y'' = x'p'$$

and the second equation is already what is required. The only remaining point is the uniqueness of y'', but since the derived pair of equations implies the above pair by composing with i', this is clear. ∎

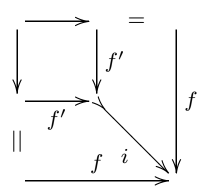
Now the defining property of a regular category is called an exactness property, since pullbacks are a “left limit” while coequalizers are a “right limit” and in general any sort of unity between these two opposites is called “exactness” in a terminology that goes back through homological algebra and through algebraic topology to Poincaré’s consideration of integrability conditions for differential forms. There is a second important property of exactness which is true in any topos and as well in any *equationally definable* category of algebras, and this is the so-called first isomorphism theorem for congruence relations. Now a congruence relation in a category of algebras is really just an equivalence relation which is “in the category”, so the usual categorical terminology is “equivalence relation”. There are five facts which are true without any exactness conditions:

Proposition 2.10:

1) In any category, a morphism f is a monomorphism iff the following square is a pullback (= is cartesian)

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

2) If $f = f'i$ with i a monomorphism, then the pullback of f by f is the same as the pullback of f', i.e. in the diagram



the little square is cartesian iff the big one is.

3) If f is any morphism for which the pullback of f by f exists and if p_1, p_2 are respectively the pullback and structural projection for this pullback, then the pair $A' \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$ is an MRST (Mono, Reflexive, Symmetric, Transitive) pair in the sense of the following definition

$$M)\ \forall x, x'[xp_1 = x'p_1\ \&\ xp_2 = x'p_2 \Longrightarrow x = x']$$

Thus we may write $a_1 \underset{p}{\equiv} a_2$ *iff* $< a_1, a_2 >\in A'$

$$iff\ \exists a'[a_1 = a'p,\ \&\ a_2 = a'p_2]$$

for any $T \underset{a_2}{\overset{a_1}{\rightrightarrows}} A$

R) $a \underset{p}{\equiv} a$ *for any* $T \xrightarrow{a} A$

S) $a_1 \underset{p}{\equiv} a_2 \Longleftrightarrow a_2 \underset{p}{\equiv} a_1$

T) $a_1 \underset{p}{\equiv} a_2$ *&* $a_2 \underset{p}{\equiv} a_3 \Rightarrow a_1 \underset{p}{\equiv} a_3$

4) If f is a regular epimorphism and if a pullback p_1, p_2 of f by f exists, then f is the coequalizer of p_1, p_2.

5) If the pullback p_1, p_2 of a morphism f by itself exists and if the coequalizer p of p_1, p_2 exists, then there exists a morphism i with $f = pi$.

Proof: 1) and 2) are obvious, and 3) is clear, once we note that

$$a_1 \underset{p}{\equiv} a_2 \Longleftrightarrow a_1 f = a_2 f$$

For the statement 4), suppose that a_1, a_2 are two morphisms of which f is a coequalizer. Then $a_1 \underset{p}{\equiv} a_2$, i.e. $\exists a'[a_1 = a'p_1$ and $a_2 = a'p_2]$. Now suppose that g is any morphism with $p_1 g = p_2 g$. We must show that there is a unique q with $g = fq$. But $a_1 g = (a'p_1)g = a'(p_1 g) = a'(p_2 g) = (a'p_2)g = a_2 g$, and hence there is such a q, since f is the coequalizer of a_1, a_2; the uniqueness is anyway clear since f is an epimorphism. But since this holds for any g, we have shown that f is the coequalizer of p_1, p_2.
5) is also clear since f "co-equalizes" p_1, p_2. ∎

$$\begin{tikzcd}
 & A' \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
T \arrow[ur, dashed, "a_1"] \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, dashed, "q"] \\
 & & S
\end{tikzcd}$$

Definition: *Any pair* $A' \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ *of morphisms satisfying the conditions MRST of the proposition above is said to form an* **equivalence relation** *in* $\mathcal{C}$.

Exercise 2.11 The conditions RST can be made "effective" as follows

(R) $\Longleftrightarrow \exists A \xrightarrow{d} A'[dp_1 = 1_A$ & $dp_2]$

(S) $\Longleftrightarrow \exists A' \xrightarrow{s} A'[sp_1 = p_2$ & $sp_2 = p_1$ & $ss = 1_{A'}]$

If the pullback $A'' \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A'$ of p_2 by p_1 exists, (so in particular $\alpha p_2 = \beta p_1$), then

(T) $\Longleftrightarrow \exists A'' \xrightarrow{\gamma} A'[\gamma p_1 = \alpha p_1$ & $\gamma p_2 = \beta p_2$

Now the pullback of a morphism by itself is often called its kernel pair, and we have seen that the kernel pair of any morphism is an equivalence relation. However, in general, even if all pullbacks and coequalizers exist, the kernel pair of the coequalizer of an equivalence relation p_1, p_2 may be "bigger" than p; a good example of this phenomenon is the pair

$$\mathbf{Z} \times \mathbf{Z} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{Z} \ \text{ where } \ \begin{array}{l} p_1(x,k) = x \\ p_2(x,k) = x + kn \end{array}$$

in the category of *torsion-free* abelian groups. Thus we are led to the second exactness property alluded to above:

Definition: *A regular category $\mathcal{C}$ is called an exact category if the coequalizer of any equivalence-relation-pair exists and if any equivalence relation is the kernel pair of some morphism.*

Exercise 2.12 In any exact category a diagram

$$\underset{p_2}{\overset{p_1}{\rightrightarrows}} \xrightarrow{f}$$

with the properties that p_1, p_2 is an equivalence relation with f as coequalizer also is a kernel pair diagram. (It is thus a "non abelian short exact sequence").

The virtue of exact categories stems from the fact that the *necessary* condition for a pair

$$\cdot \rightrightarrows A$$

to be a kernel pair of some $A \to ?$, namely that it should be an equivalence relation (i.e. MRST), is also *sufficient*, and that this condition is entirely in terms of maps with *codomain* A. We will return to exact categories later, but for the next few pages we will discuss some implications of a virtue of *regular* categories which is in some sense dual to the virtue of exact categories just mentioned; namely, a necessary condition entirely in terms of maps with domain A for a given $A \to ?$ to be a *coequalizer* of some pair $\cdot \rightrightarrows A$, which for regular categories is also sufficient. First we will further clarify some properties of subclasses of a category which "generate" the whole.

Proposition 2.11: *If a class $\mathcal{U}$ of objects weakly generates a category $\mathcal{C}$, then any morphism which is both $\mathcal{U}$-surjective and also a regular mono, is in fact an isomorphism.*

Proof: Suppose j is an equalizer of f_1, f_2. If $f_1 \neq f_2$ then there is an object U in $\mathcal{U}$ and an element y defined over U with $yf_1 \neq yf_2$. But if j is also $\mathcal{U}$-surjective, then there is an x with $xj = y$, and so $yf_1 = (xj)f_1 = x(jf_1) = x(jf_2) = yf_2$; hence actually $f_1 = f_2$. But then j, being the equalizer of two equal maps, is an isomorphism, as was to be shown.

$$\begin{tikzcd} & U \arrow[dl, dashed, "x"'] \arrow[dr, "y"] & & \\ \cdot \arrow[rr, "j"'] & & \cdot \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \cdot \end{tikzcd}$$

Corollary: *If every monomorphism in $\mathcal{C}$ happens to be regular, then any weak generating class is also a strong generating class.*

Remark: We will see that in a topos every mono is indeed regular (and that equalizers exist); hence for a topos a weak generating class is the same as a strong one. In most categories for which we are interested in generating classes, equalizers exist; but we will be interested in many categories for which *not* every monomorphism is an equalizer. Hence "in general" strong implies weak, but not conversely, for generating classes.

Fact: In the category of topological spaces and continuous mappings, the one-point space weakly generates (since monos are just injective mappings on the underlying sets), but does not strongly generate (since a continuous bijection need not be a homomorphism; consider, e.g. for any non-discrete space the map to it from the associated discrete space). "*Explanation*" of this fact: not every monomorphism is regular (an equalizer tends to have the *subspace* topology).

Now we will discuss to what extent an epimorphism is $\mathcal{U}$-surjective. As we have seen for the example of the category of commutative rings, there is no hope of proving a general theorem to the effect that every epi is surjective; hence we limit ourselves to considering *regular* epis. One of the basic facts of sheaf theory is that the regular epis there are surjective on "stalks", but *not* on "sections"; however, there is a characterization of such maps in terms of a somewhat more complicated "surjective"property in which there intervenes a "covering" of the original open set. This basic theorem of sheaf theory can in fact be proved for any *regular* category, which we do below.

Definition: *Let U be an object in a category $\mathcal{C}$ and let $\mathcal{T}$ be a class of morphisms all having codomain U (but various codomains). Say that $\mathcal{T}$* **covers** *U iff for* **any proper subject** *of U, there is an "element" in $\mathcal{T}$ which is* **not** *a member of the subobject, i.e. for any $W \xrightarrow{j} U$ which is a monomorphism, but not an isomorphism, there exists a morphism t in $\mathcal{T}$ for which $t \notin j$ i.e. t does* not *factor as $t = sj$.*

Remark: If $\mathcal{C}$ has equalizers and $\mathcal{T}$ covers U, then $\mathcal{T}$ is *jointly epic* i.e. for any $U \underset{y_2}{\overset{y_1}{\rightrightarrows}} Y$, if $ty_1 = ty_2$ for all t in $\mathcal{T}$, then $y_1 = y_2$. Now our previous simple idea of a map f being $\mathcal{U}$-surjective involved the existence of liftings x for any given y in a diagram

$$\begin{tikzcd} & {} \arrow[d, "f"] \\ U \arrow[ur, dashed, "x"] \arrow[r, "y"'] & {} \end{tikzcd}$$

with U in $\mathcal{U}$. However, as remarked above, in sheaf theory this is too simple. This is also true in ordinary predicate logic, since most first-order theories do not have "Skolem functions" and are not even "rich in constants"; explicitly, the *truth* of

$$\exists x[xf = y]$$

for a given definable f and definable element y does *not* imply the existence of a *definable* element x which f maps to y. But the idea of passing to coverings works also in logic, which is one interpretation of Henkin's proof of the completeness theorem. First we consider changing U along a map t and then looking for "partial liftings modulo t"

$$\begin{tikzcd} U' \arrow[r, dashed, "x"] \arrow[d, "t"'] & A \arrow[d, "y"] \\ U \arrow[r, "y"'] & B \end{tikzcd}$$

However, for given f and y the mere existence of one such commutative square says nothing; for example, we might have $U' = 0$. We have to demand the existence of many such commutative squares.

Definition: *Let $\mathcal{U}$ be a class of objects in a category $\mathcal{C}$ and let f and y be any two morphisms with a common codomain. Denote by $\mathcal{U}(f, y)$ the class y of all those morphisms t in C having the following two properties*

l) domain of t is an object of $\mathcal{U}$

2) there exists a morphism x for which

$$xf = ty$$

(in particular, the codomain of any t in $\mathcal{U}(f, y)$ is the same object as the domain of y). The sheaf-theoretic interpretation is that t is the passage to a "smaller open set", ty is the restriction of y to a smaller open set, and $\mathcal{U}(f, y)$ is the class of all those smaller open sets on which y becomes a value of f. One logical interpretation is that t is the passage to a "deeper stage of knowledge" and that $\mathcal{U}(f, y)$ is the class of all those deeper stages of knowledge for which we know how to construct an element of A which f maps to the known element y. (The latter interpretation (as well as many purely mathematical examples, e.g. from algebraic geometry) shows that we should not *restrict the morphisms*

t a priori to be e.g. epimorphisms. Two elements y_1, y_2 which are known at a certain stage U may not *be known to be* equal *until a* deeper *stage of knowledge is reached!)*

Theorem: *Let $\mathcal{C}$ be a regular category and let $\mathcal{U}$ be a strong generating class for $\mathcal{C}$. Then a morphism f is a regular epimorphism iff for any $U \xrightarrow{y} B$ with U in $\mathcal{U}$, the class $\mathcal{U}(f, y)$ covers U.*

Proof: Suppose $A \xrightarrow{f} B$ is a regular epimorphism and that $U \xrightarrow{f} B$. Form the *pullback*

$$\begin{tikzcd} A' \arrow[r, "q"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ U \arrow[r, "y"'] & B \end{tikzcd}$$

in which we know that f' is also a regular epimorphism.

Now let $W \overset{j}{\rightarrowtail} U$ be any subobject and consider *its* pullback by f'.

$$\begin{tikzcd} A'' \arrow[r, "j'"] \arrow[d, "f''"'] & A' \arrow[r, "a"] \arrow[d, "f'"] & A \arrow[d, "f"] \\ W \arrow[r, tail, "j"'] & U \arrow[r, "y"'] & B \end{tikzcd}$$

Then of course j' is also a monomorphism, but notice that

Lemma: *In a regular category, if the pullback of a mono j by a regular epi f' is iso, then j itself is iso.*

Hence, if j is not an isomorphism, then j' is not an isomorphism either, and therefore, since $\mathcal{U}$ strongly generates, there exists U' in $\mathcal{U}$ and $U' \xrightarrow{a'} A'$ such that $a' \notin j'$. Then we can define $t = a'f'$ and $x = a'a$, and since the above diagram commutes, we have that $xf = a'af = a'f'y = ty$, proving that t belongs to $U(f, y)$. If $t \in W$, then $t = sj$ for some j and hence

$$a'f' = sj$$

and, therefore, since the left square is a pullback, we have $a' \in j'$, contracting the choice of a'. Thus $t \notin j$. Thus for every mono-noniso j we have constructed *at* in $\mathcal{U}(f, y)$ with $t \notin j$; in other words, $\mathcal{U}(f, y)$ covers U, as was to be shown, except for the

Proof of the Lemma: Suppose

$$\begin{tikzcd} A'' \arrow[r, phantom, "="] \arrow[d, "f''"'] & A' \arrow[d, "f'"] \\ W \arrow[r, "j"'] & U \end{tikzcd}$$

is a pullback diagram with the right side f' a regular epi and the top an isomorphism (which without loss of generality we may assume is the identity - why?) We must show that if j is a monomorphism, then j is also an isomorphism. We know that f' is the coequalizer of its kernel pair $K \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A'$. Then $\pi_1 f'' = \pi_2 f f''$, since by our supposition we may cancel j. This implies that there is a unique $U \xrightarrow{k} W$ for which $f'' = f'k$. Then we have

$$f'(kj) = (f'k)j = f''j = f' = f'.idU$$

from which

$$kj = id_U$$

since f' is certainly an epimorphism. On the other hand,

$$f''(jk) = (f''j)k = f'k = f'' = f''.id_W$$

But using the regularity of the category, f'' is a regular epimorphism, being the pullback along j of a regular epimorphism f'; hence in particular f'' is an epimorphism so we may cancel in the last calculation to obtain

$$jk = id_W$$

Thus $k = j^{-1}$, so that j is an isomorphism, completing the proof.

We must still prove the converse part of the theorem. Thus we assume again that U strongly generates regular C, but now that f has the property that for all y defined over $U \in U, U(f, y)$ covers the domain of y: we must show that f is a regular epimorphism. In any case f can be factored $f = pi$ with p regular epi and i mono; we must show that i is an isomorphism. It suffices to show that i is U-surjective, by definition of the fact that U strongly generates, so let $U \xrightarrow{y} B$ be an element defined over $U \in U$ of the codomain of i and consider the pullback j of i along y. If j is an isomorphism, then y lifts along i, so we would be done; if j is not an isomorphism, then since by hypothesis $U(f, y)$ covers U, there is $U' \in U$ with t, x such that $ty = xf$. But $f = pi$ so $ty = (xp)i$. Thus j being a pullback, we have $t \in j$. Therefore i is an isomorphism, hence f a regular epimorphism, completing the proof of the theorem.

Example: Let U be the category whose objects are the open subsets of the complex plane and whose morphisms $U' \xrightarrow{t} U$ are just *inclusions* of open sets. Let A denote the (set or additive group of) all complex numbers and let B be the (set or multiplicative group of) all non-zero complex numbers. Let the morphisms $U' \xrightarrow{x} A$ be precisely the analytic functions defined in U' and let the morphisms $U \xrightarrow{y} B$ be precisely the functions analytic and non-vanishing in U. Suppose that among the morphisms $A \xrightarrow{f} B$ there is $f = \exp$, complex exponentiation, so that xf is the nonvanishing analytic function

$$z \rightsquigarrow e^{x(z)} \text{ for } z \in U'$$

There is actually a regular category containing all this, namely the usual topos of sheaves of sets on C. Thus in this example the question of the surjectivity of f is the question of the existence of the logarithm $\log(y)$. Now there is a shallow stage of our knowledge of complex functions where the logarithm does not exist, but a deeper stage where it does, since after all f is regular epi even though for a given U, y the logarithm of y will not exist as a single-valued function, yet on a fine enough covering of U a choice of $\log(y)$ can be made defined over each set U' in the covering.

As examples of regular categories, we may mention sets, groups, rings, lattice, A-modules for any ring A, compact (=quasicompact and separated) spaces, sheaves on X for any space X, any abelian category, any topos; also the opposite category (A-modules)^{op}, the opposite category (compact spaces)^{op} ($= C^*$-algebras commutative with 1), and the opposite category $\mathcal{E}^{op}$ for any topos $\mathcal{E}$. On the other hand, the category of all topological spaces is *not* regular (an example was given in class by one of the auditors). We outline below the proof that (Grp)^{op} and Ann^{op} (where Ann is the regular category of commutative associative rings with 1) are not regular. Remark that all of Groups, Rings, Ann, Lattices, A-Modules, Compact Spaces, Lie Algebras are regular categories because regular epis are just surjective homomorphisms in those cases (as may be seen from the usual theory of congruence relations) and there is a regular projective generator. Thus it will follow from the non-regularity of (Grp)^{op} and Ann^{op} that neither Grp nor Ann has a (regular) injective cogenerator.

Grp^{op} is not regular because

(1) All monomorphisms are regular and all epimorphisms are regular (Exercise). (Hint due to Eilenberg: Reduce the problem to showing that if $H \subseteq G$ is a subgroup with the property that G itself is the only *normal* subgroup of G containing H, then $H = G$. Divide that problem into two cases by considering the G-set of cosets G/H and showing that either assumption card $(G/H) = 2$ or card $(G/H) > 2$ both lead to inconsistency). Remark that a less elementary proof of this could be based on Kurosh's theorem concerning amalgamated free products.
(2) There exists a monomorphism $G' \rightarrowtail G$ where G' is not a simple group, but G is a simple group. (One example suffices here (see below), but in fact there is a theorem that *any* group G' can be embedded monomorphically into a simple group, in marked *contrast* to the category of *abelian* groups, where

there is only a small set of simple objects.)
(3) For any regular category, if the opposite category is also regular, then any nontrivial regular subobject of a simple object is again simple.

Proof: Here by a trivial object we mean a subobject of the terminal object, and by a simple object S we mean a nontrivial one with the property that for any regular epimorphism $S \to S''$, either S'' is trivial, or the given map $S \to S''$ is actually an isomorphism. Now suppose $S' \rightarrowtail S$ is a regular mono with S' nontrivial and S simple. To show that S' is also simple, suppose $S' \twoheadrightarrow S''$ is any regular epi and consider the pushout

$$\begin{array}{ccc} S' & \twoheadrightarrow & S'' \\ \downarrow & & \downarrow \\ S & \twoheadrightarrow & S''' \end{array}$$

Now, $S \to S'''$ is in any case a regular epi (and by the coregularity hypothesis on the category, $S'' \rightarrowtail S'''$ is a regular mono), hence, since S is simple, S''' is either trivial, in which case S'' is also trivial, or else $S \twoheadrightarrow S'''$ is an isomorphism, which clearly implies that $S' \to S''$ is a monomorphism and hence also an isomorphism (since by hypothesis it was regular epi, and we have previously seen that reg epi and mono $\Longrightarrow$ iso). Thus any $S' \twoheadrightarrow S''$ satisfies the alternative, i.e. S' is again simple. ∎

(Ann)^{op} is not regular. Here the above argument will not work, since the conclusion of (3) is in fact true for Ann: a simple object is a *field*, and of course a subobject $D \rightarrowtail K$ of a field in general is an integral domain; however, it is easy to see that if the inclusion $D \rightarrowtail K$ is in fact the equalizer of two ring homomorphisms $K \rightrightarrows A$, then D is also a field. Thus we need another method. For that, recall that a *flat* morphism $A \to A'$ of commutative rings means one having the property that for any monomorphic A-linear map $X \rightarrowtail Y$ of A-modules, the induced A'-linear map $A' \underset{A}{\otimes} X \to A' \underset{A}{\otimes} Y$ is also a monomorphism. To make use of this definition, we need a method for translating the study of modules into the study of certain rings. There is in fact such a method which is actually useful in many different contexts. Namely, given any A-module X, consider the direct sum $A \oplus X$ as A-modules and on it define a multiplication by

$$< a, x > \cdot < b, y > = < ab, ay + bx >$$

in other words, X becomes an ideal in the A-algebra $A \oplus X$ with the property that the product of any two elements of X is zero. In particular, if $X = A$, then $A \oplus X = A[d]$ = the ring of "dual numbers" over A, which is the A-algebra obtained by adjoining to A one indeterminate d with the condition

$d^2 = 0$. If $X \to Y$ is any A-linear map, then the induced $A \oplus X \to A \oplus Y$ is a homomorphism of A-algebras, and in particular

! if $X \rightarrowtail Y$ is a monomorphic A-linear map of A-modules, then $A \oplus X \to A \oplus Y$ is a regular monomorphism of commutative rings; in fact, the latter is the equalizer of the two ring homomorphisms $A \oplus Y \rightrightarrows A \oplus (Y/X)$ induced by the quotient map and the zero map $Y \rightrightarrows Y/X$!

Now, in general, pushouts in Ann of $A \to B$ along $A \to A'$ are computed as $A' \otimes_A B$ and, in particular, the inclusion $A \rightarrowtail B$ of a subring is a regular monomorphism iff given any element b in B, if $1 \otimes b = b \otimes 1$ in the ring $B \otimes_A B$, then $b \in A$. Now the last condition seems to lead to somewhat complicated conditions (?) in terms of the subring $A \rightarrowtail B$ itself, but fortunately we only need the "less concrete" definition $A \rightarrowtail B$ is a regular monomorphism of commutative rings iff there is some commutative ring C and some pair $B \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} C$ of ring homomorphism (preserving 1 of course), such that for all b, $b \in A$ iff $b\varphi_1 = b\varphi_2$.

(1) A morphism $A \to A'$ of commutative rings is flat iff for any *regular monomorphism* $B_1 \rightarrowtail B_2$ of A-algebras (i.e. $B_1 \rightarrowtail B_2$ is an equalizer in Ann and fits into a given commutative triangle

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B_1 \arrow[rr, tail] & & B_2 \end{tikzcd}$$

in Ann), we have that $A' \otimes_A B_1 \to A' \otimes_A B_2$ is at least a *monomorphism.*

Proof: Suppose $A \to A'$ is not flat. Then, since any equalizer $B_1 \rightarrowtail B_2$ of A-algebras is in particular an equalizer of A-modules, the condition in (1) holds. Conversely, if $A \to A'$ is not flat, there is some inclusion $X_1 \rightarrowtail X_2$ of A-modules which does not remain monomorphic upon extending scalars to A' by tensoring. But then by taking $B_i = A \oplus X_i \quad i = 1, 2$ we obtain a counterexample to our condition, noticing that

$$A' \oplus_A (A \oplus X_i) \cong (A' \oplus_A A) \oplus (A' \oplus_A X_i) \cong A' \oplus (A' \oplus_A X_i)$$

is again such a "generalized algebra of dual numbers", but over A', with maps compatible with the identifications. ∎

(2) In a regular category, the pullback along any $S' \to S$ of a commutative

triangle

$$\begin{tikzcd} W \arrow[rr, twoheadrightarrow] \arrow[dr] & & T \arrow[dl] \\ & S & \end{tikzcd}$$

where the upper map is a regular epi but the legs arbitrary, is again a triangle

$$\begin{tikzcd} W' \arrow[rr, twoheadrightarrow] \arrow[dr] & & T' \arrow[dl] \\ & S' & \end{tikzcd}$$

with the same property over S', as is easily seen by factoring $T \to S$ into regular epi followed by mono and noting that pulling back in three steps $W \twoheadrightarrow T \twoheadrightarrow I \rightarrowtail S$ along $S' \to S$, $T' \to I$, and $T' \to T$ is equivalent to pulling back in one step $W \to S$ along $S' \to S$. Thus by dualizing, we see that in Ann, the following holds.

(3) If $A \to A'$ is any ring homomorphism which is not flat and if $X_1 \rightarrowtail X_2$ is any monomorphic A-linear map of A-modules for which $X_1' \to X_2'$ is *not* monomorphic, where $X_i' = A' \underset{A}{\otimes} X_i$, then if we form the "trivial extensions"

$$A_i = A \oplus X_i$$

$$A_i' = A' \oplus X_i'$$

discussed above, we obtain a *pushout* diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_1' \arrow[d] \\ A_2 \arrow[r] & A_2' \end{tikzcd}$$

in Ann where the left side is a *regular monomorphism* but the right side is *not* even a *monomorphism.* For example, if in a ring K we have an element t which is neither invertible, nor a zero divisor, then taking $L = K/_{tK}$ we get a pushout diagram

$$\begin{tikzcd} K[d] \arrow[r] \arrow[d] & L[d] \arrow[d] \\ K[d] \arrow[r] & L[d] \end{tikzcd}$$

by taking the horizontal maps to be the one induced, the left vertical one determined by $d \rightsquigarrow td$ and the right vertical one determined by $d \rightsquigarrow 0$.

Again, the left is regular mono, but the right not even mono. Perhaps the simplest example is $K = \mathbf{Z}, t = 2$. Then our counterexample diagram is

$$\begin{tikzcd} \mathbf{Z} \times \mathbf{Z} \arrow[r] \arrow[d, tail, "id\times 2"'] & \mathbf{Z}_2 \times \mathbf{Z}_2 \arrow[d, "id\times 0"] \\ \mathbf{Z} \times \mathbf{Z} \arrow[r] & \mathbf{Z}_2 \times \mathbf{Z}_2 \end{tikzcd}$$

as abelian groups (of course *as rings* they are not direct products, rather the multiplication in all four corners is such that $<0,1>^2 = <0,0>$). To be sure that the non-regularity also occurs for Ann_k^{op}, where Ann_k is the category of commutative algebras over a field k, take $K = k[t]$ the polynomial ring. Then our counterexample diagram is the diagram of $k[t]$-algebras

$$\begin{tikzcd} d \arrow[d, mapsto] & k[t][d] \arrow[r, "t\to 0"] \arrow[d] & k[d] \arrow[d] & d \arrow[d, mapsto] \\ dt & k[t][d] \arrow[r, "t\to 0"'] & k[d] & 0 \end{tikzcd}$$

* As remarked above, the construction of the commutative ring $A[X] = A \oplus X$ where A is a commutative ring and X is an A-module is important in many contexts. Note that $B = A[X]$ has a *structural map* $B \xrightarrow{p} A$ (kill X) which is a ring homomorphism and has a section ζ which is also a ring homomorphism. Moreover, the pullback of p with itself

$$B \underset{A}{\times} B \cong A[X \oplus X]$$

has a canonical ring homomorphism

$$B \underset{A}{\times} B \xrightarrow{\alpha} B$$

(add elements of X) which commutes with the structural maps and for which ζ is a *two-sided unit* i.e.

$$\begin{tikzcd} B \arrow[r, "<id{,}\zeta>"] \arrow[d, "<\zeta{,}id>"'] \arrow[dr, "id"] & B \underset{A}{\times} B \arrow[d, "\alpha"] \\ B \underset{A}{\times} B \arrow[r, "\alpha"'] & B \end{tikzcd}$$

commutes (over A, now in the sense of Ann, not in the usual ring theoretic sense of Ann^{op}). *Conversely*, if $B \xrightarrow{p} A$ is *any* ring over A with a given section ζ and given "binary operation over A" α for which ζ is a two-sided unit, it follows that $B = A \oplus X$ where X is the (ideal) kernel of p *and* that α agrees with the addition inside B. Thus, *in particular, α is commutative, associative (on each fiber of p) and has an "inverse"* (negation in X); otherwise said, the object B, p of Ann$/A$ together with the morphism ζ, α, of Ann$/A$ determine an *abelian group object* in the category Ann$/A$. Thus we have an equivalence of categories

$$Ab(\mathrm{Ann}/A) \cong A - \mathrm{Mod}$$

More generally, if $\mathcal{C}$ is any category with a terminal object 1 and cartesian products $\times$, we can define the notion of an abelian group object $< C, \zeta, \alpha, \theta >$ in $\mathcal{C}$ to be an object C together with $\mathcal{C}$-morphisms $1 \xrightarrow{\zeta} C$, $C \times C \xrightarrow{\alpha} C$, $C \xrightarrow{\theta} C$ satisfying the commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "C"] \arrow[d, "f\times C"'] \arrow[dr, phantom, "i_d"] & C \times C \arrow[d, "\alpha"] \\
C \times C \arrow[r, "\alpha"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
(C \times C) \times (C \times C) \arrow[r, "\sim", "\mu"'] \arrow[d, "\alpha\times\alpha"'] & (C \times C) \times (C \times C) \arrow[d, "\alpha\times\alpha"] \\
C \times C \arrow[r, "\alpha"'] & C \times C
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "<C,\theta>"'] & 1 \arrow[d, "\zeta"] \\
C \times C \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

where μ is the canonical isomorphism interchanging the two middle factors. [There is actually a very general lemma to the effect that if the objects of $\mathcal{C}$ already have a group structure (not necessarily commutative), then the last two diagrams follow from the first and ζ, α, θ are uniquely determined by C; this same lemma implies that the fundamental group $\Pi_1(X)$ of an H-space X (e.g. a topological group) is always commutative, that the addition in the Lie algebra Lie (G) of an algebraic group G agrees with the "infinitesimal multiplication" in $G[d]$, and also the fact stated above for $\mathcal{C} = \mathrm{Ann}/A$ that α agrees with addition in X. The exact statement of the lemma is as follows: Suppose that on an abstract set M there are given two binary operations o and $*$ each having a *two-sided neutral element* e_o, e_*. Now any sort of algebraic structure on M induces a "coordinate-wise" structure on $M \times M$; in particular, $M \times M$ has itself a binary operation $(M \times M) \times (M \times M) \xrightarrow{(*)} M \times M$ induced by (say) $*$. Suppose that o is a *homomorphism* with respect to $(*)^{(2)}, *$, i.e. that

$$\begin{tikzcd}
M \times M \times M \times M \arrow[r, "o \times o"] \arrow[d, "\mu"'] \arrow[dd, bend right=60, "(*)^{(2)}"'] & M \times M \arrow[dd, "*"] \\
M \times M \times M \times M \arrow[d, "* \times *"'] & \\
M \times M \arrow[r, "o"'] & M
\end{tikzcd}$$

commutes (since $\mu = \mu^{-1}$, it is the same to say that $*$ is a homomorphism with respect to $o^{(2)}, o$). Then $*$ and o are actually the same mapping and moreover, they are both associative and commutative.]

If $\mathcal{E}$ is any category with pullbacks, then for each A in $\mathcal{E}$, $\mathcal{E}/A$ has a terminal object and binary cartesian products (and conversely!) Thus we are led to define the category

$$A - \mathrm{Mod}(\mathcal{E}) = Ab(\mathcal{E}/A)$$

where the notion of homomorphisms of abelian group objects in a category $\mathcal{C} = \mathcal{E}/A$ is obvious. There is a theorem of Tierney (proof in Barr's book SLN 236) that if $\mathcal{E}$ is a regular category with effective equivalence relations, then for any A in $\mathcal{E}$, A-Mod as just defined is an *abelian* category. Somewhat more surprisingly, there is the fact (exploited by Jon Beck in his Columbia University thesis on cohomology theories) that the definition is correct! That is, for many categories $\mathcal{E}$, the category A-Mod just defined is really (equivalent to) the category of A modules for A in $\mathcal{E}$ in the usual, reasonable sense. For example, if $\mathcal{E} =$ Grp , calculate (again by taking kernel of structural projection) that a G-module is really a G-module. Similarly for $\mathcal{E} =$ Lie algebras, $\mathcal{E} =$ Jordan algebras, etc. *

2. Finite (inverse or projective) limits in a category $\mathcal{E}$

By *empty* product we just mean a terminal object 1. By binary (cartesian) product we mean a *right* adjoint to the diagonal functor

$$\mathcal{E} \underset{\triangle}{\overset{\times}{\leftrightarrows}} \mathcal{E} \times \mathcal{E}$$

i.e. a natural bijection of "sets" of morphisms

$$\frac{X \to Y_1 \times Y_2}{X \to Y_1, \quad X \to Y_2}$$

mediated by natural adjunction morphisms

$$Y_1 \times Y_2 \xrightarrow{\pi_i} Y_i$$

$$X \xrightarrow{\delta} X \times X$$

called projections and diagonal morphisms respectively. The unique morphism $X \xrightarrow{f} Y_1 \times Y_2$ for which $f\pi_i = f_i$ for given $X \xrightarrow{f_i} Y_i$ $i = 1, 2$ is denoted by $f =< f_1, f_2 >$ so that

$$< f_1, f_2 > \pi_i = f_i$$

and for any $A \xrightarrow{x} X$

$$x < f_1, f_2 >=< xf_1, xf_2 >$$

On the other hand, for given $X_i \xrightarrow{fi} Y_i$ $i = 1, 2$ (note that there are now *two* X's, possibly equal) the unique morphism

$$X_1 \times X_2 \to Y_1 \times Y_2$$

such that

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r] \arrow[d, "\pi_i"'] & Y_1 \times Y_2 \arrow[d, "\pi_i"] \\
X_i \arrow[r, "f_i"'] & Y_i
\end{tikzcd} \qquad i{=}1{,}2$$

is denoted by $f_1 \times f_2$, i.e.

$$f_1 \times f_2 =< \pi_i f_1, \pi_2 f_2 >$$

where the projections are those for $X_1 \times X_2 \to X_i$.

If $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ $i = 1, 2$ then

$$(f_1\, g_1) \times (f_2\, g_2) = (f_1 \times f_2)(g_1 \times g_2)$$

If we define ternary cartesian product as a right adjoint to the diagonal

$$\mathcal{E} \to \mathcal{E} \times \mathcal{E} \times \mathcal{E}$$

then we have canonical (i.e. natural, and uniquely determined by compatibility with projections) isomorphisms

$$Y_1 \times (Y_2 \times Y_3) \cong Y_1 \times Y_2 \times Y_3 \cong (Y_1 \times Y_2) \times Y_3$$

and so, in particular, "associativity" for binary cartesian product. Also

$$Y \times 1 \cong Y \cong 1 \times Y$$

Moreover, we have a natural "commutativity" isomorphism

$$Y_1 \times Y_2 \xrightarrow{s} Y_2 \times Y_1$$

determined by

$$s\pi_1 = \pi_2$$

$$s\pi_2 = \pi_1$$

so that, in particular,

$$ss = id_{Y_1 \times Y_2}$$

(by abuse of notation, since there are really two different morphisms s). But even if $Y_1 = Y_2 = Y$,

$$s \neq id$$

unless $Y \rightarrowtail 1$. In a similar vein

$$Y \xrightarrow{\delta} Y \times Y$$

is an isomorphism iff $\pi_1 = \pi_2$ iff every morphism $Y \to Z$ is a monomorphism; i.e. a "naturally idempotent" object must be a subobject of 1, although of course there may be many "unnaturally idempotent" objects, e.g. precisely the infinite sets (or 0,1) in the case $\mathcal{E} = \mathcal{S}$ (sets). Combining the associativity and commutativity isomorphisms, we get the natural isomorphism

$$(Y_1 \times Y_2) \times (Y_3 \times Y_4) \xrightarrow{\mu} (Y_1 \times Y_3) \times (Y_2 \times Y_4)$$

utilized in the previous section for discussing associative-commutative operations on an object $C = Y_1 = Y_2 = Y_3 = Y_4$ and homomorphisms between such.

For a given object S, the functor $S \times (\ )$ may be viewed as a functor

$$\mathcal{E}/S \longleftarrow \mathcal{E}$$

since there is the canonical projection $S \times X \xrightarrow{\pi_S} S$ and for any $X \xrightarrow{f} Y$,

$$\begin{tikzcd} S \times X \arrow[rr, "S \times f"] \arrow[dr, "\pi_S"'] & & S \times Y \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd}$$

commutes and hence may be viewed as a morphism in $\mathcal{E}/S$. Of course, in general not all morphisms in $\mathcal{E}/S$ are of the form $S \times f$, not even if we look only for morphisms in $\mathcal{E}/S$ between objects of the form $S \times X$; in fact, a morphism

$$S \times X \to S \times Y \text{ in } \mathcal{E}/S$$

determines and is determined by a morphism

$$S \times X \to Y \text{ in } \mathcal{E}$$

which often may be reasonably interpreted as an S-indexed family of morphisms $X \to Y$. That is certainly valid in a *cartesian closed* category, which means one with finite products in which $(\ ) \times X$ has a right adjoint, so that there is another functor Y^X and a natural bijection

$$\frac{S \times X \to Y}{S \to Y^X}$$

We will return to cartesian closed categories later [ ], but as a first approximation to the remark about families of morphisms, note that if we are given $S \times X \xrightarrow{f} Y$ then for each $1 \xrightarrow{s} S$, we can construct

$$\begin{tikzcd} X \arrow[r, "s \times X"] \arrow[dr, "f_s"'] & S \times X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

and, in particular, if 1 is a generator for $\mathcal{E}$ different $f \neq f'$ give rise to distinct families

$$X \xrightarrow{f_s} Y \quad s \in S$$

$$X \xrightarrow{f'_s} Y \quad s \in S$$

of morphisms from X to Y. On the other hand, an arbitrary family

$$X \xrightarrow{f_s} Y$$

indexed by all $1 \xrightarrow{s} S$ will not necessarily come from a single $S \times X \to Y$ even if 1 generates, since that would in general involve "smoothness" of the parameterization $s \rightsquigarrow f_s$ itself (think e.g. of topological spaces). Essentially, the same remarks apply if we consider instead of 1 a generating class $\mathcal{U}$ and consider both "indices" $U \xrightarrow{s} S$ and "arguments for f" $U \xrightarrow{x} X$ defined over $U \in \mathcal{U}$.

Proposition 2.12: *(graphs) If $\mathcal{E}$ has (binary) cartesian products, then every morphism f can be factored into a split monomorphism (=retract) γ_f followed by a projection.*

Proof: In

$$\begin{tikzcd} & X \\ X \arrow[ur, "id_X"] \arrow[r, "<X,f>"] \arrow[dr, "f"'] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

define $\gamma_f =< X, f >$.

Corollary: *A morphism $X \to Y$ is the "same thing" as a* section *s (i.e. $sp = id_X$) of the morphism $E \to X$ where $E = X \times Y, p = \pi_X$. Or again, a morphism $X \to Y$ in $\mathcal{E}$ is the "same thing" as a global element of a special object E of the category $\mathcal{E}/X$.*

Proof: By a global (or eternal) element of an object E of a category $\mathcal{E}'$ with a terminal object, we mean a morphism from that terminal object to E. But the terminal object of $\mathcal{E}' = \mathcal{E}/X$ is just the identity morphism 1_X from $\mathcal{E}$, since for *any* object $E \xrightarrow{p} X$ of $\mathcal{E}/X$ there is exactly one morphism $E \to 1_X$ in $\mathcal{E}/X$, i.e. exactly one way of completing the diagram

$$\begin{tikzcd} E \arrow[rr, dashed] \arrow[dr, "p"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

in $\mathcal{E}$, namely with p. ∎

Remark: Thus a global element of an arbitrary object in $\mathcal{E}/X$, i.e. a (global) section of an arbitrary given morphism $E \xrightarrow{p} X$ in $\mathcal{E}$ is a natural *generalization* of the notion of a morphism $X \to$? in $\mathcal{E}$, a point of view much used in topology, differential equations, etc. It is actually also used in logic, where we may consider terms that are only definable after adjoining "new constants of type X" - for more on that see the proposition below. Note that the generalization has *two* aspects, the one already discussed (*families* of maps instead of one, new constants, etc.) and also the fact that E in $\mathcal{E}/X$ is more general than a "constant" object $X \times Y$. We might also ask how much more general is this second aspect, i.e. what characteristic properties (or structure) will a morphism $E \xrightarrow{p} X$ have if it happens to be the projection from a product $E = X\times$? ? The "constancy" can be expressed as follows: Consider any "element" $A \xrightarrow{x} X$ and the "fiber of p over x", i.e. the pullback (assume it exists)

$$\begin{tikzcd} E_X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "x"'] & X \end{tikzcd}$$

Now a diagram of the form

$$\begin{tikzcd} A \times Y \arrow[r, "x \times Y"] \arrow[d, "\pi_A"'] & X \times Y \arrow[d] \\ A \arrow[r, "x"'] & X \end{tikzcd}$$

is *always* a pullback (in any category with products) (Exercise). Thus if $E = X \times Y, p = \pi_X$, we must have a canonical isomorphism $E_x \cong A \times Y$. Since the right hand side is independent of x, we can for a given pair $A \underset{x_2}{\overset{x_1}{\rightrightarrows}} X$ compose one canonical isomorphism with the inverse of the other to obtain

$$E_{x_1} \xrightarrow{\theta^A_{x_1,x_2}} E_{x_2}$$

a family of isomorphisms, which is now a structure on E, p which no longer mentions Y as desired. This family of isomorphisms is not wholly arbitrary, but, by uniqueness of morphisms into pullbacks, must satisfy the coherence conditions (sometimes called "cocycle conditions")

$$\theta^A_{x,x} = id_{E_x} \quad \text{for } A \xrightarrow{x} X$$

$$\theta^A_{x_1,x_2}\theta^A_{x_2,x_3} = \theta^A_{x_1,x_3} \quad \text{for } A \overset{x_1, x_2, x_3}{\rightrightarrows} X$$

(which clearly imply $\theta^A_{x_2,x_1} = \left(\theta^A_{x_1,x_2}\right)^{-1}$).

Moreover, there is a compatibility condition for arbitrary change of A, $A' \xrightarrow{a} A$; in fact, we introduced various A only to enable us to think of "elements" - there is actually a universal choice of A, namely $A = X \times X$ and two canonical choices of x, namely

$$X \times X \xrightarrow{x=\pi_i} X \quad i = 1, 2$$

There are thus two canonical ways of pulling the given object of $\mathcal{E}/X$ back into $\mathcal{E}/X \times X$, and if the given object happened to come from $\mathcal{E}$ by the functor $X \times (\ )$, then between the two resulting objects E_1, E_2 of $\mathcal{E}/X \times X$ there will be one isomorphism $\theta_{1\,2}$ (and its inverse $\theta_{2\,1}$) satisfying one commutative triangle which can be considered to live in $\mathcal{E}/X \times X \times X$. Such an isomorphism θ is called *descent data* for an object E, p of $\mathcal{E}/X$; such is thus seen to be necessary for "descending" from $\mathcal{E}/X$ down to $\mathcal{E}$; we need data, not just a property of p, since in general there may be several non isomorphic Y giving isomorphic $X \times Y$, even when the latter is considered as an object over X with fixed projection. Descent data may also be considered as a "groupoid object" in the sense to be explained presently, or again simply as an equivalence relation on E which agrees well with the total equivalence relation $X \times X \rightrightarrows X$ on X. If $\mathcal{E}$ is a regular category with effective equivalence relations and if $X \to 1$ is a *regular epimorphism*, then one can show a one-to-one (suitably functorial) equivalence between objects Y of $\mathcal{E}$ and objects-over X-equipped-with-descent-data $< E, p, \theta >$. These questions are classified by "nonabelian cohomology" and are of special interest for the category $\mathcal{E} = Ann_k^{op}$, where the question becomes: given algebras (or modules, or quadratic forms, or Lie algebras...) over a commutative k-algebra X, do they come by extension $X \underset{k}{\otimes}(\ )$ of scalars from similar structures defined over k itself? If so, in how many ways, etc?

Noting that

$$\mathcal{E}/S \xleftarrow{S\times(\,)} \mathcal{E}$$

is, under the identification $\mathcal{E} \cong \mathcal{E}/1$, actually the pullback along the unique $S \to 1$, we see that the whole discussion above for an *object* $S = X$ in $\mathcal{E}$ can be generalized or relativised to a discussion for a *morphism* $S \to T$ in a category $\mathcal{F}$ with pullbacks by taking $\mathcal{E} = \mathcal{F}/T$ whereupon $\mathcal{E}/S = (\mathcal{F}/T)/S \cong \mathcal{F}/S$ the last equivalence just being between two different interpretations of the role of the given $S \to T$. Similarly, the following considerations can automatically be relativised to the consideration of "relative infinite products along a map" on a topos or compactly generated Hausdorff spaces or to "Weil restrictions" for Ann^{op}.

A section of $E \xrightarrow{p} X$ may be thought of also as an element of an infinite product $\prod_{x\in X} E_x$. Let us denote the "set" of such sections by $\Gamma_X(E)$ (it also can be considered as the set of global elements of an object in $\mathcal{E}/X$).

Exercise 2.13 If $\mathcal{E}$ has pullbacks, $X \xrightarrow{f} Y$, then for any E in $\mathcal{E}/Y$, if we denote its pullback along f by $f^*(E)$ in $\mathcal{E}/Y$, there is an induced mapping

$$\Gamma_Y(E) \xrightarrow{\Gamma_f} \Gamma_X(f^*(E))$$

(just consider the diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"'] \arrow[u, bend left, dashed, "s'"] & Y \arrow[u, bend left, "s"] \end{tikzcd}$$

and use the universal mapping property of pullbacks to construct s' given s) which is functorial with respect to composition of f, g. (This exercise is a first step toward an "element-free" version of an exercise from Lesson 1 [ ]. The further step will be made, and the relation of $\mathcal{E}$ to "sets" both clarified and also greatly generalized, when we consider $\mathcal{E}$ which is *based* in a suitable sense on another category-with-pullbacks denoted by $\mathcal{S}$.

It is a classical experience in the case $\mathcal{E} = Ann_k^{op} = Ann^{op}/k$ that in passing to an "extension ring" $k \to k'$ we obtain "actual elements" of kinds that were only "potentially" there before (e.g. $i^2 = -1, d^2 = 0, 1/f$ etc). Of course $k' \underset{k}{\otimes} (\ )$, the "passage", is in the opposite category just pulling back. Also in logic, it may be possible to prove in a theory $\mathcal{E}$ the existence of elements satisfying a formula X without being able to construct explicitly such an element, but this can be formally remedied by passing to a stronger theory $\mathcal{E}/X$ in which we "postulate X", i.e. we consider the variable in the formula X to be a new constant and take X itself as a new axiom. Under the assumption

$$\mathcal{E} \vdash \exists x X(x)$$

which we have made, we can show that the theory $\mathcal{E}/X$ is a faithful or "conservative" extension of the theory $\mathcal{E}$ in the sense that no formula of $\mathcal{E}/X$ which comes from $\mathcal{E}$ (i.e. which does not mention the new constant) can be proved in $\mathcal{E}/X$ unless it could already be proved in E. This theorem of logic is actually a special case of the following proposition: The assumption $\mathcal{E} \vdash \exists x X(x)$ is translated in our more general setting into the assumption that $X \to 1$ is a regular epimorphism; thus in the ring example mentioned above, it typically means that $k \to k'$ is a faithfully flat extension and hence only applies to the case $1/f$ for $f \in k$ non degenerate. (I say "typically" because there are actually several standard ways of enlarging Ann_k^{op} to a regular category, "faithfully flat" being the slogan for one of them.)

Proposition 2.13: *Let X be an object in a category $\mathcal{E}$ with products. Then after X is pulled back to $\mathcal{E}/X$, it has a global element δ_X which arises canonically from the construction. The pulling back functor*

$$\mathcal{E}/X \xleftarrow{X \times (\ )} \mathcal{E}$$

preserves terminal objects and pullbacks and equalizers if they exist, and preserves regular epimorphisms if $\mathcal{E}$ is a regular category; it also preserves any *coproducts or coequalizers which may exist in $\mathcal{E}$ (indeed if $\mathcal{E}$ is a topos, it also preserves the "power set" operation, function spaces, and truth-value objects as we shall see later [ ], so that in general the pulling back functor preserves whatever "logical structure" which may exist). If $\mathcal{E}$ is a regular category and if $X \longrightarrow\!\!\!\triangleright\, 1$ is a regular epimorphism, then the pulling back functor is faithful and reflects isomorphisms. (The latter property is called "conservative" in category theory, so that fortunately in this case categorical and logical terminology agree.)*

Proof: The pullback of X itself along $X \to 1$ is just $X \times X \xrightarrow{\pi_1} X$, and as the notation in the statement suggests, the diagonal morphism δ_X (=graph of the identity morphism if you will) is the section of π_1, i.e. the "global element" in $\mathcal{E}/X$ of X pulled back over X which arises canonically from the construction. Pullbacks, equalizers, and any sort of colimit are all the same in $\mathcal{E}/X$ as in $\mathcal{E}$; products and terminal objects are different, but $X \times (\ )$ transforms the old ones into the new ones, i.e.

$$\begin{tikzcd}
 & X \times (Y_1 \times Y_2) \arrow[dl] \arrow[dr] & \\
X \times Y_1 \arrow[dr] & & X \times Y_2 \arrow[dl] \\
 & X &
\end{tikzcd}$$

is a pullback in any $\mathcal{E}$ with products, but pullback in $\mathcal{E}$ is product in $\mathcal{E}/X$. By two previous statements, the ambiguity in the notation 1_X (identity morphism of X in $\mathcal{E}$ versus terminal object in $\mathcal{E}/X$) is only apparent and

$$\begin{tikzcd}
X \times 1 \arrow[rr, "\sim"] \arrow[dr, "\pi_1"'] & & X \arrow[dl, "1_X"] \\
 & X &
\end{tikzcd}$$

We have already remarked that pulling back along a regular epimorphism in a regular category reflects isomorphisms, and we can invoke the

Lemma: *Between categories which have equalizers, a functor* which preserves equalizers *is conservative iff it is faithful.*

Alternatively, we can prove the faithfulness directly as follows. If p is regular epic and f_1, f_2 have the same pullback f in

$$
\begin{tikzcd}
 & X \times Y' \arrow[dddrrr, "\pi_{Y'}"] \arrow[ddd, "\pi_X"] & & & \\
X \times Y \arrow[ur, "f"] \arrow[ddr, "\pi_X"'] \arrow[ddrrr, "\pi_Y"] & & & & \\
 & & & & \\
 & & & & Y' \arrow[dd] \\
 & {} \arrow[drrr, "p"'] & & Y \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] \arrow[dr] & \\
 & & & & 1
\end{tikzcd}
$$

then (actually $f = X \times f_i$) we have

$$\pi_X f_i = f \pi_{Y'} \quad i = 1, 2$$

and hence

$$f_1 = f_2$$

since π_Y, as the pullback of the regular epimorphism p, is a (regular) epimorphism and hence can be cancelled. ∎

Note that in general projections are not necessarily epimorphisms, since 1 may have many proper subobjects; for example, in $\mathcal{E} = \mathcal{F}/T$, the subobjects of 1 are all the subobjects in $\mathcal{F}$ of T. By the same relativization principle, taking the domain of p to be 1 in the above is not really a loss of generality, since we are making the proof for *all* regular categories, including those of the form $\mathcal{F}/T$; the actual direct proof would just slightly complicate the notation in the above diagram, replacing products by pullbacks.

We now investigate some of the basic relations between different species of finite limits. We have already remarked that cartesian squares, pulling back along a given morphism, and products in a category of the form $\mathcal{E}/X$ are really just three different aspects (all useful) of the same basic universal mapping property, and that the inverse images of a subobject along a morphism and the intersection of two subobjects of an object are special cases of the same. In particular, it is clear that

Proposition 2.14: *If $\mathcal{E}$ has pullbacks and a terminal object, then it has binary cartesian products. For a more "concrete" interpretation of pullbacks generally, consider the proof of*

Proposition 2.15: *If $\mathcal{E}$ has equalizers and binary products, then it has pullbacks.*

Proof: Given $Y_i \xrightarrow{f_i} S \quad i = 1, 2$ consider the equalizer of

$$\begin{tikzcd}
 & Y_1 \arrow[dr, "f_1"] & \\
Y_1 \times Y_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & S \\
 & Y_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Thus by slight abuse of language, we may "always" consider that the pullback of f_1 with f_2 is that subobject of the product $Y_1 \times Y_2$ on which $f_1 = f_2$. (More exactly, $\{< y_1, y_2 > | f_1(y_1) = f_2(y_2)\}$ i.e. the slogan does not mean that we apply the wrong f_i to the wrong y_i even if we happen to be in the case $Y_1 = Y_2$). ∎

Exercise 2.14 Still a fourth interpretation of pullback is a *fibered product*, since for any $A \xrightarrow{s} S$ we have, in a suggestive notation for the pullback,

$$\left(Y_1 \underset{S}{\times} Y_2\right)_s \cong (Y_1)_s \underset{A}{\times} (Y_2)_s$$

In particular, if $A = 1$, the right hand side is an "ordinary" product in $\mathcal{E}$.

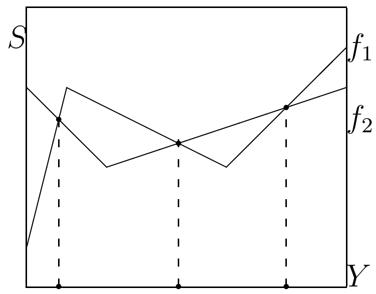
Proposition 2.16: *If $\mathcal{E}$ has binary products and binary intersections, then it has pullbacks. (This form has the clearest relation to the case of a syntactical theory $\mathcal{E}$, since it means that if we have appropriate rules for substitution and combination of (free) variables in formulas and also appropriate rules for the conjunction $(\wedge)$ of two formulas having the same complex of free variables, then we can construct all possible finite limits (the terminal object corresponds to the empty set of free variables, i.e. to any provable closed formula)).*

Proof: Here we are assuming that, although the intersection operation is applied only to subobjects, it has the universal mapping property with respect to arbitrary morphisms (not only with respect to other subobjects of the same object as in "lattice theory"). This said, let $Y \xrightarrow{f_i} S$ be given and consider the *intersection of their graphs* (as stated in a previous proposition, graphs of morphisms are even retracts, so certainly subobjects of the product).

$$\begin{tikzcd}
P \arrow[r, "k_2"] \arrow[d, "k_1"'] & Y \arrow[d, "\gamma_2"] \\
Y \arrow[r, "\gamma_1"'] & Y \times Y
\end{tikzcd} \qquad \gamma_i =< Y, f_i >$$

Using the fact that maps into a product are equal iff they have the same projections on *each* factor, we see first that $k_1 = k_2$, and then that $k = k_1 = k_2$ has the universal mapping property with respect to any $X \xrightarrow{y} Y$ with $yf_1 = yf_2$. ∎

There is a very suggestive picture for the above proposition showing clearly that the intersection of the two graphs should be isomorphic to that subobject of the domain on which the morphisms are equal:



Definition: *An arbitrary pair of morphisms* $M \overset{d_0}{\underset{d_1}{\rightrightarrows}} I$ *in a category will sometimes be called a "diagram scheme" (or "directed graph with possible multiple edges and loops") in that category. For example, if M, I are* **finite abstract sets** *and d_0, d_1 are mappings, we speak of a "finite abstract diagram scheme", I is called the set of vertices and M the set of arrows (or directed edges) of the scheme, while d_0 assigns to each arrow m its "beginning vertex" and d_1 its "ending vertex".*

Proposition 2.17: *(Finite Limits) Let $\mathcal{E}$ be a category having finite products and equalizers. Suppose* $\mathbf{M} =< M, I, d_0, d_1 >$ *is a finite abstract diagram scheme and suppose that to each vertex i of* $\mathbf{M}$ *we have associated an object Y_i of $\mathcal{E}$ and that to every arrow m of* $\mathbf{M}$ *we have associated a morphism t_m of $\mathcal{E}$ in such a way that for all i, j, m*

$$Y_i \xrightarrow{t_m} Y_j \quad \textit{iff } i = (m)d_0 \textit{ and } j = (m)d_1$$

Then there exists an object Y and morphisms $p_i : Y \to Y_i, \quad i \in I$ which *satisfies*

$$\begin{tikzcd} & Y_i \arrow[dd, "t_m"] \\ Y \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & \\ & Y_j \end{tikzcd} \qquad \begin{array}{c} m \in M, \\ (m)d_0 = i, \\ (m)d_1 = j \end{array}$$

and which moreover satisfies the following universal property. Given any object X of $\mathcal{E}$ and any family of morphisms $f_i : X \to Y_i$ (indexed by I) satisfying $f_j = f_i t_m$ for all "$i \xrightarrow{m} j$ in $\mathbf{M}$", there exists a unique morphism f in $\mathcal{E}$ for which $f_i = f p_i$ for all $i \in I$.

$$\begin{tikzcd} & & Y_i \arrow[dd, "t_m"] \\ X \arrow[r, dashed] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & Y \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & \\ & & Y_j \end{tikzcd}$$

i.e.

$$\frac{X \to Y}{< X \to Y_i >_{i \in I} \ \ \textit{compatible with } t}$$

Proof: By associativity, we may use a symbol Π for finitely iterated cartesian product without ambiguity except for unique isomorphisms. Consider the equalizer

$$Y \xrightarrow{e} \prod_{i \in I} Y_i \underset{u}{\overset{t}{\rightrightarrows}} \prod_{m \in M} Y_{(m)d_1}$$

of those two morphisms t, u which are uniquely defined by the equations

$$t\pi_m = \pi_i t_m \text{ for all } m \in M, i = (m)d_0$$

$$u\pi_m = \pi_j \text{ for all } m \in M, j = (m)d_1$$

and then define p_i by

$$p_i = e\pi_i \quad i \in I.$$

Now suppose given any X, f_i satisfying

$$\begin{tikzcd} & Y_i \arrow[dd, "t_m"] \\ Y \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\ & Y_j \end{tikzcd} \qquad \begin{array}{c} \text{all } m \in M, \text{ putting} \\ i = md_0 \\ j = md_1 \end{array}$$

Then (without the last diagram even) there is a unique $< f >$ such that $< f > \Pi_i = f_i, i \in I$ by the universal mapping property of the product $\prod_I$. But not using the last diagram, we have that

$$< f > t\pi_m =< f > \pi_i t_m = f_i t_m = f_j$$

and in any case

$$< f > u\pi_m =< f > \pi_j = f_j$$

for all $m \in M$ (taking $j = (m)d_1$). Hence by the universal mapping property of $\prod_M$, we have

$$< f > t =< f > u$$

But then by the universal mapping property of the equalizer e of t, u there is a unique f for which holds the equation

$$fe =< f >$$

Now we need only show that f also satisfies the *family* of equations at the end of the statement of the proposition, and is also uniquely determined by that, but this follows by a similar calculation. ∎

Example: The finite limits of diagrams of the forms

$$\begin{tikzcd} \cdot \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \\ & \cdot & & \cdot & \end{tikzcd}$$

and

$$\begin{tikzcd} \cdot \arrow[loop right] \end{tikzcd}$$

occur particularly frequently.

Remark: The list of general remarks about finite limits in regular categories is quite extensive, as it includes quite a bit of universal algebra, some of the essential and centrally important features of which are in the next two lessons. For the moment, consider the following simple but fundamental observation. From a one-sided point of view, one might ask, "why bother to have products at all, since mapping into them can immediately be reduced to mapping into the factors?" The answer is, of course, that having products, we can also consider mappings *out* of them.

Given morphisms of that sort

$$Y_i \times Y_j \xrightarrow{\alpha} Y_k$$

are considered as *algebraic structure* on the Y_i's. Note that each element of the domain of α divides naturally into two, giving rise to rich possibilities for producing the values of α; this effect is even more pronounced in case $Y_i = Y_j = Y_k = A$. Structures $< A, a >$ of the type $A \times A \xrightarrow{\alpha} A$ include the group objects and the monoid objects in $\mathcal{E}$, which clearly is a notion capable of reflecting a significant part of the structure of the world.

3. Images and "Existence"; Regular categories and Relational Composition

Now let us return to the alternative description of regular categories which was promised earlier. We make the general assumption that $\mathcal{E}$ is a category having finite limits.

Definition: *A morphism p is* **extremal** *iff for all morphisms a, i, if $p = ai$ and i is a monomorphism, then i is an isomorphism.*

Proposition 2.18: *Any extremal morphism is an epimorphism.*

Proof: Suppose $pg_1 = pg_2$ and let i be the equalizer of g_1, g_2. Then $\exists : a$ $[p = ai]$, so i is an isomorphism, which means $g_1 = g_2$. ∎

Proposition 2.19: *A morphism which is both a monomorphism and an extremal epimorphism, is in fact an isomorphism.*

Proof: Taking $a = id$ in the definition, the morphism is a factorization of itself. ∎

Proposition 2.20: *An extremal epimorphism p has the stronger property of being "othogonal in the sense of Kelly to the class of all monomorphisms i' of $\mathcal{E}$", i.e. if i' is* **any** *monomorphism of $\mathcal{E}$ (unrelated to the domain or codomain of p in general) and if a', b' are any two morphisms such that $a'i' = pb'$, then* *there is a (unique) morphism y such that $a' = py$ and $yi' = b'$.*

$$(*) \qquad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "a'"'] & \cdot \arrow[d, "b'"] \arrow[dl, dashed, "y"'] \\ \cdot \arrow[r, tail, "i'"'] & \cdot \end{tikzcd}$$

Conversely, if p has this more general property, then it is extremal.

Proof: The converse is trivial (take $i' = i, b' = id, a' = a$ in $(*)$; then $y = i^{-1}$. If we are given any commutative square as in $*$, then we can take the pullback of i' with b', getting a *cartesian* square inside the given square, in which i, as the pullback of the monic i, is itself monic. Since the original square is commutative, there is a unique a such that $ab = a', ai = p$. Since p is by hypothesis extremal, we have that i is an isomorphism, so we can define y by $y = i^{-1}b$. But then

$$\begin{gathered} py = pi^{-1}b = aii^{-1}b = ab = a' \\ \text{and} \\ yi' = i^{-1}bi' = i^{-1}ib' = b' \end{gathered} \qquad \begin{tikzcd} \cdot \arrow[rr, "p"] \arrow[dr, "a"] \arrow[dd, "a'"'] & & \cdot \arrow[dd, "b'"] \\ & \cdot \arrow[ur, "i"'] \arrow[dl, "b"] & \\ \cdot \arrow[rr, tail, "i'"'] & & \cdot \end{tikzcd}$$

∎

The property $(*)$ in the above proposition is equivalent to a more intuitive one (which is slightly more complicated, involving more morphisms); the following corollary may be summed up in the slogan "Extremal images are functorial".

Corollary: *Suppose a morphism f is factored $f = p_f i_f$ into an extremal epimorphism p_f followed by a monomorphism i_f, and suppose another morphism g is similarly factored $g = p_g i_g$. Suppose further that we are given "a morphism from f to g in the category $\mathcal{E}^2$", i.e. a pair of morphisms a'', b'' of $\mathcal{E}$ for which $a''g = fb''$. Then there is a unique induced morphism "from the image of f to the image of g", making both of the following squares commutative.*

$$\begin{tikzcd} \cdot \arrow[r, "p_f"] \arrow[d, "a''"'] & \cdot \arrow[r, tail, "i_f"] \arrow[d, dashed] & \cdot \arrow[d, "b''"] \\ \cdot \arrow[r, "p_g"'] & \cdot \arrow[r, tail, "i_g"'] & \cdot \end{tikzcd}$$

Proof: We need only take $a' = a''p_g, b' = i_f b''$ in the previous proposition. ∎

Exercise 2.15 It is trivial that, conversely, such a *functorial* image factorization in $\mathcal{E}$ is possible only in case the epics involved in the factorization are extremal. (Hint: Take most of the morphisms in the last diagram to be identity morphisms.)

Since (in a category having finite limits) "extremal" is equivalent to property $(*)$, we can show easily that the class of extremal (epi)morphisms has some elementary closure properties.

Corollary: *Every isomorphism is extremal, the composition of any two extremal morphisms is extremal, and if any composition fg is extremal, it follows that the second factor g is also extremal.*

Proof: We show the corresponding properties for the class of morphisms satisfying $(*)$ in *any* category. If both f and g satisfy $(*)$, suppose the composition fg is up against an arbitrary monomorphism i' by means of a', b'', i.e. the following big square is commutative.

$$
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "a'"'] & \cdot \arrow[r, "g"] \arrow[dl, dashed, "y_1"'] & \cdot \arrow[d, "b''"] \arrow[dll, dashed, "y"] \\
\cdot \arrow[rr, tail, "i'"'] & & \cdot
\end{tikzcd}
$$

Since f is extremal, if we take $b' = gb''$ in $(*)$, there is y_1 with $a' = fy_1$ and $y_1 i' = gb''$; this puts g up against i' by means of y_1, b'', so there is (g being also extremal) another morphism y with $y_1 = gy, yi' = b''$. But then we also have

$$a' = fy_1 = f(gy) = (fg)y$$
$$yi' = b''$$

so that fg satisfies $(*)$ so is in particular extremal. On the other hand, suppose that fg is extremal and oppose g to an arbitrary monic i' by means of a'', b'.

$$
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[dr, "a'"'] & \cdot \arrow[d, "a''"] & \cdot \arrow[r, "g"] & \cdot \arrow[d, "b'"] \arrow[dl, dashed, "?\ y"'] \\
& \cdot & \cdot \arrow[r, tail, "i'"'] & \cdot
\end{tikzcd}
$$

Define $a' = fa''$. Then $a'i' = fa''i' = s(fg)b'$, so by the supposition y exists with

$$a' = (fg)y$$

$$yi' = b'$$

To complete the proof that g satisfies $(*)$ (and is hence extremal) we need to show that

$$a'' \underset{?}{=} gy$$

as well; *but this follows from the fact that i' is monic*, (note that in general f is not epic), since

$$a''i' = gb' = gyi'$$

∎

Exercise 2.16 $sp = 1_Y \Rightarrow p$ extremal; *split* epics are preserved by pullback.

It is of some interest to note that in a category having good images the arbitrary monomorphisms are precisely the morphisms which are "co-orthogonal in the sense of Kelly" to the class of all extremal (epi)morphisms. Here by "co-orthogonal" we mean simply the *converse* relation to the relation $(*)$ in the *same* category $\mathcal{E}$, and not a notion in the opposite category $\mathcal{E}^{op}$ (which would involve the relation $(*)$ itself in $\mathcal{E}$, but applied to arbitrary epimorphisms versus co-extremal (mono)morphisms of $\mathcal{E}$.)

Proposition 2.21: *Let every morphism in $\mathcal{E}$ have a factorization into an extremal (epi)morphism, followed by a monomorphism. Suppose a morphism f has the property that for all extremal (epi)morphisms p and for any morphisms a', b' with $a'f = pb'$, there is a morphism y making the two triangles below commutative*

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "a'"'] & \cdot \arrow[d, "b'"] \arrow[dl, dashed, "y"'] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

Then f is a monomorphism.

Proof: Let $f = pi$ be a factorization of f as assumed, and take $a' = id, b' = i$. Then there is a morphism y with

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "id"'] & \cdot \arrow[d, "i"] \arrow[dl, dashed, "y"'] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd} \qquad \begin{aligned} id &= py \\ yf &= i \end{aligned}$$

But since p is both an extremal epimorphism and (by the first equation) a split monomorphism, it is (for two different reasons) an isomorphism; but then $f = y^{-1_i}$ is, as the composition of two monomorphisms, a monomorphism. ∎

Example: In the category top_2 of Hausdorff topological spaces, an epimorphism is any continuous map whose image set is dense in the codomain space, but an extremal (epi)morphism is a topological quotient. A monomorphism is any one-to-one continuous mapping, but a co-extremal (mono) morphism is the inclusion of a closed subspace. Both kinds of factorizations $\text{Epi}^{(\text{extr})}$. Mono and Epi $\text{Mon}^{(\text{coextr})}$ exist. The pullback of an extremal epimorphism along a continuous mapping need not be extremal, however.

"Extremal" suggests adjointness, and indeed there is another sense in which a factorization of a morphism f into extremic followed by monic is "extremal", namely that the *monic* part is the *smallest subobject* of the codomain through which the f can be factored in any way.

Proposition 2.22: *Suppose $f = pi$ where f is extremal epic and i is monic. If $f = f'i'$ is any factorization of f with i' monic, then $i \subseteq i'$ as subobjects of the codomain of f.*

Proof: Consider the diagram

$$\begin{tikzcd} \cdot \arrow[r, two heads, "p"] \arrow[d, "f'"'] & \cdot \arrow[d, tail, "i"] \arrow[dl, dashed, "y"'] \\ \cdot \arrow[r, tail, "i'"'] & \cdot \end{tikzcd}$$

which is commutative since either way around equals f. Since p is extremal, there is a y which proves that $i \subseteq i'$. ∎

The morphism y also proves that "the values of f' only depend on the fibers of p (= the fibers of f)"; such a y could also be constructed if we knew that p is a coequalizer, since the assumption $f = f'i'$ implies that f' "coequalizes" the kernel pair of f. In fact, it is easy to see that a coequalizer is extremal, and conversely, we are working toward the theorem that, *if* extremal epics are preserved by pullback (and there are enough of them in the sense that factorization exists) then all extremal epics are in fact regular epics.

Let us denote by $\mathcal{P}_{\mathcal{E}}(X)$ the category (preordered set) of all the subobjects of X in the category $\mathcal{E}$. Since inclusions of subobjects of X are in particular commutative triangles over X, there is always a full and faithful inclusion functor

$$\mathcal{P}_{\mathcal{E}}(X) \to \mathcal{E}/X$$

Corollary: *Suppose that in the category $\mathcal{E}$, every morphism with codomain Y has a factorization into an extremal (epi) morphism followed by a monomorphism. Then the inclusion functor has a left adjoint.*

$$\begin{tikzcd} \mathcal{P}_{\mathcal{E}}(Y) \arrow[r, shift right] & \mathcal{E}/Y \arrow[l, shift right, "Im"'] \end{tikzcd}$$

Proof: Define $Im(f)$ to be the monic part of a factorization as assumed. Since for an object i' of $\mathcal{E}/Y$ (which comes, say, from $\mathcal{P}_{\mathcal{E}}(Y)$, a morphism $f \xrightarrow{f'} i'$ is just (by definition of $\mathcal{E}/Y$) a morphism f' of $\mathcal{E}$ for which $f = f'i'$, the adjointness relation

$$\frac{f \to i' \text{ in } \mathcal{E}/Y}{Im(f) \xrightarrow{\subset} i' \text{ in } \mathcal{P}_{\mathcal{E}}(Y)}, \qquad \begin{array}{l} f \in \mathcal{E}/Y \\ i' \in \mathcal{P}_{\mathcal{E}}(Y) \end{array}$$

is just a restatement of the proposition. ∎

Corollary: *(Extremal) images are unique up to isomorphism.*

Proof: This follows also from the proposition on the functorality of images, since taking $a' = id_X, b' = id_Y$ we get comparison morphisms in both directions, which must be inverse to each other, since the epics and monics in the diagram force uniqueness also of the composites. ∎

Let us now make the assumption that every morphism in $\mathcal{E}$ can be factored into an extremal (epi)morphism followed by a monomorphism, then we can make the

Definition: *Let* $X \xrightarrow{f} Y$ *be any morphism. Then define a functor*

$$\mathcal{P}_{\mathcal{E}}(X) \to \mathcal{P}_{\mathcal{E}}(Y)$$

denoted by $\exists_f$ *as follows: Given any subobject* i *of* X*, factor the composite morphism* i *followed by* f *into an extremal epimorphism* p *followed by a monomorphism* i_1*,*

$$\begin{tikzcd} \bullet \arrow[r, "p"] \arrow[d, "i"'] & \bullet \arrow[d, "i_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

then forget p *and set*

$$\exists_f[i] = i_1$$

which is, of course, a subobject of Y*.*

Exercise 2.17 This is a functor; i.e. if $i \underset{X}{\subseteq} i'$ then $\exists_f[i] \underset{Y}{\subseteq} \exists_f[i']$. However, $\exists_f$ usually does not preserve intersections of subobjects; $\exists_f(1_X) = 1_Y$ iff f itself is an extremal epimorphism.

This definition is obviously a generalization of Im, indeed another reasonable notation would be $Im_f[i] = \exists_f[i] = Im(if)$, the "image of i under f". On the other hand, the special Im notion was defined for each Y and could be denoted by Im_Y: the latter provides a natural "projection" onto the operation $\exists_f$ from another functor defined at the level of arbitrary "families" (= "objects over given objects"), not only subobjects. Namely, for any $X \xrightarrow{f} Y$, define the functor

$$\mathcal{E}/X \xrightarrow{\sum_f} \mathcal{E}/Y$$

$$f_1 \rightsquigarrow f_1 f$$

to be simply composition with f; then the diagram

$$\begin{tikzcd}
\mathcal{E}/X \arrow[r, "\sum_f"] \arrow[d, "Im_X"'] & \mathcal{E}/Y \arrow[d, "Im_Y"] \\
\mathcal{P}_{\mathcal{E}}(X) \arrow[r, "\exists_f"'] & \mathcal{P}_{\mathcal{E}}(Y)
\end{tikzcd}$$

is commutative, up to equivalence of subobjects of Y, as is seen from the two possible interpretations of the diagram

$$\begin{tikzcd}
{} \arrow[d, two heads] & \\
f_1 \arrow[r, tail] \arrow[d, tail, "i_1"'] & {} \arrow[d, tail] \\
{} \arrow[r, "f"'] & {}
\end{tikzcd}$$

and using the facts that the composition of extremics is extremic and that image factorizations (in our extremal sense!) are unique.

Exercise 2.18 $\sum_f$ is the left adjoint to pulling back f^* (this is just a rephrasing of the universal mapping property of pullbacks). If $E \xrightarrow{f_1} X$ is an object over X and $A \xrightarrow{y} Y$ is an "element" of Y, then give a sense to the formula for fibers:

$$(\sum_f E_y) = \sum_{xf=y} E_x$$

interpreting $\{x|xf = y\}$ as $f^*(y) = y^*(f)$.

Since we are also assuming that pullbacks exist, there is a functor

$$\mathcal{P}(X) \xleftarrow{f^*} \mathcal{P}(Y)$$

in the direction opposite to that of $\exists_f$, defined by restricting the general pullback functor f^* from $\mathcal{E}/Y$ to $\mathcal{P}(Y)$, noting that, since pullbacks of monics are monics, the values of the restriction will always lie in $\mathcal{P}(X)$, and abusing notation to denote the restriction still by f^*; i.e. the restricted f^* is the inverse image operator.

Proposition 2.23: *$\exists_f$ is the left adjoint of f^*.*

Proof: We must show a natural bijection

$$\frac{i \subseteq f^*[j] \text{ in } \mathcal{P}(X)}{\exists_f[i] \subseteq j \text{ in } \mathcal{P}(Y)}$$

which reduces, since the categories involved are posets, to an if-and only-if condition. Suppose first that $\exists_f[i] \subseteq j$. That is, we have $if = p\exists_f[i]$ where p is extremal and $\exists_f[i]$ is, like j, a monic with codomain Y, but moreover, we have a morphism u with $\exists_f[i] = uj$. But then $if = puj$, so by the universal mapping property of the pullback, $f^*[j]$, there is a (unique) t with $i = tf^*[j]$ and $pu = tf'$ (where f' is the fourth side of the cartesian square); but the first of the last two equations already shows that $i \subseteq f^*[j]$. Conversely, suppose that we have a morphism t with $i = tf^*[j]$; we will construct u.

$$\begin{tikzcd}
\cdot \arrow[rr, "p"] \arrow[dr, dashed, "t"] \arrow[ddr, tail, "i"'] & & \cdot \arrow[dr, dashed, "u"] \arrow[ddr, tail, "\exists_f[i]"' near start] & \\
& \cdot \arrow[rr, tail, "f'"] \arrow[d, tail, "f^*[j]"] & & \cdot \arrow[d, tail, "j"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

By our supposition $i = tf^*[j]$, it follows that $if = tf'j$ since the cartesian square is in particular commutative; but $if = p\exists_f i$ by definition of our generalized image ("existential quantifier") so that we have a commutative square

$$\begin{tikzcd}
\cdot \arrow[r, "p"] \arrow[d, "tf'"'] & \cdot \arrow[d, tail, "\exists_f[i]"] \arrow[dl, dashed, "u"'] \\
\cdot \arrow[r, tail, "j"'] & \cdot
\end{tikzcd}$$

which again is precisely the situation in which we can apply the functorial property $(*)$ of extremal morphisms to obtain a morphism u which proves in particular that $\exists_f[i] \subseteq j$. ∎

We might sum up part of the discussion so far by saying: "If images are functorial, then they are adjoint".

By very definition of the inverse image operator f^* as the restriction of the pullback functor f^*, we have commutativity of the diagram

$$
\text{(1)} \qquad
\begin{tikzcd}
\mathcal{E}/X & \mathcal{E}/Y \arrow[l, "f^*"'] \\
\mathcal{P}(X) \arrow[u, hook] & \mathcal{P}(Y) \arrow[l, "f^*"] \arrow[u, hook]
\end{tikzcd}
$$

of functors for any morphism $X \xrightarrow{f} Y$ of $\mathcal{E}$. Taking left adjoints in the above diagram, we have, as already remarked, also commutativity of the diagram

$$
\text{(2)} \qquad
\begin{tikzcd}
\mathcal{E}/X \arrow[r, "\sum_f"] \arrow[d, "Im_X"'] & \mathcal{E}/Y \arrow[d, "Im_Y"] \\
\mathcal{P}(X) \arrow[r, "\exists_f"'] & \mathcal{P}(Y)
\end{tikzcd}
$$

for a morphism f in a category having adjoint images (note that adjoint images automatically involve *extremal* epics). There is a third diagram which we might reasonably ask to commute, and indeed which is needed for really serious theorems about the relationships between pullbacks and images (e.g. for discussing composition of relations, etc.); however, this third diagram is not valid in all cases. It is

$$
\text{(3)} \qquad
\begin{tikzcd}
\mathcal{E}/X \arrow[d, "Im_X"'] & \mathcal{E}/Y \arrow[l, "f^*"'] \arrow[d, "Im_Y"] \\
\mathcal{P}(X) & \mathcal{P}(Y) \arrow[l, "f^*"]
\end{tikzcd}
$$

There is a close analogy here with ring theory: we can discuss (1) multiplication and (2) addition at great length, but all serious results in the presence of both depend on (3) the distributive law. In fact, concerning the distributive law, it is possible to find a "reason" for its truth, namely the *existence* of exponentiation (at least in a categorical setting) - we may hope to also find a "reason" for the regularity of a category in terms of the existence of other things - but for the moment we consider the axiom itself and some consequences:

Regularity Axiom: $\mathcal{E}$ not only has finite limits and extremal images, but moreover the pullback along any morphism f of any extremal (epi)morphism p is again an extremal (epi)morphism $f^*(p)$.

We have stated the above axiom in terms of cartesian squares in $\mathcal{E}$ itself; in terms of the associated categories and pullback functor f^* $\mathcal{E}/X \longleftarrow \mathcal{E}/Y$, the axiom is equivalent to the statement that if E is an object of $\mathcal{E}/Y$ whose map to the terminal object 1_Y is extremal, then $f^*(E)$ is an object of $\mathcal{E}/X$ whose map to the terminal object 1_X is extremal. The axiom implies that the whole image factorization of an *arbitrary* morphism g is preserved by pulling back along f, since we can always consider the pullback diagram

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & Y' \arrow[d] \\
\cdot \arrow[r, "f'"'] \arrow[d] & \cdot \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

to be obtained by pulling back in two stages (we in fact apply the axiom to the operation of pulling back along f'), and images are unique. In other words, (considering as before that an image "is" the monic part of an image factorization), the image of $f^*(g)$ is f^* of the image of g, i.e. diagram (3) above.

In particular, pulling back along a projection $X \times Y \to Y$ just amounts to forming $X \times (\ )$ so that one consequence of the axiom is

Proposition 2.24: *If p is an extremal (epi)morphism and X is any object, then $X \times p$ is also an extremal epic.*

On the other hand, pulling back along a monic must also preserve extremal epics, so (since a cartesian square can always be viewed as a pullback in two different ways) we have

$$\begin{tikzcd}
\cdot \arrow[r, two heads] \arrow[d, tail] & \cdot \arrow[d, tail, "j"] \\
A \arrow[r, two heads, "q"'] & B
\end{tikzcd}$$

which shows that $\exists_q [q^*[j]] = j$, *i.e.*

Proposition 2.25: *If q is an extremal epic, then*

$$\mathcal{P}(A) \xrightarrow{\exists_q} \mathcal{P}(B)$$

is surjective; in fact, it is split by q^*.

Exercise 2.19 Give an example for abstract sets to show that there may be many $i \subseteq q^*[j]$ which also have the property $\exists_q [i] = j$. Hint: take q to be a projection from the product of two *non-empty* sets.

Exercise 2.20 Show that the analogue of the last proposition does not hold for $\sum_q : \mathcal{E}/A \to \mathcal{E}/B$ unless q is actually a *split* epic.

Proposition 2.26: *The last two propositions taken together are equivalent to the axiom (hence to (3)).*

Proof: By the graph construction, every morphism f can be factored into a monomorphism followed by a projection.

Our aim is to show that any category satisfying our regularity axiom is actually a regular category as previously defined, i.e. that extremal morphisms are actually coequalizers, provided they are preserved by pullbacks. Since a coequalizer is always in particular the coequalizer of its kernel pair, the essential lemma clarifies the confrontation of extremal epics and equivalence relations (under the assumption of the regularity axiom). Recall that the kernel pair of a morphism f (or the equivalence relation R_f determined by f), is defined as the pullback of f with itself. If we compare the kernel pair of f with the kernel pair of a composite fg, we see that both R_f and R_{fg} may be considered as subobjects of $X \times X$, where X is the domain of f, and moreover that there is always an inclusion $R_f \subseteq R_{fg}$ of these subobjects, (since any pair $A \overset{x_1}{\underset{x_2}{\rightrightarrows}} X$ identified by f is also identified by fg; in particular, we may take $A = R_f, x_i =$ the two projections, hence the morphism (inclusion) of R_f into the pullback R_{fg}).

Lemma: *If f is an extremal epic (in a category satisfying our regularity axiom) and if $R_f = R_{fg}$ as subobjects $X \times X$, then g is a monomorphism.*

Proof: It suffices to show that the two structural maps $R_g \rightrightarrows Y$ are equal. But using only the first hypothesis, it is clear from the pullback diagram

$$\begin{tikzcd}
R_f \arrow[dr] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
& R_{fg} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & X \arrow[d] \\
& {} \arrow[r] \arrow[d] & R_g \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

that

$$R_{fg} \longrightarrow R_g$$

is an (extremal) epic, so it suffices to prove that the composites

$$R_{fg} \longrightarrow R_g \rightrightarrows Y$$

are equal. By the second hypothesis, it thus suffices to show that the composites

$$R_f \rightarrowtail R_{fg} \twoheadrightarrow R_g \rightrightarrows Y$$

are equal. But from the above diagram these latter composites are seen to be the same as

$$R_f \rightrightarrows X \xrightarrow{f} Y$$

which are equal by definition of kernel pair ∎

Theorem: *(Joyal) The regularity axiom implies that any extremal epic p is the coequalizer of its kernel pair.*

Proof: Consider the kernel pair and any f for which $\pi_1 f = \pi_2 f$

$$\begin{tikzcd} R_p \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "p"] \arrow[dr, "f"'] & P \\ & & Y \end{tikzcd}$$

Since p is already epic, we need only show the existence of a morphism $P \xrightarrow{h} Y$ with $f = ph$; the method will be in essence to construct first the *graph* of h as $\exists_{p\times Y}[< X, f >]$. That is, we consider the following diagram

$$\begin{tikzcd} X \arrow[r, tail, "<X{,}f>"] \arrow[d, two heads, "q"'] & X \times Y \arrow[d, "p\times Y"] \\ \cdot \arrow[r, tail, "j"] \arrow[dr, "\theta"'] & P \times Y \arrow[d, "\pi_P"] \\ & P \end{tikzcd}$$

in which the main point will be to prove that the composite θ *is an isomorphism* (which is equivalent to the condition that the subobject j of $P \times Y$ is the graph of some morphism, indeed the graph of the needed h). We show that θ is both (1) an extremal epic and (2) a monic. For the first,

$$q\theta =< X, f > (p \times Y)\pi_P =< p, f > \pi_P = p$$

and hence θ, as the second factor in a composition yielding an extremal epic p, its itself extremal epic. For the second, we will use the lemma, i.e. it suffices to show $R_q = R_q\theta$; but as we have just seen $p = q\theta$, so it suffices to show $R_q = R_p$. Since j is monic $R_q = R_{qj}$, so we aim to show

$$(?)R_{qj} = R_p$$

Now

$$R_{qj} = R_{<p,f>} = R_p \cap R_f$$

the last being a general fact about the equivalence relations of any pair of morphisms with the same domain. On the other hand, our hypothesis $\pi_1 f = \pi_2 f$ on f just means that $R_p \subseteq R_f$, so

$$R_{qj} = R_p \cap R_f = R_p$$

as required. Thus θ is an isomorphism. If j is a subobject of a product whose projection on the first factor P is an isomorphism θ, then j is actually the graph of the morphism $h = \theta^{-1_j}\Pi_Y$ where Π_Y is the projection on the second factor; in our case we must verify the commutativity of a triangle involving h so defined:

$$ph = p\theta^{-1_j}\pi_Y = q\theta\theta^{-1_j}\pi_Y = qj\pi_Y =< p, f > \pi_Y = f$$

∎

Thus a category satisfies the regularity axiom iff it is a regular category as defined previously (and has a terminal object 1). The construction of h in the proof of the theorem may be considered as the construction of a *composite relation*, the inverse of p followed by f. In fact, the regularity of a category $\mathcal{E}$ may be considered to be due precisely to the existence of an associated category Rel ($\mathcal{E}$) of relations, as we will indicate. The category Rel($\mathcal{E}$) has the same objects as the category $\mathcal{E}$, but as morphisms has

$$Rel(\mathcal{E})(X, Y) = \mathcal{P}_{\mathcal{E}}(X \times Y)$$

thus Rel($\mathcal{E}$) has a significant aspect of structure which $\mathcal{E}$ lacks, namely an *order relation* on the "hom sets" corresponding to the inclusion of relations; since the composition of relations will be defined in such a way as to preserve this order relation, it is convenient to consider that $Rel(\mathcal{E})$ is not merely a category, but a simple kind of "two-dimensional" category. We have $\mathcal{E} \subset Rel(\mathcal{E})$ since every morphism of $\mathcal{E}$ can be considered as a relation by means of its graph (in particular identity morphisms correspond to "equality" or "diagonal" relations) and composition of relations is to be defined in such a way as to correspond (when applied to graphs) to the composition of $\mathcal{E}$-morphisms. "Conversely" those relations which are actually $\mathcal{E}$-morphisms can be characterized in terms of composition of relations, inclusion of relations, identity relations, and the operation of forming *inverse* relations

$$\mathcal{P}_{\mathcal{E}}(X \times Y) \xrightarrow{(\,)^{-1}} \mathcal{P}_{\mathcal{E}}(Y \times X)$$

Moreover, every relation is the composition of the inverse of an $\mathcal{E}$-morphism followed by an $\mathcal{E}$-morphism, and indeed such *decomposition* of relations can be accomplished in a *minimal* manner. Thus in principle, if we are given a two-dimensional category with all the structure just outlined, we should be able to recover a category $\mathcal{E}$ such that the given 2-category is equivalent to $Rel(\mathcal{E})$.

The first technical problem to be overcome in carrying out the above program is to define composition of relations and prove that it is associative; as already indicated, this seems to require again the *regularity* of $\mathcal{E}$. Let us denote by

$$X \overset{\alpha}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;} Y$$

relations from X to Y; thus such an α can always be considered as a third object, equipped with two $\mathcal{E}$-morphisms, to X and Y respectively, which are *jointly monic* (the construction of $Y \overset{\alpha^{-1}}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;} X$ in these terms simply amounts to reversing the order in which we consider these two "projections"). If $Y \overset{\beta}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;} Z$ is another relation, then for the case of sets we have the well-known definition

$$(\alpha * \beta)(x, z) \equiv \exists y[\alpha(x, y) \wedge \beta(y, z)]$$

for the composite relation $X \overset{\alpha*\beta}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;} Z$; guided by our experience with this case, we make the

Definition:

$$\alpha * \beta \equiv \exists_{\pi_{XZ}}[\pi^*_{XY}[\alpha] \cap \pi^*_{YZ}[\beta]$$

(as subobjects of $X \times Z$) in any category having inverse images (pullbacks) and direct images. Here all three projections π which occur have as domain the triple product $X \times Y \times Z$ and the indicated codomains. Note that π_{XZ} is the projection which "forgets" the factor Y; thus taking image along it is to be considered intuitively as "existential quantification with respect to the variable $y \in Y$".

The presence of the two inverse image operators in the definition takes account of a fact which is less clear in the notation using "variables"; namely, that we intersect α, β only after first considering them as relations with three variables.

The proof of the associativity of the operation defined above depends on some kind of "distributivity" of existential quantification with respect to conjunction; such does not hold without condition, even in sets, but it does hold if the "variables" involved are sufficiently "independent" of each other. The most general "independence" condition of this type is that we consider *cartesian* diagrams, and in this case we can prove the needed "distributivity" or commutativity of the two operations in any *regular* category.

Lemma: *Suppose*

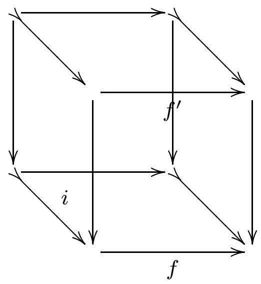
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "u"'] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a cartesian (pullback) square in a regular category and suppose i is a subobject of X. Then

$$\exists_{f'}[u^*[i]] \equiv v^*[\exists_f[i]]$$

as subobjects of Y'.

Proof: Consider the cube



in which the bottom is the image factorization of if and in which the right side and the back are pullbacks. The top left arrow exists since the front is a pullback by hypothesis; then the left square must also be a pullback, since we already have the *composite* pullback. By regularity, the top arrow in the back is also a regular (extremal) epic, hence the top square is also an image factorization.

Remark: In the above proof we used a general fact about pullbacks which perhaps we should have explicitly stated earlier, namely the

Sublemma on Cartesian Squares: Suppose that in the commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "f_1'"] \arrow[d, "w"'] & \cdot \arrow[r, "f_2'"] \arrow[d, "u"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "f_1"'] & \cdot \arrow[r, "f_2"'] & \cdot \end{tikzcd}$$

the whole rectangle and the right square are both cartesian. Then the left square is also cartesian.

Proof: Let $xf_1 = x'u$. Then certainly also $x(f_1f_2) = (x'f_2')v$; hence since w is the pullback of v along f_1f_2, there is a unique $\overline{x}$ such that $\overline{x}w = x, \overline{x}(f_1'f_2') = x'f_2'$. However, since also u is the pullback of v along f_2, there is a unique

t with $tu = xf_1$ and $tf_2' = x'f_2'$. Since both $t = \overline{x}f_1'$ and also $t = x'$ both satisfy the last two equations, we must have $\overline{x}f_1' = x'$. But since x, x' were arbitrary, this shows that we also have that w is the pullback of u by w, since the needed uniqueness of Z clearly propagates from the rectangle to the left square. ∎

Subremark: The hypothesis in the above sublemma on cartesian squares can clearly be weakened; we need only the uniqueness, not necessarily the existence condition (for cartesian squares) in the right-hand square; we might call this weakened condition the "subcartesian" property, since it means that the induced map $< u, f_2' >$ into the pullback of v along f_2 is monic.

Exercise 2.21 As a special case of the lemma, we have for any $X \xrightarrow{f} Y$ in a regular category, any subobject i of X and subobject j of Y, that

$$\exists_f[f^*[j] \cap i] \equiv j \cap \exists_f[i]$$

Exercise 2.22 In the category of sets

$$y \in \exists_f[i] \text{ iff } \exists x[x \in i \wedge xf = y]$$

where $\exists x$ may be interpreted as $\exists_{\pi_Y}$ where $\pi_Y : X \times Y \to Y$. Show that the same fact holds in a regular category in the sense that

$$\exists_f[i] \equiv \exists_{\pi_Y}[\pi_X^*[i] \cap \gamma_f]$$

(where γ_f is the subobject $< X, f >$ of $X \times Y$). Thus in a regular category the general $\exists_f$ may be expressed in terms of the case where f is a projection, which in turn may be thought of as the case of the "usual existential quantification with respect to a variable".

To show the associativity of relational composition, we will use again the technique of comparing the "large" categories $\mathcal{E}/A$ with the "small" ones $\mathcal{P}_{\mathcal{E}}(A)$ by means of the natural functors Im_A. This time we take A explicitly as a product and we define a composition for "pre-relations" which is clearly associative; we then show, using regularity, that the composition of relations is functorially the result of applying the image operators to it. Actually, our consideration of pre-relations (or "matrices" as we shall prefer to call them) amounts to the construction of another 2-dimensional category whose "hom-categories" do *not* reduce to posets, but will not need here all the technicalities of this observation. Denote by

$$Mat(\mathcal{E})(X, Y) = \mathcal{E}/X \times Y$$

in which we will often think of an object as simply an arbitrary pair $X \longleftarrow E \longrightarrow Y$ of $\mathcal{E}$-morphisms. Pursuing the analogy whereby objects of

$\mathcal{E}/A$ are thought of as A-indexed families of $\mathcal{E}$-objects, we now have doubly-indexed families, hence "matrices"; more explicitly, if $A \xrightarrow{x} X$, $A \xrightarrow{y} Y$ are two "elements" of the indexing objects, then the simultaneous pullback (or intersection of the fibers) may be denoted by E_{xy}. Matrices suggest matrix multiplication, and indeed it is easy to give a sense to

$$(E * F)_{xz} = \sum_y E_{xy} \times E_{yz}$$

by using our formal interpretation of $\sum\limits_y$ as being simply the forgetting of the part of the indexing done by Y. That is, we define a functor

$$Mat(\mathcal{E})(X,Y) \times Mat(\mathcal{E})(Y,Z) \xrightarrow{*} Mat(\mathcal{E})(X,Z)$$

as follows: Given any diagram

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

form the pullback (fibered product over Y) of the two middle morphisms, consider the outer composite morphisms as the structural morphisms for $E * F$, and forget the morphism from $E * F$ to Y. It is an immediate consequence of the definition of pullbacks in any category that this operation so defined is *associative* up to canonical natural isomorphism, and moreover, that the objects of $\mathcal{E}/X \times X$ corresponding to

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X \end{tikzcd}$$

act as identities (again up to canonical natural isomorphism) with respect to the operation *. (Or, in other terms, the "distributivity" of our formal $\sum$ with respect to fibered products is a tautology in any category with pullbacks, which shows that the formal $\sum$ can only be closely related to coproducts in a category $\mathcal{E}$ for which the latter not only exist, but also satisfy distributivity (e.g. a topos but not an abelian category).)

Now we want to show that for regular $\mathcal{E}$

$$\begin{tikzcd} Mat(\mathcal{E})(X,Y) \times Mat(\mathcal{E})(Y,Z) \arrow[r, "*"] \arrow[d, "Im_{X\times Y} \times Im_{Y \times Z}"'] & Mat(\mathcal{E})(X,Z) \arrow[d, "Im_{X \times Z}"] \\ Rel(\mathcal{E})(X,Y) \times Rel(\mathcal{E})(Y,Z) \arrow[r] & Rel(\mathcal{E})(X,Z) \end{tikzcd}$$

is a commutative diagram of functors (up to equivalence of subobjects of $X \times Z$) for any three objects X, Y, Z. Since Im_A is a one-sided inverse for the inclusion $\mathcal{P}(A) \hookrightarrow \mathcal{E}/A$, we get as a consequence the *associativity of relational* composition; another consequence is that the relational composition may be computed in the following (sometimes simpler) way: Given $X \overset{\alpha}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} Y \overset{\beta}{\longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;} Z$, consider the (jointly monic) pairs of morphism $X \longleftarrow E_\alpha \longrightarrow Y \longleftarrow E_\beta \longrightarrow Z$ and form their "matrix product" = *pre-relational* product = fibered product over Y

$$E_\alpha * F_\beta = E_\alpha \underset{Y}{\times} F_\beta = G$$

whose resulting structural morphisms $X \longleftarrow G \longrightarrow Z$ will in general *not* be jointly monic, but we can take its *image* in $X \times Z$, and by the above diagram we get the *relational* product, i.e.

$$\alpha * \beta = Im_{X \times Z}(E_\alpha \underset{Y}{\times} F_\beta)$$

Now, in order to prove the above functorial diagram, we complicate our very simple definition of pre-relational ("matrix") composition by analyzing it into three steps so as to make more explicit the analogy with relational composition. Namely, given $E \in \mathcal{E}/X \times Y, F \in \mathcal{E}/Y \times Z$, *first* transform them both into objects of the same type $\mathcal{E}/X \times Y \times Z$ by applying the *two* different operations $E \rightsquigarrow E \times Z,\ F \rightsquigarrow X \times F$, with the obvious structural morphisms. Now it is a very easy calculation to see that for any T the pairs of morphisms $T \to E,\ T \to F$ such that

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ E \arrow[dr] & & F \arrow[dl] \\ & Y & \end{tikzcd}$$

(i.e. commuting with the respective halves of the structure of E, F) are in natural one-to-one correspondence with the pairs of morphisms $T \to E \times Z$, $T \to X \times F$ such that

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ E \times Z \arrow[dr] & & X \times F \arrow[dl] \\ & X \times Y \times Z & \end{tikzcd}$$

(i.e. commuting with all the structure of the objects $E \times Z$, $X \times F$ of $\mathcal{E}/X \times Y \times Z$ constructed in the first step.) Thus as a *second* step we may form the fibered product

$$(E \times Z) \underset{X \times Y \times Z}{\times} (X \times F)$$

and know that by the foregoing remark what we have is really just

$$E \underset{Y}{\times} F$$

which only differs from our pre-relational product $E * F$ in that it is being considered as an object over $X \times Y \times Z$, whereas the latter is an object over $X \times Z$. This leads to the *third* step, namely to forget the middle structural map, or formally to apply the functor $\sum_{\pi_{XZ}}$, where $\pi_{XZ} : X \times Y \times Z \to X \times Z$. Since the *relational* product is constructed by three precisely analogous steps, the theorem that $Im : Mat(\mathcal{E}) \to Rel(\mathcal{E})$ preserves composition may thus be analyzed into three separate commutative diagrams as follows:

Notation: Given X, Y, Z write

$$A = X \times Y \qquad B = Y \times Z \qquad C = X \times Z$$

$$D = X \times Y \times Z$$

and let

$$A \times Z \xrightarrow{\pi_A} A, \qquad X \times B \xrightarrow{\pi_B} B$$

$$D \xrightarrow{\pi} C$$

be the obvious projections. This notation will be used merely to translate the following slightly more general proposition into the corollary in which we are actually interested. Note that $A \times Z = D = X \times B$.

Proposition 2.27: *Let A, B, C, D, XZ be objects in a regular category $\mathcal{E}$, let $D \xrightarrow{\pi} C$ be a morphism, and let $A \times Z \xrightarrow{\pi_A} A$, $X \times B \xrightarrow{\pi_B} B$ be the indicated projections. Then the following diagrams of functors are commutative up to equivalence of subobjects.*

$$\begin{tikzcd}
\mathcal{E}/A \arrow[r, "(\,)\times Z"] \arrow[d, "Im_A"'] \arrow[dr, phantom, "(1)"] & \mathcal{E}/A \times Z \arrow[d, "Im_{A\times Z}"] \\
\mathcal{P}_{\mathcal{E}}(A) \arrow[r, "\pi_A^*"'] & \mathcal{P}_{\mathcal{E}}(A \times Z)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{E}/B \arrow[r, "X\times(\,)"] \arrow[d, "Im_B"'] \arrow[dr, phantom, "(1)"] & \mathcal{E}/X \times B \arrow[d] \\
\mathcal{P}_{\mathcal{E}}(B) \arrow[r, "\pi_B^*"'] & \mathcal{P}_{\mathcal{E}}(X \times B)
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{E}/D \times \mathcal{E}/D \arrow[r, "(\,) \underset{D}{\times} (\,)"] \arrow[d, "Im_D \times Im_D"'] \arrow[dr, phantom, "(2)"] & \mathcal{E}/D \arrow[d, "Im_D"] \\
\mathcal{P}_{\mathcal{E}}(D) \times \mathcal{P}_{\mathcal{E}}(D) \arrow[r, "\cap"'] & \mathcal{P}_{\mathcal{E}}(D)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{E}/D \arrow[r, "\sum_\pi"] \arrow[d, "Im_D"'] & \mathcal{E}/C \arrow[d, "Im_C"] \\
\mathcal{P}_{\mathcal{E}}(D) \arrow[r, "\exists_\pi"'] & \mathcal{P}_{\mathcal{E}}(C)
\end{tikzcd}
$$

Proof:

(1) The two forms are proved the same way, starting from the observation that for any $E \xrightarrow{p} A$, or $F \xrightarrow{q} B$ the diagram

$$
\begin{tikzcd}
E \arrow[d, "p"'] & E \times Z \arrow[l] \arrow[d, "p \times Z"] \\
A & A \times Z \arrow[l, "\pi_A"]
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
F \arrow[d, "q"'] & X \times F \arrow[l] \arrow[d, "X \times q"] \\
B & X \times B \arrow[l, "\pi_B"]
\end{tikzcd}
$$

is a pullback.

(2) Given $G_1 \to D,\ G_2 \to D$ factor and pullback as follows

$$
\begin{tikzcd}
G_1 \underset{D}{\times} G_2 \arrow[r] \arrow[d] \arrow[dr] & {} \arrow[r] \arrow[d] & G_2 \arrow[d] \\
\bullet \arrow[r, tail] \arrow[d, tail] & \bullet \arrow[r, tail] \arrow[d] \arrow[dr] & \bullet \arrow[d, tail] \\
G_1 \arrow[r] & \bullet \arrow[r, tail] & D
\end{tikzcd}
$$

The diagonal is clearly the image factorization of $G_1 \underset{D}{\times} G_2 \to D$, and the lower right square is an intersection by construction.

(3) has already been proved before. Again, it uses the diagram

$$
\begin{tikzcd}
{} \arrow[dd, "\text{given}"'] & {} \arrow[d] & \\
& \bullet \arrow[r] \arrow[d, tail] & \bullet \arrow[d, tail] \\
{} & {} \arrow[r] & {}
\end{tikzcd}
$$

the fact that extremal (now regular) epics compose, and uniqueness of image. ∎

Corollary: *For any*

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

we have

$$Im_{x\times Y}(E) * Im_{Y\times Z}(F) = Im_{X\times Z}(E \times_Y F)$$

where $*$ denotes relational composition.

Proof: Interpret the proposition using the preceding "notation" and compose the three diagrams. There results the formula

$$\exists_{\pi_{XZ}}\,[\pi^*_{XY}[Im_{XY}(E)] \cap \pi^*_{YZ}[Im_{YZ}(F)]]$$

$$\cong Im_{XZ}\left[\sum \pi_{XZ}[(E \times Z) \underset{XYZ}{\times} (X \times F)]\right]$$

But the left-hand side is simply the definition of the relational composition on the left in the statement of the corollary, and as previously demonstrated, the part of the right-hand side occurring inside the Im_{XZ} is simply a three-stage analysis of the construction of $E \underset{Y}{\times} F$. ∎

Corollary: *Relational composition is associative.*

Proof: For $X \overset{\alpha}{\longmapsto} Y \overset{\beta}{\longmapsto} Z \quad Im_{X\times Y}(\alpha) = \alpha$ and $Im_{Y\times Z}(\beta) = \beta$. Thus from the formula above

$$\alpha * \beta = Im(\alpha \underset{Y}{\times} \beta)$$

If, moreover, $Z \overset{\gamma}{\longmapsto} W$ is any third relation, then

$$\begin{aligned} (\alpha * \beta) * \gamma &= Im(\alpha \underset{Y}{\times} \beta) * Im(\gamma) \\ &= Im((\gamma \underset{Y}{\times} \beta) \underset{Z}{\times} \gamma) \end{aligned}$$

by another application of the foregoing corollary. But as already remarked, pre-relational composition is obviously associative

$$(\alpha_Y \beta) \underset{Z}{\times} \gamma \cong \alpha \underset{Y}{\times} (\beta \underset{Z}{\times} \gamma)$$

∎

Two important special classes of relations are those of the

Definition: *By a* **partial morphism** $X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$ *is meant a pair consisting of a monic morphism* $X' \rightarrowtail X$ *(the "domain" of the partial morphism) and an arbitrary morphism* $X' \to Y$. *By an* **everywhere-defined relation** $X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$ *is meant a pair* $X \longleftarrow E \longrightarrow Y$ *of morphisms which are not only jointly monic, but for which also the first* $X \longleftarrow E$ *is extremal epic.*

Since a pair $X \longleftarrow \quad \longrightarrow Y$ is certainly jointly monic if one of the morphisms is by itself monic, a partial morphism is a special kind of relation; composing two partial morphisms is simpler than composing general relations since from the pullback

$$\begin{tikzcd}
 & & \cdot \arrow[dl, tail] \arrow[dr] & & \\
 & \cdot \arrow[dl, tail] \arrow[dr] & & \cdot \arrow[dl, tail] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

it is clear that their composition as *pre*-relations is already again a partial morphism (so in particular again a relation); one has: $Dom(fg) = Dom(f) \cap f^{-1}(Dom(g))$ for partial morphisms f, g.

Since for any pre-relation $X \longleftarrow E \longrightarrow Z$ in which $Im_X(E) = X, Im_{XZ}(E)$ is an everywhere-defined relation, it is also clear that in a regular category the composition of two everywhere-defined relations is again an everywhere-defined relation.

Moreover, in a regular category, a relation is the graph of an ordinary morphism iff it is both everywhere-defined and also a partial morphism, since in that case the projection to the first factor is an isomorphism.

Proposition 2.28: *In a regular category a relation* $X \xrightarrow{\alpha} Y$ *is*
a partial morphism iff $\alpha^{-1} * \alpha \subseteq 1_Y$
everywhere-defined iff $1_X \subseteq \alpha * \alpha^{-1}$
the graph of a (unique) morphism iff both the above inclusions are valid.

Proof: If $\alpha =< i, f' >$ is a partial morphism, then the composite relation $\alpha^{-1} * \alpha$ is the image in $Y \times Y$ of the outer maps in the pullback diagram

$$\begin{tikzcd}
 & & X' \arrow[dl, equal] \arrow[dr, equal] & & \\
 & X' \arrow[dl, "f'"'] \arrow[dr, tail, "i"] & & X' \arrow[dl, tail, "i"'] \arrow[dr, "f'"] & \\
Y & & X & & Y
\end{tikzcd}$$

but since these two outer maps are *equal*, the image is contained in the diagonal 1_Y of $Y \times Y$. Conversely, suppose $\alpha =< p, q >$ is any relation with $\alpha^{-1} * \alpha \subseteq 1_Y$; this means we have a commutative diagram

$$\begin{tikzcd}
 & & {} \arrow[dl, "p_1"'] \arrow[d, "h"] \arrow[dr, "p_2"] & & \\
 & {} \arrow[dl, "q"'] \arrow[dr, "p"] & Y \arrow[dll, "id"] \arrow[drr, "id"'] & {} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y & & X & & Y
\end{tikzcd}$$

i.e.

$$p_1 q = h = p_2 q$$

$$p_1 p = p_2 p$$

which implies, since $< p, q >$ is monic, that $p_1 = p_2$; but the pullback p_1, p_2 of a morphism p with itself has $p_1 = p_2$ iff p is monic; hence α is a partial morphism. Now suppose that $\alpha =< p, q >$ everywhere-defined relation and consider

$$\begin{tikzcd}
 & & R_q \arrow[dl] \arrow[dr] & & \\
 & E \arrow[dl, "p"'] \arrow[dr, "q"] & & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & Y & & X
\end{tikzcd}$$

where R_q is actually the equivalence relation determined by q; $\alpha * \alpha^{-1} = Im_{X\times X}(R_q \rightrightarrows E \xrightarrow{p} X)$. Since equivalence relations are always reflexive, there is $E \to R_q$ projecting to the identity on both factors E. Thus we have a commutative square

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d, "R_q"'] & X \arrow[d, "\delta_X"] \arrow[dl, dashed] \\
\alpha * \alpha^{-1} \arrow[r, tail] & X \times X
\end{tikzcd}$$

in which the diagonal arises from the basic factorization property of extremal epics and proves the claimed reflexivity of $\alpha * \alpha^{-1}$. Conversely, if $1_X \subseteq \alpha * \alpha^{-1}$, then p, as the last factor in a composition which yields an extremal epic, is certainly itself extremal epic. As already remarked previously, if $\alpha =< p, q >$ is a relation in which p is an isomorphism, then $\alpha \equiv< X, f >$ is an equivalence of subobjects, where $f = p^{-1} q$ ∎

Proposition 2.29: *If α is the graph of a morphism f, then*

$$\begin{array}{ll} f \text{ is monic} & \text{iff } 1_X = \alpha * \alpha^{-1} \\ f \text{ is extremal} & \text{iff } 1_Y = \alpha^{-1} * \alpha \end{array}$$

Proof: Since α is the graph of a morphism, we have $\alpha^{-1}\alpha \subseteq 1_Y,\ 1_X \subseteq \alpha * \alpha^{-1}$ by the previous proposition. f is monic iff α^{-1} is a partial morphism iff $\alpha * \alpha^{-1} \subseteq 1_X$ iff $1_X = \alpha * \alpha^{-1}$. f is extremal epic iff α^{-1} is everywhere-defined iff $1_Y \subseteq \alpha^{-1} * \alpha$ iff $1_Y = \alpha^{-1} * \alpha$. ∎

Proposition 2.30: *Any relation α can be represented as the composition*

$$\alpha \equiv p^{-1} * q$$

of the inverse of the graph of a morphism, followed by the graph of a morphism. Moreover, this can be accomplished in a "best way" in the sense that if f, g are (the graphs of) any two morphisms for which

$$f^{-1} * g \subseteq \alpha$$

then there is a unique (graph of a) morphism h such that

$$f = h * p$$

$$g = h * q$$

Proof: If $\alpha =< p, q >$ is a jointly monic pair, then

$$\begin{tikzcd} & & E \arrow[dl, equal] \arrow[dr, equal] & & \\ & E \arrow[dl, "p"'] \arrow[dr, equal] & & E \arrow[dl, equal] \arrow[dr, "q"] & \\ Y & & E & & Y \end{tikzcd}$$

is clearly the representation as claimed; if $f^{-1} * g \subseteq \alpha$, then, since $f^{-1} * g$ is just Im_{XY} of

$$\begin{tikzcd} & F \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

so we can apply the adjointness property of Im to complete the proof. ∎

Proposition 2.31: *A pair of morphisms p, q is the pullback of the pair of morphisms u, v iff p, q is the minimal decomposition (as in the preceding* *proposition)* $u * v^{-1} = p^{-1} * q$ *of the composite relation* u *followed by the inverse of* v*.*

Proof: I claim that a square

$$\begin{tikzcd} {} \arrow[rr, "g"] \arrow[dr, "f"'] & & {} \arrow[d, "v"] \\ & {} \arrow[r, "u"] & {} \end{tikzcd}$$

of *morphisms* (not of general relations) is commutative iff

$$f^{-1} * g \subseteq u * v^{-1}$$

namely, if $fu = gv$, then

$$f^{-1} * g \subseteq f^{-1} * g * v * v^{-1} \equiv f^{-1} * f * u * v^{-1} \subseteq u * v^{-1}$$

the first inclusion because v is everywhere-defined and the last because f is (in particular a partial) morphism. Conversely, if we assume $f^{-1} * g \subseteq u * v^{-1}$, then by a similar calculation, we get $g * v \subseteq f * u$; however, since *inversion* reverses composition, preserves inclusion, and is inverse to itself, the *same* assumption implies also that

$$g^{-1} * f \subseteq v * u^{-1}$$

which again by the same kind of calculation, gives $f * u \subseteq g * v$; thus $f * u \equiv g * v$ as relations, which implies $fu = gv$ as morphisms. Having established the claim, the proposition is immediate from the previous proposition. ∎

Remark: The condition $p^{-1} * q \equiv u * v^{-1}$ is thus by itself already stronger than the mere commutativity $pu = qv$ of the square; in fact, $f^{-1} * g \equiv u * v^{-1}$ means that $< f, g >$ maps "onto" the pullback of u, v. The strength of the condition is also indicated by the

Exercise 2.23 Suppose $p^{-1} * q \equiv u * v^{-1}$ (all four morphisms). Make the additional assumption that u is monic. Show that v extremal epic implies p extremal epic (i.e. $v^{-1} * v = 1 \Rightarrow p^{-1} * p = 1$) by purely formal relational calculations without invoking the universal mapping property of the previous two propositions.

The terminal object and image factorization of $\mathcal{E}$ can also be described in the relational formalism:

Exercise 2.24 If $\alpha \subseteq \beta$ and β is single-valued, then α is also single-valued. If $\alpha \subseteq \beta$ and α is everywhere-defined, then β is also everywhere-defined. If $\alpha \subseteq \beta$ and both are (graphs of) morphisms, then $\alpha \equiv \beta$. The terminal object

1 is characterized by the fact that the largest relation $X \longrightarrow\!\!\!\!\!\!\mapsto 1$ is a morphism, for any X.

Exercise 2.25 Given any morphism $X \xrightarrow{f} Y$, consider the $1 \xleftarrow{t^{-1}} X \xrightarrow{f} Y$ and the associated relation $X \overset{t^{-1}*f}{\longmapsto} Y$. The *minimal* decomposition $1 \xleftarrow{p} I \xrightarrow{g} Y$ (as in the penultimate proposition) of the relation $t^{-1} * f$ thus receives a morphism h from the pair t, f. The h is extremal epic and the q monic and thus $f = hq$ is the extremal image factorization of f.

Thus we have been able to express all the ingredients of our discussion of regular categories in terms of the relational calculus, which leads naturally to the converse

Problem: Formulate the "obvious" axioms on a two-dimensional category $\mathcal{R}$ with inversion, so that $\mathcal{R} = Rel(\mathcal{E})$ for a unique *regular* category $\mathcal{E}$. In particular, the regularity axiom itself (not only the monic case as in the exercise) should be a formal consequence of the associativity, etc. of composition and the adjointness (minimality) of $\mathcal{E}^{-1} * \mathcal{E}$ decompositions in $\mathcal{R}$, where $\mathcal{E} \subseteq \mathcal{R}$ is defined by the obvious two conditions as in the proposition on partial morphisms and everywhere-defined relations. A functor $\mathcal{E} \to \mathcal{E}'$ which preserves finite inverse limits and extremal (=regular) epics, is equivalent to a "functor" $\mathcal{R} \to \mathcal{R}'$ which preserves inclusion, composition, inversion, and minimal decompositions. Use this axiomatic theory of relations to give a simple treatment of both the *construction* and the example below by starting with the *relations* of the appropriate category and *deducing* that the morphisms form a regular category as claimed.

Construction: For any regular category $\mathcal{E}$ there is another category $Q(\mathcal{E})$ and a functor $\mathcal{E} \to Q(\mathcal{E})$ such that (Q for "quotient")

1. Every equivalence relation in $Q(\mathcal{E})$ has a coequalizer whose kernel pair is the given relation (effectivity).

2. $Q(\mathcal{E})$ is also a regular category.

3. $\mathcal{E} \to Q(\mathcal{E})$ preserves finite $\varprojlim$ and extremal epics.

4. $\mathcal{E} \to Q(\mathcal{E})$ is an equivalence of categories iff equivalence relations are already effective in $\mathcal{E}$; more generally, a universal mapping property for functors satisfying 1, 2, 3.

Note that an equivalence relation on X may be considered as a relation $X \overset{R}{\longmapsto} X$ satisfying $1_X \subseteq R,\ R^{-1} = R,\ RR = R$. The objects of $Q(\mathcal{E})$ are pairs $< X, R >$ where X is an object of $\mathcal{E}$ and R is an $\mathcal{E}$-equivalence relation on X. A morphism

$$< X, R > \xrightarrow{\alpha} < Y, S >$$

of $Q(\mathcal{E})$ is a *relation* $X \overset{\alpha}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ of $\mathcal{E}$ satisfying $R \subseteq \alpha * \alpha^{-1}, \alpha^{-1} * \alpha \subseteq S$. (Verify easily that $Q(\mathcal{E})$ is then a *category.*) In particular, R itself defines a morphism $< X, 1_X > \rightarrow < X, R >$ which plays the role of the quotient map in $Q(\mathcal{E})$. A *relation* $< X, R > \overset{\alpha}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} < Y, S >$ in the sense of $Q(\mathcal{E})$ is a relation in the sense of $\mathcal{E}$ satisfying $R\alpha S \subseteq \alpha$; this "clearly" gives a *2-dimensional category* $Rel(Q(\mathcal{E}))$.

Example: (Actually, every small regular category is equivalent to one of the types to be described.) Consider any ordered pair consisting of any **theory** formulated in the first-order predicate calculus (classical or intuitionistic), and a **distinguished subclass of the formulas** of the theory which is closed with respect to substitution of variables, conjunction, and existential quantification. We will construct a regular category $\mathcal{T}$ from this data. Recall that the formulas of the theory are constructed from a set of "atomic formulas" (or "relational symbols") with finite numbers of argument variables and in fact constitute all meaningful combinations of the atomic formulas by means of the connectives $\wedge, \vee, \Rightarrow, \neg, \exists, \forall$; we assume that the theory of equality is part of the theory; the axioms of the theory are a given set of formulas which are *closed* (i.e. all variables bound by $\forall$ or $\exists$) and $\vdash \phi$ means that ϕ is a closed formula which can be deduced from the axioms by the rules of inference. The distinguished subclass is only supposed to be closed with respect to $\wedge, \exists$ and substitution and to contain "equality" and will usually not consist only of closed formulas; we will denote by single variables such as x the *finite string of all the free variables* of a formula $\phi(x)$ (usually one from the distinguished subclass). We allow the theory to have an arbitrary set of "sorts" (i.e. not necessarily only one "universe of discourse"; of course, if there were only a finite set of sorts, we could consider it to be one by forming disjoint union); thus the variables would, in principle, be labelled somehow to indicate which sort they range over and/or each sort itself may be considered as a particular atomic formula. Now we say that an *object* of $\mathcal{T}$ is simply any formula of our distinguished class. For example, "the" terminal object is any true formula such as $x = x$ and the cartesian product of $\phi_1(x_1)$ with $\phi_2(x_2)$ is simply the conjunction $\phi_1(x_1) \wedge \phi(x_2)$ (with twice as many *free* variables; actually, this description only applies in case x_1, x_2 are *disjoint* sets of variables, otherwise we first change them all to make them disjoint).

By a *relation* $\phi_1 \longrightarrow\!\!\!\!\!\!\!\!+\;\; \phi_2$ of $\mathcal{T}$ is meant any distinguished formula $\psi(x_1, x_2)$ whose free variables include those of ϕ_1, ϕ_2 (assumed disjoint for simplicity) and for which

$$\vdash \forall x_1 \forall x_2 [\psi(x_1, x_2) \Rightarrow \phi(x_1) \wedge \phi(x_2)]$$

By a *morphism* $\phi_1 \to \phi_2$ of $\mathcal{T}$ is meant any ψ as above for which also

$$\vdash \forall x_1 [\phi_1(x_1) \Rightarrow \exists ! x_2 \psi(x_1, x_2)]$$

We claim that $\mathcal{T}$ is then a regular category, with moreover the property that any subobject ϕ' of ϕ is isomorphic to one with the same free variables x as ϕ, in which form we have

$$\vdash \forall x [\phi'(x) \Rightarrow \phi(x)]$$

Any model of the theory will lead to a *functor*

$$\mathcal{T} \xrightarrow{M} \mathcal{S}$$

into the category of sets which preserves finite $\varprojlim$ and extremal epics; in general such a functor might be called a "weak model" since in general we cannot assure that it preserves the axioms which involve logical operators other than $\wedge, \exists$. (However, a good collection of "weak models" can be used as the "stages of knowledge" in constructing a Kripke-style intuitionistic model; also, in case the logic is classical and our distinguished class consists of all formulas and we assume, moreover, that M preserves finite sups ($\vee$) of subobjects, then M will be a model in the usual sense.) Still more generally, we can obviously consider "weak models" with values in any regular category $\mathcal{E}$, not only in sets, in which context the identity functor on $\mathcal{T}$ itself clearly serves as the *universal* (weak) model of the original theory. There is an intrinsic reason for considering a distinguished subclass of formulas, namely the need to account for the notion of morphism of models. That is, if we say that a morphism $M \xrightarrow{f} M'$ of models is a mapping (i.e. a family of mappings, one for each sort) such that

$$\phi_M(x) \Rightarrow \phi_{M'}(f(x))$$

for each (n-tuple of elements) x of M and for each "atomic" formula ϕ, then it follows that the same holds for all formulas obtained from the atomic formulas by means of the operations $\wedge, \vee, \exists$ (but *not* for the operations $\forall, \Rightarrow, \neg$). This shows that a natural transformation defined on all $\mathcal{T}$

$$\begin{array}{ccc} & M & \\ & \longrightarrow & \\ \mathcal{T} & \downarrow^{f} & \mathcal{E} \\ & \longrightarrow & \\ & M' & \end{array}$$

is the correct notion of "morphism of models". Actually, these (and other) considerations suggest that instead of regular categories we should consider "regular categories with stable $\vee$", meaning regular categories with the additional property that any finite family of subobjects of an object has a *supremum*, and that these "unions" are (like images) *preserved by pulling back* along an arbitrary morphism, and moreover between such categories we should consider functors $\mathcal{E} \to \mathcal{E}'$ which preserve finite $\varprojlim$, images, and finite sups. We do not consider in detail this theory, as there is a still more "stable" notion, namely that of *pretopos.* A pretopos has finite $\varprojlim$, finite coproducts $(+)$ which are "universal and disjoint", coequalizers of equivalence relations, and the property that *every* epimorphism is universal effective; pretoposes can be "presented" somewhat as in the present example, except that as suggested above, the distinguished class is also required to be closed with respect to $\vee$. On the other hand, there is a one-to-one correspondence between pretoposes and certain toposes, namely, the *coherent* ones (the latter being a "finiteness" condition on a topos which arose quite naturally in algebraic geometry (e.g. sh(spec A) is coherent for any commutative ring A, as is the "Zariski topos" and also the "étale topos"), but as these remarks indicate, is also intimately connected with the geometric approach to intuitionistic logic and its model theory).

Finally, we remark that one technical advantage of having all equivalence relations effective in a regular category, is that the problem of "descent" can be, at least theoretically, solved:

Exercise 2.26 Suppose $\mathcal{E}$ is a regular category in which every equivalence relation has a coequalizer, and suppose X is an object for which $X \to 1$ is extremal epic (i.e. for which $X \times X \rightrightarrows X \to 1$ is a coequalizer diagram). Denote by X^2 the object of $\mathcal{E}/X$ which is $X \times X$ equipped with the projection on the first factor as structural morphism. For any object E of $\mathcal{E}/X$, denote by $X^2 \underset{X}{\times} E \xrightarrow{p_1} E$ the projection for the indicated product in $\mathcal{E}/X(=$ a certain pullback in $\mathcal{E}$ itself). By descent data for E in $\mathcal{E}/X$, we mean any morphism $X^2 \underset{X}{\times} E \xrightarrow{p_2} E$ in $\mathcal{E}$, such that $< p_1, p_2 >$ is an *equivalence relation* on E (considered as an object of $\mathcal{E}$). If p_2 is descent data for E and p_2' is descent data for E' then by a *morphism of descent data* $E, p_2 \to E', p_2'$ is meant any morphism $E \xrightarrow{f} E'$ of $\mathcal{E}/X$ such that

$$
\begin{tikzcd}
X^2 \underset{X}{\times} E \arrow[r] \arrow[d, "<p_1,p_2>"'] & X^2 \underset{X}{\times} E' \arrow[d, "<p_1',p_2'>"] \\
E \times E \arrow[r, "f \times f"'] & E' \times E'
\end{tikzcd}
$$

in $\mathcal{E}$ (i.e. an inclusion involving either inverse or direct images). This defines the category $\mathcal{D}(\mathcal{E}, X)$ of descent data over X. Show that under the stated hypothesis $X \to 1$ on X, the functor

$$\mathcal{E} \to \mathcal{D}(\mathcal{E}, X)$$

defined by $F \rightsquigarrow X \times F$ is an equivalence of categories, with inverse given by passing to the quotient $E, p_2 \rightsquigarrow E/X^2 \underset{X}{\times} E$.

The relationship of regular categories having quotients to cohomology, both abelian and non-abelian, is further indicated by

Exercise 2.27 Suppose $\mathcal{E}$ is a regular category (with 1 as usual) in which every equivalence relation is a kernel pair. Then the category Grp $(\mathcal{E})$ of group objects in $\mathcal{E}$ has the same properties. The category Ab$(\mathcal{E})$ of *abelian* group objects in $\mathcal{E}$ is an *abelian* category (by the preceding sentence, it is the same to say that any *additive* category which satisfies the regularity axiom and in which equivalence relations are kernel pairs, is in fact an abelian category). Thus, for example, all the possible definitions of abelian groups and homomorphisms within any given first-order theory (say the theory of commutative rings, or number theory) form an abelian category. Here a "definition" should be a triple of formulas, one to define the "elements" of the group, one to define the addition, and one to define the "equality" relation. In fact, we may even limit ourselves to any distinguished class of formulas closed with respect to $\exists, \wedge$ for the possible definitions and still get an abelian category.

Lesson 3

1. Introduction to Yoneda's lemma and Kan extension

The central goal of this lesson will be to explain the contradiction between *small* categories $\mathbf{C}$ and *cocomplete* categories $\mathcal{X}$, the basic formula of which is the equivalence

$$Fun(\mathbf{C}, \mathcal{X}) \simeq Adj_{\mathrm{Right}}(\mathcal{X}, \mathcal{S}^{\mathbf{C}^{op}})$$

between arbitrary functors

$$\mathbf{C} \xrightarrow{F} \mathcal{X}$$

and arbitrary *adjoint pairs* of functors

$$\mathcal{X} \underset{F_*}{\overset{F^*}{\rightleftarrows}} \mathcal{S}^{\mathbf{C}^{op}} \qquad F^* \dashv F_*$$

between $\mathcal{X}$ and the category of *presheaves* on $\mathbf{C}$ (= the category of "left-$\mathbf{C}$-sets"). The correspondence in one direction is

$$F_*(X)(C) = \mathcal{X}(F(C), X)$$

$$F^*(T) = F \underset{\mathbf{C}}{\otimes} T = \varinjlim_{t \in \mathbf{C}/T} F(C_t)$$

where the last notation will be explained in detail presently. In the other direction, F is the restriction of F^* along the full and faithful Yoneda inclusion

$\mathbf{C} \hookrightarrow \mathcal{S}^{\mathbf{C}^{op}}$. Many fundamental mathematical constructions are special cases of the above for certain $\mathbf{C}, F, \mathcal{X}$; for example, with $\mathcal{X}$-topological spaces and continuous maps, $\mathbf{C}$ = finite well-ordered sets and order preserving maps, F = standard simplices (= one point space, interval, triangle, solid tetrahedron, etc.) then F_* is the singular semi-simplicial complex of a topological space and F^* is the geometric realization of a semi-simplicial set. On the other hand, the above correspondence is also the fundamental tool or starting point of nearly all investigations into the structure of large categories $\mathcal{X}$, where one tries to find a $\mathbf{C}$ and F such that F_* is full and faithful and hence so that the objects of $\mathcal{X}$ can be represented "concretely" (i.e. abstractly) as *certain* contravariant set-valued functors on a small category $\mathbf{C}$; this program has been most successful in two cases, namely universal algebra (where $\mathbf{C}$ can be taken to be a suitable category of finitely-generated algebras, and in topos theory (general sheaf theory), in which $\mathbf{C}$ is a "site" (e.g. a basis for the open sets of a topological space). In fact, these two cases can be more or less characterized by the slogan F^* preserves finite $\varprojlim$ in topology and F_* preserves filtered $\varinjlim$ in algebra, the intersection of these two cases being algebraic geometry = geometric logic. But these more refined "exactness" conditions will be studied in the next lesson; here we concentrate on the basic correspondence itself.

The notions *small* category and *cocomplete* category can perhaps best be understood as two kinds of additional structure that a general category might have, these being in fact two specific kinds of relationships with the category $\mathcal{S}$ of abstract sets. One of the main themes of this course is to investigate the extent to which the "base" category $\mathcal{S}$ can be usefully generalized, to a topos or generally to a regular category – in fact, the basic notions of a category $\mathbf{C}$ *in* a category $\mathcal{S}$ and of the category $\mathcal{S}^{\mathbf{C}^{op}}$ of $\mathcal{S}$-valued presheaves in $\mathcal{S}$ on $\mathbf{C}$ will be meaningfully defined for any category $\mathcal{S}$ which has pullbacks. On the other hand, the notion of a large category $\mathcal{X}$ over $\mathcal{S}$ has not yet received a final definition and thus will vary according to need, as will consequently the conditions required on a category $\mathcal{S}$ to support it. In fact, in order to study this question carefully we will not immediately go to the most general case, but consider first two classical cases of the Yoneda construction, due to Dedekind and Cayley, and see to what extent they can be generalized; these concern representing an arbitrary ordered set by *inclusions* between some of the *subsets* of an appropriate set, respectively representing an arbitrary monoid (e.g. group) by *composition* among some of the *endomorphisms* of an appropriate set, and using modern ideas we can show that these classical constructions are universally the most natural ones in a very far-reaching sense which helps to explain many other constructions. Another useful interpretation of the Dedekind-Cayley-Yoneda inclusion $\mathbf{C} \hookrightarrow \mathcal{S}^{\mathbf{C}^{op}}$ is that it amounts to a *completion* of $\mathbf{C}$; in fact, that basic formula above considered for a variable $\mathcal{X}$ shows that $\mathcal{S}^{\mathbf{C}^{op}}$ is the universal co-complete category containing $\mathbf{C}$. This is not unrelated to, e.g. the construction of the reals from the rationals, but the extent to which a *two-sided* completion can be obtained is really a question of "exactness properties" which will be treated in more detail in the next lesson.

There is also a third special case of importance, which when $\mathcal{S}$ = sets is just the case $\mathbf{C}$ = a discrete category; however, when $\mathcal{S}$ is "general", the description of the structure relating $\mathcal{X}$ to $\mathcal{S}$ which is necessary for formulating the above basic formula in this discrete case will amount to an axiomatic description of the notion of "family" which will have geometric applications. It is difficult to attach a specific historical name to this third special case, perhaps since when $\mathcal{S}$ = sets it reduces to the "banal" formulas

$$F_*(X)_i = \mathcal{X}(F_i, X) \quad i \in I = \mathbf{C}$$

$$F^*(T) = \sum_{i \in I} F(T_i)$$

(actually, even when $\mathcal{S}$ = sets these formulas are not entirely banal from the logical point of view, since their usual interpretation via the identification $\mathcal{S}/I \cong \mathcal{S}^I$ precisely invokes questions of inaccessible cardinals, so in particular of regular cardinals and of the extent of the "replacement schema", etc.). In case $\mathcal{S} = \mathcal{E}$ is a "general" category, was a general principle already formulated before 1960 by Grothendieck, by the writer, and others, namely that to study an object X of an arbitrary "kind" $\mathcal{E}$ is equivalent to the study of an object $\mathcal{E}/X$ of the fixed kind Cat; around the same time Bénabou, the writer, and others had begun to develop the formalism whereby small categories $\mathbf{C}$ and "profunctors" are treated as structured "matrices" in $\mathcal{S}$, and had noticed that the same formalism could be applied to more general categories (the basis for this has already been touched upon in Lesson 2); since 1965 the writer has insisted that an axiomatic theory of "families" is the necessary complement to the Eilenberg-Kelly theory of *closed* categories, and since 1969 the necessary basis for such a theory of families has been available in the form of the elementary topoi, although to my knowledge such a theory itself has never been published. Actually, in their original paper on categories, Eilenberg and Mac Lane had given a representation theorem for arbitrary categories $\mathbf{C} = \mathcal{E}$ which when made more "precise" leads *either* to the Yoneda embedding $\mathbf{C} \hookrightarrow \mathcal{S}\mathbf{C}^{op}$ *or* to the Grothendieck principle $\mathcal{E} \hookrightarrow Cat/\mathcal{E}[X \rightsquigarrow \mathcal{E}/X]$; inseparably connected with the present circle of ideas is in fact *this* contradiction just cited (between the two kinds of refinement of the representation theory) which leads to the theory of fibered categories and to a functor

$$Cat/\mathbf{C} \to \mathcal{S}^{\mathbf{C}^{op}}$$

whose adjoint is from a narrow logical point of view just the replacement scheme of set theory, but which in fact captures an aspect undreamt of by such a point of view, namely a first approximation to a theory of the internal contradictions (a certain category $/\mathbf{C}$) which gave rise to a certain mode of development (a given left $\mathbf{C}$-set). Thus again it is clearly of importance not to limit these considerations to $\mathcal{S}$ = abstract sets.

Let us close this introduction by making somewhat more explicit the workings of the basic formula on page 1 in the case where also $\mathcal{X}$ is of the form $\mathcal{S}^{\mathbf{D}^{op}}$ for some small category $\mathbf{D}$ (in fact many times the "first approximation" to a geometrical or algebraic functor is calculated in just this way, which may be considered as a formalism of "$\mathbf{D}$ - $\mathbf{C}$ bimodules" or "*pro*functors $\mathbf{C} \longrightarrow\!\!\!\!\!\!\mapsto \mathbf{D}$ "). The Yoneda embedding

$$\mathbf{C} \hookrightarrow \mathcal{S}^{\mathbf{C}^{op}}$$

sends an object C of $\mathbf{C}$ to the left $\mathbf{C}$-set whose A-th stage is the set $\mathbf{C}(A, C)$ of $\mathbf{C}$ morphisms from A to C and whose $\mathbf{C}$-action is just $x \rightsquigarrow ax$ for any element x of the just-named set and any morphism $A' \xrightarrow{a} A$ in $\mathbf{C}$. If we seriously consider Yoneda's embedding as an inclusion, i.e. identify any C notationally with the left $\mathbf{C}$-set just described, then Yoneda's lemma states that for any left $\mathbf{C}$-set T we may identify its elements at stage C with $\mathcal{S}^{\mathbf{C}^{op}}$-morphisms $C \to T$ and its internal $\mathbf{C}$-action with compositions $C' \to C \to T$ of $\mathcal{S}^{\mathbf{C}^{op}}$-morphisms. Thus the defining property of $\mathcal{S}^{\mathbf{C}^{op}}$–morphisms $T \xrightarrow{f} T'$, namely that they are homomorphisms with regard to the respective $\mathbf{C}$-actions, becomes a special case of the *associativity* of composition

$$\begin{tikzcd}
 & C \arrow[dd, "x"] \arrow[dr, "xf"] & \\
C' \arrow[ur, "u"] \arrow[dr, "ux"'] & & T' \\
 & t \arrow[ur, "f"'] &
\end{tikzcd}$$

of $\mathcal{S}^{\mathbf{C}^{op}}$–morphisms, namely the special case where actually $u \in \mathbf{C}$. Now, in case $\mathcal{X} = \mathcal{S}^{\mathbf{D}^{op}}$, a functor $\mathbf{C} \xrightarrow{F} \mathcal{S}^{\mathbf{D}^{op}}$ is fairly clearly equivalent to a "$\mathbf{D}$ - $\mathbf{C}$ bimodule", i.e. for each pair of objects D, C there is a set DFC and for morphisms $D' \xrightarrow{w} D$ of $\mathbf{D}$, $C \xrightarrow{u} C'$ of $\mathbf{C}$ there are mappings

$$DFC \longrightarrow D'FC$$

$$f \rightsquigarrow wf$$

$$DFC \longrightarrow DFC'$$

$$f \rightsquigarrow fu$$

which are identities if the morphisms are, and which satisfy

$$\begin{aligned} (wf)u &= w(fu) \\ (w_1 w)f &= w_1(wf) \quad D'' \xrightarrow{w_1} D' \\ f(uu_1) &= (fu)u_1 \quad C' \xrightarrow{u_1} C'' \end{aligned}$$

Now in case F is as just described and T is any left-$\mathbf{C}$-set, we can describe quite explicitly the left $\mathbf{D}$-set $F \underset{\mathbf{C}}{\otimes} T$ called for on page 1 as follows: every one of its elements is represented by a triple

$$f \underset{C}{\otimes} t$$

where $f \in DFC$ and t is an element of the C-th stage of T (in fact such represents an element at stage D of $F \underset{C}{\otimes} T$; however, there must be identifications

$$fu \underset{C'}{\otimes} t' \equiv f \underset{C}{\otimes} ut'$$

between these representing symbols whenever $C \xrightarrow{u} C'$ in $\mathbf{C}$. These identifications force "associativity" in "diagrams" of the sort

$$\begin{tikzcd} & C \arrow[dd, "u"'] \arrow[dr] & \\ D \arrow[ur, "f"] \arrow[dr] & & T \\ & C' \arrow[ur, "t'"'] & \end{tikzcd}$$

which combine F and T and in which we have taken the liberty of denoting even the elements of F by arrows (in fact it is easy to make a "triangular matrix" category containing $\mathbf{D}$, $\mathbf{C}$ and F in which these arrows become morphisms). The action of $\mathbf{D}$ on $F \underset{\mathbf{C}}{\otimes} T$ must clearly be

$$w(f \underset{C}{\otimes} u) \underset{\text{def}}{=} (wf) \underset{C}{\otimes} u$$

and indeed it is then easy to verify the adjointness $F^* \dashv F_*$, i.e.

$$(F \underset{\mathbf{C}}{\otimes} T, X) \cong (T, Hom_{\mathbf{D}}(F, X))$$

claimed on page 1 (the indicated set of **D**-homomorphisms on the left is naturally isomorphic to the indicated set of **C**-homomorphisms on the right). If also $\mathbf{D} \xrightarrow{G} \mathcal{S}^{\mathbf{E}^{op}}$ is considered as bimodule, then the "profunctorial product of F with G", namely the bimodule $G \underset{\mathbf{D}}{\otimes} F$ whose corresponding adjoint pair of big functors is the composition of the adjoint pairs of G and of F, is the "matrix" product modulo the conflicts in **D**-actions in the middle, i.e.

$$E(G \underset{\mathbf{D}}{\otimes} F)C = \sum_D EGD \times DFC \Bigg/ \begin{array}{l} g_1 w \otimes f_2 \equiv g_1 \otimes w f_2 \\ \text{for} \\ D_1 \xrightarrow{w} D_2 \text{ in } \mathbf{D} \end{array}$$

i.e. a formalism of representing triples and identifications can be developed just as for $F \underset{C}{\otimes} T$. Then surely $(G \otimes F) \otimes T \cong G \otimes (F \otimes T)$ all T so that, taking adjoints

$$\begin{tikzcd} \mathcal{X}' \arrow[rr, "(G\otimes F)_*"] \arrow[dr, "G_*"'] & & \mathcal{S}^{\mathbf{C}^{op}} \\ & \mathcal{X} \arrow[ur, "F_*"'] & \end{tikzcd}$$

as claimed, where $\mathcal{X}' = \mathcal{S}^{\mathbf{E}^{op}}, \mathcal{X} = \mathcal{S}^{\mathbf{D}^{op}}$.

Any little functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$ can be considered as a profunctor by composing it with the Yoneda embedding for **D**; the resulting bimodule

$$DFC = \mathbf{D}(D, f(C))$$

But there is in this case also an induced profunctor in the *opposite* direction

$$Cf^{-1}D = \mathbf{D}(f(C), D)$$

which turns out to be *adjoint* to f in an appropriate sense; conversely any pair of adjoint bimodules arises from a functor under the very weak condition on **D** that "idempotents split". We have clearly here a strong analogue or generalization of the "relational" calculus, in which sets are replaced by small categories and "truth values" are replaced by sets; it can also be used in an analogous way, e.g. compositions $f^{-1} \otimes g$ of profunctors under certain conditions may again be functors and in general can be "decomposed" etc.

Remark that in order to make the basic formula of page 1 an equivalence of categories as claimed, it is necessary to define morphisms of adjoint pairs as natural transformations between the *left* adjoints, so that $F_1 \to F_2$ iff $F_1^* \to F_2^*$.

Let us now consider some special cases of the theory just outlined.

2. Dedekind's construction for partially ordered sets

Let $\mathbf{C} = \mathbf{P}$ be a set equipped with a reflexive and transitive relation $\leq$.

Definition: *By a* **crible** T *in* $\mathbf{P}$ *(also called an "order ideal") is meant any subset* T *of* P *satisfying the condition*

$$t' \leq t, t \in T \Rightarrow t' \in T$$

The cribles in $\mathbf{P}$ are considered also as a partially-ordered set by means of the inclusion $T_1 \subseteq T_2$ of subsets of $\mathbf{P}$. Let us denote by $\mathbf{2}^{\mathbf{P}^{op}}$ the ordered set of all cribles in $\mathbf{P}$.

Proposition 3.1: *(Dedekind - Yoneda) For any element* p *of* $\mathbf{P}$

$$T_p = \{x | x \in \mathbf{P} \text{ and } x \leq p\}$$

is a crible in $\mathbf{P}$. *For any crible* T *of* $\mathbf{P}$ *and any element* p *of* $\mathbf{P}$*, we have*

$$p \in T \Longleftrightarrow T_p \subseteq T$$

and in particular for any two elements p_1, p_2 *of* P*,*

$$p_1 \leq p_2 \Longleftrightarrow T_{p_1} \subseteq T_{p_2}$$

Thus $p \rightsquigarrow T_p$ *defines a full and faithful embedding* $\mathbf{P} \hookrightarrow \mathbf{2}^{\mathbf{P}^{op}}$ *of partially ordered sets.*

Proof: Exercise. We may call a special crible of the form T_p a *representable* one.

Proposition 3.2: *If a family of elements* p_i $i \in I$ *of* $\mathbf{P}$ *happens to have an infimum*

$$p = \inf_{i \in I} \quad p_i$$

in $\mathbf{P}$*, then the Dedekind-Yoneda embedding preserves it, i.e.*

$$T_p = \bigcap_{i \in I} T_{p_i}$$

However, such a statement is not true for suprema; in particular, no T_p *is empty (even* T_0 *for 0 = a possible smallest element of* $\mathbf{P}$*), but the empty subset of* $\mathbf{P}$ *is always the smallest element of* $\mathbf{2}^{\mathbf{P}^{op}}$ *(this is the case of suprema of the empty family indexed by* $I = 0$*).*

Proof: Exercise. ∎

Note that for empty *infima*, the statement says that if 1 is a greatest element of $\mathbf{P}$, then $T_1 = \mathbf{P}$. But in any case $\mathbf{2}^{\mathbf{P}^{op}}$ has suprema (and thus is a "completion" of $\mathbf{P}$ (in a one-sided sense)):

Proposition 3.3: *If $T_i, i \in I$ is any family of cribles (representable or not), then*

$$T = \bigcup_{i\in I} T_i$$

is again a crible (usually not representable). Every crible T is the union of a canonical system $T_t, t \in T$ of representable cribles, namely the family of cribles represented by all the elements of T itself.

Proof: If $x \leq y \in \bigcup_{i\in I} T_i$, then $y \in T_i$, for some i, hence $x \in T_i$, since T_i is a crible; but then also $x \in \bigcup_{i\in I} T_i$, which proves that the union is again a crible. For the second assertion, it is clear that $\bigcup_{t\in T} T_t \supseteq T$, since $t \in T_t$; for the converse inclusion, suppose $x \in \bigcup_{t\in T} T_t$, then $x \in T_t$, for some $t \in T$, i.e. $x \leq t$, for some $t \in T$, which implies $x \in T$ since T is a crible. ∎

The second assertion of the foregoing proposition means that $\mathbf{2}^{\mathbf{P}^{op}}$ is "generated" (with respect to the operation of forming arbitrary unions) by its subset $\mathbf{P} \hookrightarrow \mathbf{2}^{\mathbf{P}^{op}}$; hence any mapping defined on $\mathbf{2}^{\mathbf{P}^{op}}$ which preserves this operation is certainly determined by its restriction to $\mathbf{P}$. Conversely, such a mapping can be prescribed arbitrarily on $\mathbf{P}$, provided only that it is order-preserving; more exactly,

Proposition 3.4: *Let $\mathcal{X}$ be any (co-)complete partially-ordered set, i.e. any family of elements $x_i, i \in I$ of $\mathcal{X}$ has a supremum in $\mathcal{X}$. Let $F : \mathbf{P} \to \mathcal{X}$ be any order-preserving mapping, i.e. $p_1 \leq p_2$ in $\mathbf{P}$ implies $F(p_1) \leq F(p_2)$ in $\mathcal{X}$. Then the mapping $F^* : \mathbf{2}^{\mathbf{P}^{op}} \to \mathcal{X}$ defined by the formula*

$$F^*(T) = \sup_{t\in T} F(t)$$

preserves suprema, i.e.

$$F^*(\bigcup_{i\in I} T_i) = \sup_{i\in I} F^*(T_i)$$

for any family $T_i, i \in I$ of cribles on $\mathbf{P}$, and also the restriction on F^* (along the Dedekind-Yoneda embedding) to $\mathbf{P}$ is again F itself, i.e.

$$F^*(T_p) = F(p) \text{ all } p \in P$$

(Moreover, F^* is the only mapping with these two properties, as already remarked.)

Proof: Recall that $\sup\limits_{i\in I} x_i$ is characterized by the equivalence

$$\frac{\sup\limits_{i\in I} x_i \leq x}{x_i \leq x \text{ for all } i \in I}$$

for any $x \in \mathcal{X}$. From this it is easy to establish a general associativity formula for iterated suprema in any partially-ordered set $\mathcal{X}$ in which suprema exist: namely, if $J_i, i \in I$ is a family of index sets and $\{x^i_j, j \in J_i\}, i \in I$ is a family of families of elements of $\mathcal{X}$ indexed by them, then

$$\sup_{j\in\bigcup\limits_{i\in I} J_i} x^i_j = \sup_{i\in I} \sup_{j\in J_i} x^i_j.$$

Thus in particular we have

$$\begin{aligned} F^*(\bigcup_{i\in I} T_i) &= \sup_{t\in\bigcup\limits_{i\in I} T_i} F(t) \\ &= \sup_{i\in I} \sup_{t\in T} F(t_i) \\ &= \sup_{i\in I} F^*(T_i) \end{aligned}$$

Another general fact about suprema is that if a family $x_i, i\in I$ of elements of $\mathcal{X}$ has the property that there exists $i_1 \in I$ such that

$$x_i \leq x_{i_1} \text{ for all } i\in I$$

then

$$\sup_{i\in I} x_i = x_{i_1}$$

Thus in particular we have for a *representable* crible that

$$F^*(T_p) = \sup_{t\in T_p} F(t) = F(p)$$

since p is the greatest element of T_p and hence $F(p)$ is the greatest element of the family $\{F(t)|t \in T_p\}$ since F is order-preserving. ∎

For complete ordered sets there is no difficulty in showing that one-sided continuity is equivalent to having an adjoint; in particular, in our case we have

Proposition 3.5: *With the notation of the foregoing proposition, define for each element x of $\mathcal{X}$ a subset of* $\mathbf{P}$ *as follows*

$$F_*(x) = \{p|F(p) \leq x\}$$

Then $F_*(x)$ is a crible in $\mathbf{P}$, and the mapping F_* is the right adjoint of the mapping F^*, i.e.

$$\frac{T \subseteq F_*(x)}{F^*(T) \leq x}$$

for any crible T of $\mathbf{P}$ and any element x of $\mathcal{X}$ (where the horizontal bar may be read in this case simply as "iff").

Proof: Suppose $p' \leq p$ and $F(p) \leq x$. Then since F is order-preserving, we have $F(p') \leq F(p) \leq x$ which implies that p', like p, is an element of $F_*(x)$; thus $F_*(x)$ is a crible. Suppose T is any crible and $T \subseteq F_*(x)$; then we must show that $F^*(T) \leq x$. But $F^*(T) = \sup\limits_{t \in T} F(t)$, and so $F^*(T) \leq x$ is equivalent to the condition $F(t) \leq x$ for all $t \in T$; the supposition is that

$$t \in \{p | F(p) \leq x\} \text{ for all } t \in T$$

which clearly implies $F(t) \leq x$ for all $t \in T$, and hence we have shown $F^*(T) \leq x$. Conversely, suppose that $F^*(T) \leq x$, i.e. that $F(t) \leq x$ for all $t \in T$, then for any $t \in T$, we have $t \in F_*(x)$, i.e. $T \subseteq F_*(x)$. Thus the adjointness condition is proved.

For any adjoint pair $\mathcal{X} \rightleftarrows \mathbf{2}^{\mathbf{P}^{op}}$, the left adjoint part F^* must preserve sups, and hence the whole adjoint pair must arise from an F as above, completing the proof of this "miniature" version of the basic formula. ∎

A special case ($\mathbf{C}$ = poset) of the basic formula discussed on page 1 of this lesson is the extension of the foregoing results from two values $\mathbf{2}$ = {false, true} to arbitrary set-values, i.e. the extension from cribles $\mathbf{2}^{\mathbf{P}^{op}}$ to "inverse systems" of sets $\mathcal{S}^{\mathbf{P}^{op}}$ parameterized by $\mathbf{P}$ (but we have not necessarily assumed that $\mathbf{P}$ is "directed") and from complete posets $\mathcal{X}$ to cocomplete categories $\mathcal{X}$ (the assumption that $\mathcal{X}$ has $\varinjlim$ over all posets is not really any more general than the assumption that $\mathcal{X}$ has $\varinjlim$ over any small abstract diagram scheme, since coproducts are $\varinjlim$s over discrete posets and pushouts are $\varinjlim$s over a certain three-element poset, and we can apply the dual of a proposition from Lesson 2 in which the finiteness of the products in reality played no role in the proof), with the accompanying extension from order-preserving mappings F to arbitrary "direct systems" (not necessarily *directed*)$F : \mathbf{P} \to \mathcal{X}$ in $\mathcal{X}$.

We can formulate this special case now as an

Exercise 3.1 Let $\{P_1 \rightrightarrows P_0\} = \mathbf{P}$ be a poset (P_0 is the set of elements and P_1 the *order relation*). By an inverse system T of sets over $\mathbf{P}$ is meant a family of sets with an action, i.e. formally a triple consisting of a set T of elements, a mapping $T \to P_0$ specifying "at which stage" an element of T

lives, and a mapping $P_1 \underset{P_0}{\times} T \to T$ which specifies the "action" or "transitions" or "development" in T and which is subject to the following three conditions. The pullback $P_1 \underset{P_0}{\times} T$ is supposed to be taken with respect to the *second* ("greater") of the two projections $P_1 \rightrightarrows P_0$, so that a typical element of $P_1 \underset{P_0}{\times} T$ is a triple $< p', p, t >$ such that $p' \leq p$ and such that t is an element of T which lives at stage p; $P_1 \underset{P_0}{\times} T$ has its own structural map to P_0, namely the one induced by the first ("smaller") projection $P_1 \to P_0$, which consists of forgetting p, t, but remembering p'. The first condition on the action is that

$$\begin{tikzcd} P_1 \underset{P_0}{\times} T \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & P_0 & \end{tikzcd}$$

commutes, which expresses that the development considered as a mapping assigns to t the element from which it came at the earlier time p' (the transitions considered as mappings point backward, hence the name "inverse" system). The other two conditions to which the action of $\mathbf{P}$ on T are subject, are an identity law and an associative law corresponding to the reflexivity and transitivity properties of $\mathbf{P}$ itself. A crible on P is just that special case of an inverse system for which $T \rightarrowtail P_0$ is monic. Cribles may also be identified as the subobjects of the terminal object in the category $\mathcal{S}^{\mathbf{P}^{op}}$, in which a morphism $T \to T'$ is a mapping $T \to T'$ which satisfies

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & T' \arrow[dl] \\ & P_0 & \end{tikzcd}$$

(so that it may also be considered as a "family" of mappings, one for each stage p) and which also commutes with the respective actions. Now each inverse system T gives rise also to a new partially ordered set $\mathbf{P}/T$ whose elements are the elements of T itself, but in which the ordering is defined as follows:

$t' \leq t$ iff $p' \leq p$ and the action of $\mathbf{P}$ on T takes t to t',
where p' (respectively p)
are the stages at which t' (respectively t) live.

Moreover, there is a canonical "direct" system of *representable cribles* indexed by T, namely it assigns T_p to t where p is the stage at which t lives (one might say that T_p is simply, *from the point of view of* t, the entire past, i.e. the set of all times at which t has lived). Now the category $\mathcal{S}^{\mathbf{P}^{op}}$ has arbitrary $\varinjlim$, and moreover every object T is canonically the direct limit of representable cribles

$$T = \varinjlim_{t \in \mathbf{P}/T} T_p \text{ (where } t \rightsquigarrow p)$$

More generally, if $\mathcal{X}$ is "any" category, then by a "direct" $\mathbf{P}$-system F in $\mathcal{X}$ is meant a family

$$F(p), \qquad p \in P_0$$

of objects of $\mathcal{X}$ and a family

$$F(p') \to F(p), \qquad p' \leq p \text{ in } \mathbf{P}$$

of morphisms of $\mathcal{X}$ subject to the identity law and to the associative law

$$\begin{tikzcd} F(p'') \arrow[rr] \arrow[dr] \arrow[r, phantom, "(\mathcal{X})", shift right=12] & & F(p) \\ & F(p') \arrow[ur] & \end{tikzcd} \qquad p'' \leq p' \leq p \text{ in } \mathbf{P}$$

If $\mathcal{X}$ is co-complete, we can form

$$F^*(T) = \varinjlim_{t \in \mathbf{P}/T} F(t)$$

for any direct system F in $\mathcal{X}$ and any inverse system T of sets (both over $\mathbf{P}$). If $\mathcal{X}$ has "small homsets", then the set

$$F_*(X) = \sum_{p \in \mathbf{P}} \mathcal{X}(F(p), X)$$

has in a natural way the structure of an inverse system of sets, for any given object X of $\mathcal{X}$. The functors F^* and F_* are *adjoint* for any given direct system F in $\mathcal{X}$. The correspondence $F \rightsquigarrow <F_*, F^*>$ is bijective (up to isomorphism) for any given poset $\mathbf{P}$ and cocomplete locally small category $\mathcal{X}$. To sum up, for any given poset $\mathbf{P}$, the representable cribles form the generic one among all the possible direct systems on $\mathbf{P}$ defined in all the possible cocomplete locally small categories, indeed a "far-reaching" formulation of the universality and naturalness of Dedekind's construction. This exercise could be continued along the lines of the introduction. In particular, by combining the "exactness" concepts of Lesson 2 with the considerations of this lesson (as we will do in Lesson 4) we can obtain a reasonable theory of *two-sided* completions. More particularly, as we hope to discuss in a later lesson, if $\mathbf{P} = Q^{op}$ is the set of positive rational numbers ordered by $\geq$, we can consider the sub-category $\mathcal{R} \subseteq \mathcal{S}^{\mathbf{P}^{op}}$ of those inverse systems T satisfying the semi-continuity condition

$$pT \xrightarrow{\sim} \varprojlim_{q>p} qT \quad p \in \mathbf{P}$$

where pT denotes the p-th stage of T. Then $\mathcal{R}$ is a topos which represents the notion of semi-continuous real-valued function, whose "points" are just the nonnegative real numbers (including ∞) , and which as a "closed category" provides a very effective basis for a functorial development of the theory of *metric spaces.*

However, in the present lesson we are more interested in how the "base category" $\mathcal{S}$ (in which $\mathbf{P}$ itself lives and on which $\mathcal{X}$ is "based") can be generalized, and this question is already of interest in the case where we limit ourselves to cribles T in $\mathcal{S}$ (i.e. even without the extension to inverse systems) and to "posets" $\mathcal{X}$ over $\mathcal{S}$. Certainly, the correct axiomatic formulation of the concept of "families" will allow at least this "minature" version of the basic constructions to be performed "over $\mathcal{S}$." We will return to this question after describing the other "classical" case of the Yoneda-Kan formula, due to Cayley.

3. Cayley's construction for monoids

Actually, Cayley's representation theorem is usually stated for a group, and indeed there are certain simplifications in that case, particularly in the interpretation of F^* in terms of orbits. However, the basic construction works just as well without the assumption of inverses.

Let $\mathbf{C} = \mathbf{M} = <M, \eta, \mu>$ be any monoid in $\mathcal{S}$. Thus M is a set, $1 \xrightarrow{\eta} M$ is a unit element, and $M \times M \xrightarrow{\mu} M$ is an associative multiplication for which η is an identity element on both sides.

Definition: *By a* **left M-set** T *is meant any set* T *together with a left* $\mathbf{M}$*-action* $M \times T \xrightarrow{\alpha} T$ *for which the diagrams*

$$\begin{tikzcd} M \times M \times T \arrow[r, "M\times\alpha"] \arrow[d, "\mu\times T"'] & M \times T \arrow[d, "\alpha"] \\ M \times T \arrow[r, "\alpha"'] & T \end{tikzcd} \qquad \begin{tikzcd} 1 \times T \arrow[r, "\eta\times T"] \arrow[dr, "\cong"'] & M \times T \arrow[d, "\alpha"] \\ & T \end{tikzcd}$$

are commutative, i.e. for which

$$\begin{aligned} (m_1m_2)t &= m_1(m_2t) \qquad && m_i \in M, \quad t \in T \\ \eta t &= t && t \in T \end{aligned}$$

where we often write $mt \underset{def}{=} <m, t>\alpha$ *and let* α *be "understood". By a homomorphism* $T_1 \xrightarrow{h} T_2$ *of left* $\mathbf{M}$*-sets is meant any mapping (morphism of* $\mathcal{S}$*)*

which makes

$$\begin{tikzcd} M \times T_1 \arrow[r, "M \times h"] \arrow[d, "\alpha_1"'] & M \times T_2 \arrow[d, "\alpha_2"] \\ T_1 \arrow[r, "h"'] & T_2 \end{tikzcd}$$

commutative, i.e. for which

$$m(t\ h) = (mt)h \qquad m \in M, \quad t \in T_1$$

Let us denote by $\mathcal{S}^{\mathbf{M}^{op}}$ the category of all left-$\mathbf{M}$ sets and homomorphisms.

Proposition 3.6: *(Cayley-Yoneda) setting $T = M, \alpha = \mu$ defines a particular left $\mathbf{M}$-set, called the representable one (also often called "$\mathbf{M}$ acting on itself by left translation"). For any left $\mathbf{M}$-set T, there is a one-to-one correspondence*

$$\frac{1 \quad \to \quad T \quad (\mathcal{S})}{M \quad \to \quad T \quad (\mathcal{S}^{\mathbf{M}^{op}})}$$

between elements of t of T and homomorphisms h from the representable one to T, defined by $t = \eta h$ (taking the value of h at the unit element of $\mathbf{M}$ considered as a certain element of M). In particular, every element m of $\mathbf{M}$ defines by right *multiplication a homomorphism $M \to M$ of left $\mathbf{M}$-sets, and every homomorphism $M \to M$ is* right *multiplication by a uniquely determined element m. Thus the representable left $\mathbf{M}$-set defines a full and faithful embedding $\mathbf{M} \hookrightarrow \mathcal{S}^{\mathbf{M}^{op}}$ of $\mathbf{M}$ considered as a category with one object into the large category of all possible left $\mathbf{M}$-objects in $\mathcal{S}$.*

Proof: The associative law for μ is clearly the same diagram as that for α in the case $T = M$ and the fact that η is a *left* unit element for μ is the identity law for α. If t is an element of an arbitrary left-$\mathbf{M}$-set T, then define a mapping

$$M \xrightarrow{h_t} T$$

by the formula

$$mh_t = mt \qquad\qquad m \in M$$

i.e. by the diagram

$$\begin{tikzcd} M \times 1 \arrow[r, "M \times t"] & M \times tT \arrow[d, "\alpha"] \\ M \arrow[u, "\cong"] \arrow[r, "h_t"'] & T \end{tikzcd}$$

Then we have

$$(nm)h_t = (nm)t = n(mt) = n(mh_t) \qquad\qquad n \in \mathbf{M}, \quad m \in M$$

(which could also be expressed by a diagram), i.e. h_t is a homomorphism. Conversely, given any homomorphism $M \xrightarrow{h} T$, we can evaluate it at the distinguished element η of M to obtain $t_h = \eta h : 1 \to T$. Now we show these two processes are inverse to each other:

On the one hand, $t_{h_t} = \eta h_t = \eta t = t$ by the identity law for α; on the other hand, for any $m \in M$, $mh_{t_h} = mt_h = m(\eta h) = (m\eta)h = mh$, since η is a right identity element for μ, hence $h_{t_h} = h$ for any *homomorphism* $M \to T$ (this calculation could also easily be done by a diagram without using elements of M). Now we must consider the special case $T = M$: the foregoing says then that for any element t of M, *right* multiplication by t is an endomorphism of M considered as a *left* M-set, and that every such endomorphism is in fact right multiplication by a uniquely determined element of M (the "every" in the last clause is a strengthening of the usual statement of Cayley's representation theorem for abstract groups, in which the left action of $\mathbf{M}$ on M is not taken into account). ∎

Since all the above proof can be expressed by commutative diagrams, we actually used only the existence of finite products in $\mathcal{S}$, so the theorem as stated is true for any category with products: The essential mentions of "elements" $1 \xrightarrow{\eta} M$, $1 \xrightarrow{t} T$ may be interpreted as morphisms from the terminal object; we did not use the condition that 1 is a generator. The unit element of a monoid object $\mathbf{M}$ in any category with products is defined on the *terminal* object 1: but in the absence of enough maps from 1 we would like to strengthen the statement about the "elements" t:

Exercise 3.2 Let $\mathbf{M}$ be a monoid object in any category with finite products, let T be a left $\mathbf{M}$-object in the same category, and let A be any object in the same category (no extra structure given on A). Show that the composite

$$M \times (M \times A) \cong (M \times M) \times A \xrightarrow{\mu \times A} M \times A$$

makes $M \times A$ into a left $\mathbf{M}$-object. Show that there is a one-to-one correspondence

$$\frac{A \xrightarrow{t} T}{M \times A \xrightarrow{h} T}$$

between arbitrary morphisms t of the category (into a left M-object T) and $\mathbf{M}$-*homomorphisms* h, defined in one direction by $t = (\eta \times A)h$. In algebraic language, this means that $M \times A$ is the *free* left-$\mathbf{M}$ "set" with "set" of generators A.

We now investigate some "cocompleteness" properties which, for example, the category of left $\mathbf{M}$-sets has.

Definition: *Let $\mathcal{S}$ be any category having finite products and let $\mathcal{X}$ be a* *category. By a* **right $\mathcal{S}$-action** *on $\mathcal{X}$ is meant a triple consisting of a functor*

$$\mathcal{X} \times \mathcal{S} \xrightarrow{\otimes} \mathcal{X}$$

and two natural isomorphisms of composite functors

$$\begin{tikzcd} \mathcal{X} \times \mathcal{S} \times \mathcal{S} \arrow[r, "\otimes \times \mathcal{S}"] \arrow[d, "\mathcal{S} \times \otimes"'] & \mathcal{X} \times \mathcal{S} \arrow[d, "\otimes"] \\ \mathcal{X} \times \mathcal{S} \arrow[r, "\otimes"'] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X} \times 1 \arrow[r, "\mathcal{X} \times 1"] \arrow[dr] & \mathcal{X} \times \mathcal{S} \arrow[d, "\otimes"] \\ & \mathcal{X} \end{tikzcd}$$

which are coherent, i.e. there is given a way of "multiplying" an object X of $\mathcal{X}$, and there are given isomorphisms

$$\begin{aligned} X \otimes (S_1 \times S_2) &\cong (X \otimes S_1) \otimes S_2 \\ X \otimes 1 &\cong X \end{aligned}$$

in $\mathcal{X}$ which are **natural** *when the indicated objects are varied along morphisms of their respective categories and whose higher composites resulting from various bracketings of the products in $\mathcal{S}$ of finite strings S_1, S_2, S_n, are consistent.*

Example: Let $\mathcal{S}$ be the category of abstract sets and suppose that $\mathcal{X}$ has coproducts of families of objects indexed by sets. Define

$$X \otimes S = \sum_S X$$

to be the coproduct of the *constant* family X indexed by S, for any object X of $\mathcal{X}$. The isomorphisms and coherence follow from the universal mapping properties defining infinite coproducts.

Example: Let $\mathcal{S}, \mathcal{X}$ be any two categories having finite products and let $\mathcal{S} \xrightarrow{K} \mathcal{X}$ be any functor which preserves the products in the sense that the canonical induced morphisms

$$K(1) \to 1$$

$$K(S_1 \times S_2) \to K(S_1) \times K(S_2)$$

in $\mathcal{X}$ are actually isomorphisms. Define

$$X \otimes S = X \times K(S)$$

Then the associativity isomorphisms of the above definition and their coherence follow from the universal mapping property of finite products.

Definition: *By* **reflexive coequalizers** *in a category we mean simply the usual notion of coequalizers*

$$E \underset{f_1}{\overset{f_0}{\rightrightarrows}} X \xrightarrow{q} Q$$

except applied in the restricted situation where the given data $E \rightrightarrows X$ *(of which we consider the coequalizer* q*) is assumed to have the special property that there exists also a "diagonal" morphism* $X \xrightarrow{d} E$ *satisfying the two equations* $df_i = 1_X \quad i = 0, 1$. *Thus we may consider the condition on a category that "there exist reflexive coequalizers" on a functor that it "preserves reflexive coequalizers", etc.*

Remark: Reflexivity of coequalizer data means basically that the relation which E induces on X is reflexive (though it may not be transitive). The reason for introducing the concept here is that it plays a technical role in our extension of the Cayley-Yoneda-Kan theorems to more general categories. However, there is a more general reason for considering the concept, namely that in many familiar categories such as groups, R-modules, Lie algebras, etc. (but *not* in sets, monoids, or lattices), every reflexive relation is already an equivalence relation! (This is actually part of the theoretical reason why in the first list of categories one can use normal subgroups, or ideals instead of equivalence relations and also one can use kernels instead of kernel pairs, etc), namely we have

Exercise 3.3 Let $\mathcal{A}$ be an equationally defined category of algebras in which among the definable operations in the algebras there is at least one ternary operation θ for which the following two identities are consequences of the defining identities for $\mathcal{A}$

$$\theta(x, x, z) = z$$

$$\theta(x, z, z) = x$$

Let A be any algebra A in $\mathcal{A}$ and suppose that $R \subseteq A \times A$ is any subalgebra of the product algebra which is reflexive (i.e. for any $a_1, \ldots, a_n, b_1, \ldots, b_n$ in A and any n-ary definable operation of $\mathcal{A}$, if $< a_i, b_i >\in R$ for $i = 1, \ldots, n$, then also $< \varphi(a_1, \ldots, a_n), \varphi(b_1, \ldots, b_n >\in R$ and for any a in A, we have $< a, a >\in R$). Then R is actually also symmetric and transitive and hence a congruence relation in the sense of $\mathcal{A}$. Hint: If $< a, b >\in R$, then also $< a, a >\in R$ and $< b, b >\in R$, so that θ applied to the triple $(< a, a >, < a, b >, < b, b >)$ is in R. But operations (such as θ) on a product (such as $A \times A$) are computed co-ordinate-wise. Similarly, if $< a, b >\in R$ and $< b, c >$ in R, then θ can be applied to the triple $(< a, b >, < b, b >, < b, c >)$ of elements of $A \times A$. There is even a converse theorem due to Malcev: If a category $\mathcal{A}$ of algebras, defined by operations and identities has the property that every reflexive subalgebra of any "square" algebra is symmetric and transitive, then there must exist among the definable operations of three arguments at least one θ satisfying the two stated identities (proved by considering finitely generated free algebras and *certain* reflexive relations and deducing from the hypothesis the existence of an element θ of the free algebra $F_{\mathcal{A}}(x, y, z)$ on three generators). In all the usual examples from algebra, we can take $\theta(x, y, z) = x - y + z$ (we don't need commutativity of the group operation, or even all of associativity, to prove the two identities, in fact, the theorem applies to certain classes of "loops").

Definition: *Suppose $\mathcal{S}$ is a category with finite products and reflexive co-equalizers, which satisfies the condition that $S \times (\ )$ preserves reflexive co-equalizers for any object S of $\mathcal{S}$. (This assumption is similar to regularity, except that it is more general in one direction, since we only consider that products, not necessarily pullbacks, preserve the co-equalizers in question, but more restrictive in another direction, since we require that products preserve not only coequalizers of equivalence relations (as in a regular category), but any reflexive coequalizer diagram*

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] & X \arrow[l, dashed, bend left] \arrow[r, "q"] & Q \end{tikzcd}$$

in $\mathcal{S}$, i.e. for any object S, the induced diagram $S \times E \rightrightarrows S \times X \to S \times Q$ again has the universal mapping property of coequalizers (i.e. for any $S \times X \to Y$ which has equal composites $S \times E$, it can be uniquely factored across $S \times Q$ by a $S \times Q \to Y$).) Then by a **monoidally $\mathcal{S}$-cocomplete** *category $\mathcal{X}$ we will mean one which is equipped with a right $\mathcal{S}$-action $X \otimes S$, which has reflexive coequalizers, and moreover for which the "tensor" product preserves reflexive coequalizers in each variable separately.*

Remark: The terminology just introduced (which is not standard) is meant to suggest three different (but related) aspects. For one thing, the hypotheses are appropriate for our present study of monoid *objects* and monoid actions on objects in the categories $\mathcal{S}$ and $\mathcal{X}$. But also at the level of the *categories* themselves, the weak "cocompleteness" axiom on $\mathcal{X}$ which we are considering has clearly the nature of an "action" $\otimes$ of the "monoid" $< \mathcal{S}, 1, \times >$ and moreover the fragment of genuine cocompleteness which we postulate (the reflexive coequalizers) are required to be compatible with this "monoidal" structure; again, as the first above example shows, the sense in which $\otimes$ can be thought of as a "colimit" (it actually is not, of course, in general) is *monoidal* in the sense that we take "S-fold coproducts" of one object X.

Proposition 3.7: *If $\mathcal{S}$ is a category with finite products and reflexive coequalizers which are preserved by products (i.e. by each functor $S \times (\ )$), and if $\mathbf{M} =< M, \eta, \mu >$ is any monoid object of $\mathcal{S}$, then the category $\mathcal{S}^{\mathbf{M}^{op}}$ of left*

M-*objects in* $\mathcal{S}$ *is a monoidally* $\mathcal{S}$-*cocomplete category.*

Proof: We define a functor $\mathcal{S}^{\mathbf{M}^{op}} \times \mathcal{S} \xrightarrow{\otimes} \mathcal{S}^{\mathbf{M}^{op}}$ as follows. Let T be any left **M**-object with action α and let S be any unadorned object. Define the left **M**-object $T \otimes S$ to be, as an object, just $T \times S$, but with the left action of **M** defined by the composite

$$M \times (T \otimes S) \xrightarrow{\simeq} (M \times T) \quad S \xrightarrow{\alpha \times S} T \times S = T \otimes S$$

i.e.

$$T \otimes S = < T \times S,\ \alpha \times S >$$

(neglecting in the last equality the associativity isomorphism for the triple $\times$). Note that "the action of **M** on $T \otimes S$ is the identity on the second factor". The associativity axiom for $\alpha \times S$ with respect to μ and the left identity axiom for $\alpha \times S$ with respect to η, follows easily from the corresponding axioms for α, so that $T \otimes S$ is indeed a left **M**-object. Moreover, if $T_1 \to T_2$ is an **M**-homomorphism and $S_1 \to S_2$ is a morphism of $\mathcal{S}$, then $T_1 \otimes S_1 \to T_2 \otimes S_2$ is an **M**-homomorphism, and this preserves composition (since $\times$ does) and hence $\otimes$ is a functor. Finally, $\otimes$ satisfies the associativity and identity (coherent) isomorphisms for a right S-action, since $\times$ satisfies similar properties which immediately "lift". Now we must show that $\mathcal{S}^{\mathbf{M}^{op}}$ has reflexive coequalizers. [This is relatively easy here, since unlike less banal situations in algebra where more than just unary operations are involved, colimits in the category $\mathcal{S}^{\mathbf{M}^{op}}$ of "unary" algebras tend to be computed in the same way as for the "underlying sets" in $\mathcal{S}$ - of course, in reality even this case uses our exactness assumption on $\mathcal{S}$.] Let

$$\begin{tikzcd} R \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & T \arrow[l, dashed, bend left=60, "d"] \end{tikzcd}$$

be any (reflexive) data consisting of left **M**-objects and **M**-homomorphisms. Let $T \xrightarrow{a} Q$ be the coequalizer of f_0, f_1 in *the sense of* $\mathcal{S}$ (i.e. ignoring for the moment the actions α_R, α_T). We must show that there is a unique way of defining a left **M**-action α_Q on Q such that q becomes an **M**-homomorphism. By the assumed exactness of $\mathcal{S}$,

$$M \times R \underset{M \times f_1}{\overset{M \times f_0}{\rightrightarrows}} M \times T \xrightarrow{M \times q} M \times Q$$

is again a coequalizer diagram in $\mathcal{S}$. Moreover, in the diagram

$$\begin{tikzcd} M \times R \arrow[r, shift left, "M \times f_0"] \arrow[r, shift right, "M \times f_1"'] \arrow[d, "\alpha_R"'] & M \times T \arrow[r, "M \times q"] \arrow[d, "\alpha_T"] & M \times Q \arrow[d, "\alpha_Q"] \\ R \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & T \arrow[r, "q"'] & Q \end{tikzcd}$$

the i-th square (i=0,1) on the left commutes since f_i ($i = 0, 1$) is an $\mathbf{M}$-homomorphism, and since q is a coequalizer we have in particular that $f_0 q = f_1 q$. Thus

$$(M \times f_0)\alpha_T q = \alpha_R f_0 q = \alpha_R f_1 q = (M \times f_1)\alpha_T q$$

which says that the morphism $\alpha_T q$ has equal compositions with the two morphisms $M \times f_i$ ($i = 0, 1$). Therefore, since $\mathbf{M} \times q$ is the coequalizer of these two, there exists a unique morphism α_Q for which the right square commutes. This already says that q is an $\mathbf{M}$-homomorphism, except that one still has to show that Q equipped with α_Q really is a left-$\mathbf{M}$-object, i.e. that α_Q satisfies the associativity and left identity axioms with respect to μ and η; this proof is left to the reader, except to remark that in addition to another use (for $S = M \times M$ instead of $S = M$) of our exactness condition on S, the final crucial step is the *uniqueness* of maps out of coequalizers (i.e. the fact that coequalizers are epic). Finally, we have to verify that $\mathcal{S}^{\mathbf{M}^{op}} \times S \xrightarrow{\otimes} S$ preserves (reflexive) coequalizers in each variable separately, but (again the details are left to the reader) this follows from the fact that coequalizers are always compatible with coequalizers, plus the assumption on S that products are compatible with coequalizers. [Note that the *commutativity* of the product $\times$ in the category $\mathcal{S}$ is also involved; e.g. in particular for $\mathbf{M} = 1$, we have the fact that the exactness condition for $S \times (\ )$ on $\mathcal{S}$ is equivalent to one for $(\ ) \times S$. ∎

Remark: In case $\mathcal{S}$ = sets, one may picture $T \otimes S$ as a disjoint sum of S copies of T which do not interact with each other under the action.

Remark: In the above proposition the reflexivity played no role - we could as well have considered "any" interesting class of coequalizers in the assumption and in the conclusion. But reflexivity in particular does play a role in the following

Theorem: *Let $\mathcal{S}$ be a category with finite products and reflexive coequalizers preserved by the products and let $\mathbf{M} = \langle M, \eta, \mu \rangle$ be a monoid object in $\mathcal{S}$. Let M denote $\mathbf{M}$ acting on itself by left translation, i.e. the generic example of a left $\mathbf{M}$-object. Then for any left $\mathbf{M}$-object T with action α, the following is a reflexive coequalizer diagram in the category $\mathcal{S}^{\mathbf{M}^{op}}$:*

$$\begin{tikzcd}
M \otimes (M \times T) \arrow[rr, "M\otimes\alpha"] \arrow[dr, "\cong"'] & & M \otimes T \arrow[r, "\alpha"] & T \\
& (M \otimes M) \otimes T \arrow[ur, "\mu\otimes T"'] & &
\end{tikzcd}$$

(The morphism which proves the reflexivity is $M \otimes (\eta \times T)$.)

Remark: Note that the occurrences of T in the coequalizer *data* are simply T considered as an object *without* action, the tensor products being endowed with action solely by virtue of the multiplication in M (of course, the action α on T appears in another guise in the coequalizer data, i.e. in specifying how the spread out $M \otimes T$ is to be glued back together to yield T as an **M**-object).

Proof: First we show that the four morphisms really are homomorphisms. $M \otimes \alpha$ and $M \otimes (\eta \times T)$ are homomorphisms by the functorality of $\otimes$ as defined in the previous theorem, and the same is true for $\mu \times T$ provided we note that μ itself is a homomorphism *when* its domain is taken as $M \otimes M$ (μ is not a homomorphism *when* its domain is taken as $M \times M$, though the latter also has sense as an **M**-object). Using elements, the proof that μ is a homomorphism is as follows:

$$(m < a, b >)\mu = (< ma, b >)\mu = (ma)b = m(ab) = m(< a, b > \mu);$$

a diagrammatic proof is just as easy. That $M \otimes (\eta \times T)$ proves reflexivity follows from the identity laws for α and μ. The fact that α is a homomorphism is just another statement of the associative law for α, though again the fact that we are using $M \otimes T$ (and not, say, $M \times T$ in the sense of $\mathcal{S}^{\mathbf{M}^{op}}$) is important:

$$\begin{tikzcd} M \times (M \otimes T) \arrow[r, "M\times\alpha"] \arrow[d, "\alpha_{M\otimes T} \cong \mu \times T"'] & M \times T \arrow[d, "\alpha_T = \alpha"] \\ M \otimes T \arrow[r, "\alpha"'] & T \end{tikzcd}$$

Now we must show that α is the coequalizer of $M \otimes \alpha$ and $\mu \times T$ in the sense of the category $\mathcal{S}^{\mathbf{M}^{op}}$. The morphism $\eta \times T : T \cong 1 \times T \to M \times T$ is, by the identity law for α, a splitting for α, so that α is certainly an epimorphism in the sense of $\mathcal{S}$, and hence an epimorphism in the sense of $\mathcal{S}^{\mathbf{M}^{op}}$ (recall that $M \otimes T$ has $M \times T$ as its underlying $\mathcal{S}$-object). If T', α' is any other left **M**-object of $\mathcal{S}$, and $M \otimes T \xrightarrow{h} T'$ is any $\mathcal{S}$-morphism, we can define an $\mathcal{S}$-morphism $T \xrightarrow{h'} T'$ by $h' = (\eta \times T)h$. Since α is epic, the theorem will follow from the following fact: If (1) h is a homomorphism, and if (2) $(M \otimes \alpha)h = (\mu \times T)h$, then h' is also a homomorphism and $h = \alpha h'$. This fact is proved by the following two calculations: First, the assumption $(M \times \alpha)h = (\mu \times T)h$ implies that

$$\begin{aligned} \alpha h' &= \alpha(\eta \times T)h = (\eta \times \alpha)h = (\eta \times (M \times T))(M \times \alpha)h \\ &= (\eta \times M \times T)(\mu \times T)h = h \end{aligned}$$

i.e. that the needed triangle is commutative. Second, the two assumptions together imply that

$$\begin{aligned}(M \times h')\alpha' &= (M \times ((\eta \times T)h))\alpha' = (M \times \eta \times T)(M \times h)\alpha' \\ &= (M \times \eta \times T)(\mu \times T)h = h\end{aligned}$$

and hence that $\alpha h' = (M \times h')\alpha'$, i.e. that h' is a homomorphism. ∎

Remark: In case $\mathbf{M}$ is a group in sets, the above representation of T as a quotient of a sum of copies of $\mathbf{M}$ can be "commuted" to obtain T as a sum (one for each "orbit") of quotients of $\mathbf{M}$ (e.g. if $\mathbf{M}$ is a *cyclic* group, T is actually a sum of fixed points and copies of $\mathbf{M}$ itself). However, even in sets, if $\mathbf{M}$ is a monoid, but not a group, two orbits Mt_1 and Mt_2 may intersect and still not be equal, so that the above theorem is all that can be asserted in general; e.g. if $\mathbf{M}$ = nonnegative natural numbers under addition, then an $\mathbf{M}$-action on T is determined entirely by one endomorphism α_1 of T and if e.g. T = 3-element set, we can take α_1 as follows:

$$t_1\alpha_1 = t_2\alpha_1 = t_3\alpha_1 = t_3 \qquad \begin{tikzcd} t_1 \arrow[dr] & & t_2 \arrow[dl] \\ & t_3 & \end{tikzcd}$$

Definition: *Suppose $\mathcal{S}$ is a category with products, $\mathbf{M}$ a monoid object in $\mathcal{S}$, and $\mathcal{X}$ a category equipped with a right $\mathcal{S}$-action denoted by $\otimes$. Then by a* **right M-object** *in $\mathcal{X}$ is meant a pair consisting of an object F in $\mathcal{X}$ and a morphism $F \otimes M \xrightarrow{\beta} F$ in $\mathcal{X}$ satisfying the commutativity of the following two diagrams in*

$$\begin{tikzcd} F \otimes (M \times M) \arrow[rr, "F\otimes\mu"] \arrow[d, "\cong"] & & F \otimes M \arrow[dd, "\beta"] \\ (F \otimes M) \otimes M \arrow[d, "\beta\otimes M"'] & & \\ F \otimes M \arrow[rr, "\beta"'] & & P \end{tikzcd} \qquad \begin{tikzcd} F \otimes 1 \arrow[r, "F\otimes\eta"] \arrow[dr, "\cong"'] & F \otimes M \arrow[d, "\beta"] \\ & F \end{tikzcd}$$

(We will sometimes use the usual abuse of notation whereby F stands for the pair F, β). By a homomorphism $F_1 \xrightarrow{h} F_2$ of right $\mathbf{M}$-objects in $\mathcal{X}$, is meant any morphism of $\mathcal{X}$ satisfying the commutativity of the diagram

$$\begin{tikzcd} F_1 \otimes M \arrow[r, "h\otimes M"] \arrow[d, "\beta_1"'] & F_2 \otimes M \arrow[d, "\beta_2"] \\ F_1 \arrow[r, "h"'] & F_2 \end{tikzcd}$$

composing these in the obvious way, we obtain a category $\mathcal{X}^{\mathbf{M}}$. [Note that we are also using an abuse of notation whereby $\mathcal{X}$ stands for the "pair" (again an abuse, actually a quadruple) $\mathcal{X}$, $\otimes$ (associativity natural isomorphism, identity natural isomorphism).]

Example: Let $\mathcal{S}$ be sets and $\mathbf{M}$ be an ordinary monoid and let $\mathcal{X}$ be any category. By a right $\mathbf{M}$-object in $\mathcal{X}$ is meant any functor $\mathbf{F} : \mathbf{M} \to \mathcal{X}$ where $\mathbf{M}$ is considered as a category which has *one* object η (which is one explanation of the term "monoid") whose (endo)morphisms are the elements of M which are composed by means of μ; in other words, one object F of $\mathcal{X}$ (the value of the functor at the one object of $\mathbf{M}$) is given, and (the "morphism part" of the functor) to each element m of M is assigned an endomorphism β_m of F in $\mathcal{X}$, subject to the axioms $\beta_\eta = 1_F, \beta_{m_1m_2} = \beta_{m_1}\beta_{m_2}$. This definition is not yet of the form of the above definition, but if we assume moreover that $\mathcal{X}$ has S-fold coproducts of objects for any set S, then the family $\{\beta_m | m \in M\}$ can be interpreted as a single morphism $\sum_M F \xrightarrow{\beta} F$.

Proposition 3.8: *M itself, considered as a left $\mathbf{M}$-object, is an object of $\mathcal{X} = \mathcal{S}^{M^{op}}$; but this object is actually a right $\mathbf{M}$-object in $\mathcal{S}^{\mathbf{M}^{op}}$.*

Proof: Exercise, using the associative law for μ, and noting that we consider $M \otimes M$ (not for example $M \times M$) in $\mathcal{S}^{\mathbf{M}^{op}}$, where the first copy of M has the action μ, but where the second copy of M has no action.

Definition: *Suppose $\mathcal{S}$ is a category with finite products and reflexive coequalizers which are preserved by the products and suppose that $\mathcal{X}, \mathcal{Y}$ are two monoidally $\mathcal{S}$-cocomplete categories. By a monoidally $\mathcal{S}$-cocomplete functor $F^* : \mathcal{Y} \to \mathcal{X}$ is meant* (0) *a functor which* (1) *preserves reflexive coequalizers and* (2) *is equipped with a further structure consisting of a coherent natural isomorphism of functors*

$$F^*(Y) \otimes S \xrightarrow{\approx} F^*(Y \otimes S) \qquad Y \in \mathcal{Y}, \qquad S \in \mathcal{S}$$

(*where the first $\otimes$ is the right $\mathcal{S}$-action given on $\mathcal{X}$ and the second $\otimes$ is the right $\mathcal{S}$-action given on $\mathcal{Y}$.*)

Remark: In those examples where $\otimes$ is an iterated coproduct, the natural transformation in condition (2) automatically *exists* canonically, and means that it has an inverse; but in the general case where we have not assumed any universal mapping property for our $\otimes$, the natural isomorphism has to be *given*. The "coherence" means that the diagrams of the following two forms must be assumed to be commutative, where θ is the structural isomorphism given for F (as mentioned above), where a' (a respectively) denotes the given associativity isomorphism for the external tensor product of $\mathcal{Y}$ ($\mathcal{X}$ respectively), and where i' (respectively i) denotes the "unit" isomorphism given for $\mathcal{Y}$ (respectively $\mathcal{X}$):

$$
\begin{tikzcd}
& F^*(Y\otimes(S_1\times S_2)) \arrow[r, "\theta_{Y,S_1\times S_2}"] \arrow[dl, "F^*(a'_{Y,S_1,S_2})"'] & F^*(Y)\otimes(S_1\times S_2) \arrow[dr, "a_{F^*(Y),S_1,S_2}"] & \\
F^*((Y\otimes S_1)\otimes S_2) \arrow[drr, "\theta_{Y\otimes S_1,S_2}"'] & & & (F^*(Y)\otimes S_1)\otimes S_2 \\
& & F^*(Y\otimes S_1)\otimes S_2 \arrow[ur, "\theta_{Y,S_1\otimes S_2}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
F^*(Y\otimes 1) \arrow[rr, "\theta_{Y,1}"] \arrow[dr, "F^*(i'_Y)"'] & & F^*(Y)\otimes 1 \arrow[dl, "i_{F^*(Y)}"] \\
& F^*(y) &
\end{tikzcd}
$$

Definition: *With the hypotheses of the preceding definition, by a morphism* $F_1^*\to F_2^*$ *of monoidally* $\mathcal{S}$*-cocontinuous functors* $\mathcal{Y}\underset{F_2^*}{\overset{F_1^*}{\rightrightarrows}}\mathcal{X}$ *is meant any natural transformation* $F_1^*\xrightarrow{\varphi}F_2^*$ *of functors which has the property that for any* $Y\in\mathcal{Y}, S\in\mathcal{S}$ *the following diagram commutes in*

$$
\begin{tikzcd}
F_1^*(Y)\otimes S \arrow[r, "\sim"] \arrow[d, "\varphi_{Y\otimes S}"'] & F_1^*(Y\otimes S) \arrow[d, "\varphi_{Y\otimes S}"] \\
F_2^*(Y)\otimes S \arrow[r, "\sim"'] & F_2^*(Y\otimes S)
\end{tikzcd}
$$

where the upper and lower lines are the natural isomorphisms given as part of the structure of F_1^*, F_2^* *respectively. [In cases where* $\otimes$ *has some good universal mapping property (so that the horizontal natural maps are canonical, as mentioned before), then the commutativity of the above square will also be automatic, so that* any *natural transformation in the naive sense will count as a morphism* $F_1^*\to F_2^*$.*] Denote the* category *of all monoidally* $\mathcal{S}$*-cocontinuous functors* $\mathcal{Y}\to\mathcal{X}$ *and morphisms between them by*

$$\mathcal{S}\text{-}\mathit{Fun}_{\to\otimes}(\mathcal{Y},\mathcal{X})$$

[The arrow to suggest both " $\varinjlim$ *" - preservation in "general" and (reflexive) coequalizer preservation in particular.]*

By the previous theorem, if $\mathcal{Y}=\mathcal{S}^{\mathbf{M}^{op}}$, then every $T\in\mathcal{Y}$ is canonically representable using the operations $\otimes$ and (reflexive) coequalizer in terms of the canonical object M of $\mathcal{Y}$; hence in this case, any functor $\mathcal{Y} \to \mathcal{X}$ which preserves those two operations is already determined by its value at that one object; this value is in reality just a right $\mathbf{M}$-object in $\mathcal{X}$. *Conversely*, any right $\mathbf{M}$-object in $\mathcal{X}$ can be extended to a cocontinuous functor on all of $\mathcal{S}^{\mathbf{M}^{op}}$, more precisely

Theorem: *Let $\mathcal{S}$ be a category with finite products and reflexive coequalizers which are preserved by the products, $\mathbf{M}$ a monoid in it, and $\mathcal{X}$ a monoidally $\mathcal{S}$-cocomplete category, then there is an equivalence of categories*

$$\mathcal{X}^{\mathbf{M}} \xrightarrow{\sim} \mathcal{S}\text{-}Fun_{\to\otimes}(\mathcal{S}^{\mathbf{M}^{op}}, X)$$

given by

$$F \rightsquigarrow (T \rightsquigarrow F \underset{\mathbf{M}}{\otimes} T)$$

where the new "mixed" tensor product "over $\mathbf{M}$" is defined in the proof below. In terms of this mixed tensor product in the case $\mathcal{X} = \mathcal{S}^{\mathbf{M}^{op}}$, the previous theorem is just the "identity law"

$$M \underset{\mathbf{M}}{\otimes} T \xrightarrow{\sim} T \ \textit{all}\ T \in \mathcal{S}^{\mathbf{M}^{op}}$$

Proof: Before defining the mixed tensor product, we define the functor which is (quasi)inverse to it: First, note that if $\mathcal{Y} \xrightarrow{F} \mathcal{X}$ is any monoidally $\mathcal{S}$-cocontinuous functor, then there is an induced functor $\mathcal{Y}^{\mathbf{M}} \xrightarrow{F^{*\mathbf{M}}} \mathcal{X}^{\mathbf{M}}$ (in fact, also again monoidally $\mathcal{S}$-cocontinuous) which to any right $\mathbf{M}$-object G, γ in $\mathcal{Y}$ associates the right $\mathbf{M}$-object of $\mathcal{X}$ whose underlying $\mathcal{X}$-object is $F^*(G)$ and whose $\mathbf{M}$-action is given by the composite $\mathcal{X}$-morphism

$$F^*(G) \otimes M \xleftarrow{\sim} F^*(G \otimes M) \xrightarrow{F^*(\gamma)} F^*(G)$$

where the first part is the inverse of the structural isomorphism given for F^*. Second, in the case $\mathcal{Y} = \mathcal{S}^{\mathbf{M}^{op}}$, we have M itself by the preceding proposition as a standard right $\mathbf{M}$-object in the category $\mathcal{S}^{\mathbf{M}^{op}}$ of left $\mathbf{M}$-objects in $\mathcal{S}$, so $F^*(M)$ is an object of $\mathcal{X}^{\mathbf{M}}$ determined by $\mathcal{S}^{\mathbf{M}^{op}} \xrightarrow{F^*} \mathcal{X}$. Except for several verifications, this defines a functor

$$\mathcal{S}\text{-Fun}_{\to\otimes}(\mathcal{S}^{\mathbf{M}^{op}}, \mathcal{X}) \longrightarrow \mathcal{X}^{\mathbf{M}}$$

called "restriction along the Yoneda embedding". We now define the quasi-inverse of this functor. Given any right $\mathbf{M}$-object F, β in $\mathcal{X}$, let T, α be any left $\mathbf{M}$-object in the base category $\mathcal{S}$. Then in $\mathcal{X}$ we have the following coequalizer diagram

$$\begin{tikzcd} F \otimes (M \times T) \arrow[d, "\cong"'] \arrow[dr, "F\otimes\alpha"] & & \\ (F \otimes M) \otimes T \arrow[r, "\beta\otimes T"'] & F \otimes T \arrow[r, "q"] & F \underset{\mathbf{M}}{\otimes} T \end{tikzcd}$$

(since $(F \otimes \eta) \otimes T$ shows that the data is reflexive) defining a new object $F \underset{\mathbf{M}}{\otimes} T$ of $\mathcal{X}$ (in general without any action remaining). It is clearly bifunctorial in F and T and hence can be considered as a functor-valued functor of F. Since coequalizers commute with each other, $F \underset{\mathbf{M}}{\otimes} (\ )$ preserves reflexive coequalizers. Moreover, because of the assumed "distributivity" of the given tensor structure and product structure with respect to reflexive coequalizers, one can prove that the associativity isomorphisms induce structural isomorphisms

$$F^*(T \otimes S) = F \underset{\mathbf{M}}{\otimes} (T \otimes S) \xrightarrow{\sim} (F \underset{\mathbf{M}}{\otimes} T) \otimes S \equiv F^*(T) \otimes S$$

which are natural and coherent as T ranges through $\mathcal{S}^{\mathbf{M}^{op}}$ and S through $\mathcal{S}$. Thus $F^* \underset{def}{=} F \underset{\mathbf{M}}{\otimes} (\ )$ is a monoidally $\mathcal{S}$-cocontinuous functor for given F in $\mathcal{X}^M$. Also, if we vary F along homomorphisms, there will be induced natural transformations which commute with the structural isomorphisms and hence are morphisms of monoidally $\mathcal{S}$-cocontinuous functors. The fact that $F \underset{\mathbf{M}}{\otimes} M \cong F$ for any F may be proved in the same way as in the theorem (for β on the left instead of α on the right). Finally, if F^* is any monoidally $\mathcal{S}$-cocontinuous functor $\mathcal{Y} \to \mathcal{X}$, then we can show that there is an isomorphism

$$F^*(G \underset{\mathbf{M}}{\otimes} T) \cong F^*(G) \underset{\mathbf{M}}{\otimes} T$$

natural as G varies through $\mathcal{Y}^{\mathbf{M}}$ and T through $\mathcal{S}^{\mathbf{M}^{op}}$; thus in particular for $\mathcal{Y} = \mathcal{S}^{\mathbf{M}^{op}}$ we have

$$F^*(T) = F^*(M \underset{\mathbf{M}}{\otimes} T) \cong F^*(M) \underset{\mathbf{M}}{\otimes} T = F \underset{\mathbf{M}}{\otimes} T$$

The last two sentences show that the two functors constructed are inverse to each other up to natural isomorphism. ∎

In order to obtain a theorem of the form

$$\mathcal{X}^{\mathbf{M}} \cong \mathcal{S}\!-\!Adj_R(\mathcal{X}, (\mathcal{S}^{\mathbf{M}^{op}})$$

for $\mathbf{M}$ a monoid in $\mathcal{S}$ and $\mathcal{X}$ suitably based on $\mathcal{X}$, we need to have further structure on both $\mathcal{S}$ and $\mathcal{X}$ which will in fact imply that "taking right adjoint"

$$\mathcal{S}\text{-Fun}_{\to\otimes}(\mathcal{S}^{\mathbf{M}^{op}}, \mathcal{X}) \longrightarrow \mathcal{S}\!-\!Adj_R(\mathcal{X}, (\mathcal{S}^{\mathbf{M}^{op}})$$

is an equivalence of categories. The essential aspects of this further structure are those found in the theory of *closed categories*, a theory of considerable significance for functional analysis, homological algebra, and algebraic geometry. Before outlining those features of the theory of closed categories needed to complete our present discussion of monoid actions, we insert a clarifying

Remark and Example: *None* of the definitions, theorems, and calculations in the present section ("Cayley") have depended on the assumption that the products in $\mathcal{S}$ are *cartesian* products (i.e. have diagonal projection morphisms and a universal property)! That is, we could have taken $\mathcal{S}$ to be a "monoidal" category, i.e. equipped with a bifunctor $\mathcal{S} \times \mathcal{S} \xrightarrow{\otimes} \mathcal{S}$, a unit object K (instead of 1) and coherent associativity, and left and right unit isomorphisms (in which, moreover, there are reflexive coequalizers preserved by this "product" in each variable separately. For example, taking for $\mathcal{S}$ the category of abelian groups and for the product the usual tensor product of two abelian groups, we find that a monoid $\mathbf{M}$ (with respect to this product) is nothing but an arbitrary *ring*, and that in place of $\mathcal{S}^{\mathbf{M}^{op}}$ we are considering the category of left $\mathbf{M}$-modules. If $\mathcal{X}$ is a cocomplete (in the usual sense) *additive* category, it is not difficult to define a tensor product $X \otimes A$ of an object times an abelian group, so that the usual notion of "right M-module in $\mathcal{X}$" can be expressed in the same way as we have defined $\mathcal{X}^{\mathbf{M}}$. Then our theorem is again valid, i.e. a right $\mathbf{M}$-module in $\mathcal{X}$ can be uniquely "extended" to a cocontinuous ($\Rightarrow$ additive) functor from ordinary left $\mathbf{M}$-modules into $\mathcal{X}$, for any ring $\mathbf{M}$.

Definition: *A* **closed category** $\mathcal{S}$ *is a category with a product* $\otimes$ *as discussed just above and moreover a functor*

$$Hom : \mathcal{S}^{op} \times \mathcal{S} \to \mathcal{S}$$

with a natural equivalence

$$\mathcal{S}(A \otimes B, C) \cong \mathcal{S}(A, Hom(B, C))$$

[This, of course, implies by general properties of adjoint functors that $(\ ) \otimes B$ *will preserve any coequalizer that may exist.] By a tensored* $\mathcal{S}$*-based category* $\mathcal{X}$ *is meant an* $\mathcal{S}$*-monoidally cocomplete category with moreover an* $\mathcal{S}$*-valued functor*

$$Hom_{\mathcal{X}} : \mathcal{X}^{op} \times X \to \mathcal{S}$$

with a natural equivalence

$$\mathcal{X}(X \otimes S, X') \cong \mathcal{S}(S, Hom_{\mathcal{X}}(X, X')$$

[So, in particular, taking $S =$ *unit object of* $\mathcal{S}$*, we get that* $Hom_{\mathcal{X}}$ *really is a "strong" version of* $\mathcal{X}(,)$ *and moreover that there is a morphism* $Hom_{\mathcal{X}}(X, X') \otimes Hom_{\mathcal{X}}(X', X'') \to Hom_{\mathcal{X}}(X, X'')$ *in* $\mathcal{S}$ *which represents composition in* $\mathcal{X}$*.]*

Exercise 3.4 If $\mathcal{S}$ is a closed category having reflexive equalizers (dualize the definition) and coequalizers and $\mathbf{M}$ is a monoid in $\mathcal{S}$ (relative to the given not-necessarily-cartesian product in $\mathcal{S}$), then $\mathcal{S}^{\mathbf{M}^{op}}$ is a tensored $\mathcal{S}$-based category. Hint: Define $Hom_{\mathbf{M}}$ as the equalizer

$$\begin{tikzcd} Hom_{\mathbf{M}}(T,T') \arrow[r, tail] & Hom(T,T') \arrow[r, "{(\alpha,T')}"] \arrow[dr, "{(T,\hat{\alpha}')}"'] & Hom(M\otimes T,T') \\ & & Hom(T,Hom(M,T')) \arrow[u, "\cong"'] \end{tikzcd}$$

where $\hat{\alpha}'$ corresponds to α' via the closed structure, and where the indicated isomorphism is a (provable) "strong" version of the closed structure.

Exercise 3.5 If F is a right $\mathbf{M}$-object in $\mathcal{X}$, then $Hom_{\mathcal{X}}(F,X)$ is a left $\mathbf{M}$-object in $\mathcal{S}$ for any object X. $Hom_{\mathcal{X}}(F,-)$ is $\mathcal{S}$-strongly right adjoint to $F \underset{\mathbf{M}}{\otimes} (\ )$. [Supply definition of the last notion.]

4. Large categories with an $\mathcal{S}$-atlas of families and small categories and functors internal to $\mathcal{S}$

These topics will not be considered in all detail due to lack of time. [It is planned to include a more thorough treatment in a later version of these notes.] The second topic can be read independently of the first, and is also more basic, since most examples of large categories with an $\mathcal{S}$-atlas are in fact subcategories of internal functor categories for some small category internal to $\mathcal{S}$. In contrast to the previous section on monoids, the cartesian nature of the base category $\mathcal{S}$ will be essential here. [This could be alleviated by passing to a still useful level of greater generality in which two roles of $\mathcal{S}$ are in fact divided among two categories: a cartesian $\mathcal{S}$ to provide the models for atlases on large categories and the object - "sets" for small categories, and a not-necessarily-cartesian closed category $\mathcal{V}$ to serve as the recipient of the hom-functors for large and small categories; of course $\mathcal{S}$ and $\mathcal{V}$ need to be strongly connected to each other to allow a useful theory, and an important aspect of this connection is that $\mathcal{V}$ should be equipped with an $\mathcal{S}$-atlas; thus the present section in which we restrict ourselves to the cartesian case is also an instructive preliminary to such a more general theory.]

We will assume that the base category $\mathcal{S}$ has pullbacks, a terminal object 1, and for each morphism $S' \xrightarrow{\sigma} S$ a right adjoint

$$\mathcal{S}/S' \xrightarrow{\Pi_\sigma} \mathcal{S}/S$$

to the pulling-back functor

$$\mathcal{S}/S' \xrightarrow{\sigma^*} \mathcal{S}/S'$$

That is for each pair of morphisms

$$\begin{tikzcd} B \arrow[d] & \\ S' \arrow[r, "\sigma"'] & S \end{tikzcd}$$

there is a morphism

$$\begin{tikzcd} \prod_\sigma(B) \arrow[d] \\ S \end{tikzcd}$$

with the property that for any morphism

$$\begin{tikzcd} A \arrow[d] \\ S \end{tikzcd}$$

there is a natural bijection

$$\frac{A \to \prod_\sigma (B)[\text{ over } S]}{\sigma^* A \to B \quad [\text{ over } S']}$$

In particular, there is a canonical "evaluation" morphism

$$\sigma^*(\prod_\sigma (B)) \to B$$

over S', and the natural bijection is equivalent to a universal property of the evaluation. In the case $S = 1$, $\prod_{S'} (B)$ is an object of $\mathcal{S}$ which may be considered as the "object sections of $B \to S'$ "; in particular, if $B = \sigma^* Y = S' \times Y$ with the projection, $Y^{S'} = \prod_{S'} (S' \times Y)$ is the "object of morphisms from S' to Y ", so that $\mathcal{S}$ is actually a cartesian closed category. But more, in the general case S' may be considered as an object in $\mathcal{S}/S$ with help of σ, and the composite functor, first σ^* then $\prod_\sigma$, is the operation "raising to the power S' in the sense of $\mathcal{S}/S$". In fact

Exercise 3.6 For a category $\mathcal{S}$ with pullbacks, all the functors $\prod_\sigma$ exist iff all the categories $\mathcal{S}/S$ are cartesian closed.

As a general principle, the *effective* way of guaranteeing that an exactness property holds, is to have the existence of a suitable adjoint functor. In our present $\mathcal{S}$ the existence of the "internal infinite products" Π implies that pulling back will preserve any kind of colimit that might exist; for example, if equivalence relations have coequalizers in $\mathcal{S}$, then $\mathcal{S}$ is automatically regular.

For the discussion to follow it will be convenient to introduce now two more categories $\mathcal{S}^{\swarrow\searrow}$ and $\mathcal{S}^{\neq}$ constructed from $\mathcal{S}$. Both have as objects all possible diagrams

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ I & & J \end{tikzcd}$$

in $\mathcal{S}$. In $\mathcal{S}^{\swarrow\searrow}$ the morphisms are the obvious ones, namely all commutative diagrams of the form

$$\begin{tikzcd} & S \arrow[rr, "g"] \arrow[dl] \arrow[ddr] & & S' \arrow[dl] \arrow[ddr] & \\ I \arrow[rr, "\sigma"] & & I' & & \\ & & J \arrow[rr, "\tau"] & & J' \end{tikzcd}$$

(In case $\mathcal{S}$ = sets, an object of $\mathcal{S}^{\swarrow\searrow}$ may be considered as an I by J matrix S_{ij} of arbitrary sets, for arbitrary sets I and J, and a morphism consists of a pair of mappings $I \xrightarrow{\sigma} I'$, $J \xrightarrow{\tau} J'$ together with a matrix of mappings

$$S_{ij} \xrightarrow{g_{ij}} S'_{\sigma(i),\tau(j)} \quad i \in I, \quad j \in J.)$$

The category $\mathcal{S}^{\swarrow\searrow}$ has two canonical functors

$$\begin{tikzcd} & \mathcal{S}^{\swarrow\searrow} \arrow[dl] \arrow[dr] & \\ \mathcal{S} & & \mathcal{S} \end{tikzcd}$$

defined by forgetting the S, g, but remembering the I, σ, respectively the J, τ. Note that the matrix multiplication of Lesson 2 may be considered as one functor over $\mathcal{S} \times \mathcal{S}$

$$\begin{tikzcd} \mathcal{S}^{\swarrow\searrow} \underset{\mathcal{S}}{\times} \mathcal{S}^{\swarrow\searrow} \arrow[rr, "*"] \arrow[dr] & & \mathcal{S} \arrow[dl] \\ & \mathcal{S} \times \mathcal{S} & \end{tikzcd}$$

where the fiber-product category has as objects all possible diagrams

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ I & & J & & K \end{tikzcd}$$

with the forgetful functor remembering J and K, and where the matrix multiplication is defined without indices by taking the pullback $S \underset{J}{\times} T$ and using the composite structural morphisms to J and K.

The other category $\mathcal{S}^{\neq}$ with matrices as objects has less obvious morphisms; here a morphism from

$$\begin{tikzcd} & S' \arrow[dl] \arrow[dr] & \\ I' & & J' \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ I & & J \end{tikzcd}$$

is any triple [note reverse direction of the I-map].

$$I' \xleftarrow{\alpha} I, J' \xrightarrow{\beta} J$$

$$\alpha^*(S') \xrightarrow{g} \beta^*(S)$$

for which

$$\begin{tikzcd} \alpha^*(S') \arrow[rr, "g"] \arrow[dr] & & \beta^*(S) \arrow[dl] \\ & I \times J' & \end{tikzcd}$$

Thus a morphism in $\mathcal{S}^{\neq}$ from a primed matrix into a unprimed one is a backwards morphism on the row-indices, a forward one on the column and a family

$$S'_{\alpha(i),j'} \xrightarrow{g_{ij'}} S_{i,\beta(j')}$$

of morphisms on the entries. We have forgetful functors

$$\begin{tikzcd} & \mathcal{S}^{\neq} \arrow[dl] \arrow[dr] & \\ \mathcal{S}^{op} & & \mathcal{S} \end{tikzcd}$$

and hence taking duals in all three forgetful functors

$$\begin{tikzcd} & (\mathcal{S}^{\neq})^{op} \arrow[dl] \arrow[dr] & \\ \mathcal{S} & & \mathcal{S}^{op} \end{tikzcd}$$

where the first is still remembering I and the second J.

We now can sketch the definition of the structure required for a category $\mathcal{X}$ which is based on $\mathcal{S}$ both atlas-wise and hom-wise and which moreover has "$\mathcal{S}$-coproducts and $\mathcal{S}$-products (infinite)". [Again having the "$\mathcal{S}$-products" which are not really needed is a simple effective way of making automatic the needed exactness properties when we assume moreover that coequalizers exist; naturally, most examples which are cocomplete will also have products anyway.] Actually, we use the symbol $\mathcal{X}$ to denote a much bigger category than just the objects and morphisms of $\mathcal{X}$ in the naive sense; we will use $\mathcal{X}[1]$ to denote the latter category, but for any $I \in \mathcal{S}, \mathcal{X}$ will also contain a category $\mathcal{X}[I]$ declared to be the category of "I-indexed families" of objects and morphisms of $\mathcal{X}$ (i.e. of $\mathcal{X}[1]$). In general, the idea is that if X is an I-indexed family of objects and if Y is a J-indexed family of objects, then the general morphism in $\mathcal{X}$ from X to Y is a morphism $I \xrightarrow{\sigma} J$ of $\mathcal{S}$, together with an "I-indexed family of morphisms of $\mathcal{X}[1]$" $X_i \to Y_{\sigma(i)}$; but here also the indexing of the family of morphisms is to be thought of as "smooth" in the sense determined by $\mathcal{S}$, the whole $\mathcal{X}$, and their given relation. To make this reasonable we must give structure and axioms, which from a set-theoretic point of view mean that the closure conditions holding for families include substitutions and relative infinite sums and products, or from the geometrical "atlas" point of view mean that we have covariant and contravariant coordinate transformations as well as "inverses" for the charts.

First, we express the family of categories $I \rightsquigarrow \mathcal{X}(I)$ in finite form, using the formalism of fibered categories. That is, we take as data a category $\mathcal{X}$ and a "projection" functor $\mathcal{X} \to \mathcal{S}$ equipped with a *cleavage.* If $I \in \mathcal{S}$, then $\mathcal{X}[I]$ is the (nonfull) subcategory of $\mathcal{X}$ whose objects and morphisms are all those whose "projection" is I or identity morphism of I ; we might call it the category of I-fold objects and morphisms of $\mathcal{X}[1]$. The cleavage means that for any pair $I' \xrightarrow{\sigma} I, X$ in which I is the projection of X

$$\begin{tikzcd} & X \arrow[d, squiggly] \\ I' \arrow[r, "\sigma"'] & I \end{tikzcd}$$

we are given a morphism σ_X of $\mathcal{X}$ whose projection is σ and whose codomain is X; denote the domain of σ_X by $\sigma^* X$ is I'.

$$\begin{tikzcd} \sigma^* X \arrow[r, "\sigma_X"] \arrow[d, phantom, ""] & X \\ I' \arrow[r, "\sigma"'] & I \end{tikzcd}$$

The cleavage is subject to a universal property and a coherence property: If

$X' \xrightarrow{f} X$ is any morphism of $\mathcal{X}$ with projection σ (the domain of f can be any object of projection I') then there is a unique morphism $X' \to \sigma^* X$ whose projection is the identity of I' and for which the triangle below commutes

$$\begin{tikzcd}
X' \arrow[dr, dashed] \arrow[drr, "f"] \arrow[ddr, squiggly, bend right] & & \\
& \sigma^* X \arrow[r, "\sigma_X"'] & X \arrow[d, squiggly] \\
& I' \arrow[r, "\sigma"'] & I
\end{tikzcd}$$

Moreover, if $I'' \xrightarrow{\overline{\sigma}} I$, then the canonical

$$\overline{\sigma}^*(\sigma^* X) \to (\overline{\sigma} s)^* X$$

is an isomorphism, and these isomorphisms are coherent [sometimes called "cocycle condition"]. X is an I-fold family of objects of $\mathcal{X}[1]$ and the I'-fold family $\sigma^* X$ arises from substituting σ into X; in particular $X \in \mathcal{X}[I]$ and if $1 \xrightarrow{i} I$ then $i^* X \in \mathcal{X}[1]$ and we may write $X_i \underset{def}{=} i^* X$; then in general

$$(\sigma^* X)_{i'} \xrightarrow{\sim} X_{i'}, \sigma.$$

Given a fibered category $\mathcal{X}$ over $\mathcal{S}$ and an object $J \in \mathcal{S}$ we can construct another fibered category $\mathcal{X}^J$ over $\mathcal{S}$ by taking the pullback category

$$\begin{tikzcd}
\mathcal{X}^J \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{S} \arrow[r, "(\,)\times J"'] & \mathcal{S}
\end{tikzcd}$$

in which we have

$$\mathcal{X}^J[I] \cong \mathcal{X}[I \times J] \text{ for all } I \in \mathcal{S}$$

Thus we automatically have an atlas of families for $\mathcal{X}^J$. We leave to the reader to see that $\mathcal{X}^J$ will also inherit the further three aspects of structure below.

The fundamental "coordinate transformation" is a given functor over $\mathcal{S}$

$$\begin{tikzcd}
\mathcal{X} \underset{\mathcal{S}}{\times} \mathcal{S}^{\swarrow\searrow} \arrow[rr, "\otimes"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

for which we also have given coherent associativity and unit isomorphisms, i.e.

$$\mathcal{X}[I] \times \mathcal{S}/I \times J \xrightarrow{\otimes} \mathcal{X}[J]$$

and

$$\begin{aligned} X \otimes (S * T) &\cong (X \otimes S) \otimes T \\ X \otimes 1_I &\cong X \end{aligned}$$

coherently. In the case where $\mathcal{X}[1]$ has small coproducts and $\mathcal{S}$ = sets, we can have

$$(X \otimes S)_j = \sum_i X_i \cdot S_{ij}$$

where $X_i \cdot S_{ij}$ is also an iterated coproduct.

To define the second further aspect, consider first the new fibered category $\mathcal{X}_\Pi$ in which the objects are those of $\mathcal{X}$ but in which a morphism $Y \to Y'$ is a pair $< \beta, f >$ where $J' \xrightarrow{\beta} J$ in $\mathcal{S}$ (J being a projection of Y and J' the projection of Y') and where

$$\beta * Y \xrightarrow{f} Y'$$

is a morphism of $\mathcal{X}$ whose projection is $1_{Y'}$. The projection for $\mathcal{X}_\Pi$ goes to $\mathcal{S}^{op}$, not to $\mathcal{S}$, but of course $\mathcal{X}_\Pi^{op}$ has a projection functor to $\mathcal{S}$.

Exercise 3.7 Interpret $\mathcal{X}_\Pi$ in the case of $\mathcal{X} = \mathcal{S}^2$ by giving an alternate definition using the operator Π of $\mathcal{S}$ instead of pullback. Discuss cleavages and cocleavages for $\mathcal{X}_\Pi$ in this case.

Now we require further the existence of a strong hom-functor over $\mathcal{S} \times \mathcal{S}$

$$\begin{tikzcd} \mathcal{X}_\Pi^{op} \times \mathcal{X} \arrow[rr, "(\ ,\ )"] \arrow[dr] & & \mathcal{S}^{\swarrow\searrow} \arrow[dl] \\ & \mathcal{S} \times \mathcal{S} & \end{tikzcd}$$

characterized being its right adjoint to $\otimes$. That is

$$\mathcal{X}[I]^{op} \times \mathcal{X}[J] \xrightarrow{(\ ,\ )} \mathcal{S}/I \times J$$

and there is a natural bijection

$$\frac{X \otimes S \to Y \text{ in } \mathcal{X}[J]}{S \to (X, Y) \text{ in } \mathcal{S}/I \times J}$$

for $X \in \mathcal{X}[I]$. Taking $I = J, S = 1_I$, we get that the morphisms $X \to Y$ in $\mathcal{X}$ are in natural one-to-one correspondence with the *diagonal sections* of

(X, Y) in $\mathcal{S}$, justifying the name "hom". Taking $S \xrightarrow{=} (X, Y)$, we deduce a canonical evaluation morphism

$$X \otimes (X, Y) \to Y \text{ in } \mathcal{X}[J]$$

for $X \in \mathcal{X}[I]$. Using the latter twice and again the adjointness, we get a strong composition morphism

$$(X, Y) * (Y, Z) \to (X, Z) \text{ in } \mathcal{S}/I \times K$$

Also we get the composite functor

$$\mathcal{X}[I]^{op} \times \mathcal{X}[I] \xrightarrow{(\ ,\ )} \mathcal{S}/I \times I \xrightarrow{\triangle^*} \mathcal{S}/I \xrightarrow{\prod_I} S$$

as an "ordinary" strong hom on each fiber.

Exercise 3.8 For $\mathcal{S}$ = sets, $(X, Y)_{ij} = \mathcal{X}[1](X_i, Y_j)$.

Exercise 3.9 For $\mathcal{S}$ (i.e. $\mathcal{X} = \mathcal{S}^2$), if $X \xrightarrow{p} I$ in $\mathcal{X}[I], Y \xrightarrow{q} J$ in $\mathcal{X}[J]$ show that $(X, Y) = \prod_{p \times J}(X \times q)$.

The third further aspect is a functor over $\mathcal{S}$

$$\begin{tikzcd}
(\mathcal{S}^{\uparrow})^{op} \underset{\mathcal{S}^{op}}{\times} \mathcal{X} \arrow[rr] \arrow[dr] & & \mathcal{X} \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

characterized by being the right adjoint (for fixed S rather than for fixed X as in the case of (,)) to $\otimes$, i.e.

$$(\mathcal{S}/I \times J)^{op} \times \mathcal{X}[J] \to \mathcal{X}[I]$$

denoted by $S, Y \rightsquigarrow Y^S$ with a natural bijection

$$\frac{X \to Y^S \quad \text{in } \mathcal{X}[I]}{X \otimes S \to Y \text{ in } \mathcal{X}[J]}$$

If $I \xrightarrow{\sigma} J$ and we take S to be σ viewed as a matrix, then $Y^S \cong \sigma^* Y$. In the case $\mathcal{S}$ = sets, $\mathcal{X}[1]$ complete, we have

$$(Y^S)_i = \prod_j Y_j S_{ij}$$

where the exponent inside the product sign represents iterated infinite product. This third aspect is not needed for the following result.

A basic *example* has $\mathcal{X}[I] = \mathcal{S}/I$; to put this in the above form, we need only take $\mathcal{X} = \mathcal{S}^{\underline{2}}$ with the functor "codomain" as projection, and the cleavage arises from any choice of *pullbacks* in $\mathcal{S}$.

Exercise 3.10 Define $\mathcal{X}$ and projection so as to have

$$\begin{array}{lcl} \mathcal{X}[I] & = & Ab(\mathcal{S}/I) \quad \text{or} \\ \mathcal{X}[I] & = & Ann(\mathcal{S}/I) \end{array}$$

Remark: If $\mathcal{S}$ has finite colimits, it would also be natural to require

$$\begin{array}{llll} \mathcal{X}[0] & = & 1 & \\ \mathcal{X}[I_1 & + & I_2] \xrightarrow{\sim} \mathcal{X}[I_1] \times \mathcal{X}[I_2] & \text{(defined by ( )}^* \text{ of injections)} \\ \mathcal{X}[Q] & \cong & \text{Descent } (\mathcal{X}[I], R, Q) & \text{where } R \text{ is an equivalence relation on } I \\ & & & \text{with coequalizer } Q. \end{array}$$

Exercise 3.11 If $\mathcal{X} = \mathcal{S}^{\underline{2}}$, i.e. $\mathcal{X}[I] = \mathcal{S}/I$ where $\mathcal{S}$ has finite coproducts and coequalizers of equivalence relations, show that the above holds iff

0 is "strict"
$+$ is "disjoint"
equivalence relations are *effective*

Proposition 3.9: $\mathcal{S}Adj_R(\mathcal{X}, \mathcal{S}^J) \cong \mathcal{X}[J]$. *Instead of a proof, we give the definition of the left hand side, which should be nearly sufficient. Indeed, the present "discrete J" case of our basic formula should be true practically by definition of the rich structure we have assumed on $\mathcal{X}$. The fundamental fact for the proof is that $1_J \in \mathcal{S}^J[J]$ is a "basis" for "J-dim space $\mathcal{S}^J$".*

First, we recall the definition of a morphism of fibered categories; it is of course a functor which preserves the fibers as in

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "F^*"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

but, moreover, it should preserve the cleavages in the sense that the canonical morphism

$$F_*(\sigma^* X) \to \sigma^*(F_* X)$$

should be an isomorphism in $\mathcal{X}$ for any morphism σ in $\mathcal{S}$ and any object X which projects to the codomain of σ. Of course, a functor F^* going the other way will be required to preserve cleavages too. When both $\mathcal{X}, \mathcal{Y}$ have $\otimes$, we would naturally require of F^* that it be equipped with coherent natural isomorphisms (required to project to identities)

$$F^*(Y) \otimes S \xrightarrow{\sim} F^*(Y \otimes S)$$

Applying these in the case $S = \mathcal{Y}(Y, Y')$ and taking F^* of the evaluation, we get coherent morphisms (usually not isomorphisms)

$$\mathcal{Y}(Y, Y') \to \mathcal{X}(F^*Y, F^*Y')$$

which we express by saying that F^* is an $\mathcal{S}$-functor. If F_* is another $\mathcal{S}$-functor in the other direction, it is possible to formulate and require $\mathcal{S}$-*adjointness*

$$\mathcal{Y}(Y, F_*X) \cong \mathcal{X}(F^*Y, X)$$

where these isomorphisms in $\mathcal{S}^{\swarrow\searrow}$ are also required to project to identities in $\mathcal{S} \times \mathcal{S}$. Finally, if we also have the infinite $\mathcal{S}$-products in $\mathcal{X}$ we can formulate that F_* preserves them by given coherent isomorphisms and by a different but analogous argument it follows that F_* is also an $\mathcal{S}$-functor.

Large versus Internal

For the elementary formal part of the discussion of internal categories $\mathbf{C}$ and internal functors (= "left $\mathbf{C}$-objects") T, it is only necessary to assume that *pullbacks* exist in $\mathcal{S}$. Notice that for fixed $I \in \mathcal{S}$, the "multiplication of square matrices" is a (non-commutative!) bifunctor

$$(\mathcal{S}/I \times I) \times (\mathcal{S}/I \times I) \xrightarrow{\ *\ } \mathcal{S}/I \times I$$

which (because of the uniqueness of pullbacks), is equipped with a coherent associativity natural isomorphism, with the identity matrix 1_I as unit object (again up to coherent isomorphism, since pulling back in either direction along the identity morphism is equivalent to the identity functor). Moreover, (as a second aspect of the "global" bifunctoriality of $\mathcal{S}^{\swarrow\searrow} \underset{\mathcal{S}}{\times} \mathcal{S}^{\swarrow\searrow} \xrightarrow{\ *\ } \mathcal{S}^{\swarrow\searrow}$ over $\mathcal{S} \times \mathcal{S}$ as discussed in earlier paragraphs) if $I \xrightarrow{\sigma} I'$ and if we are given commutative squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ I \times I \arrow[r, "\sigma\times\sigma"'] & I' \times I' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[d] & B' \arrow[d] \\ I \times I \arrow[r, "\sigma\times\sigma"'] & I' \times I' \end{tikzcd}$$

then there is an induced commutative square

$$\begin{tikzcd} A * B \arrow[r, "f^*g"] \arrow[d] & A' * B' \arrow[d] \\ I \times I \arrow[r, "\sigma\times\sigma"'] & I' \times I' \end{tikzcd}$$

Now the idea of a category $\mathbf{C}$ *in* $\mathcal{S}$ is that it has an "object of objects" I and an "object of morphisms" C which is equipped with structural morphisms ("domain" and "codomain") making C a $I \times I$ matrix [such that if i, j are "elements" of I then $C_{ij} = \mathbf{C}(i,j) = hom_{\mathbf{C}}(i,j) =$ "object of morphisms from i to j"] plus a *composition* morphism $C * C \xrightarrow{\mu} C$ and *unit* morphism $1_I \xrightarrow{\eta} C$, both over $I \times I$ [the "over" is expressed by four commutative diagrams in $\mathcal{S}$] such that the associative and unit laws following hold for μ and η

$$\begin{tikzcd} C * C * C \arrow[r, "C*\mu"] \arrow[d, "\mu*C"'] & C * C \arrow[d, "\mu"] \\ C * C \arrow[r, "\mu"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "1_I*\eta"] \arrow[d, "\eta*1_I"'] \arrow[dr, "id"] & C * C \arrow[d] \\ C * C \arrow[r, "\mu"'] & C \end{tikzcd}$$

[We have suppressed mention of the canonical associativity and unit isomorphisms for * and 1_I.] Thus, a category in $\mathcal{S}$ may be denoted by $\mathbf{C} = < I, C, \eta, \mu >$, where it is understood that C is an $I \times I$ matrix, or strictly as a diagram in $\mathcal{S}$ as $\mathbf{C} =< I, C, d_0, d_1, q_0, q_1, \eta, \mu >$ where $C \underset{d_1}{\overset{d_0}{\rightrightarrows}} I$ are the domain and codomain operations making C a matrix and $C * C \underset{q_1}{\overset{q_0}{\rightrightarrows}} C$ are the structural maps for the pullback defining the matrix product so that in particular

$$C * C \underset{q_1 d_1}{\overset{q_o d_0}{\rightrightarrows}} I$$

are the structural maps for $C*C$ as an $I\times I$ matrix. The object $C*C$ of $\mathcal{S}$ may be considered either as the "set" of all composable pairs of morphisms in $\mathbf{C}$ or as the "set" of commutative triangles of morphisms in $\mathbf{C}$ [intuitively, these two "sets" are certainly canonically isomorphic]; in the latter interpretation, not only q_0, q_1 but also μ are "projections", since if

$$\begin{tikzcd} & \cdot \arrow[dr, "v"] & \\ \cdot \arrow[ur, "u"] \arrow[rr, "w"'] & & \cdot \end{tikzcd}$$

is a *commutative* triangle, then q_0 of it is u, q_1 of it is v, and μ of it is w.

The notion of a category $\mathbf{C}$ in a category $\mathcal{S}$ can be made more "concrete" (i.e. abstract) if we suppose that we have a generating subcategory $\mathcal{A}$ of $\mathcal{S}$ (better, an adequate subcategory in the sense of the next section) and we consider that the hom sets $\mathcal{S}(A, X)$ of $\mathcal{S}$ "are" abstract sets (more formally, we consider that $\mathcal{S}$ is equipped with part of the structure of a "large category with an atlas" over the base category $\mathcal{S}_0$ of abstract sets). Then an $\mathcal{S}$-morphism $A \to I$ is called "an object of $\mathbf{C}$ defined over A" and an $\mathcal{S}$-morphism

$A \xrightarrow{u} C$ is called "a morphism of $\mathbf{C}$ defined over A"; if $A \underset{v}{\overset{u}{\rightrightarrows}} C$ are two morphisms of $\mathbf{C}$ defined over A and if $ud_0 = vd_1$ then since $C * C = C \underset{I}{\times} C$ is a pullback of d_0 with d_1, there is a unique $A \xrightarrow{t} C * C$ with $tq_0 = u$ and $tq_1 = v$ and hence we may define the *composition of* u *followed by* v *in* C as

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[dr, "u \cdot v"'] & C * C \arrow[d, "\mu"] \\ & C \end{tikzcd}$$

The associativity and unit laws of μ, η imply that for any fixed A, the foregoing defines an "actual" (i.e. in the sense of $\mathcal{S}_0$) category $\mathbf{C}[A]$. Moreover, it is clear that if $A' \xrightarrow{a} A$, composition in $\mathcal{S}$ gives rise to an "actual" (again in the sense of *abstract* small categories) *functor* $\mathbf{C}[A] \xrightarrow{C[a]} \mathbf{C}[A']$ and that this process is functorial with respect to composites $A'' \xrightarrow{\overline{a}} A' \xrightarrow{a} A$. Thus $\mathbf{C} \in Cat(\mathcal{S})$ gives rise to a functor

$$\mathcal{A}^{op} \to Cat(\mathcal{S}_0)$$

[and the resulting process is actually a functor

$$Cat(\mathcal{S}) \to Cat(\mathcal{S}_0)^{\mathcal{A}^{op}}$$

which will be full and faithful if $\mathcal{A}$ is adequate in $\mathcal{S}$].

* A philosophical explanation of the preceding paragraph is as follows: $\mathcal{S}$ represents all the things in the world and their interactions and $\mathcal{A}$ represents the consciousness of investigators $\mathcal{A}$ who, armed with the theory of abstract sets $\mathcal{S}_0$, are constructing an $\mathcal{S}_0$-atlas of $\mathcal{S}$. An important form of the internal contradictions which things-in-their motion may have, has been found to be that of a small category $\mathbf{C}$, so the investigators have deepened their theory to include the theory of small abstract categories Cat ($\mathcal{S}_0$). When A struggles with $\mathbf{C}$, this is reflected theoretically as $\mathbf{C}[A]$. The existence of social practice, i.e. the fact that A, A' represent organized groups of investigators who struggle $A \rightleftarrows A'$ with each other, and that $\mathcal{A}$ is adequate in $\mathcal{S}$ implies that

$$\varinjlim_{i \in \mathcal{A}/\mathbf{C}} \mathbf{C}[A_i]$$

is a correct and complete theory of $\mathbf{C}$. Thus also mathematics does not conform to the philosophy of subjective idealism given by Bishop Berkeley, Poincaré, and others which says that no correct theory of a thing $\mathbf{C}$ can be developed, since different investigators A, A' may see it differently; we see that subjective idealism is founded on denial of the progressive role of struggle between investigators, leading to the promotion of the theory that $\mathcal{A}$ is a discrete category.

A valid criticism of our theory is that it is objective idealism, since in most examples we consider that $\mathcal{A} \in Cat(\mathcal{S}_0)$, rather than that "$\mathcal{S}_0 \in Cat(\mathcal{A})$"; this criticism is difficult to overcome within the present state of mathematics and the attempts to resolve the just-mentioned contradiction by developing it into a sequence of "universes" is surely at best only a partial answer. A second criticism is that we have left out of explicit account the nature of the internal contradictions of the A's which have also developed theories of logic or of quantity or of algebro-geometric structure $\mathcal{V}_0$, that $\mathcal{A}$ may also struggle with a $\mathbf{C}$ in a second sense $\mathcal{S}(\mathbf{C}, A)$, and that this leads to theories of function algebras and convolution algebras and the contradiction between those, which are also important for a complete theory of the *motion and development* of the things in $\mathcal{S}$. This omission is only partly due to the fundamental nature of the present state of mathematics, since the basis for a formalism of *quantitative* change was already developed before 1871 by Euler, d'Alembert, Bernoulli, Fourier, Hamilton, Maxwell and others, and thus we can hope to give in a later continuation of this course some of the general features of an analysis of $\mathcal{S}$ which involves both $\mathcal{S}_0$ and a $\mathcal{V}_0$. [A formalism of *qualitative* change must involve a more profound consideration of the internal contradictions in things which could be perhaps partly reflected by representing $\mathcal{S}$ itself as something like a "2-dimensional category" rather than just a category. The known physical facts which should be reflected in such a conjectured mathematical theory are the transformation of time and space into each other in a way more profound than that reflected in the notion of velocity itself, the transformation of radiation and ponderable matter into each other, and the transformation of microscopic motion and macroscopic motion into each other in a way more profound (e.g. conduction in metals) than that reflected in the theory of convection and heat transfer in itself. In the writer's opinion the development of such a mathematical theory of the dialectics of qualitative change in $\mathcal{S}$ will have to be part of a process of qualitative change not limited to mathematics alone.] *

An obvious mathematical explanation of the process $A \rightsquigarrow \mathbf{C}[A]$ is that it is similar to the $A \rightsquigarrow \mathcal{X}[A]$ discussed earlier in this lesson. We leave it to the reader to formulate a general notion of $\mathcal{S}$-based categories which have atlases and hom-functors and to show that both the internal $\mathcal{S}$-categories $\mathbf{C}$ and the "$\mathcal{S}$-bicomplete" ones $\mathcal{X}$ "are" special cases, as well as to show that all three types of functors with domain $\mathbf{C}$ to be discussed below, give rise naturally to (not continous) morphisms of such general $\mathcal{S}$-categories. Note that, in contrast to the theory of "universes", we have demonstrated (even without introducing a $\mathcal{V}$) that there is a *qualitative* (as well as quantitative) difference between large and small $\mathcal{S}$-based categories: The objects of $\mathcal{X}[1]$ form only an *abstract* class, though this is partly counteracted by the atlas structure of $\mathcal{X}$, whereas the objects of $\mathbf{C}$ form an object of the not-necessarily abstract (i.e. objects of $\mathcal{S}$ may have structures *like* topological or algebraic spaces) category $\mathcal{S}$; this means in particular that $\mathcal{X}$ can be complete without being trivial, whereas for the $\mathcal{S}$ known to the writer (and certainly by a known theorem for $\mathcal{S}_0$), if $\mathbf{C}$ is complete or cocomplete, then $\mathbf{C}$ must reduce to a poset object in $\mathcal{S}$.

Exercise 3.12 Formally, $\mathbf{C} = < I, C, \eta, \mu >$ is a poset object in $\mathcal{S}$ means that $C \xrightarrow{<d_0,d_1>} I \times I$ is a monomorphism (η is then the reflexive law and μ is the transitive law in this case). Show that if $\mathcal{S}$ is any regular category, the full inclusion functor

$$Poset(\mathcal{S}) \hookrightarrow Cat(\mathcal{S})$$

has a left adjoint. (Recall that the image factorization is connected with $\exists$, and that the set of objects of an arbitrary category can be partially ordered by saying that $X \leq Y$ iff there exists at least one morphism $X \to Y$). If σ denotes the reflection functor just discussed, in $\mathbf{C} \in Cat(\mathcal{S})$ and if A is a regular *projective* object of $\mathcal{S}$, then the induced bijection

$$\sigma_{\mathcal{S}_0}(\mathbf{C}[A]) \to (\sigma_{\mathcal{S}}(\mathbf{C}))[A]$$

of abstract sets is actually an isomorphism of (ordinary abstract) posets. However, the last is not necessarily true for an arbitrary object A, reflecting the fact that the connection between images and existence is not mechanical (recall the discussion in Lesson 2 concerning *local* nature of existence in a general $\mathcal{S}$); try to find an example to illustrate this point about the poset reflection of $\mathbf{C}$ as seen by an A, perhaps the following example or some even simpler one such as $\mathcal{S}$=permutations of finite orbit. [In case the definition of the morphisms in $Cat(\mathcal{S})$ is not obvious, see below.]

Example: Let K be a commutative ring (say $\mathbf{Z}$ or a field) in the category $\mathcal{S}_0$ of abstract sets and let $\mathcal{A}$ be the (small $\mathcal{S}_0$-) category of finitely presented affine schemes over K, i.e. $\mathcal{A}$ is simply the opposite category of the category of all commutative K-algebras of the form $K[t_1, \ldots, t_n]/\mathcal{J}$ for some $n \in \mathbf{N}$, some ideal $\mathcal{J}$, and all homomorphisms of K-algebras between them. A foundation for algebraic geometry consists of the construction of an intermediate category

$$\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{S}_0^{\mathcal{A}^{op}}$$

[where the last is just the category of all covariant set-valued functors on the category of finitely-presented commutative K-algebras and the composite inclusion is the Yoneda embedding which in this context is sometimes called "spec"] such that $\mathcal{S}$ is a category with good closure properties (e.g. Π) and such that $\mathcal{S}$ contains not only the affine spaces, but also projective space, Grassmann manifolds and flag manifolds. Now the latter spaces can all be constructed using the closure properties of $\mathcal{S}$ from a *particular* category object $\mathbf{P} \in Cat(\mathcal{S})$, "the category of projective modules", which we now describe. To give $\mathbf{P}$ is equivalent to giving for each finitely presented A an ordinary small category $\mathbf{P}[A] \in Cat(\mathcal{S}_0)$ and for each ring homomorphism $A \xrightarrow{\varphi} A'$ (which preserves the K-scalars) an ordinary functor $\mathbf{P}[A] \xrightarrow{P[\varphi]} \mathbf{P}[A']$ such that composite φ's are taken to composite functors. The idea is that $\mathbf{P}[A]$ should be equivalent to the category of finitely generated projective modules over the ring A and that $\mathbf{P}[\varphi]$ should be the extension of scalars $(\ )\underset{A}{\otimes} A'$; however, in this case we can rigorously avoid the coherence problems with the tensor product by representing projective modules by idempotent matrices. Thus formally $\mathbf{P}[A]$ has as *objects* the set

$$\left\{ e \in A^{n\times n} \,\middle|\, n \in \mathbf{N} \sum_j e_{ij}e_{jk} = e_{ik} \quad \text{all } i,k = 1,\ldots,n \right\}$$

and as *morphisms* the set

$$\left\{ < e, a, e' > \,\middle|\, \begin{array}{l} e, e' \text{ objects of } \mathbf{P}[A] \text{ of some dimensions } n, n' \\ a \in A^{n\times n'} \text{ such that in the sense of matrix} \\ \qquad\qquad\quad \text{multiplication } ea = a = ae' \end{array} \right\}$$

and as *composition* μ_A simply the multiplication of the matrices a; η_A assigns to an object e the morphism $< e, e, e >$ which under μ will act as the identity. Since any ring homomorphism φ can be applied to all the entries of a matrix and preserves the addition and multiplication involved in definition of matrix multiplication, it is clear that $\mathbf{P}[\varphi]$ is defined and functorial.

* There is a unit object $1 \to \mathbf{P}$ defined by assigning to each A the 1×1 identity matrix e and a tensor product functor $\mathbf{P} \times \mathbf{P} \xrightarrow{\otimes} \mathbf{P}$ and direct sum functor $\mathbf{P} \times \mathbf{P} \xrightarrow{\oplus} \mathbf{P}$ (both morphisms of $Cat(\mathcal{S})$ as well as a functor $\mathbf{P}^{op} \times P \xrightarrow{\text{Hom}} \mathbf{P}$. Since $\mathcal{S}$ is cartesian closed, we may form for any $X \in \mathcal{S}$ the "category of vector bundles" $\mathbf{P}^X$ and the tensor product and direct sum lift to this. We could even say that a morphism $X \xrightarrow{f} Y$ in $\mathcal{S}$ is $\mathbf{P}$-proper if the induced functor $\mathbf{P}^X \longleftarrow \mathbf{P}^Y$ has a right adjoint in the sense of the 2-dimensional category $Cat(\mathcal{S})$. There will be natural maps

$$K(\mathbf{P}^X[A]) \to K(\mathbf{P}^X)[A]$$

comparing external with internal Grothendieck rings. Similarly for Picard groups, which are derived from the "line bundle" subcategory $\mathbf{P}_1$ of $\mathbf{P}$ which is just the part consisting of those $A \xrightarrow{\ell} \mathbf{P}$ for which the canonical internal map $\ell \otimes Hom(\ell, A) \to A$ is an isomorphism. *

Note in this example that the usual underlying set functor for finitely presented commutative K-algebras is actually a single ring object R of $\mathcal{S}$, called the (algebraic) line; it contains as subobjects both the multiplicative group and the "infinitesimal" space $D \subseteq R$ where $D[A] = \{a \in A | a^2 = 0\}$; for any $X, X^D \to X$ (induced by $1 \xrightarrow{0} D$) is the *tangent bundle* of X which is an R-module in the category $\mathcal{S}/X$; some of the R-modules in $\mathcal{S}/X$ correspond to morphisms $X \to \mathbf{P}$ in $\mathcal{S}$.

* For any regular category $\mathcal{S}$ in which equivalence relations are effective, the idea of "descent" determines a subcategory $Champ(\mathcal{S}) \subseteq Cat(\mathcal{S})$ which has been used by Giraud and others to unify the facts that a scheme is locally affine, a vector bundle is locally trivial, an Azuwaya algebra is locally a matrix algebra, etc. Formally $\mathbf{C}$ is a champ iff whenever $X \xrightarrow{p} Y$ is a regular epimorphism in $\mathcal{S}$ with kernel pair $E \rightrightarrows X$, any functor $\mathbf{E} \to \mathbf{C}$ is actually induced by a map from Y to the objects of $\mathbf{C}$. [Here $\mathbf{E}$ is the obvious category object $< X, E, refl., trans. >$. In the case of $\mathcal{S}_0^{\mathcal{A}^{op}}$, $\mathcal{A} =$ finitely presented affine schemes $/K$ discussed above, for some choices of $\mathcal{S}$ the projective module category $\mathbf{P}$ is not only in $\mathcal{S}$, but actually a champ in $\mathcal{S}$; conversely, a plausible choice of a foundation for algebraic geometry would be the smallest $\mathcal{S} \subseteq \mathcal{S}_0^{\mathcal{A}^{op}}$ which is a topos, which contains $\mathcal{A}$, and for which $\mathbf{P} \in Champ(\mathcal{S})$. *

There are two notions of internal functor. If $\mathbf{C}, \mathbf{D}$ are two categories *in* $\mathcal{S}$, then a functor $\mathbf{C} \xrightarrow{f} \mathbf{D}$ is a pair $f_0 : I \to J, f_1 : C \to D$ of morphisms of $\mathcal{S}$ (where $I =$ obj $(\mathbf{C})$, $J =$ obj (D), $C =$ Mor (C), $D =$ Mor $(\mathbf{D})$ for which the following three diagrams are commutative

$$
\begin{tikzcd}
C \arrow[r, "f_1"] \arrow[d] & D \arrow[d] \\
I \times I \arrow[r, "f_0 \times f_0"'] & J \times J
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "f_0"] \arrow[d, "\eta"'] & J \arrow[d, "\eta"] \\
C \arrow[r, "f_1"'] & D
\end{tikzcd}
$$

$$
\begin{tikzcd}
C * C \arrow[r, "f_1 * f_1"] \arrow[d, "\mu"'] & D * D \arrow[d, "\mu"] \\
C \arrow[r, "f_1"'] & D
\end{tikzcd}
$$

Exercise 3.13 Define the notion of natural transformation φ between two functors

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "\downarrow\varphi"] & \mathbf{D}
\end{tikzcd}
$$

so as to give Cat ($\mathcal{S}$) defined above its two-dimensional structure.

Exercise 3.14 Say that any pair

$$M \underset{d_1}{\overset{d_0}{\rightrightarrows}} I$$

of morphisms of $\mathcal{S}$ is a "diagram scheme" and define the notion of morphism between two such to be a pair consisting of a morphism $I \to I'$ on "vertices" and another $M \to M'$ on "arrows", such that beginnings d_0 and ends d_1 of arrows are preserved. There is a forgetful functor

$$Cat(\mathcal{S}) \to \text{Diagr sch}(\mathcal{S})$$

if it has a *left adjoint* $\mathcal{L}$, say that $\mathcal{S}$ satisfies the "axiom of infinity". [The idea is the "set" of all meaningful strings of arrows from M, of whatever finite length, is an object of $\mathcal{S}$; in particular, $\mathcal{L}(1 \rightrightarrows 1) = N$ will play the role of the natural numbers with $\mu =$ addition.] A category object $\mathbf{C} =< I, C, \eta, \mu >$ is called *discrete* iff $C = I, d_0 = d_1 = \mu = id_I$. Clearly, the notion of discrete category defines a full inclusion $\mathcal{S} \to Cat(\mathcal{S})$; show that the latter has a left adjoint Π_0 if $\mathcal{S}$ is a category having reflexive coequalizers. [Hint: take the coequalizer

$$C \underset{d_1}{\overset{d_0}{\rightrightarrows}} I \longrightarrow \Pi_0(C)$$

thus splitting $\mathbf{C}$ into "components".] If $\mathcal{S}$ satisfies the axiom of infinity, then the coequalizer of any pair $M \rightrightarrows I$ is isomorphic to $\Pi_0(\mathcal{L}(M \rightrightarrows I))$, and if $\mathcal{S}$ moreover is a regular category, then for any pair

$$M \underset{d_1}{\overset{d_0}{\rightrightarrows}} I$$

which has a symmetry $M \xrightarrow{s} M$ with $sd_0 = d_1, s^2 = id_M$, the smallest equivalence relation on I "containing M" is just the *image* of

$$C \xrightarrow{<\tilde{d}_0, \tilde{d}_1>} I \times I$$

where $< I, C, \tilde{d}_0, \tilde{d}_1, \eta, \mu >= \mathcal{L}(M \underset{d_1}{\overset{d_0}{\rightrightarrows}} I)$.

The second kind of internal functor is an $\mathcal{S}$-valued one, of which we consider the contravariant case. If $\mathbf{C} =< I, C, \eta, \mu >\in Cat(\mathcal{S})$, then $\mathbf{T} \in \mathcal{S}^{\mathbf{C}^{op}}$ iff $\mathbf{T} =< T, p, \alpha >$ where $T \xrightarrow{p} I$ and $C * T \xrightarrow{\alpha} T$ *over* I satisfying unit and associative laws. A natural transformation (or homomorphism of left $\mathbf{C}$-actions) $T \xrightarrow{h} T'$ is any morphism of $T \to T'$ for which

$$\begin{tikzcd} T \arrow[rr, "h"] \arrow[dr, "p"'] & & T' \arrow[dl, "p'"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} C * T \arrow[r, "C*h"] \arrow[d, "\alpha"'] & C * T' \arrow[d, "\alpha'"] \\ T \arrow[r, "h"'] & T' \end{tikzcd}$$

are commutative; such are the morphisms of the catgegory $\mathcal{S}^{\mathbf{C}^{op}}$.

Exercise 3.15 The forgetful functor $\mathcal{S}^{\mathbf{C}^{op}} \to \mathcal{S}/I$ has $C * (\ )$ as a left adjoint.

Exercise 3.16 (Kan extensions of $\mathcal{S}$-valued functors with small domains) If $\mathcal{S}$ has internal products Π and coequalizers, then for any $\mathbf{C} \xrightarrow{f} \mathbf{D}$ in $Cat(\mathcal{S})$ the induced $\mathcal{S}^{\mathbf{D}^{op}} \to \mathcal{S}^{\mathbf{C}^{op}}$ has both right and left adjoints, [equalizer of a Π and coequalizer of a $\sum$ where internal $\sum$ is just composition].

Remark: In particular, when $\mathbf{D} = 1$, these adjoints are denoted by

$$\varprojlim_{\mathbf{C}} : \mathcal{S}^{\mathbf{C}^{op}} \to \mathcal{S}; \qquad \varinjlim_{\mathbf{C}} : \mathcal{S}^{\mathbf{C}^{op}} \to \mathcal{S}$$

(often we would replace $\mathbf{C}$ by $\mathbf{C}^{op}$ especially for $\varinjlim$).

Exercise 3.17 If $M \underset{d_1}{\overset{d_0}{\rightrightarrows}} I$ is a diagram scheme, define directly and simply a category $\Delta(\mathbf{M}, \mathcal{S})$ which has a forgetful functor

$$\Delta(\mathbf{M}, \mathcal{S}) \xrightarrow{U_M} \mathcal{S}/I$$

and for which there is an equivalence of categories $\Delta(\mathbf{M}, \mathcal{S}) \cong \mathcal{S}^{\mathcal{L}(\mathbf{M})^{op}}$ in case the axiom of infinity holds for $\mathcal{S}$. Show that if internal product Π exists in $\mathcal{S}$, the axiom of infinity is equivalent to the somewhat simple axiom that for all $\mathbf{M} \in \text{Diagrsch}(\mathcal{S}), U_{\mathbf{M}}$ has a left adjoint. [Hint: develop a notion of *bi*action and use an extension (Yoneda) of the Cayley idea that for any category $\mathbf{C}$ (against which we might want to test the freeness of a proposed free category), there is an explicit ample supply of actions.]

Finally, if $\mathcal{X}$ is an $\mathcal{S}$-bicomplete category with an $\mathcal{S}$-atlas, and if $\mathbf{C} = < I, C, \eta, \mu > \in Cat(\mathcal{S})$ define the objects of $\mathcal{X}[\mathbf{C}]$ to be pairs F, β where $F \in \mathcal{X}[I]$ and $F \otimes C \xrightarrow{\beta} F$ is a morphism of $\mathcal{X}[I]$ satisfying an associative law with μ and a unit law with η, and define then the morphisms of $\mathcal{X}[\mathbf{C}]$ in the obvious way.

Note that if $X \in \mathcal{X}[I]$, then the composition of $\mathcal{X}$ induces a morphism $(X,X) * (X,X) \xrightarrow{\mu} (X,X)$ of $\mathcal{S}/I$ so that the atlas structure implies that $\mathcal{X}$ is covered by *full* subcategories which are "small" (i.e. internal to $\mathcal{S}$).

Exercise 3.18 $\mathcal{S}^{\mathbf{C}^{op}}[I] \underset{def}{=} \mathcal{S}(I \times \mathbf{C})^{op}$ defines a large category with an $\mathcal{S}$-atlas which we also denote by $\mathcal{S}^{\mathbf{C}^{op}}$.

Exercise 3.19 $\mathcal{S}-Adj_R(\mathcal{X}, \mathcal{S}^{\mathbf{C}^{op}}) \cong \mathcal{X}[\mathbf{C}]$ if $\mathcal{X}$ has coequalizers.

Remark: In case $\mathcal{S}$ satisfies the axiom of infinity, it may be useful in some cases to consider a further closure condition on the $\mathcal{S}$-atlas structure for a large category $\mathcal{X}$ and in particular a still stronger axiom on $\mathcal{S}$ so that $\mathcal{X} = \mathcal{S}^{\underline{2}}$ will satisfy the closure condition. The problem is similar to that involved in the Fraenkel-Skolem strengthening of Zermelo set theory (which, it should be emphasized, plays usually very *little* role in the practice of analysis, algebra, topology, etc). Our theory of bicompleteness of $\mathcal{X}$ with respect to an atlas says roughly that any families which exist have coproducts and products, but one could consider stronger assumptions on the existence of families parameterized by the natural-numbers-object $N \in \mathcal{S}$. In the case of $\mathcal{X} = \mathcal{S}^{\underline{2}}$, one could say that our axioms of infinity guarantee the existence of the iteration of "linear" functors (matrices and profunctors), but not the existence of iterations of functors which are "too big to be well approximated by linear functors" such as the *power set functor* P; thus, in particular, the stronger closure axiom would give an object E in $\mathcal{X}[N] = \mathcal{S}/N$ having the properties

$$E = \sum_{n=0}^{\infty} P^n(S)$$

for any given $S \in \mathcal{S}$, and so in particular for $S = N$ we would have an object "of cardinality aleph sub omega". We emphasize again that such objects do not occur in usual mathematics, being "bigger" than, e.g. the set of all operators on Hilbert spaces, etc.; however, they are sometimes convenient in treating, e.g. model theory of higher-order theories.

5. Adequacy and density

In l959 Isbell defined a subcategory $\mathbf{C}$ of $\mathcal{X}$ with inclusion functor F to be *adequate* in $\mathcal{X}$ iff the functor $F_* : \mathcal{X} \to \mathcal{S}^{\mathbf{C}^{op}}$ is *full and faithful.* For example, the full subcategory determined by $\mathbf{Z}[t], \mathbf{Z}[x_1, x_2]$ is adequate in the category of all commutative rings since addition and multiplication can be presented by morphisms $\mathbf{Z}[t] \overset{\cdot}{\underset{+}{\rightrightarrows}} \mathbf{Z}[x_1, x_2]$ in $\mathbf{C}$ which act on the sets

$$\begin{aligned} F_*(A)_1 &= \mathcal{X}(\mathbf{Z}[t], A) \cong A \\ F_*(A)_2 &= \mathcal{X}(\mathbf{Z}[x_1, x_2], A) \cong A \times A \end{aligned}$$

and hence any natural transformation $F_*(A) \to F_*(A')$ must preserve addition and multiplication since it commutes with the action. Similarly, the full subcategory determined by $A, A \oplus A$ is adequate in the category of all A-modules. For a dual example, let $\mathcal{X}$ be the *opposite* of the category of all compact (Hausdorff) topological spaces and continuous mappings and let $\mathbf{C}$ be the full subcategory of $\mathcal{X}$ determined by $[0,1]$ and $[0,1]^N$; then a slight modification of the usual theory of rings of continuous functions implies that $\mathbf{C}$ is adequate in $\mathcal{X}$, i.e. any pair of operators, $Conts(Y,[0,1]) \xrightarrow{h_1} Conts(X,[0,1])$. $Conts(Y,[0,1]^N) \xrightarrow{h_N} Conts(X,[0,1]^N)$ which commutes with all the operations induced by all continuous $[0,1]^N \xrightarrow{\theta} [0,1]$ is actually induced by a unique continuous $X \to Y$ [it is easy to see that actually $h_N = h_1^N$, since in particular among the θ's are all the projections; but θ may also be any convex combination $t \rightsquigarrow \sum_{i=0} \lambda_1 t_i$ or any multiplication $t \rightsquigarrow t_i t_j$ or constantly 1, or $t \rightsquigarrow \sup\{t_i | i \in I\}$ for any finite $I \subset N$ etc.].

Isbell also considered the following interesting example: Let $\mathcal{X} = \mathcal{S}_0^{op}$ be the opposite of the category of sets and let $\mathbf{C}$ be the full submonoid of all endomorphisms of a countable set N; then $\mathbf{C}$ is adequate iff there are no measurable cardinals in S_0! That is, suppose X and Y are sets and $N^Y \xrightarrow{h} N^X$ is a mapping such that for every mapping $N \xrightarrow{\theta} N$ the diagram

$$\begin{tikzcd} N^Y \arrow[r, "h"] \arrow[d, "\theta^Y"'] & N^X \arrow[d, "\theta^X"] \\ N^Y \arrow[r, "h"] & N^X \end{tikzcd}$$

is commutative; h is clearly equivalent to a mapping $X \to: \dot{Y} = \mathrm{Hom}_{N^N}(N^Y, N)$ and moreover there is a canonical inclusion ("evaluation") $Y \to \dot{Y}$ whose members may be called the "fixed" elements of $\dot{Y}$; there is no known set Y large enough to have any non-fixed elements in $\dot{Y}$. By contrast, the simplest example of adequacy is that the one-element set is adequate in the category $\mathcal{S}_0$ of abstract sets itself.

Later Ulmer and Gabriel spoke of a functor $F : \mathbf{C} \to \mathcal{X}$ as *dense* iff for any object X of $\mathcal{X}$, the canonical morphism

$$\varinjlim_{x \in F/X} F(C_x) \to X$$

is an isomorphism. Here the category F/X has as objects the pair $< C, x >$ where C is an object of $\mathbf{C}$ and $F(C) \xrightarrow{x} X$ in $\mathcal{X}$, and as morphisms $< C, x > \to < C', x' >$ the morphisms u of $\mathbf{C}$ for which $x = F(u)x'$; by abuse of notation we write $x =< C, x >$ and $C = C_x$. For example, the rational numbers are dense in the real numbers, since for any real number X the relation

$$\sup_{\substack{C \leq X \\ C \in Q}} C \leq X$$

is actually an equality. Already implicit in Isbell's paper was the

Proposition 3.10: *The following three conditions on a functor $F : \mathbf{C} \to \mathcal{X}$ are equivalent for any small category $\mathbf{C}$ and any locally small cocomplete category $\mathcal{X}$:*

i) *F is adequate*

ii) *F is dense*

iii) *The composite $\mathcal{X} \xrightarrow{F_*} \mathcal{S}^{\mathbf{C}^{op}} \xrightarrow{F^*} \mathcal{X}$ is canonically equivalent to the identity functor, i.e. adjunctions $F^*F_*X \to X$ are all isomorphisms.*

Proof: The equivalence of i) and iii) is a special case of a general fact about adjoint functors, while the equivalence of ii) and iii) results from recalling that

$$F^*T = \varinjlim_{t \in \mathbf{C}/T} C_t$$

for $T \in \mathcal{S}^{\mathbf{C}^{op}}$ where C_t means the representable functor assigned by Yoneda to C_t, then comparing the $\varinjlim$ in the definition of density with this one in the case $T = F_*X = C \rightsquigarrow (FC, X)$. ∎

Remark: The equivalence of i) and ii) can actually be shown even if $\mathcal{X}$ does necessarily have all small $\varinjlim$. On the other hand, there is a sort of converse:

Proposition 3.11: *If $\mathcal{X}$ is a full subcategory of $\mathcal{S}^{\mathbf{C}^{op}}$ where $\mathbf{C}$ is a small category, and if the inclusion functor has a left adjoint F^*, then $\mathcal{X}$ is both cocomplete and complete.* [*Thus in particular a category with $\varinjlim$ which has a small adequate subcategory also has $\varprojlim$, but the present proposition is also often the most convenient way to prove that a category has $\varinjlim$.*]

Proof: Since the inclusion functor F_* is full and faithful and has a left adjoint F^*, the composite functor

$$\mathcal{X} \xhookrightarrow{F_*} \mathcal{S}^{\mathbf{C}^{op}} \xrightarrow{F^*} \mathcal{X}$$

is naturally equivalent to the identity functor on $\mathcal{X}$, and an object $T \in \mathcal{S}^{\mathbf{C}^{op}}$ is isomorphic to an object of the subcategory $\mathcal{X}$ iff the adjunction homomorphism $T \to F_*(F^*(T))$ is an isomorphism. If $X_i, i \in \mathbf{I}$ is a system of objects of $\mathcal{X}$ over an "index" category $\mathbf{I}$, then an easy calculation shows that

$$\varprojlim_{i \in \mathbf{I}} F_*(X_i) \quad \text{in } \mathcal{S}^{\mathbf{C}^{op}}$$

is isomorphic to an object of $\mathcal{X}$, and that

$$F^*(\varinjlim_{i \in \mathbf{I}} F_*(X_i)) \quad \text{in } \mathcal{X}$$

has the universal property of colimit when tested against compatible families $X_i \xrightarrow{x_i} X' \quad i \in \mathbf{I}$ of morphisms *in* $\mathcal{X}$. ∎

Remark on Proposition 3.11: The $\varinjlim$ and $\varprojlim$ indicated are computed in the sense of $\mathcal{S}^{\mathbf{C}^{op}}$, which in turn is just in the *set-theoretic* sense in $\mathcal{S}$ at each component $C \in \mathbf{C}$. Thus $\varprojlim$ in $\mathcal{X}$ can be understood set-theoretically if $\mathcal{X}$ has an adequate small subcategory, and so can $\varinjlim$ provided we have a good grasp of the particular functor F^* involved. The proof shows more than the proposition states: not only does $\mathcal{X}$ have $\varprojlim$ and $\varinjlim$ but the inclusion F_* preserves $\varprojlim$ and the reflector F^* preserves $\varinjlim$. If one is only interested in existence, there is the more general

Exercise 3.20 (Isbell) Suppose $\mathcal{Y}$ is any bicomplete category and suppose that Φ is any endofunctor of $\mathcal{Y}$ for which $\Phi \circ \Phi \cong \Phi$ (Φ is not necessarily the composition of two adjoints). Let $\mathcal{X}$ be the maximal subcategory of $\mathcal{Y}$ for which the restriction of Φ to $\mathcal{X}$ is naturally isomorphic to the identity. [For example suppose we have a natural transformation $1_Y \xrightarrow{t} \Phi$ such that $\Phi(t_Y)$ is an isomorphism for all Y, and let $\mathcal{X}$ be the full subcategory of all Y for which t_Y itself is an isomorphism.] Assume that $\mathcal{X}$ is closed with respect to splitting of idempotents, i.e. if $Y \in \mathcal{X}$ and $Y \xrightarrow{\varphi} Y$ with $\varphi\varphi = \varphi$, there exist $Y \xrightarrow{p} X \xrightarrow{i} Y$ all in $\mathcal{X}$ such that $\varphi = pi$ and $ip = 1_X$. Then $\mathcal{X}$ is also bicomplete.

General Remark on Adequacy: This very important general property may require deep analysis to verify in an arbitrary particular case because of its transcendental character: "Every" abstract set-theoretical mapping $F_*(X) \to F_*(Y)$ which is $\mathbf{C}$-natural is required to come from a (unique) *actual* $\mathcal{X}$-morphism $X \to Y$. Fortunately, in two important classes of particular cases, adequacy is *implied by* a more elementary condition on a subcategory $\mathbf{C} \overset{F}{\hookrightarrow} \mathcal{X}$: If $\mathcal{X}$ is an algebraic category, it suffices that $\mathbf{C}$ is a strongly generating family for $\mathcal{X}$ which consists of "finitely generated" objects; if $\mathcal{X}$ is a topos, over $\mathcal{S}_0$, it suffices that $\mathbf{C}$ be a generating family.

The elementary operation of splitting idempotents is tightly related to the study of functor categories $\mathcal{S}^{\mathbf{C}^{op}}$ (as well as to the study of vector bundles, projective modules, manifolds, etc.).

Exercise 3.21 (Freyd) If $\mathbf{C}$ has either equalizers or coequalizers, then it has splitting of idempotents. *Any* functor $\mathbf{C} \to \mathbf{D}$ preserves the splitting of any idempotents which happen to split in $\mathbf{C}$. Any category $\mathbf{C}$ can be embedded in a category $\mathbf{C}^\sharp$ in which idempotents split, and in such a way that for any category $\mathbf{D}$ in which idempotents split, any functor $\mathbf{C} \to \mathbf{D}$ can be extended (uniquely up to natural equivalence) to a functor $\mathbf{C}^\sharp \to \mathbf{D}$; explicitly $\mathbf{C}^\sharp$ has as objects the idempotents e in $\mathbf{C}$, and a morphism $e \xrightarrow{a} e'$ is a morphism of $\mathbf{C}$ for which $ea = a = ae'$. For example, if $\mathbf{C}$ is the category of open sets in Euclidean spaces and smooth mappings, $\mathbf{C}^\sharp$ is equivalent to the category of smooth manifolds which have good tubular neighbourhoods; the classification of $\mathbf{D}$-valued functorial invariants of the latter reduces to the classification of $\mathbf{D}$-valued functorial invariants of the former. Since $\mathcal{S}$ has splitting of idempotents the induced $\mathcal{S}^{(\mathbf{C}^\sharp)^{op}} \to \mathcal{S}^{\mathbf{C}^{op}}$ is a natural equivalence (i.e. both Kan adjoints are quasi-inverse to it). If $\mathcal{S} = \mathcal{S}_0$ is the category of abstract sets, then an object X of $\mathcal{S}_0^{\mathbf{C}^{op}}$ is representable by an object of $\mathbf{C}^\sharp$ iff X is projective and $\mathcal{S}_0^{\mathbf{C}^{op}} \xrightarrow{nat(X,-)} \mathcal{S}_0$ preserves all small coproducts $\sum_i T_i$.

Lesson 4

1. Functorial semantics of algebraic theories

In this lesson we give an outline of some of the main aspects of the writer's 1963 thesis of the same title, taking some account of later contributions by Isbell, Freyd, Bénabou, Linton, Beck, André, Eilenberg, Gabriel-Ulmer, Wraith and others in the field. Isbell and Bénabou had in fact also found independently some of the writer's results. Much of Lesson 4 can be viewed as a deeper analysis of the basic Yoneda-Kan formula (see introduction to Lesson 3) in which certain exactness conditions are taken into account. For such a deepened analysis of the same basic formula in which different exactness conditions are taken into account, see Lesson 5 which outlines the theory of topoi. As references we may mention the writer's 1968 paper in Springer Lecture Notes vol. 61, Wraith's 1970 Aarhus Lecture Notes vol. 22, and Isbell's 1972 paper in the American Journal of Mathematics.

Algebraic theories and algebraic categories are an invariant formulation of the 1935 "Universal Algebra" of Birkhoff, and, like the latter, rest on the fact that groups, rings, modules, lattices, etc. have some properties in common. Such a general theory is not banal as might appear, since it led for example to the notion of morphism between algebraic theories which is effective in studying the particularity of the interaction between the examples and which generalizes the usual "change-of rings" formalism to a non-linear situation.

Platonic idealism, while mistaken on the relation between consciousness and matter, contains an aspect of truth when applied within the realm of ideas; for example, the doctrine of algebraic theories begins with the affirmation that there exists as a definite mathematical object the "perfect idea of a group" $\mathbf{A}$ of which the idea of any particular group is "merely an imperfect representation". (Indeed, it is a simple further step to form as a mathematical object the "idea of an algebraic theory" of which any particular algebraic theory $\mathbf{A}$ is a representation, as was carried out as a part of Bénabou's 1966 dissertation); of course, even here Platonic idealism is wrong on the order of development, since the ideas of several particular groups were concentrated from practice well before the idea of a group in general was concentrated from the practice of mathematicians around 1800.

2. Colimit preservation by modified Yoneda embedding and commutation of filtered colimits with finite limits

These are the two elementary facts on which much of the general doctrine of algebraic theories is based.

Exercise 4.1 If $\mathbf{A}$ is a small category with a certain kind of $\varprojlim$, e.g. equalizers, pullbacks, products, etc., then the Yoneda embedding $\mathbf{A} \to \mathcal{S}^{\mathbf{A}^{op}}$ preserves these $\varprojlim$.

On the other hand, the Yoneda embedding is not likely to preserve any kind of colimits with the exception of the special coequalizers involved in splitting idempotents. However, the idea of the Yoneda embedding may still be used in replacing $\mathcal{S}^{\mathbf{A}^{op}}$ by any subcategory $\mathcal{X}$ of $\mathcal{S}^{\mathbf{A}^{op}}$ which contains $\mathbf{A}$; since the property of being a colimit is a universal property, they often change when we restrict the "universe", and in particular we can find $\mathcal{X}$ in which colimits are different from those in $\mathcal{S}^{\mathbf{A}^{op}}$, but agree with those in $\mathbf{A}$. We formulate the notions below for the case of finite coproducts, but for later reference in Lesson 5 note that the argument applies for any specified set of kinds of colimits.

Definition: *Let $\mathbf{A}$ be a small category in which finite coproducts exist (i.e. $\mathbf{A}$ has an initial object and a* **left** *adjoint to the diagonal functor $\mathbf{A} \to \mathbf{A} \times \mathbf{A}$). Denote by $Alg(\mathbf{A}, \mathcal{S})$, (or just $Alg(\mathbf{A})$ when $\mathcal{S}$ is understood) the full subcategory of $\mathcal{S}^{\mathbf{A}^{op}}$ determined by all those T for which*

$$T_0 \xrightarrow{\sim} 1$$

$$T_{n*m} \xrightarrow{\sim} T_n \times T_m$$

are bijections in $\mathcal{S}$ where 0 is the initial object of $\mathbf{A}$, where n, m are arbitrary objects of $\mathbf{A}$, and where $*$ denotes coproduct in $\mathbf{A}$.

Examples: If $\mathbf{G}_{Ab}$ is the opposite of the category of all finite rectangular matrices with entries in $\mathbf{Z}$, with matrix multiplication as composition, then $Alg(\mathbf{G}_{Ab}) \cong Ab$, the category of abelian groups. If $\mathbf{B}$ is the opposite of the category of all finite "truth tables", i.e. the opposite of the category of all *mappings* between all finite powers of a two-element set, then $Alg(\mathbf{B}) \cong$ Boole, the category of all Boolean algebras. If $\mathbf{S}$ is the category of all *finite* sets, then $Alg(\mathbf{S}) \cong \mathcal{S}$ itself. These examples suggest another

Definition: *An* **algebraic theory** *(in the narrow sense) is any category* $\mathbf{C}$ *which has a given structure of finite cartesian products (with given projections) and in which there is a distinguished object A such that every object of* $\mathbf{C}$ *is one in the sequence*

$$\begin{aligned} A^0 &= 1 \\ A^{n+1} &= A^n \times A, \quad n \in \mathbf{N} \cong ob(\mathbf{C}) \end{aligned}$$

A morphism $A^n \xrightarrow{\theta} A$ *in* $\mathbf{C}$ *is called an* **n-ary operation** *of* $\mathbf{C}$*, and in particular a 0-ary operation* $1 \to A$ *is also called a* **constant** *of* $\mathbf{C}$*; clearly any morphism in* $\mathbf{C}$ $A^n \to A^m$ *is uniquely an m-tuple of n-ary operations* $< \theta_1, \dots, \theta_m >$*. If φ is an m-ary operation and ψ is an n-ary operation, we say that* $< \theta_1, \dots, \theta_m > \varphi = \psi$ *is an* **identity** *of* $\mathbf{C}$ *iff the obvious triangle commutes. If* $T \in Alg(\mathbf{C}^{op}) \subset \mathcal{S}^{\mathbf{C}}$*, then clearly T carries the object A into a certain set which is naturally equipped with certain operations indexed by the "operations" in* $\mathbf{C}$ *which satisfy (as commutative diagrams in $\mathcal{S}$) all the "identities" in* $\mathbf{C}$ *(in general the algebra T will satisfy more identities than just those valid in* $\mathbf{C}$*). The* **underlying set functor** *for* $\mathbf{C}$ *is just the functor* $Alg(\mathbf{C}^{op}) \to \mathcal{S}$ *which evaluates any T at the object A. By a* **morphism** *(in the narrow sense) of algebraic theories is meant any functor* $\mathbf{C}' \to \mathbf{C}$ *between algebraic theories which* preserves the finite product structure *(and which* preserves the distinguished object A*). Thus we get a category* $\mathcal{F}_{alg}$ *of algebraic-theories-and-their-morphisms (in the narrow sense). If* $\mathbf{f} : \mathbf{C}' \to \mathbf{C}$ *is a morphism in* $\mathcal{T}_{alg}$*, there is an induced* **algebraic functor between algebraic categories** $\mathbf{f}^b : Alg(\mathbf{C}) \to Alg(\mathbf{C}')$ *which preserves the underlying sets.*

We will prove below a theorem which implies that "every algebraic functor has a left adjoint" (examples of these left adjoints are universal enveloping algebras for Lie algebras, abelianization of groups, and extension of scalars for modules).

If $\mathcal{R}$ is the category of all unital (not necessarily commutative) rings (itself an algebraic category), there is a full and faithful functor $Mat : \mathcal{R} \to \mathcal{T}_{alg}$ (all rectangular matrices) such that $Alg(Mat(\mathcal{R})) \cong R =$ Mod for all $R \in \mathcal{R}$ and such that for any $\mathbf{C} \in \mathcal{T}_{alg}$, $Alg(\mathbf{C})$ is **abelian** iff $\mathbf{C}$ is isomorphic to a value of Mat. If $\mathcal{M}$ is the category of all monoids in $\mathcal{S}$ (actually $\mathcal{M} \cong Alg(\mathbf{C})$

where $\mathbf{C}(A^n, A) = \sum_{i=0}^{\infty} n^i$) then the functor $\mathcal{T}_{alg} \to \mathcal{M}, \mathbf{C} \rightsquigarrow \mathbf{C}(A, A)$ has a full-and-faithful left adjoint Ψ such that $\mathcal{S}^{\mathbf{M}^{op}} \cong Alg(\Psi(\mathbf{M}))$ for all $\mathbf{M} \in \mathcal{M}$.

Proposition 4.1: *If* $\mathbf{A}$ *is any small category having finite coproducts, then the Yoneda embedding preserves finite coproducts* when *considered as a functor* $\mathbf{A} \to Alg(\mathbf{A})$. *(Later [ ] we will show that* $Alg(\mathbf{A})$ *has all* $\varinjlim$ *.)*

Proof: (Essentially the same will work if we consider any other kind of colimits, modifying of course the definition of "Alg" accordingly.) First we note that all the values of the Yoneda embedding do actually lie in the subcategory $Alg(\mathbf{A})$ of $\mathcal{S}^{\mathbf{A}^{op}}$. Let us denote by $A(n)$ the representable functor corresponding to the object n of $\mathbf{A}$, so that

$$A(n)_m = \mathbf{A}(m, n) \text{ all } m \in obj(\mathbf{A})$$

Then for any n_1, n_2 and for any $T \in Alg(\mathbf{A})$, we have by Yoneda's Lemma that, if (,) indicates a set of natural transformations,

$$(A(n_1 * n_2), T) \cong T_{n_1 * n_2} \cong T_{n_1} \times T_{n_2} \cong (A(n_1), T) \times (A(n_2), T)$$

which shows that $A(n_1 * n_2)$ has *within* $Alg(\mathbf{A})$ in a natural way the universal property of a coproduct of $A(n_1)$ with $A(n_2)$. Similarly, $A(0)$ is an initial object of $Alg(A)$ since $(A(0), T) \cong T_0 \cong 1$ for all T. ∎

Theorem 1: *If* $\mathcal{X}$ *is any locally small large category, having small* $\varinjlim$ *and if* $\mathbf{A}$ *is any small category having finite coproducts, then there is an equivalence of categories*

$$Adj(\mathcal{X}, Alg(\mathbf{A})) \cong Coalg(\mathbf{A}, \mathcal{X})$$

where $F : \mathbf{A} \to \mathcal{X}$ *is a "coalgebra" iff it preserves finite coproducts.*

Proof: If F is a coalgebra in the above sense, then for any $X \in \mathcal{X}$, $F_*(X) \in Alg(\mathbf{A})$ since if 0 denotes the initial object of $\mathbf{A}$

$$F_*(X)_0 = \mathcal{X}(F(0), X) = \mathcal{X}(0, X) = 1$$

and

$$\begin{aligned} F_*(X)_{n_1 * n_2} &\cong \mathcal{X}(F(n_1 * n_2), X) \cong \mathcal{X}(F(n_1) * F(n_2), X) \\ &\cong \mathcal{X}(F(n_1), X) \times (F(n_2), X) \cong F_*(X)_{n_1} \times F_*(X)_{n_2} \end{aligned}$$

for any two objects n_1, n_2 of $\mathbf{A}$, where we have used $*$ to denote also the coproduct in $\mathcal{X}$. In the other direction, given an adjoint pair $\mathcal{X} \underset{F_*}{\overset{F^*}{\rightleftarrows}} Alg(\mathbf{A})$ then the corresponding F is by definition the composition

$$A \xrightarrow{\text{Yoneda}} Alg(A) \xrightarrow{F^*} \mathcal{X}$$

but the first preserves finite coproducts by the above proposition and the second preserves all colimits since it is a left adjoint. Thus $F \in Coalg(\mathbf{A})$. ∎

In another notation, if T is an $\mathbf{A}$-algebra (in sets) and F is an $\mathbf{A}$-coalgebra in $\mathcal{X}$, then $F \underset{A}{\otimes} T \in \mathcal{X}$. The Yoneda embedding $\mathbf{A} \xrightarrow{A} Alg(\mathbf{A})$ is a standard $\mathbf{A}$-coalgebra in the category of $\mathbf{A}$-algebras, and

$$\mathbf{A} \underset{A}{\otimes} T \cong T \qquad \text{all } T$$

Unfortunately, we can't yet apply this theorem to get the adjoints to algebraic functors since we have to prove first that $\mathcal{X} = Alg(\mathbf{B})$ is cocomplete; in fact, our argument will actually go the other way, and use also the second elementary fact in the title of this section.

Definition: *If* $\mathbf{X} \in Cat(\mathcal{S})$ *say that* $\mathbf{X}$ *is* **filtered** *iff*

1) *For any two objects* X_1, X_2 *of* $\mathbf{X}$ *there exists an object* X' *and two morphisms of* $\mathbf{X}$

$$\begin{tikzcd} X_1 \arrow[dr, dashed] & \\ & X' \\ X_2 \arrow[ur, dashed] & \end{tikzcd}$$

(In particular, $\mathbf{X}$ *is non-empty.)*

2) *For any two morphisms* x_1, x_2 *of* $\mathbf{X}$ *having equal domains and equal codomains, there exists a morphism* x' *in* $\mathbf{X}$ *with*

$$x_1 x' = x_2 x'$$

$$\begin{tikzcd} \cdot \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & \cdot \arrow[r, dashed, "x'"] & \cdot \end{tikzcd}$$

(This definition is strictly rigorous for $\mathcal{S} = \mathcal{S}_0 =$ *category of abstract sets; for a topos* $\mathcal{S}$*, replace the existential conditions by the conditions that a certain two morphisms of* $\mathcal{S}$ *derived from the structure of* $\mathbf{X}$ *are epimorphisms of* $\mathcal{S}$*. It should be clear that* $\mathbf{X}$ *is filtered iff for any* **finite** *diagram scheme* $M \rightrightarrows I$ *and any realization* X, x *of it in* $\mathbf{X}$*, there exists an object* X' *of* $\mathbf{X}$

and morphisms $X_i \xrightarrow{x_i'} X'$ *such that*

$$\begin{tikzcd} X_{i_1} \arrow[dd, "x_m"'] \arrow[dr, "x'_{i_1}"] & \\ & X' \\ X_{i_2} \arrow[ur, "x'_{i_2}"'] & \end{tikzcd}$$

for any $m : i_1 \to i_2$ *in* M.)

Definition: $\mathbf{D} \in Cat(\mathcal{S})$ *is called* **finitely-generated** *iff* $\mathbf{D}$ *has a finite set of morphisms* I *and there exists a finite set of morphisms* M *in* $\mathbf{D}$ *such that* $\mathcal{L}(M \rightrightarrows I) \to \mathbf{D}$ *is a regular epimorphism of* $\mathcal{S}$ *on morphisms, i.e. for* $\mathcal{S} = \mathcal{S}_0$ *every morphism of* $\mathbf{D}$ *can be represented as the composition of a finite string of morphisms from* M. *Four important examples (and by Lesson 2 "sufficient" in the case of* $\mathcal{S}_0$) *of finitely generated categories* $\mathbf{D}$ *are the finite ones*

for which $\varprojlim_{\mathbf{D}}$ *are respectively terminal object, binary cartesian products, equalizers, and pullbacks.*

Theorem 2: *If* $\mathbf{X}$ *is a filtered category,* $\mathbf{D}$ *a finitely generated category, and* $\Phi \in \mathcal{S}^{\mathbf{D}^{op}\times\mathbf{X}}$, *then the canonical morphism in* $\mathcal{S}$

$$\varinjlim_{X\in\mathbf{X}} \varprojlim_{D\in\mathbf{D}} \Phi(D,X) \to \varprojlim_{D\in\mathbf{D}} \varinjlim_{X\in\mathbf{X}} \Phi(D,X)$$

is always an isomorphism, provided $\mathcal{S}$ *is a topos.*

Proof: In Kock-Wraith; well-known and easy to verify in the case $\mathcal{S} = \mathcal{S}_0$. The basic fact for the proof is that if $\mathbf{X}$ is filtered, then for the coequalizer which defines $\varinjlim_{\mathbf{X}}$, the equivalence relation is generated in only two steps (rather than the usual unbounded finite number of steps).

Remark: If we consider Φ in the above theorem as a "bimodule" in the sense of the introduction of Lesson 3, and consider the tensor product $\otimes_{\mathbf{X}}$ discussed there, then we could say that a filtered $\mathbf{X}$ is flat.

Theorem 3: (André) If $\mathbf{A}', \mathbf{A}$ are categories having finite coproducts and if $\mathbf{A}' \xrightarrow{f} \mathbf{A}$ is a functor which *preserves finite coproducts*, then for any *finite-product-preserving* $T' : (A')^{op} \to \mathcal{S}$, the Kan extension

$$A \underset{A'}{\otimes} T' : \mathbf{A}^{op} \to \mathcal{S}$$

is *also* product-preserving. In other words, the left Kan extension restricted to $Alg(\mathbf{A}')$ has its values in $Alg(\mathbf{A})$ and hence is a left adjoint $\mathbf{f}_!$ to $\mathbf{f}^b$.

$$\begin{tikzcd} Alg(\mathbf{A}') \arrow[r, dashed, shift left, "\mathbf{f}_!"] \arrow[d, hook] & Alg(\mathbf{A}) \arrow[l, shift left, "\mathbf{f}^b"] \arrow[d, hook] \\ \mathcal{S}^{\mathbf{A}'^{op}} \arrow[r, shift left, "F^*"] & \mathcal{S}^{\mathbf{A}^{op}} \arrow[l, shift left, "\mathcal{S}^{\mathbf{f}^{op}}"] \end{tikzcd}$$

$f_!$ is the restriction of F^* where F is the composite

$$\mathbf{A}' \xrightarrow{\mathbf{f}} \mathbf{A} \xrightarrow{\text{Yoneda}} \mathcal{S}^{\mathbf{A}^{op}}$$

Proof: Under the assumptions on $\mathbf{A}', \mathbf{A}, \mathbf{f}, T,$ the colimit in

$$\mathbf{A}' \underset{A}{\otimes} T = F^*(T) = \varinjlim_{t \in \mathbf{f}/T} F(A'_t) = \varinjlim_{t \in \mathbf{f}/T} f(n'_t) \text{ (taken in } \mathcal{S}^{\mathbf{A}^{op}})$$

is sufficiently directed when evaluated at any $n \in obj(\mathbf{A})$. ∎

Corollary: *Algebraic functors have left adjoints.*

Another corollary will be that $Alg(\mathbf{A})$ is cocomplete, but to see this we will first consider the notion of the *free* category-with-finite-coproducts $\mathcal{F}(\mathbf{C})$ generated by an arbitrary small category $\mathbf{C}$.

The basic formula of Lesson 3,

$$\mathcal{S}-Fun(\mathbf{C}, \mathcal{X}) = \mathcal{X}(\mathbf{C}) \cong \mathcal{S}Adj(\mathcal{X}, \mathcal{S}^{\mathbf{C}^{op}}) \cong \mathcal{S}\text{-Fun}_{\to\otimes}(\mathcal{S}^{\mathbf{C}^{op}}, \mathcal{X})$$

for any cocomplete $\mathcal{X}$, may be interpreted to say that $\mathcal{S}^{\mathbf{C}^{op}}$ is the *free $\mathcal{S}$-category-with-all-small colimits* generated by $\mathbf{C}$. When we consider some reasonable restricted class of colimits, a suitable subcategory $\mathcal{F}(\mathbf{C})$ of $\mathcal{S}^{\mathbf{C}^{op}}$ will have the same universal property with respect to the more general class of categories $\mathcal{X}$ having (at least) colimits of the restricted kind and functors $\mathcal{F}(\mathbf{C}) \to \mathcal{X}$ which preserve the colimits of the restricted kind. Here we are interested specifically in *finite coproducts*, and the appropriate $\mathcal{F}(\mathbf{C}) \subseteq \mathcal{S}^{\mathbf{C}^{op}}$ is just the full subcategory consisting of all finite sums of representable functors. (Since we consider $\mathcal{S}^{\mathbf{C}^{op}}$ as a large category equipped with an $\mathcal{S}$-atlas, it will actually be possible to realize $\mathcal{F}(\mathbf{C})$ as a small category in $Cat(\mathcal{S})$ provided $\mathcal{S}$ satisfies the axiom of infinity.) The objects of $\mathcal{F}(\mathbf{C})$ are all finite strings of objects of $\mathbf{C}$, or more formally $obj(\mathcal{F}(\mathbf{C}) = \mathcal{W}(obj(\mathbf{C}))$, where $\mathcal{W}$ = free monoid functor (=restriction of the free category functor $\mathcal{L}$ to the subcategory $\mathcal{S} \hookrightarrow$ Diagr sch $(\mathcal{S})$ consisting of diagram schemes with exactly one vertex ($I = 1$, but $M \in \mathcal{S}$ arbitrary)) and a morphism

$$C_0 C_1, \ldots C_{n-1} \to C'_0 C'_1, \ldots C'_{m-1}$$

in $\mathcal{F}(\mathbf{C})$ is a pair consisting of a mapping $n \xrightarrow{\sigma} m$ and a family $C_i \xrightarrow{u_i} C'_{\sigma(i)}$, $i < n$ of morphisms of $\mathbf{C}$.

Proposition 4.2: *For any* $\mathbf{C} \in Cat(\mathcal{S})$

$$Alg(\mathcal{F}(\mathbf{C})) \cong \mathcal{S}^{\mathbf{C}^{op}}$$

Now, if $\mathbf{C}$ already has finite coproducts, there is a canonical finite-coproduct-preserving functor

$$\mathcal{F}(\mathbf{C}) \xrightarrow{*} \mathbf{C}$$

which is left adjoint to the inclusion $\mathbf{C} \to \mathcal{F}(\mathbf{C})$ (which is defined for any $\mathbf{C}$ by considering words of length 1). The functor $*$ takes the "formal" coproducts in $\mathcal{F}(\mathbf{C})$ back into the original ones in $\mathbf{C}$.

Theorem 4: *The full inclusion*

$$Alg(\mathbf{A}) \hookrightarrow \mathcal{S}^{\mathbf{A}^{op}}$$

has a left adjoint for any category $\mathbf{A}$ *which has finite coproducts.*

Proof: The composite

$$Alg(\mathbf{A}) \hookrightarrow \mathcal{S}^{\mathbf{A}^{op}} \cong Alg(\mathcal{F}(\mathbf{A}))$$

is just $(\ )^b$ of $*$. ∎

Corollary: $Alg(\mathbf{A})$ *is cocomplete.*

Proof: $\mathcal{S}^{\mathbf{A}^{op}}$ is, and we can apply the theorem near the end of Lesson 3. ∎

Exercise 4.2 It is obvious from the definition of $Alg(\mathbf{A})$ that it has $\varprojlim$ since $\mathcal{S}^{\mathbf{A}^{op}}$ does, and $\varprojlim$ always commutes with finite products.

Theorem 5: *If* $\mathbf{A}' \xrightarrow{f} \mathbf{A}$ *is any finite-coproduct-preserving functor, then* $Alg(A') \xrightarrow{f_*} Alg(\mathbf{A})$ *, and in particular* $Alg(A) \hookrightarrow \mathcal{S}^{\mathbf{A}^{op}}$*, preserves filtered colimits.*

Exercise 4.3 Apply the explicit construction of $\mathcal{F}(\mathbf{C})$ to the case where $\mathbf{C}$ is a monoid to obtain an explicit construction of the functor Ψ which assigns to each monoid the algebraic theory of its actions.

3. Structure as the adjoint of semantics and the characterization of algebraic categories

By *semantics* as a functor we mean the process $\mathbf{A} \rightsquigarrow Alg(\mathbf{A})$ which assigns to each theory its (category of possible) meaning (s). [Here restricted to the "algebraic" case.] There are two possible ways of making precise its codomain. If we limit ourselves to objects and morphisms in $\mathcal{T}_{alg}$, semantics is a (full-and-faithful) functor

$$\mathcal{T}_{alg}^{op} \to CAT/\mathcal{S}$$

where the last is the "category" whose objects are "arbitrary" large categories $\mathcal{X}$ equipped with an "arbitrary" functor $\mathcal{X} \xrightarrow{U} \mathcal{S}$ and whose morphisms are any functors Φ such that

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \mathcal{X}' \arrow[dl, "U'"] \\ & \mathcal{S} & \end{tikzcd}$$

is commutative. With an inoffensive restriction on the "arbitrariness", this semantics functor has a left adjoint called *structure*, such that the composite gives a best algebraic approximation to an "arbitrary" category $\mathcal{X}$ equipped with an "underlying set functor" U; explicitly U gives rise to a sequence $\mathcal{X} \xrightarrow{U^n} \mathcal{S}$ of functors defined by $U^n(X) = U(X)^n$ = the cartesian power in $\mathcal{S}$, and an algebraic theory $\mathbf{A}_U$ is defined by

$$\mathbf{A}_U(A^n, A) = Nat_{\mathcal{X}}(U^n, U)$$

provided only that we assume the latter are (representable by) sets in $\mathcal{S}$. Thus $\mathbf{A}_U$ consists of all the operations θ which can be defined simultaneously on all the values of U *naturally*, i.e. so that all morphisms of $\mathcal{X}$ become (under U) homomorphisms with respect to θ. If in particular $\mathbf{A} \in \mathcal{T}_{alg}$ and $\mathcal{X} = Alg(\mathbf{A}^{op})$ with $U = (T \rightsquigarrow T(A) = T_1)$ then $\mathbf{A} \xrightarrow{\sim} \mathbf{A}_U$, by Yoneda's Lemma.

On the other hand, if we consider as "theories" arbitrary objects of $Cat_*(\mathcal{S})$ = all small categories with finite coproducts and all functors which preserve finite coproducts, then semantics is

$$Cat_*^{op} \to ALG$$

where objects of ALG are "arbitrary" categories $\mathcal{X}$ having all colimits and a morphism of ALG is any pair $\mathcal{X} \underset{f^b}{\overset{f_*}{\rightleftarrows}} \mathcal{X}'$ of functors such that f_* is the *left* adjoint of f^b and f^b preserves all *filtered* $\varinjlim$ as well as all regular epimorphisms. Again, a suitable restriction on the objects of ALG leads to an adjoint *structure* functor (explained below), and semantics is full and faithful *when* restricted to those $\mathbf{A} \in Cat_*$ which have moreover splitting of idempotents. In either case, the adjunction $\mathcal{X} \xrightarrow{\Phi} sem\ str(\mathcal{X})$ is an equivalence iff $\mathcal{X}$ is an *algebraic category* (in the appropriate sense) and by studying Φ we can characterize intrinsically such $\mathcal{X}$.

Definition: *If $\mathcal{X}$ is a cocomplete $\mathcal{S}$ based category, an object $X \in \mathcal{X}$ is said to be* **finitely-presented** *iff for any system $X_i, i \in \mathbf{I}$ where $\mathbf{I}$ is a* **filtered** *category, the natural morphism*

$$\mathcal{X}(X, \varinjlim_{i\in\mathbf{I}} X_i) \longleftarrow \varinjlim_{i\in\mathbf{I}} \mathcal{X}(X, X_i)$$

in $\mathcal{S}$ is actually an isomorphism, i.e. iff the functor $\mathcal{X}(X,-) : \mathcal{X} \to \mathcal{S}$ preserves filtered colimits. It can be shown that if $\mathcal{X} = Alg(\mathbf{A})$ where $\mathbf{A}$ has finite coproducts, then $\mathcal{X}$ is finitely-presented iff there exists a finite presentation, i.e. iff there exists a coequalizer diagram

$$A_R \rightrightarrows A_G \longrightarrow X$$

in $\mathcal{X}$ where A_R, A_G are (objects representable by) objects from $\mathbf{A} \subset Alg(\mathbf{A})$. The further "inoffensive restriction" to be made on $\mathcal{X}$ in order that $\mathcal{X} \in ALG$ is that there is only a small set of finitely-presented objects, i.e. that there exists $J \in \mathcal{S}$ and $X \in \mathcal{X}[J]$ such that every finitely-presented object of $\mathcal{X}$ is isomorphic to X_j for some $j \in J$. For any such (cocomplete) $\mathcal{X}$, let $str(\mathcal{X})$ be the full subcategory of $\mathcal{X}$ determined by "all" (up to isomorphism) objects which are both finitely-presented and **regular projective**. *$\mathcal{X}$ is called* **quasi-algebraic** *iff $str(\mathcal{X})$ strongly generates $\mathcal{X}$, and $\mathcal{X}$ is called* **algebraic** *(in the wide sense) iff furthermore equivalence relations are effective in $\mathcal{X}$.*

Example: If $\mathcal{X}$ is the category of torsion-free abelian groups, then $str(\mathcal{X}) \cong Mat(\mathbf{Z})$; $\mathcal{X}$ is quasi-algebraic but not algebraic.

Theorem 6: *The small category $str(\mathcal{X})$ has finite coproducts and splitting of idempotents and if $\mathcal{X} \underset{f^b}{\overset{f_*}{\rightleftarrows}} \mathcal{X}'$ is a morphism of ALG, $f \underset{def}{=} f_* \mid str(\mathcal{X})$ takes $str(\mathcal{X})$ into $str(\mathcal{X}')$ and preserves finite coproducts. There is a natural equivalence between morphisms $\mathbf{A} \to str(\mathcal{X})$ in $Cat_*(\mathcal{S})$ and morphisms*

$\mathcal{X} \to Alg(\mathbf{A}) = sem(\mathbf{A})$ of ALG; in particular, if $f \in Cat_*(\mathcal{S})$ is given, the f^b, f_* corresponding constitute a morphism of ALG, i.e. $f_* \dashv f^b$ and f^b preserves filtered $\varinjlim$ and regular epis. The canonical $\mathbf{A} \to str(sem\mathbf{A})$ is an equivalence iff $\mathbf{A}$ is moreover closed with respect to splitting of idempotents, and the canonical $\mathcal{X} \to \mathrm{sem}\,\mathrm{str}(\mathcal{X})$ is an equivalence iff $\mathcal{X}$ is moreover an algebraic category in the wide sense.

Proof: See Gabriel-Ulmer SLN 221 and the writer's article in SLN61 for most. The statement for an adjoint pair $\mathcal{X} \underset{f^b}{\overset{f_*}{\leftrightarrows}} \mathcal{X}'$ in which f^b preserves filtered $\varinjlim$ and regular epis (if $\mathcal{X}'$ is finitely-presented regular projective in $\mathcal{X}$, then $f_*(\mathcal{X}')$ is finitely-presented regular projective in $\mathcal{X}'$) is proved by the following two calculations:

$$\begin{aligned}
\mathcal{X}(f_*(X'), \varinjlim X_i) &= \mathcal{X}'(X', f^b, \varinjlim X_i) = \mathcal{X}'(X', \varinjlim f^b X_i) \\
&= \varinjlim \mathcal{X}'(X', f^b X_i) \; = \varinjlim \mathcal{X}(f_*(x'), X_i);
\end{aligned}$$

$$\begin{tikzcd}[column sep=small]
 & X \arrow[d] & & & f^b(X) \arrow[d] & & & f^b(X) \arrow[d] & & & X \arrow[d] \\
f_*(X') \arrow[r] & Y & \Longrightarrow & X' \arrow[r] & f^b(Y) & \Longrightarrow & X' \arrow[r] \arrow[ur, dashed] & f^b(Y) & \Longrightarrow & f_*X' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

∎

Remark: Recall from Lesson 2 that if a category has a set of generators which are regular projectives, then it is a regular category; thus any *quasi-algebraic* category is a regular category, while an *algebraic* category has moreover effective equivalence relations. While an algebraic category is to be thought of as *equationally* definable, a quasi-algebraic category is definable by the more general type of conditions (sometimes called "universal Horn sentences")

$$\begin{array}{rcl}
\theta_1(x, y \ldots) & = & \theta_1'(x, y \ldots) \\
 & \& & \qquad\qquad \Rightarrow \varphi(x, y \ldots) = \varphi'(x, y \ldots) \\
\theta_2(x, y \ldots) & = & \theta_2'(x, y \ldots) \\
 & \vdots &
\end{array}$$

(See Isbell, Rozprawy Matematyczne XXXVI, 1964.)

Example: The category $\mathcal{T}_{alg}$ of algebraic theories in the narrow sense is actually itself algebraic in the wide sense. The usual category of all modules over all rings is algebraic ("obviously") in the wide sense, but even also in the narrow sense (if we define the underlying set to be the cartesian product of two sets, namely of the underlying set of the ring times the underlying set of the module).

Example: The inclusion Groups $\rightarrowtail$ Monoids is an algebraic functor (it happens to be full and faithful even though the smaller has *more operations* instead of more equations, a fact which is difficult to account for in the classical Birkhoff theory). This inclusion, like any algebraic functor, has a left adjoint; however, unlike most algebraic functors, it also has a *right* adjoint, namely take the group of invertible elements from any monoid.

Example: The functor $Assoc/K \to Lie/K$ induces by the map of theories which interprets the Lie bracket $[x, y]$ as the operation $x{\cdot}y - y{\cdot}x$ in the theory of associative linear algebras over K, has as left adjoint the "universal enveloping algebra". Schanuel and Isbell call an inclusion $\mathbf{A} \overset{i}{\rightarrowtail} \mathbf{A}'$ of theories *pure* iff the "extension of operators" $T \to \mathbf{A}' \underset{\mathbf{A}}{\otimes} T = i_*(T)$ is a monomorphism in $Alg(\mathbf{A})$ for all $T \in Alg(\mathbf{A})$. The famous theorem of Poincaré-Birkhoff-Witt states that for a *field* K, the relation between Lie and associative algebras is pure in this sense.

Example: In the commutative square

$$\begin{tikzcd} \text{Assoc} \arrow[r] \arrow[d] & \text{Monoids} \arrow[d] \\ Ab \arrow[r] & \mathcal{S} \end{tikzcd}$$

of algebraic functors, the left adjoints are

$$\begin{tikzcd} \text{Assoc} & \text{Monoid} \arrow[l, "\text{monoid ring}"'] \\ Ab \arrow[u, "\text{tensor algebra}"] & \mathcal{S} \arrow[l, "\text{free abelian group}"] \arrow[u, "\text{free monoid}"'] \end{tikzcd}$$

and the commutativity (up to isomorphism) of that square means that the free associative ring can be calculated in two different ways. But the four morphisms of algebraic theories involved here actually participate in a further spatial property which is not true in general, namely

$$\begin{tikzcd} \text{Assoc} \arrow[d] & \text{Monoids} \arrow[l] \arrow[d] \\ Ab & \mathcal{S} \arrow[l] \end{tikzcd}$$

is *also* commutative (as is clear from the usual explicit construction of the monoid ring). This special property was analyzed by Barr-Beck as being the essence of the *distributive* law, and effectively used by them in many situations not directly connected with addition versus multiplication.

Bialgebras: By a special case of Theorem 1 concerning coalgebras in an arbitrary category $\mathcal{X}$, and as used by Freyd in his 1966 paper in Colloq. Math. (Warsaw), if $\mathbf{A}, \mathbf{B} \in Cat_*(\mathcal{S})$ then

$$Adj(Alg(\mathbf{A}), Alg(\mathbf{B})) \cong Bialg(\mathbf{A}, \mathbf{B})$$

where $Bialg(\mathbf{A}, \mathbf{B}) \cong Coalg(\mathbf{B}, Alg(\mathbf{A}) \cong Alg(\mathbf{B}, Alg(\mathbf{A})^{op})op$ consists of all $\Phi \in \mathcal{S}^{\mathbf{A}^{op} \times \mathbf{B}}$ which are product-preserving in the variable $\mathbf{A}$ and which in the variable $\mathbf{B}$ take *coproducts into coproducts in the sense of* $\mathbf{A}$ *algebras.* Thus the arbitrary adjoint pairs between algebraic categories can also be represented more concretely in the language of "theories" by considering bimodules; every morphism $\mathbf{f}$ in $Cat_*(\mathcal{S})$ gives rise to a special bimodule $\Phi(B, A) = \mathbf{A}(f(B), A)$, but the only bimodules which are of this form are those for which, in the corresponding adjoint pair, the right adjoint preserves filtered colimits and regular epis. An intermediate case arises often in practice, namely an adjoint pair of functors between algebraic categories in which the right adjoint preserves filtered $\varinjlim$ but not necessarily regular epis; the corresponding bialgebras in this case are those for which the underlying algebras are finitely presented (if they were moreover projective, we would be back to the case of morphisms in $Cat_*(\mathcal{S})$, except for splitting of idempotents).

For example, the functor Ann $\xrightarrow{\text{units}}$ Ab which assigns to a commutative ring its multiplicative group of units is representable by $\mathbf{Z}[t, t^{-l}]$, and therefore this ring also has the co-structure

$$\mathbf{Z}[t, t^{-l}] \xrightarrow{\text{“mult”}} \mathbf{Z}[t, t^{-l}] \otimes \mathbf{Z}[t, t^{-l}]$$

$$\mathbf{Z}[t, t^{-l}] \xrightarrow{\text{“}inv\text{”}} \mathbf{Z}[t, t^{-1}]$$

of an abelian group (recall that $\otimes$ is the coproduct in Ann). The left adjoint of this functor is the *group ring* functor (restricted to the commutative case).

Another example is the functor $Ann \to$ Boole which assigns to any commutative ring its Boolean algebra of idempotent elements; this functor is obviously representable by $\mathbf{Z}[t]/_{t^2-t}$.

Exercise 4.4 Show explicitly that $\mathbf{Z}[t]/_{t^2-t}$ has the co-structure of a Boolean algebra in Ann, i.e. is a Boolean algebra object in Ann^{op}. In fact, $\mathbf{Z}(t)/_{t^2-t} \cong \mathbf{Z} \times \mathbf{Z}$, which may be considered as "1 + 1" in the sense of the category $\mathcal{Y} = Ann^{op}$; moreover

$$Y \times Y \cong Y \otimes (\mathbf{Z} \times \mathbf{Z}), \quad Y \in Ann$$

i.e.

$$Y + Y \cong Y \times (1 + 1) \text{ in } \mathcal{Y}$$

In any category $\mathcal{Y}$ satisfying the last distributivity property, the coproduct $1 + 1$ of the terminal object with itself is always a Boolean algebra object in $\mathcal{Y}$. Calculate explicitly the left adjoint of

$$Ann \xrightarrow{\text{idemp}} \text{Boole};$$

if one considers a given Boolean algebra as an algebra of "measurable sets" (in a finitely-additive sense) then (Linton) the nature of this left adjoint is to take the "ring of $\mathbf{Z}$-valued step functions".

Another example (actually induced by a morphism in $Cat_*(\mathcal{S})$), which was exploited by Amitsur in the 1970 International Congress, is the construction for a given $n \in \mathbf{N}, R \in \mathcal{R}$ of the best *commutative* ring $K \in Ann$ such that $Mat_n(K)$ approximates R; this construction is the left adjoint of the functor

$$Ann \xrightarrow{Mat_n} \text{Rings}$$

assigning to commutative K the ring $Mat_n(K)$ of $n \times n$ matrices with entries from K. Since the latter functor is represented by the polynomial ring in n^2 variables, it is clear that *that* ring has the structure of a "co-ring" in Ann.