String diagrams for symmetric powers

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Introduction

This presentation is about symmetric powers in symmetric monoidal $\mathbb{Q}^+$-linear categories.

We provide a characterization of symmetric powers in terms of an algebraic structure that we call binomial graded bialgebras.

It provides some nice string diagrams.

We present various results in this framework.

Definition

A symmetric monoidal $\mathbb{Q}^+$-linear category is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ such that every hom-set $\mathcal{C}[A, B]$ is a $\mathbb{Q}^+$-module[1], and moreover:

- $- \otimes - : \mathcal{C}[A, B] \times \mathcal{C}[C, D] \to \mathcal{C}[A \otimes C, B \otimes D]$ is bilinear
- $-;- : \mathcal{C}[A, B] \times \mathcal{C}[B, C] \to \mathcal{C}[A, C]$ is bilinear

Example

- $\mathbf{Mod_R}$ for R any $\mathbb{Q}^+$-algebra (ie. a semiring[2] R together with a semiring morphism $\mathbb{Q}^+ \to R$)
- **Rel** the category of set and relations
- $\mathbf{FVec_k}$ the category of finite-dimensional vector spaces over a field k of characteristic 0
- **FRel** the category of finite set and relations
- ...

In all the reset of the presentation $\mathcal{C}$ will be a symmetric monoidal $\mathbb{Q}^+$-linear category.

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1 $\mathbb{Q}^+$ = rational numbers ≥ 0

2 semiring = ring without requiring negative numbers

For any object $A \in \mathcal{C}$ and every $n \geq 1$, we can form the n^{th} tensor power $A^{\otimes n}$.

The n^{th} symmetric power is a symmetrization of this tensor power. It can be interpreted in different ways which are all equivalents in this framework:[3]

$$\text{An equalizer of } A^{\otimes n} \underset{\cdots}{\overset{\sigma}{\rightrightarrows}} A^{\otimes n} \tag{1}$$

$$\text{A coequalizer of } A^{\otimes n} \underset{\cdots}{\overset{\sigma}{\rightrightarrows}} A^{\otimes n} \tag{2}$$

$$\text{A splitting of the idempotent} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma : A^{\otimes n} \to A^{\otimes n} \tag{3}$$

Such a splitting is given by two maps $r_n : A^{\otimes n} \to A_n$ and $s_n : A_n \to A^{\otimes n}$ such that $r_n; s_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma$ and $s_n; r_n = Id_{A_n}$.

[3] below, there is one arrow for every permutation $\sigma \in \mathfrak{S}_n$

Proposition

Given objects A and $A_n \in \mathcal{C}$, there are bijections between: an equalizer $r_n : A^{\otimes n} \to A_n$, a coequalizer $s_n : A_n \to A^{\otimes n}$ and a splitting $A^{\otimes n} \xrightarrow{r_n} A_n \xrightarrow{s_n} A^{\otimes n}$ of $\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma : A^{\otimes n} \to A^{\otimes n}$.

Example

- In $\mathbf{Mod}_R$ (R a $\mathbb{Q}^+$-algebra), the n^{th} symmetric power can be seen equivalently as the subspace $(A^{\otimes n})^{\mathfrak{S}_n}$ of vectors invariants by permutation or as the quotient $(A^{\otimes n})_{\mathfrak{S}_n}$ of $A^{\otimes n}$ by the $n!$ permutations.
- In **Rel**, the n^{th} symmetric power of A is the set $\mathcal{M}_n(A)$ of multisets of n elements in A. r_n sends any tuple $(a_1, ..., a_n)$ to the multiset $[a_1, ..., a_n]$, s_n relates any multiset $[a_1, ..., a_n]$ to all the tuples $(a_{\sigma(1)}, ..., a_{\sigma(n)})$ for every $\sigma \in \mathfrak{S}_n$. $r_n; s_n$ relates any tuple $(a_1, ..., a_n)$ to all the tuples $(a_{\sigma(1)}, ..., a_{\sigma(n)})$ for every $\sigma \in \mathfrak{S}_n$. [4]

[4] In $Rel[A, B]$, $+$ is the union of relations, and $n.1 = 1$

It provides our first definition of a family $(A_n)_{n\geq 1}$ of symmetric powers:

Definition

In a symmetric monoidal $\mathbb{Q}^+$-linear category, a permutation splitting is given by:

- a family $(A_n)_{n\geq 1}$ of objects
- a family $(r_n : A_1^{\otimes n} \to A_n)_{n\geq 1}$ of morphisms
- a family $(s_n : A_n \to A_1^{\otimes n})_{n\geq 1}$ of morphisms

such that:

- $r_n; s_n = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \sigma : A^{\otimes n} \to A^{\otimes n}$
- $s_n; r_n = Id_{A_n}$

Definition

A morphism of permutation splittings $(A_n)_{n\geq 1} \to (B_n)_{n\geq 1}$ is given by a family $(f_n : A_n \to B_n)_{n\geq 1}$ such that any of the three equivalent conditions are verified:

$$\begin{CD} A_1^{\otimes n} @>{f_1^{\otimes n}}>> B_1^{\otimes n} \\ @V{r_n^A}VV @VV{r_n^B}V \\ A_n @>>{f_n}> B_n \end{CD} \tag{4}$$

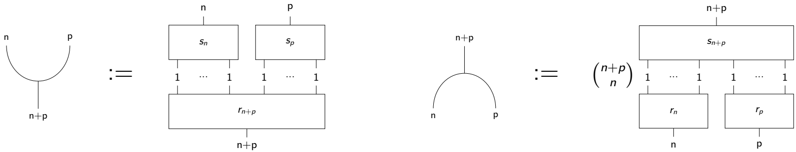
$$f_n = r_n^A ; f_1^{\otimes n} ; s_n^B \tag{5}$$

$$\begin{CD} A_n @>{s_n^A}>> A_1^{\otimes n} \\ @V{f_n}VV @VV{f_1^{\otimes n}}V \\ B_n @>>{s_n^B}> B_1^{\otimes n} \end{CD} \tag{6}$$

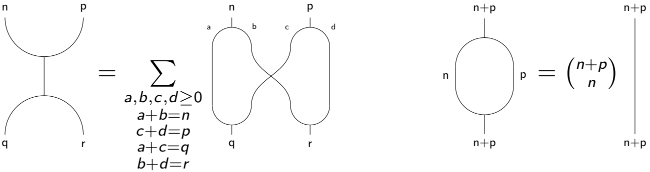
Proposition

The category of permutation splittings is isomorphic to the category of permutations splittings $((A_n)_{n\geq 1}, (r_n)_{n\geq 1}, (s_n)_{n\geq 1})$ and morphisms $f_1 : A_1 \to B_1$ (= the reduced category of permutation splittings).

Given a permutation splitting $((A_n)_{n\geq 1}, (r_n)_{n\geq 1}, (s_n)_{n\geq 1})$, we can define $(\nabla_{n,p} : A_n \otimes A_p \to A_{n+p})_{n,p\geq 1}$,$(\Delta_{n,p} : A_{n+p} \to A_n \otimes A_p)_{n,p\geq 1}$ by:

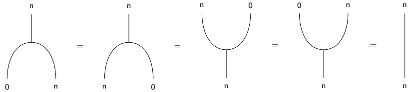


These equations are then verified: [5] [6]



[5] first one for every $n, p, q, r \geq 1$ such that $n + p = q + r$, second one for every $n, p \geq 1$

[6] where we note:



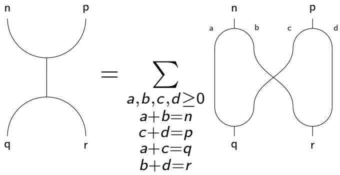
It provides our second definition of a family $(A_n)_{n\geq 1}$ of symmetric powers:

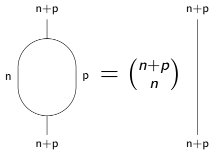
Definition

In a symmetric monoidal $\mathbb{Q}^+$-linear category[7], a binomial graded bialgebra is given by:

- a family $(A_n)_{n\geq 1}$ of objects
- a family $(\nabla_{n,p} : A_n \otimes A_p \to A_{n+p})_{n,p\geq 1}$ of morphisms
- a family $(\Delta_{n,p} : A_{n+p} \to A_n \otimes A_p)_{n,p\geq 1}$ of morphisms

such that:





Proposition

Every binomial graded bialgebra is biassociative and bicommutative.

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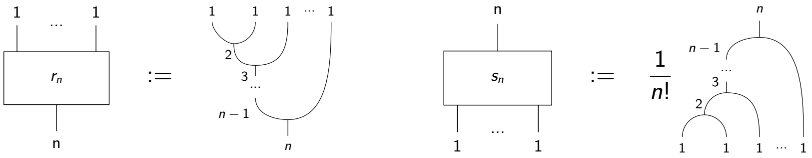
[7] Remark that it is sufficient that the hom-sets are commutative monoids to define a binomial graded bialgebra, but we will not go into this today.

Given a binomial graded bialgebra, we can define:



Proposition

Given a family $(A_n)_{n\geq 1}$ of objects, the constructions between the families $(r_n)_{n\geq 1}, (s_n)_{n\geq 1}$ which define a permutation splitting and the families $(\nabla_{n,p} : A_n \otimes A_p \to A_{n+p})_{n,p\geq 1}$,$(\Delta_{n,p} : A_{n+p} \to A_n \otimes A_p)_{n,p\geq 1}$ which define a binomial graded bialgebra provide a bijection between splitting idempotents with underlying objects $(A_n)_{n\geq 1}$ and binomial graded bialgebras with underlying object $(A_n)_{n\geq 1}$.

Corollary

In a symmetric monoidal $\mathbb{Q}^+$-linear category, a family $(A_n)_{n\geq 1}$ verifies that A_n is the n^{th} symmetric power of A_1 iff it can be equipped with a structure of binomial graded bialgebra.

Proposition

If $(f_n : A_n \to B_n)_{n \geq 1}$ is a family of morphisms between two binomial graded biagebras, these conditions are equivalent:[8]

$$\begin{array}{ccc} A_n \otimes A_p & \xrightarrow{\nabla^A_{n,p}} & A_{n+p} \\ {\scriptstyle f_n \otimes f_p}\downarrow & & \downarrow{\scriptstyle f_{n+p}} \\ B_n \otimes B_p & \xrightarrow[\nabla^B_{n,p}]{} & B_{n+p} \end{array} \tag{7}$$

$$f_n = r_n^A ; f_1^{\otimes n} ; s_n^B \tag{8}$$

$$\begin{array}{ccc} A_{n+p} & \xrightarrow{\Delta^A_{n,p}} & A_n \otimes A_p \\ {\scriptstyle f_{n+p}}\downarrow & & \downarrow{\scriptstyle f_n \otimes f_p} \\ B_{n+p} & \xrightarrow[\Delta^B_{n,p}]{} & B_n \otimes B_p \end{array} \tag{9}$$

If any of these conditions is verified, we say that $(f_n)_{n \geq 1}$ is a morphism of binomial graded bialgebras.

[8] where r_n, s_n are defined as before

Corollary

If R is a $\mathbb{Q}^+$-module, if $f : R[Y_1, ..., Y_n] \to R[X_1, ..., X_p]$ is a linear map which preserves the degree of homogeneous polynomials, then it is a morphism of bialgebras iff it is a morphism of algebras iff it is a morphism of coalgebras and there is exactly one such map for every linear map $R.Y_1 \oplus ... \oplus R.Y_n \to R.X_1 \oplus ... \oplus R.X_p$.

Example

For every $\sigma \in \mathfrak{S}_n$, the linear map $f_\sigma : R[X_1, ..., X_n] \to R[X_1, ..., X_n]$ which sends $X_1 \mapsto X_{\sigma(1)}, ..., X_n \mapsto X_{\sigma(n)}$ is such a map. A polynomial $P \in R[X_1, ..., X_n]$ is called symmetric iff it is invariant by f_σ for every $\sigma \in \mathfrak{S}_n$.

Proposition

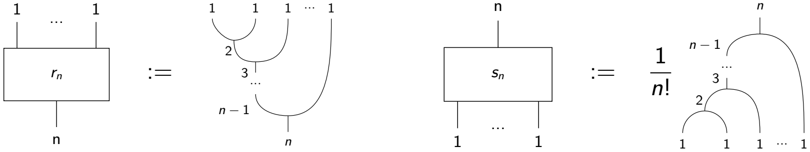
The category of binomial graded bialgebras is isomorphic to the category of permutation splittings, to the reduced category of binomial graded bialgebras and to the reduced category of binomial graded bialgebras.

- *Between the category of binomial graded bialgebras and of permutation splittings, the isomorphism sends* $(f_n)_{n\geq 1}$ *to* $(f_n)_{n\geq 1}$
- *from a non-reduced category to a reduced one, it sends* $(f_n)_{n\geq 1}$ *to* f_1
- *from a reduced category to a non-reduced one, it sends* f_1 *to* $(f_n)_{n\geq 1}$ *where* $f_n : A_n \to B_n$ *is defined by* $f_n = r_n^A; f_1^{\otimes n}; s_n^B$.

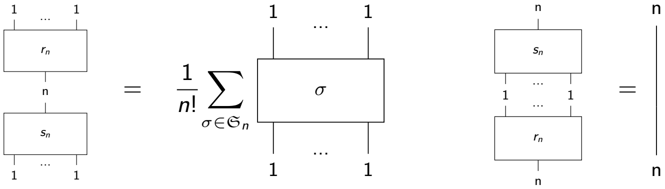
A sketch of proof

How to prove that in a binomial graded bialgebra, A_n is the n^{th} symmetric power of A_1?

We must show that if we define

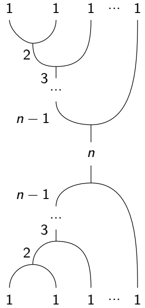


then

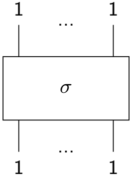


ie. we must prove that:

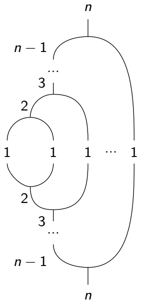
$$\frac{1}{n!}$$



$$= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n}$$

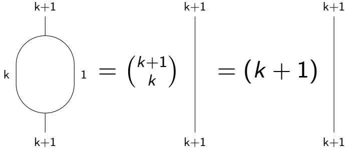


$$\frac{1}{n!}$$



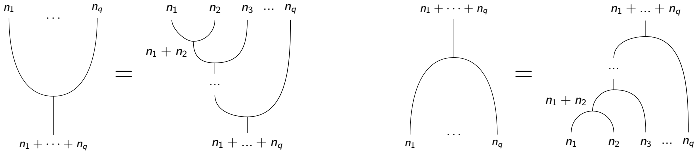
$$=$$

The second equality is not very difficult to prove. One of our axiom is that

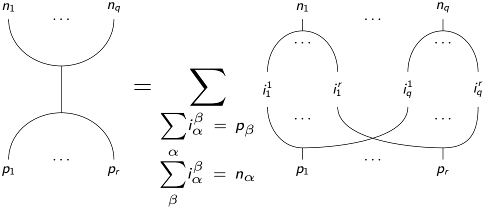


It then follows by using this for every $1 \leq k \leq n-1$.

The first one is a bit more difficult. Define

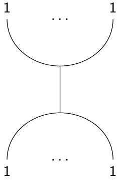


We can then prove by induction on $(q, r) \in \mathbb{N}_{\geq 2}^{\times 2}$ that

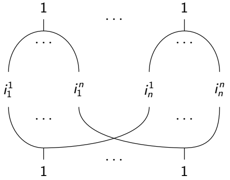


where the sum is indexed by the matrices $(i_\alpha^\beta)_{\substack{1\leq\alpha\leq q\\ 1\leq\beta\leq r}}$ such that for every $1 \leq \beta \leq r$, the sum $\sum_{1\leq\alpha\leq q} i_\alpha^\beta$ of the terms in the β^{th} line is equal to p_β and for every $1 \leq \alpha \leq q$, the sum $\sum_{1\leq\beta\leq q} i_\alpha^\beta$ of the terms in the α^{th} column is equal to n_α for some weigths $(n_1, ..., n_q) \in \mathbb{N}^q$ and $(p_1, ..., p_r) \in \mathbb{N}^r$.

We then put $q = r := n$ and deduce that



$$= \sum_{\substack{\sum_\alpha i_\alpha^\beta \,=\, 1 \\ \sum_\beta i_\alpha^\beta \,=\, 1}}$$

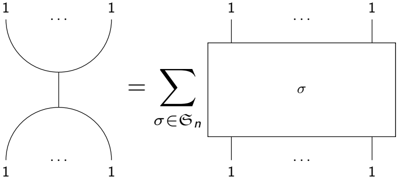


where the sum is indexed by the matrices $(i_\alpha^\beta)_{1 \leq \alpha, \beta \leq n}$ such that the sum of the terms in every column and the sum of the terms in every line is equal to 1.

For every such matrix, every input is related to exactly one output in the RHS. The α^{th} input is related to exactly one output by a path which contains a 1 and not a 0 which is the only output β such that $i_\alpha^\beta = 1$.

By definition of the multiplications and comultiplications, we can then for every entry $1 \leq \alpha \leq q$ keep only a link to this output β. We have thus replaced the RHS by a permutation. And every such matrix corresponds to exactly one permutation $\sigma \in \mathfrak{S}_n$.

We have proved that



we then multiply each side by $\frac{1}{n!}$ to obtain the desired equality.

Thank you for your attention!