COENDS OF HIGHER ARITY

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ABSTRACT. We specialise a recently introduced notion of generalised dinaturality for functors $T : (\mathcal{C}^{\mathrm{op}})^p \times \mathcal{C}^q \to \mathcal{D}$ to the case where the domain (resp., codomain) is constant, thus obtaining notions of ends (resp., coends) of higher arity, which we dub (p, q)*-ends* (resp., (p, q)*-coends*). Higher arity co/ends reduce to a certain kind of classical co/ends (that can be recovered as $(1, 1)$-co/ends), but it proves to be useful to describe some new phenomena.

The theory so determined paves the way to two interesting developments: 1) *weighted* ends and *weighted* Kan extensions, standing to ends and Kan extensions in the same relation as weighted limits stand to limits, and 2) *diagonal* constructions, where, in analogy to the passage from limits to ends, one replaces naturality by dinaturality in categorical concepts besides that of a limit; as a result, we obtain a rich theory with notions such as *diagonal* Kan extensions, similarly standing to ordinary Kan extensions as ends stand to limits.

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1. Introduction

A functor $T : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ can be thought as a generalised form of pairing defined on objects of $\mathcal{C}$, thanks to its action on morphisms, once covariantly and once contravariantly; the 'generalised quantity' $T(C, C')$ can then be 'integrated' to yield two distinct objects having dual universal properties:

c1) a *coend*, resulting by the symmetrisation along the diagonal of T, i.e. by modding out the coproduct $\coprod_{C\in\mathcal{C}} T(C, C)$ by the equivalence relation generated by the arrow functions $T(-, C') : \mathcal{C}^{\mathsf{op}}(X, Y) \to \mathcal{D}(TX, TY)$ and $T(C, -) : \mathcal{C}(X, Y) \to \mathcal{D}(TX, TY)$;

c2) an *end*, i.e. an object $\int_{\mathcal{C}} T(C, C)$ arising as an 'object of invariants' of 'fixed points' for the same action of T on arrows; by dualisation, if a coend is a quotient of $\coprod_{C\in\mathcal{C}} T(C, C)$, an end is a *subobject* of the product $\prod_{C\in\mathcal{C}} T(C, C)$.

The fact that given $T : \mathcal{B}^{\mathsf{op}} \times \mathcal{B} \times \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ this operation satisfies the commutativity rule

$$\int^A \int^B T(B, B; C, C) \cong \int^B \int^A T(B, B; C, C)$$

(called the 'Fubini rule') motivated N. Yoneda [Yon60] to adopt for them an integral-like notation and terminology.

Since, given a functor T as above, the co/end of T can be computed as a certain co/equaliser, co/ends can be regarded as just particular co/limits, associated to functors of particular variance type. Central to this reduction rule of co/ends to co/limits is the *twisted arrow category* of $\mathcal{C}$, i.e. the category of elements of the hom functor $\hom_{\mathcal{C}} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$.

The intuition of co/ends as fixed points and orbit spaces of suitable actions lends itself to many fruitful interpretation: such a functor $T : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ can be regarded as some sort of "module" that testify a two-sided action of the category $\mathcal{C}$ on the codomain $\mathcal{D}$: in fact, this is what T is, in the case where $\mathcal{C}$ is a monoid (i.e. a category with a single object) and $\mathcal{D}$ is the category of sets; T is nothing but a set with both a left and a right action of $\mathcal{C}$, i.e. a *bimodule*.

Taken to an extreme this definition works for $\mathcal{A}$-$\mathcal{B}$-bimodules and yields the 1-cells of the bicategory Prof of profunctors, introduced in [BS00] and studied in [CP89; Lor15]).

Given two bimodules $S \in {}_A\mathsf{Mod}_B$ and $T \in {}_B\mathsf{Mod}_C$ over rings A, B, C their classical tensor product ${}_A S_B \otimes_B {}_B T_C$ is defined by the cokernel

$$\bigoplus_{b\in B} S \otimes_{\mathbb{Z}} T \xrightarrow{1\otimes b - b\otimes 1} S \otimes_{\mathbb{Z}} T$$

of a map that renders the action of B on both sides bilinear; for functors $S : \mathcal{A}^{\mathsf{op}} \times \mathcal{B} \to \mathsf{Set}$ and $T : \mathcal{B}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$, such universal quotient that renders the action of morphisms $u : X \to Y$ in $\mathcal{B}$ bilinear is an instance of a *coend*.

Coend calculus, i.e. the set of rules allowing to formally manipulate integrals of the above kind in order to prove statements in category theory, has applications in as different fields as homotopy theory, functional programming, the foundations of combinatorics, and category theory.

The particular variance T is forced to have now begs the question of whether there is an analogue of the above picture (a universal property), for more general functors

$$T : \underbrace{\mathcal{C}^{\mathsf{op}} \times \cdots \times \mathcal{C}^{\mathsf{op}}}_{p \text{ times}} \times \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{q \text{ times}} \to \mathcal{D}$$

taking $p \geq 1$ contravariant arguments, and $q \geq 1$ covariant arguments, admitting the possibility that $p \neq q$ (mimicking a nomenclature of Kelly, introduced in [Kel72a], we shortly refer to these functors as having "type" $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$).

$\boxed{\text{The present paper's aim is to answer this question in the positive.}}$

Such generalised (or "(p, q)"-)co/ends exist and can be characterised as suitable particular cases of classical ("$(1, 1)$"-)co/ends. Moreover, they yield a similarly rich calculus.

Knowing that all (p, q)-coends are suitable $(1, 1)$-coends one might expect that they do not provide much new examples. This is not the case: our Section 4 are entirely devoted to providing examples. In particular, we focus on two natural frameworks harbouring "higher arity" co/ends:

OR1) The possibility to define a higher arity analogue of the Day convolution monoidal structure of [Day70a; Day70b; IK86], on the presheaf category $\mathsf{Cat}(C^{\mathrm{op}}, \mathsf{Set})$ over a monoidal category $(C, \otimes)$; classically, the Day convolution of two presheaves $\mathcal{F}, \mathcal{G} : C^{\mathrm{op}} \to \mathsf{Set}$ is defined by

$$\mathcal{F} \circledast \mathcal{G} : \int^{X,Y \in C} \mathcal{F}(X) \times \mathcal{G}(Y) \times C(-, X \otimes Y).$$

We generalise this notion into Definition 4.30: given n presheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ their *n-ary Day convolution* is the (n, n)-coend

$$(\mathcal{F}_1 \circledast_n \cdots \circledast_n \mathcal{F}_n)(A) \overset{\text{def}}{=} {}^{(n,n)}\!\!\int^{A \in C} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times C\left(-, A^{\otimes n}\right).$$

The sets of various n-ary Day convolutions, together with the convolution of order $k \leq n$, organise into an operad that we dub the *Day operad* in Example 4.31.

OR2) The object of dinatural transformations between two functors $F, G : (C^p)^{\mathrm{op}} \times C^q \to \mathcal{D}$ organise as a (p, q)-end; in the case $(p, q) = (1, 1)$ this result was first noted by [DS70]; however, we generalise even further this picture, by providing a canonical way to appropriately "resolve" a functor G of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ into a functor $\text{⅃}^{p,q}(G)$ of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ in such a way that the isomorphism

$$\mathrm{Nat}\left(\text{⅃}^{p,q}(F), G\right) \cong \mathrm{DiNat}^{(p,q)}(F, G)$$

holds, i.e. in such a way that a *dinatural* transformation between F and G amounts exactly to a *natural* transformation from $\text{⅃}^{p,q}(F)$ to G. The object so determined has a certain universal property: we study this construction in detail along Section 5, where moreover we also discover that

$$\mathrm{Nat}\left(F, \Gamma^{p,q}(G)\right) \cong \mathrm{DiNat}^{(p,q)}(F, G)$$

for a certain functor $\Gamma^{p,q}$ that is thus a right adjoint to $\text{⅃}^{p,q}$.

From this, a number of future directions can be taken:

(1) *Weighted category theory (beyond co/limits)*, [dLb]. The notion of weighted co/limit arises as the solution to a representability problem. Computing the 'conical' colimit of a diagram $D : C \to \mathcal{D}$, we aim to find an object $\mathrm{colim}(D)$ that represents the functor sending $X \in \mathcal{D}$ to the set of cocones for D, i.e. natural transformations between the 'constant at the point' functor $*$ and the functor $\mathcal{D}(D-, X)$. In a similar fashion, when we want to compute the *weighted* colimit of D we try to represent the functor

$$X \mapsto \mathsf{Cat}(C^{\mathrm{op}}, \mathsf{Set})(W, \mathcal{D}(D-, X)).$$

Now, what if we try to do the same for the other usual categorical constructions besides co/limits, such as adjunctions, Kan extensions, monads, or co/ends? For instance, what if, instead of trying to represent the functor that sends $T : C^{\mathrm{op}} \times C \to \mathcal{D}$ to the set of its co/wedges, we try to represent the functor that sends X to $\mathrm{DiNat}(W, \mathcal{D}(T, X))$?

We dub *weighted category theory* the piece of technology that addresses this problem.

(2) *Diagonal category theory*, [dLa]. In diagonal category theory, a different but still related path is taken: in analogy to the passage from limits (universal cones) to ends (universal wedges), we seek a general framework, for other categorical constructions, in which the passage from cones to wedges is meaningful. Thus, we aim to replace naturality requests with dinaturality in categorical concepts besides that of a limit, obtaining, as a result, a very rich theory with notions such as diagonal Kan extensions, adjunctions and monads, all standing to their classical counterparts as co/ends stand to co/limits.

Despite the intrinsic obstruction to compose dinatural transformations, the theory so obtained is non-trivial: it sheds light on a number of aspects of classical category theory, such as the disparity between limits and colimits assorting themselves into a triple adjunction

$$(\operatorname{colim} \dashv \Delta_{(-)} \dashv \lim) \colon \mathsf{Cat}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathcal{D}$$

whilst no such no such result exists for ends and coends: instead we have *diagonally adjoint functors*, of which ends and coends are a fundamental example, assembling into a triple $\int^A \dashv_\delta \Delta_{(-)} \dashv_\delta \int_A$.

1.1. Structure of the paper

In Section 1.2, we motivate higher arity dinaturality, showing how such a notion arises geometrically as a “diagonal” version of natural transformations between functors with domain a product category. All the material is very well-known, and this is only meant to fix notation at the outset. We borrow an intuitive explanation of what dinaturality is about from [BS10, pp. 48–50] (see also [Gav19] for a similar presentation), first recalling it in Remark 1.2 and then generalising the argument to the higher arity case in Remark 1.3.

In Section 2, we review and specialise the notion of dinaturality introduced in [San19; MS20] to the appropriate setting for considering “universal generalised dinaturality”. In detail, we first recall the notion of a (p, q)-*dinatural transformation* and study its properties, generalising results of Street–Dubuc ([DS70]) to functors of arbitrary arity. We then proceed to discuss (p, q)-dinatural transformations from constant functors, which we dub (p, q)-*wedges*, in analogy with the classical case.

In Section 3, we formulate the notion of a *higher arity co/end*. Just as ends are universal wedges, higher arity ends are universal (p, q)-wedges. After introducing them in Definition 3.1, we discuss some of their basic properties (Proposition 3.5). Then, in Section 3.2, we state and prove a Fubini rule for higher arity co/ends, generalising the classical Fubini rule for co/ends.

In Section 4, we illustrate the theory developed so far by working out a large number of examples. We study naturally-appearing instances of higher arity co/ends in category theory as well as in related areas. The machinery employed here is elementary, but provides insightful examples when applied to laying down the rules of a ‘calculus of weighted ends’, and to ‘diagonal’ category theory (see Sections 4.2.1 and 4.2.3 for reference).

In Section 5, we introduce the notion of *co/kusarigamas*. These are fundamental constructions in higher arity co/end calculus allowing us to reduce the study of (p, q)-dinaturality to that of (ordinary) naturality. Co/kusarigamas also provide us with a way to express higher arity co/ends as weighted co/limits, as well as with a higher arity version of the twisted arrow category (Section 5.3).

1.2. Geometric motivation for higher arity dinaturality

Notation 1.1 ((p, q)-products, tensor calculus notation)**.** The entire paper deals with categories that are the product of q copies of a (small) category $\mathcal{C}$, and p copies of the opposite category $\mathcal{C}^{\mathsf{op}}$, for p, q two non-negative integers: throughout the entire discussion we will denote

$$C^{(p,q)} \stackrel{\text{def}}{=} \underbrace{C^{\text{op}} \times \cdots \times C^{\text{op}}}_{p \text{ times}} \times \underbrace{C \times \cdots \times C}_{q \text{ times}}.$$

An alternative, short notation for the category $(C^{\text{op}})^p = (C^p)^{\text{op}}$ is $C^{-p} \times C^q$, but this has to be used *cum grano salis*, as some of the usual sign convention do not apply (for example, exponent of discordant sign do not add: $C^{-1} \times C^1$ is 'irreducible').

A functor having domain $C^{(p,q)}$ will be called a functor of *type* $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$.

To denote the action of a functor of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ on objects we write

$$F^{\underline{A}''}_{\underline{A}'} \stackrel{\text{def}}{=} F^{A_1,\dots,A_p}_{A_1,\dots,A_q} \stackrel{\text{def}}{=} F\left(A''_1,\dots,A''_p,A'_1,\dots,A'_q\right)$$

for (tuples of) objects $\underline{A}' \stackrel{\text{def}}{=} A'_1,\dots,A'_q \in C^q$ and $\underline{A}'' \stackrel{\text{def}}{=} A''_1,\dots,A''_p \in C^{-p}$. (See Section 1.3 below for variations and specialisations of this notation.)

This is reminding of the way in which one writes the coordinates of a tensor of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$.

We start this introductory section by recalling the notion of dinatural transformation; we borrow an argument from [BS10, pp. 48–50] that shows how dinaturality is, if not unavoidable, at least motivated by elementary considerations.

In short, one may in fact arrive at defining dinaturality by considering a 'naturality cube' and then removing its "non-diagonal" pieces.

Baez–Stay's argument can be adapted to motivate our notion of (p,q)-*dinaturality* Definition 2.1 as similarly unavoidable (see Remark 1.3).

Remark 1.2 (From Naturality Cubes to Dinaturality Hexagons)**.** Let C be a category, and consider the product category $C^{\text{op}} \times C$. This is the category whose

CC1) Objects are pairs (A,B) with $A \in C^{\text{op}}_o = C_o$ and $B \in C_o$;

CC2) Morphisms $(A,B) \to (A',B')$ are pairs $\left(\left[\begin{smallmatrix} & A' \\ g & \downarrow \\ & A \end{smallmatrix}\right], \left[\begin{smallmatrix} & B \\ f & \downarrow \\ & B' \end{smallmatrix}\right]\right)$ of morphisms of C.

Now, each morphism $f\colon A \to B$ of C, gives rise to a commutative square in $C^{\text{op}} \times C$ of the form

$$\begin{tikzcd} (B,A) \arrow[r, "{(B,f)}"] \arrow[d, "{(f,A)}"'] & (B,B) \arrow[d, "{(f,B)}"] \\ (A,A) \arrow[r, "{(A,f)}"'] & (A,B), \end{tikzcd}$$

to which we can apply functors $F,G\colon C^{\text{op}} \times C \rightrightarrows \mathcal{D}$, obtaining two naturality squares

$$\begin{tikzcd} F^B_A \arrow[r, "F^B_f"] \arrow[d, "F^f_A"'] & F^B_B \arrow[d, "F^f_B"] \\ F^A_A \arrow[r, "F^A_f"'] & F^A_B \end{tikzcd} \qquad \begin{tikzcd} G^B_A \arrow[r, "G^B_f"] \arrow[d, "G^f_A"'] & G^B_B \arrow[d, "G^f_B"] \\ G^A_A \arrow[r, "G^A_f"'] & G^A_B, \end{tikzcd}$$

from which we derive that $F^f_B \circ F^B_f = F^A_f \circ F^f_A$, and similarly for G.

A natural transformation from F to G is then a collection

$$\{\alpha^A_B\colon F^A_B \to G^A_B \mid (A,B) \in (C^{\text{op}} \times C)_o\}$$

of morphisms of $\mathcal{D}$ making the diagram

$$\begin{tikzcd}
F^A_B \arrow[r, "F^g_f"] \arrow[d, "\alpha^A_B"'] & F^{A'}_{B'} \arrow[d, "\alpha^{A'}_{B'}"] \\
G^A_B \arrow[r, "G^g_f"'] & G^{A'}_{B'}
\end{tikzcd}$$

commute for every $\left[\begin{smallmatrix} A' \\ g\downarrow \\ A \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B \\ f\downarrow \\ B' \end{smallmatrix}\right]$. When $g = f$, this reduces to

$$\begin{tikzcd}
F^B_A \arrow[r, "F^f_f"] \arrow[d, "\alpha^B_A"'] & F^A_B \arrow[d, "\alpha^A_B"] \\
G^B_A \arrow[r, "G^f_f"'] & G^A_B,
\end{tikzcd}$$

which we may rewrite as the following commutative cube, using that $(f, f) = (\mathrm{id}_B, f) \circ (f, \mathrm{id}_A)$ in $C^{\mathsf{op}} \times C$:

$$\begin{tikzcd}
F^B_A \arrow[rr] \arrow[dr] \arrow[dd] & & F^B_B \arrow[dr] & \\
& F^A_A \arrow[rr] \arrow[dd] & & F^A_B \arrow[dd] \\
G^B_A \arrow[dr] & & & \\
& G^A_A \arrow[rr] & & G^A_B
\end{tikzcd}
=
\begin{tikzcd}
F^B_A \arrow[rr] \arrow[dd] & & F^B_B \arrow[dr] \arrow[dd] & \\
& & & F^A_B \arrow[dd] \\
G^B_A \arrow[rr] \arrow[dr] & & G^B_B \arrow[dr] & \\
& G^A_A \arrow[rr] & & G^A_B
\end{tikzcd}$$

(in all arrows the action of F on its covariant or contravariant component is taken into account).

A notion of "diagonal transformation" between F and G is then a collection of morphisms of $\mathcal{D}$ from F^A_A to G^A_A; to obtain it, we should remove the "non-diagonal pieces" α^B_A and α^A_B from this cube, arriving at the diagram

$$\begin{tikzcd}
F^B_A \arrow[rr, "F^B_f"] \arrow[dr, "F^f_A"'] \arrow[dd, gray] & & F^B_B \arrow[dr, gray] \arrow[dd, "\alpha^B_B"] & \\
& F^A_A \arrow[rr, gray] \arrow[dd, "\alpha^A_A"'] & & F^A_B \arrow[dd, gray] \\
G^B_A \arrow[rr, gray] \arrow[dr, gray] & & G^B_B \arrow[dr, "G^f_B"] & \\
& G^A_A \arrow[rr, "G^A_f"'] & & G^A_B.
\end{tikzcd}$$

Writing α_A (resp. α_B) for α^A_A (resp. α^B_B) and 'flattening' the resulting diagram, we get the dinaturality hexagon

$$\begin{tikzcd}
& F^A_A \arrow[r, "\alpha_A"] & G^A_A \arrow[dr, "G^A_f"] & \\
F^B_A \arrow[ur, "F^f_A"] \arrow[dr, "F^B_f"'] & & & G^A_B \\
& F^B_B \arrow[r, "\alpha_B"'] & G^B_B \arrow[ur, "G^f_B"'] &
\end{tikzcd}$$

for $\alpha : F \Longrightarrow\!\circ\, G$.

Given functors $F, G : \mathcal{C}^{(p,q)} \to \mathcal{D}$ we seek a similar intuitive explanation for an analogue notion of (p,q)-dinaturality. Of course, as the sum $p+q$ grows bigger, it is harder and harder to visualise the underlying geometry, since we have to work in dimension $p+q+1 \geq 4$.

Before giving the general definition in Definition 2.1, we illustrate in detail the case $(p,q) = (2,1)$.

Remember that we write $\mathcal{C}^{(2,1)}$ for the category $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}^{\mathsf{op}} \times \mathcal{C}$.

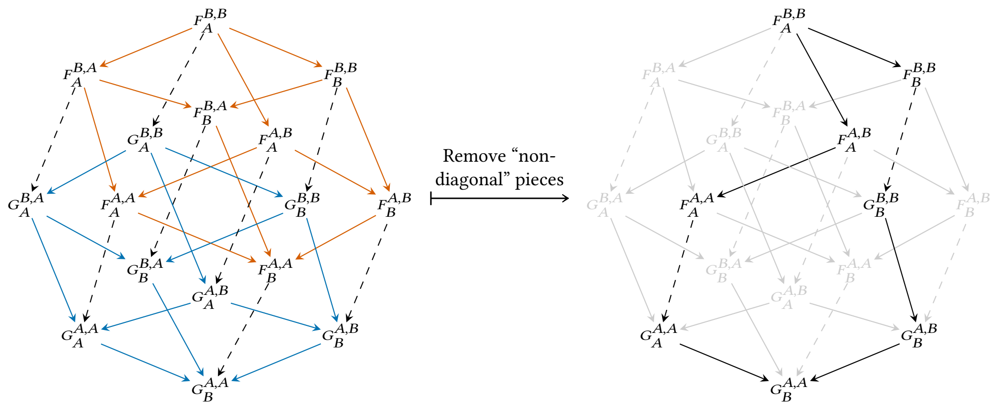
Remark 1.3 (From a naturality hypercubes to $(2,1)$-Dinaturality)**.** In a similar fashion, a morphism $f : A \to B$ induces a commutative cube in $\mathcal{C}^{(2,1)}$ which, under the action of two functors $F, G\colon \mathcal{C}^{(2,1)} \rightrightarrows \mathcal{D}$, yields two commutative cubes

$$\begin{tikzcd}
F_A^{B,B} \arrow[rr] \arrow[dr] \arrow[dd] & & F_B^{B,B} \arrow[dr] \arrow[dd, dashed] & \\
& F_A^{B,A} \arrow[rr] \arrow[dd] & & F_B^{B,A} \arrow[dd] \\
F_A^{A,B} \arrow[rr, dashed] \arrow[dr] & & F_B^{A,B} \arrow[dr, dashed] & \\
& F_A^{A,A} \arrow[rr] & & F_B^{A,A}
\end{tikzcd}
\qquad
\begin{tikzcd}
G_A^{B,B} \arrow[rr] \arrow[dr] \arrow[dd] & & G_B^{B,B} \arrow[dr] \arrow[dd, dashed] & \\
& G_A^{B,A} \arrow[rr] \arrow[dd] & & G_B^{B,A} \arrow[dd] \\
G_A^{A,B} \arrow[rr, dashed] \arrow[dr] & & G_B^{A,B} \arrow[dr, dashed] & \\
& G_A^{A,A} \arrow[rr] & & G_B^{A,A}
\end{tikzcd}$$

in $\mathcal{D}$. Now, a natural transformation α from F to G is a collection

$$\{\alpha_C^{A,B} \colon F_C^{A,B} \to G_C^{A,B} \mid (A,B) \in \mathcal{C}_o^{(2,1)}\}$$

of morphisms of $\mathcal{D}$ such that the hypercube diagram below-left is commutative:



(vermillion: the F cube; blue: the G cube). Again deleting the nodes $F_X^{Y,Z}$, $G_X^{Y,Z}$ for which X, Y, Z are not all equal, we get the above-right diagram. Flattening the result, this gives us the octagonal diagram below-left, which becomes the "(2, 1)-dinaturality hexagon" below-right upon using that $F_A^{f,f} = F_A^{A,f} \circ F_A^{f,B}$ and $G_B^{f,f} = G_B^{A,f} \circ G_B^{f,B}$:

$$\begin{tikzcd}
 & F_A^{B,B} \arrow[dl] \arrow[dr] \arrow[ddl, gray] & & & & F_A^{B,B} \arrow[dl] \arrow[dr] & \\
F_A^{A,B} \arrow[d] & & F_B^{B,B} \arrow[d] & & F_A^{A,A} \arrow[dd] & & F_B^{B,B} \arrow[dd] \\
F_A^{A,A} \arrow[d] & & G_B^{B,B} \arrow[d] \arrow[ddl, gray] & \longmapsto & & & \\
G_A^{A,A} \arrow[dr] & & G_B^{A,B} \arrow[dl] & & G_A^{A,A} \arrow[dr] & & G_B^{B,B} \arrow[dl] \\
 & G_B^{A,A} & & & & G_B^{A,A} &
\end{tikzcd}$$

Notation 1.4. The reader might have noticed, at this point, that in order not to clutter the page with too many unwanted apices and pedices, we have to establish a nifty notation to represent how functors act on tuples.

This will become even more evident as the discussion goes on: no argument in this work is more elaborate than elementary category theory. Yet, it is literally impossible to follow any intuition without a clever way to represent the action of functors of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ on objects, as well as functors having domain the functor category $\mathsf{Cat}(C^{(p,q)}, \mathcal{D})$.

1.3. Notation and preliminaries

All the basic notation for categories and functors used in this paper follows standard practice. Apart from this, and apart from what we already introduced in Notation 1.1, we need notation for:

N1) A generic tuple of objects,

$$\underline{A} \overset{\text{def}}{=} (A_1, \dots, A_n)$$

often split as the juxtaposition $\underline{A}'; \underline{A}''$ of two subtuples of length p, q,

$$\underline{A}' \overset{\text{def}}{=} (A_1, \dots, A_q), \qquad \underline{A}'' \overset{\text{def}}{=} (A_{p+1}, \dots, A_{p+q})$$

N2) As already said, the image of a split tuple $\underline{A}'; \underline{A}''$ under a functor of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, $F : C^{(p,q)} \to \mathcal{D}$ is denoted $F^{\underline{A}'}_{\underline{A}''}$: the contravariant components come first, and the covariant component second. So: contravariant components are always *left* in the typing

$$F : C^{(p,q)} \longrightarrow \mathcal{D}$$

of a functor, and *up* in its action on objects.

N3) Denoting a functor F of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ evaluated at a diagonal tuple: we write

$$F^{\boldsymbol{A}}_{\boldsymbol{A}} \overset{\text{def}}{=} F^{A,\dots,A}_{A,\dots,A},$$

where the superscript has p elements, and the subscript has q elements.

N4) Substitution of an object at a prescribed index

$$\underline{A}[X/i] \overset{\text{def}}{=} (A_1, \dots A_{i-1}, X, A_{i+1}, \dots, A_n).$$

N5) Substitution of a tuple at a prescribed tuple of indices

$$\underline{A}[X_1, \dots, X_r/i_1, \dots, i_r] \overset{\text{def}}{=} ((\underline{A}[X_1/i_1])[X_2/i_2] \cdots)[X_r/i_r].$$

Definition 1.5. The (p,q)*-diagonal functor* is the functor $\Delta_{p,q} \colon C^{\mathsf{op}} \times C \to C^{(p,q)}$ defined by

$$\Delta_{p,q} \overset{\text{def}}{=} \underbrace{\Delta^{\mathsf{op}} \times \cdots \times \Delta^{\mathsf{op}}}_{p \text{ times}} \times \underbrace{\Delta \times \cdots \times \Delta}_{q \text{ times}}.$$

Remark 1.6 (Unwinding Definition 1.5)**.** Explicitly, $\Delta_{p,q} \colon C^{\mathsf{op}} \times C \to C^{(p,q)}$ is the functor sending

UD1) An object (A, B) of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to the object $(\boldsymbol{A}, \boldsymbol{B}) \stackrel{\text{def}}{=} (A, \dots, A, B, \dots, B)$ of $\mathcal{C}^{(p,q)}$, and

UD2) A morphism $(f, g) \colon (A, B) \to (A', B')$ of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to the morphism $(\boldsymbol{f}, \boldsymbol{g}) \stackrel{\text{def}}{=} (f, \dots, f, g, \dots, g)$ of $\mathcal{C}^{(p,q)}$,

where in the expression $(\boldsymbol{A}, \boldsymbol{B})$ we have p repeated copies of A and q repeated copies of B, and similarly for $(\boldsymbol{f}, \boldsymbol{g})$.

Notation 1.7 (Mixed Products and Coproduts)**.** Let $\mathcal{C}$ be a category with finite products and coproducts. Given tuples $A_1, \dots, A_p, B_1, \dots, B_q$, we write

$$\mathsf{W}_{p,q}(\underline{A}, \underline{B}) \stackrel{\text{def}}{=} \left(\prod_{i=1}^{p} A_i, \coprod_{j=1}^{q} B_j \right),$$

$$\mathsf{M}_{p,q}(\underline{A}, \underline{B}) \stackrel{\text{def}}{=} \left(\coprod_{i=1}^{p} A_i, \prod_{j=1}^{q} B_j \right).$$

Proposition 1.8 (Adjoints to the (p, q)-Diagonal Functor)**.** If $\mathcal{C}$ has products and coproducts, then we have a triple adjunction

$$\left(\mathsf{W}_{p,q} \dashv \Delta_{p,q} \dashv \mathsf{M}_{p,q}\right) : \quad \begin{tikzcd} \mathcal{C}^{(p,q)} \arrow[r, shift left=3, "\mathsf{W}_{p,q}"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "\mathsf{M}_{p,q}"'] & \mathcal{C}^{\mathsf{op}} \times \mathcal{C}. \arrow[l, "\Delta_{p,q}" description] \end{tikzcd}$$

Proof. The proposition follows from the universal property of the co/product, as we have a string of bijections

$$\begin{aligned} \hom_{\mathcal{C}^{(p,q)}}\left((\boldsymbol{A}, \boldsymbol{B}), \Delta_{p,q}(C, D)\right) &\stackrel{\text{def}}{=} \left(\prod_{i=1}^{p} \hom_{\mathcal{C}^{\mathsf{op}}}(A_i, C) \right) \times \left(\prod_{j=1}^{q} \hom_{\mathcal{C}}(B_j, D) \right) \\ &\stackrel{\text{def}}{=} \left(\prod_{i=1}^{p} \hom_{\mathcal{C}}(C, A_i) \right) \times \left(\prod_{j=1}^{q} \hom_{\mathcal{C}}(B_j, D) \right) \\ &\cong \hom_{\mathcal{C}}\left(C, \prod_{i=1}^{p} A_i \right) \times \hom_{\mathcal{C}}\left(\coprod_{j=1}^{q} B_j, D \right) \\ &\stackrel{\text{def}}{=} \hom_{\mathcal{C}^{\mathsf{op}}}\left(\prod_{i=1}^{p} A_i, C \right) \times \hom_{\mathcal{C}}\left(\coprod_{j=1}^{q} B_j, D \right) \\ &\stackrel{\text{def}}{=} \hom_{\mathcal{C}^{\mathsf{op}} \times \mathcal{C}}\left(\left(\prod_{i=1}^{p} A_i, \coprod_{j=1}^{q} B_j \right), (C, D) \right), \\ &\stackrel{\text{def}}{=} \hom_{\mathcal{C}^{\mathsf{op}} \times \mathcal{C}}\left(\mathsf{W}_{p,q}(A_i, B_j), (C, D)\right), \end{aligned}$$

natural in $A_1, \dots, A_p, B_1, \dots, B_q, C, D \in \mathcal{C}_o$. The proof that $\Delta_{p,q}$ admits a right adjoint is dual to this one. □

We also collect a couple of standard results on generating strings of adjunctions by left/right Kan extending a given adjunction:

Lemma 1.9 (Applying Kan Extensions to an Adjunction)**.** Every adjunction

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

induces a quadruple adjunction $\mathsf{Lan}_K \dashv K^* \dashv L^* \dashv \mathsf{Ran}_L$ such that

$$\mathsf{Lan}_K \cong L^*, \qquad \mathsf{Ran}_L \cong K^*.$$

Proof. Both L and K induce triple adjunctions between $\mathsf{Cat}(\mathcal{C},\mathcal{E})$ and $\mathsf{Cat}(\mathcal{D},\mathcal{E})$. Proving that $\mathsf{Lan}_K \cong L^*$ and $\mathsf{Ran}_L \cong K^*$ would show that these are actually parts of a single quadruple adjunction, which is the stated one. That this is indeed so follows from the string of isomorphisms

$$\begin{aligned} L^*(F) &\stackrel{\text{def}}{=} F \circ L \\ &\cong \int_X \mathcal{C}\,(L(-),X) \odot F(X) \\ &\cong \int_X \mathcal{D}(-,K(X)) \odot F(X) \\ &\cong \mathsf{Ran}_K F \end{aligned}$$

natural in F. Hence $K^* \cong \mathsf{Lan}_L$. By a similar argument, $K^* \cong \mathsf{Ran}_L$, finishing the proof. □

Combining two applications of Proposition 1.8 as well as uniqueness of adjoint functors with Lemma 1.9, we get the following corollary:

Corollary 1.10. We have a quintuple adjunction

$$\left(\mathsf{Lan}_{\mathsf{M}_{p,q}} \dashv \mathsf{M}^*_{p,q} \dashv \Delta^*_{p,q} \dashv \mathsf{W}^*_{p,q} \dashv \mathsf{Ran}_{\mathsf{W}_{p,q}}\right) : \ \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}) \rightleftarrows \mathsf{Cat}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C},\mathcal{D}),$$

with natural isomorphisms

$$\begin{aligned} \mathsf{Lan}_{\Delta_{p,q}} &\cong \mathsf{M}^*_{p,q}, \\ \mathsf{Ran}_{\Delta_{p,q}} &\cong \mathsf{W}^*_{p,q}. \end{aligned}$$

2. Higher Arity Wedges

This section formalises completely the notion that we dubbed "(2,1)-dinaturality" above and presents it for general $p, q \geq 0$.

The definition of dinaturality given below is not new: it was recently introduced in Santamaria's PhD thesis [MS20; San19], building on previous work by M. Kelly [Kel72b; Kel72a] in fair more generality than the one we need.

In [MS20; San19], however, an "unbiased" arrangement of the factors in $\mathcal{C}^{(p,q)}$ is considered, in the sense that [MS20, Definition 2.4] takes into account functors $\mathcal{C}^\alpha \to \mathcal{B}$, where α is a "binary multi-index", i.e. an element in the free monoid over the set $\{\oplus,\ominus\}$, and the convention is that $\mathcal{C}^\varnothing \stackrel{\text{def}}{=} \mathsf{pt}$, the terminal category, $\mathcal{C}^\oplus \stackrel{\text{def}}{=} \mathcal{C}$, $\mathcal{C}^\ominus \stackrel{\text{def}}{=} \mathcal{C}^{\mathsf{op}}$, and $\mathcal{C}^{\alpha\uplus\alpha'} \stackrel{\text{def}}{=} \mathcal{C}^\alpha \times \mathcal{C}^{\alpha'}$.

Here instead, we adopt a different convention: a generic power $\mathcal{C}^\alpha$ is always "reshuffled" in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right. The categories $\mathcal{C}^\alpha$ and $\mathcal{C}^{(p,q)}$ so obtained are, of course, canonically isomorphic, and the tuple α is equivalent to the reshuffled tuple $(\ominus_1,\dots,\ominus_p,\oplus_1,\dots,\oplus_q)$.

2.1. Higher arity dinaturality

Let $p, q \in \mathbb{N}$ and $\mathcal{C}$ be a category. The definition of a (p,q)*-dinatural transformation* from a functor $F\colon \mathcal{C}^{(p,q)} \to \mathcal{D}$ of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ to a functor $G\colon \mathcal{C}^{(q,p)} \to \mathcal{D}$ of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ can be shortly stated as the condition that a dinaturality hexagon commutes, when filled with the conjoint action of F (resp. G) in all its contravariant and covariant components separately.

The choice of joining with a transformation just functors of opposite types deserves a bit of explanations.

As stated in 2.1, the definition could be dubbed "(p,q)-to-(q,p)" dinaturality. It sits in the middle between a rigid one (a "(p,q)-to-(p,q)" dinaturality, where F, G below have the same type) and a loose one (a "(p,q)-to-(r,s)" dinaturality, where F, G below have possibly completely different types: this is the path chosen by [San19], recalled in Definition 2.18 below).

We could have stuck with the tighter notion, but some of the characterisations we give would have become false: for example, the set of (p,q)-dinatural transformations between F, G is a (p,q)-end only with our convention (see Example 4.7).

We could have stuck with the looser one; but the definition of co/wedge given in Definition 2.10 wouldn't have changed (a constant functor can be "dummified", in the sense of Notation 3.4, to have whatever type is needed).

It must be noted that both notions allow for dinatural transformations to be composed (but, as it is well-known, the composition isn't always dinatural). On the contrary, (p,q)-to-(q,p) dinaturality does not allow to speak about composition; the reason is, again, that the definition isn't engineered to speak about composition, but instead about (co/wedges and) co/ends.

Definition 2.1. A (p,q)*-dinatural transformation* $\alpha : F \Longrightarrow\!\circ\, G$ is a collection

$$\left\{\alpha_A : F^{\overbrace{A,\dots,A}^{p\text{ times}}}_{\underbrace{A,\dots,A}_{q\text{ times}}} \to G^{\overbrace{A,\dots,A}^{q\text{ times}}}_{\underbrace{A,\dots,A}_{p\text{ times}}} \mid A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
 & F^{A_p}_{A_q} \arrow[r, "\alpha_A"] & G^{A_q}_{A_p} \arrow[dr, "G^{A_q}_{f_p}"] & \\
F^{B_p}_{A_q} \arrow[ur, "F^{f_p}_{A_q}"] \arrow[dr, "F^{B_p}_{f_q}"'] & & & G^{A_q}_{B_p} \\
 & F^{B_p}_{B_q} \arrow[r, "\alpha_B"'] & G^{B_q}_{B_p} \arrow[ur, "G^{f_q}_{B_p}"'] &
\end{tikzcd}$$

commutes.

Example 2.2. For (p,q)=$(2,1)$, a $(2,1)$-dinatural transformation is a collection

$$\left\{\alpha_A : F^{A,A}_{A} \to G^{A}_{A,A} \;\middle|\; A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{D}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the following hexagonal diagram commutes:

$$\begin{tikzcd}
 & F^{A,A}_{A} \arrow[r, "\alpha_A"] & G^{A}_{A,A} \arrow[dr, "G^{A}_{f,f}"] & \\
F^{B,B}_{A} \arrow[ur, "F^{f,f}_{A}"] \arrow[dr, "F^{B,B}_{f}"'] & & & G^{A}_{B,B} \\
 & F^{B,B}_{B} \arrow[r, "\alpha_B"'] & G^{B}_{B,B}. \arrow[ur, "G^{f}_{B,B}"'] &
\end{tikzcd}$$

Notation 2.3. We write $\mathrm{DiNat}^{(p,q)}(F,G)$ for the set of (p,q)-dinatural transformations from F to G.

Remark 2.4. The same convention of Section 1.3 applies to morphisms as well as to objects: $F^{\boldsymbol{B}}_{\boldsymbol{f}} \overset{\text{def}}{=} F^{B,\dots,B}_{f,\dots,f}$ is the morphism $F^{B,\dots,B}_{A,\dots,A} \to F^{B,\dots,B}_{A,\dots,A}$ induced by the conjoint action of f in all the covariant components of F, and similarly for $F^{\underline{F}}_{\boldsymbol{A}}, G^{\boldsymbol{A}}_{\boldsymbol{f}}$, etc.

Dinatural transformations can always be composed with *natural* ones of the appropriate arity, on the left and on the right.

Definition 2.5 (Composing dinaturals with naturals)**.** Let F and G be a functors of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, let H and K be functors of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$, let $\alpha : F \to G$ and $\beta : H \to K$ be natural transformations, and let $\theta : G \Longrightarrow\!\!\circ\, H$ be a (p,q)-dinatural transformation.

DC1) The *vertical composition of θ with α* is the (p,q)-dinatural transformation

$$\theta \circ \alpha : F \Longrightarrow\!\!\circ\, H$$

defined as the collection

$$\left\{ (\theta \circ \alpha)_A : F^{\boldsymbol{A}}_{\boldsymbol{A}} \to H^{\boldsymbol{A}}_{\boldsymbol{A}} \;\middle|\; A \in \mathcal{C}_o \right\},$$

where $(\theta \circ \alpha)_A = \theta_A \circ \alpha^{\boldsymbol{A}}_{\boldsymbol{A}}$;

DC2) The *vertical composition of β with θ* is the (p,q)-dinatural transformation

$$\beta \circ \theta : G \Longrightarrow\!\!\circ\, K$$

defined as the collection

$$\left\{ (\beta \circ \theta)_A : G^{\boldsymbol{A}}_{\boldsymbol{A}} \to K^{\boldsymbol{A}}_{\boldsymbol{A}} \;\middle|\; A \in \mathcal{C}_o \right\},$$

where $(\beta \circ \theta)_A = \beta^{\boldsymbol{A}}_{\boldsymbol{A}} \circ \theta_A$.

Remark 2.6. Note that the dinaturality in Definition 2.1, does not allow us to conclude that a natural transformation $\alpha^{\underline{A}}_{\underline{B}}$ between functors induces a dinatural family of maps by 'complete symmetrisation' $\alpha^{\underline{A}}_{\underline{B}} \mapsto \alpha^{\boldsymbol{A}}_{\boldsymbol{B}}$; in fact, this request does not even make sense, as natural transformations are defined only between functors of the same type: given F and G of variance (p,q) with $p \neq q$ (say $(p,q) = (1,2)$), we have

$$\alpha^{A}_{A} \colon F^{A}_{A,A} \to G^{A}_{A,A},$$

rather than something of the form

$$\alpha^{A}_{A} \colon F^{A}_{A,A} \to G^{A,A}_{A},$$

as required per Definition 2.1. More formally, this lack of a rule yielding a dinatural transformation from a natural transformation between functors of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ for $p \neq q$ boils down to the absence of an identity dinatural transformation.

Proposition 2.7. *$\theta \circ \alpha$ and $\beta \circ \theta$ are (p,q)-dinatural transformations.*

Proof. The (p,q)-dinaturality condition for $\theta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
 & F^A_A \arrow[rr, "\alpha^A_A"] \arrow[dr, phantom, "(1)"] & & G^A_A \arrow[rr, "\theta_A"] & & H^A_A \arrow[dr] & \\
F^B_A \arrow[ur] \arrow[rr, "\alpha^B_A"] \arrow[dr] & & G^B_A \arrow[ur] \arrow[dr] \arrow[rrrr, phantom, "(3)"] & & & & H^A_B \\
 & F^B_B \arrow[rr, "\alpha^B_B"'] \arrow[ur, phantom, "(2)"] & & G^B_B \arrow[rr, "\theta_B"'] & & H^B_B \arrow[ur] &
\end{tikzcd}$$

commutes. Since

(1) Sub-diagrams (1) and (2) commute by the naturality of α, and
(2) Sub-diagram (3) commutes by the (p,q)-dinaturality of θ,

so does the boundary diagram, and $\theta \circ \alpha$ is indeed a (p,q)-dinatural transformation.

Similarly for $\beta \circ \theta$: one considers instead

$$\begin{tikzcd}
 & \bullet \arrow[rr] & & \bullet \arrow[dr] \arrow[rr] & & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] \arrow[rrrr, phantom, "(1)"] & & & & \bullet \arrow[rr] \arrow[ur, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & & \bullet, \\
 & \bullet \arrow[rr] & & \bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] &
\end{tikzcd}$$

where again each sub-diagram commutes by either the dinaturality of θ or the naturality of β. $\square$

For the next proposition, recall the definition of the (p,q)-*diagonal functor* $\Delta_{p,q} \colon C^{\mathrm{op}} \times C \to C^{(p,q)}$ of C introduced in Definition 1.5.

Proposition 2.8 (Higher arity dinaturality via ordinary dinaturality)**.** *Let $F \colon C^{(p,q)} \to \mathcal{D}$ and $G \colon C^{(q,p)} \to \mathcal{D}$ be functors. We have a natural bijection*

$$\mathrm{DiNat}^{(p,q)}(F,G) \cong \mathrm{DiNat}^{(1,1)}\big(\Delta^*_{p,q}(F), \Delta^*_{q,p}(G)\big). \tag{2.9}$$

Proof. This is simply a matter of unwinding the definitions: since $\big(F \circ \Delta_{p,q}\big)^A_B \overset{\mathrm{def}}{=} F^{\boldsymbol{A}}_{\boldsymbol{B}}$ (and similarly for morphisms and for G), it follows that a (p,q)-dinatural transformation $F \Longrightarrow\!\!\circ\; G$ is precisely a dinatural transformation $\Delta^*_{p,q}(F) \Longrightarrow\!\!\circ\; \Delta^*_{q,p}(G)$. $\square$

2.2. Higher arity wedges

The notion of wedge (resp., cowedge) for a diagram $D : C^{\mathrm{op}} \times C \to \mathcal{D}$ arises when assuming that the domain (resp., codomain) of a dinatural transformation to/from D is constant; similarly, a (p,q)-wedge (resp., (p,q)-cowedge) for a diagram $D : C^{(p,q)} \to \mathcal{D}$ consists of a (p,q)-dinatural transformation whose domain (resp., codomain) is a constant functor $X : C^{(q,p)} \to \mathcal{D}$ of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$.

Definition 2.10. Let $D : C^{(p,q)} \to \mathcal{D}$ be a functor and let $X \in \mathcal{D}_o$.

cw1) A (p,q)-*wedge for D under X* is a (p,q)-dinatural transformation $\theta : \Delta_X \Longrightarrow\!\!\circ\; D$ from the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X to D;

cw2) A (p,q)-*cowedge for D over X* is a (p,q)-dinatural transformation $\zeta : D \Longrightarrow\!\!\circ\; \Delta_X$ from D to the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X.

Remark 2.11 (Unwinding Definition 2.10)**.**

CWU1) A (p,q)-wedge $\theta : \Delta_X \Longrightarrow\!\circ\, D$ is a collection

$$\left\{\theta_A : X \to D^A_A \ :\ A \in C_o\right\}$$

of morphisms of C such that, for each morphism $f : A \to B$ of C, the diagram

$$\begin{tikzcd} X \arrow[r, "\theta_B"] \arrow[d, "\theta_A"'] & D^B_B \arrow[d, "D^f_B"] \\ D^A_A \arrow[r, "D^A_f"'] & D^A_B \end{tikzcd}$$

commutes.

CWU2) A (p,q)-cowedge $\zeta : D \Longrightarrow\!\circ\, \Delta_X$ is a collection

$$\left\{\zeta_A : D^A_A \to X \ :\ A \in C_o\right\}$$

of morphisms of C such that, for each morphism $f : A \to B$ of C, the diagram

$$\begin{tikzcd} X & D^B_B \arrow[l, "\zeta_B"'] \\ D^A_A \arrow[u, "\zeta_A"] & D^A_B \arrow[l, "D^A_f"] \arrow[u, "D^f_B"'] \end{tikzcd}$$

commutes.

Remark 2.12. For the sake of clarity, we remind the reader that in our notation, the commutativity of the diagram in Item CWU1 of Remark 2.11 above means that, for every $f \in \mathrm{Mor}(C)$,

$$D^{f_1,\dots,f_p}_{B_1,\dots,B_q} \circ \theta_B = \theta_A \circ D^{A_1,\dots,A_p}_{f_1,\dots,f_q}$$

where $f_i \equiv f$, $A_i \equiv \operatorname{src} f, B_i \equiv \operatorname{trg} f$ are the domain and codomain of f, for every index in the relevant range, and $\theta_A : X \to D^{A,\dots,A}_{A,\dots,A}$ is a morphism in $\mathcal{D}$.

Notation 2.13. We write $\mathsf{Wd}^{(p,q)}_X(D)$ for the set of (p,q)-wedges of X over D, and similarly, $\mathsf{CWd}^{(p,q)}_X(D)$ for (p,q)-cowedges.

Proposition 2.14. Let $D : C^{(p,q)} \to \mathcal{D}$ be a functor.

WDF1) The assignment $X \mapsto \mathsf{Wd}^{(p,q)}_X(D)$ defines a presheaf

$$\mathsf{Wd}^{(p,q)}_{(-)}(D) : C^{\mathsf{op}} \to \mathsf{Set}.$$

WDF2) The assignment $X \mapsto \mathsf{CWd}^{(p,q)}_X(D)$ defines a functor

$$\mathsf{CWd}^{(p,q)}_{(-)}(D) : C \to \mathsf{Set}.$$

Proof. Item WDF1: Let $f : X \to Y$ be a morphism of C. We have a map

$$\begin{aligned} \mathsf{Wd}^{(p,q)}_f(D) \colon \mathsf{Wd}^{(p,q)}_Y(D) &\longrightarrow \mathsf{Wd}^{(p,q)}_X(D) \\ (Y \Longrightarrow\!\circ\, D) &\longmapsto \left(X \xrightarrow{f} Y \Longrightarrow\!\circ\, D\right), \end{aligned}$$

where we have used Proposition 2.7. As it is clear that this construction preserves composition and identities, we get our desired presheaf.

Item WDF2: This is dual to Item WDF1. □

Proposition 2.15. Let $D : \mathcal{C}^{(p,q)} \to \mathcal{D}$ be a functor.[1]

WDF'1) The assignment $D \mapsto \mathsf{Wd}^{(p,q)}_X(D)$ defines a functor

$$\mathsf{Wd}^{(p,q)}_X : \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D}) \to \mathcal{D}.$$

WDF'2) The assignment $D \mapsto \mathsf{CWd}^{(p,q)}_X(D)$ defines a functor

$$\mathsf{CWd}^{(p,q)}_X : \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D}) \to \mathcal{D}.$$

Proof. Let $\alpha : D \to D'$ be a natural transformation. We have a map

$$\begin{aligned} \mathsf{Wd}^{(p,q)}_X(\alpha) : \mathsf{Wd}^{(p,q)}_X(D) &\longrightarrow \mathsf{Wd}^{(p,q)}_X(D') \\ (X \Longrightarrow\!\!\circ\, D) &\longmapsto \left(X \Longrightarrow\!\!\circ\, D \overset{\alpha}{\Longrightarrow} D'\right), \end{aligned}$$

where we have used Proposition 2.7. As it is clear that this construction preserves composition and identities, we get our desired functor. □

Definition 2.16. Let $\theta : X \Longrightarrow\!\!\circ\, D$ be a (p,q)-wedge, and $\zeta : D \Longrightarrow\!\!\circ\, Y$ be a (p,q)-cowedge;

PC1) The (p,q)*-wedge post-composition natural transformation* associated to a (p,q)-wedge $\theta : X \Longrightarrow\!\!\circ\, D$ is the natural transformation

$$\theta_* : h_X \to \mathsf{Wd}^{(p,q)}_{(-)}(D)$$

consisting of the collection

$$\left\{\theta_{*,A} : h_X(A) \to \mathsf{Wd}^{(p,q)}_A(D) : A \in \mathcal{C}_o\right\},$$

where $\theta_{*,A}$ is the map

$$\begin{aligned} \mathcal{C}(X,A) &\longrightarrow \mathsf{Wd}^{(p,q)}_A(D) \\ \begin{bmatrix} A \\ f\downarrow \\ X \end{bmatrix} &\longmapsto \left(\Delta_A \xrightarrow{f} \Delta_X \Longrightarrow\!\!\circ\, D\right). \end{aligned}$$

PC2) The (p,q)*-cowedge precomposition natural transformation* associated to a (p,q)-cowedge $\zeta : D \Longrightarrow\!\!\circ\, Y$ is the natural transformation

$$\zeta^* : h^Y \to \mathsf{CWd}^{(p,q)}_{(-)}(D)$$

consisting of the collection

$$\left\{\zeta^*_A : h^Y(A) \to \mathsf{CWd}^{(p,q)}_A(D) : A \in \mathcal{C}_o\right\},$$

where ζ^*_A is the map

[1] More generally, the assignments $F, G \mapsto \mathrm{DiNat}^{(p,q)}(F,G)$ define functors

$$\begin{aligned} \mathrm{DiNat}^{(p,q)}(-,-) &: \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D})^{\mathsf{op}} \times \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D}) \to \mathsf{Set}, \\ \mathrm{DiNat}^{(p,q)}(F,-) &: \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D}) \to \mathsf{Set}, \\ \mathrm{DiNat}^{(p,q)}(-,G) &: \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathcal{D})^{\mathsf{op}} \to \mathsf{Set}. \end{aligned}$$

$$
\begin{aligned}
C(Y,A) &\longrightarrow \mathsf{CWd}^{(p,q)}_{A}(D) \\
\left[\begin{smallmatrix} Y \\ f\downarrow \\ A \end{smallmatrix}\right] &\longmapsto \left(D \Longrightarrow\!\circ\, \Delta_Y \xrightarrow{f} \Delta_A\right).
\end{aligned}
$$

The notion of dinaturality introduced in [MS20] is in fact more general, as [MS20, Definition 2.4] introduces what would be called here a (p,q)*-to-*(r,s)*-dinatural transformation*. Recall that in the setting of [MS20], the tuple of powers of C is unbiased. Their definition is as follows:

Definition 2.17. Let α, β be two multi-indices, and let $F : C^{\alpha} \to \mathcal{D}$, $G : C^{\beta} \to \mathcal{D}$ be functors. A *transformation* $\phi : F \to G$ *of type* $|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$ (with $n = |\boldsymbol{A}|$ a positive integer) is a family of morphisms in $\mathcal{D}$

$$\left(\phi_{\boldsymbol{A}} : F(\boldsymbol{A}\sigma) \to G(\boldsymbol{A}\tau)\right)_{\boldsymbol{A} \in C^n}.$$

This translates into a family $\phi_{A_1,\dots,A_n} : F(A_{\sigma 1}, \dots, A_{\sigma|\alpha|}) \to G(A_{\tau 1}, \dots, A_{\tau|\beta|}))$.

Notice that α and β are *different* multi-indices in this definition, and σ, τ need not be injective or surjective, so we may have repeated or unused variables.

Definition 2.18. Let $\phi = (\phi_{A_1,\dots,A_n}) : F \to G$ be a transformation. For $i \in \{1, \dots, n\}$, we say that ϕ is *dinatural in* A_i (or, more precisely, *dinatural in its i-th variable*) if and only if for all $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ objects of C and for all $f : A \to B$ in C the following hexagon commutes:

$$
\begin{tikzcd}
& F(\boldsymbol{A}[A/i]\sigma) \arrow[rr, "\phi_{\boldsymbol{A}[A/i]}"] && G(\boldsymbol{A}[A/i]\tau) \arrow[dr, "G(\boldsymbol{A}[A,f/i]\tau)"] & \\
F(\boldsymbol{A}[B,A/i]\sigma) \arrow[ur, "F(\boldsymbol{A}[f,A/i]\sigma)"] \arrow[dr, "F(\boldsymbol{A}[B,f/i]\sigma)"'] &&&& G(\boldsymbol{A}[A,B/i]\tau) \\
& F(\boldsymbol{A}[B/i]\sigma) \arrow[rr, "\phi_{\boldsymbol{A}[B/i]}"'] && G(\boldsymbol{A}[B/i]\tau) \arrow[ur, "G(\boldsymbol{A}[f,B/i]\tau)"'] &
\end{tikzcd}
$$

where $\boldsymbol{A}$ is the n-tuple $(A_1, \dots, A_n)$ of the objects above with an additional (unused in this definition) object A_i of C.

As far as higher arity co/wedges (i.e. higher arity dinatural transformations from/to a constant functor) are concerned, however, the notions of (p,q)-dinaturality and (p,q)-to-(r,s)-dinaturality agree and yield the same theory of higher arity co/ends.

3. Higher Arity Ends

3.1. Basic definitions

Definition 3.1. Let $D : C^{(p,q)} \to \mathcal{D}$ be a functor.

PQ1) The (p,q)*-end* of D is, if it exists, the pair $\left({}_{(p,q)}\int_{A \in C} D^{\boldsymbol{A}}_{\underline{\boldsymbol{A}}}, \omega\right)$ formed by an object

$$ {}_{(p,q)}\int_{A \in C} D^{\boldsymbol{A}}_{\underline{\boldsymbol{A}}} $$

of $\mathcal{D}$, and a (p,q)-wedge

$$\omega : {}_{(p,q)}\int_{A \in C} D^{\boldsymbol{A}}_{\underline{\boldsymbol{A}}} \Longrightarrow\!\circ\, D$$

for ${}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}$ over D, such that the (p,q)-wedge post-composition natural transformation

$$\omega_* : \mathsf{h}\left(-, {}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}\right) \Longrightarrow \mathsf{Wd}^{(p,q)}_{(-)}(D)$$

is a natural isomorphism.

PQ2) The (p,q)-*coend* of D is, if it exists, the pair $\left({}^{(p,q)}\int^{A\in C} D^{\underline{A}}_{\underline{A}}, \xi\right)$ formed by an object

$$^{(p,q)}\int^{A\in C} D^{\underline{A}}_{\underline{A}}$$

of $\mathcal{D}$, and a (p,q)-cowedge

$$\xi : D \Longrightarrow\!\circ\ {}^{(p,q)}\int^{A\in C} D^{\underline{A}}_{\underline{A}}$$

for ${}^{(p,q)}\int^{A} D^{\underline{A}}_{\underline{A}}$ under D, such that the (p,q)-cowedge post-composition natural transformation

$$\xi^* : \mathsf{h}\left({}^{(p,q)}\int^{A\in C} D^{\underline{A}}_{\underline{A}}, -\right) \Longrightarrow \mathsf{CWd}^{(p,q)}_{(-)}(D)$$

is a natural isomorphism.

We follow the customary abuse of notation of denoting the (p,q)-end of D as just the tip ${}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}$ of the terminal (p,q)-wedge ω. The object ${}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}$ can also be shortly denoted as ${}_{(p,q)}\int_{A} D$, or ${}_{(p,q)}\int D$.

Remark 3.2. The co/representability conditions of Definition 3.1 unwind as the following universal properties:

UPQ1) The (p,q)-end of D consists of a pair $\left({}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}, \omega\right)$ with
1) ${}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}$ an object of $\mathcal{D}$, and
2) ω a natural isomorphism with components

$$\omega_E : \mathcal{D}\left(E, {}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}\right) \cong \mathsf{Wd}^{(p,q)}_{E}(D).$$

The family of such morphisms of $\mathcal{D}$ is such that evaluating the isomorphism ω_E at the identity of $E = {}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}}$ gives a (p,q)-wedge

$$\left\{\omega_A : {}_{(p,q)}\int_{A\in C} D^{\underline{A}}_{\underline{A}} \to D^{\underline{A}}_{\underline{A}} \ : \ A \in C_o\right\}$$

indexed by the objects of C. This (p,q)-wedge has the following universal property:

($\star$) Given another such pair (E,θ), there exists a unique morphism $E \overset{\exists!}{\dashrightarrow} {}_{(p,q)}\int_{A} D^{\underline{A}}_{\underline{A}}$ filling the diagram

$$\begin{tikzcd}
 & E \arrow[ddl, bend right, "\theta_A"'] \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\theta_B"] & & \\
 & & {}^{(p,q)}\int_A D^{A}_{\underline{A}} \arrow[r, "\omega_B"] \arrow[dll, "\omega_A"'] & D^{B}_{\underline{B}} \arrow[dl, "D^{f}_{\underline{B}}"] \\
D^{A}_{\underline{A}} \arrow[rr, "D^{A}_{\underline{f}}"'] & & D^{B}_{\underline{A}} &
\end{tikzcd}$$

UPQ2) The (p,q)-coend of D consists of a pair $\left({}^{(p,q)}\!\int^{A\in C} D^{A}_{\underline{A}}, \xi\right)$ with
1) ${}^{(p,q)}\!\int^{A\in C} D^{A}_{\underline{A}}$ an object of $\mathcal{D}$, and
2) ξ a natural isomorphism with components

$$\xi_E : \mathcal{D}\left(E, {}_{(p,q)}\!\int_{A\in C} D^{A}_{\underline{A}}\right) \cong \mathsf{CWd}^{(p,q)}_{E}(D).$$

The family of such morphisms of $\mathcal{D}$ is such that evaluating the isomorphism ξ_D at the identity of $C = {}^{(p,q)}\!\int^{A\in C} D^{A}_{\underline{A}}$ gives a (p,q)-cowedge

$$\left\{\xi_A : D^{A}_{\underline{A}} \to {}^{(p,q)}\!\int^{A\in C} D^{A}_{\underline{A}} \;:\; A \in C_o\right\}$$

indexed by the objects of C. This (p,q)-cowedge has the following universal property:
($\star$) Given another such pair (C,ζ), there exists a unique morphism ${}^{(p,q)}\!\int^{A} D^{A}_{\underline{A}} \overset{\exists!}{\dashrightarrow} C$ filling the diagram

$$\begin{tikzcd}
 & E & & \\
 & & {}^{(p,q)}\int^A D^{A}_{\underline{A}} \arrow[ul, dashed, "\exists!"'] & D^{B}_{\underline{B}} \arrow[l, "\xi_B"'] \arrow[ull, bend right, "\zeta_B"'] \\
D^{A}_{\underline{A}} \arrow[uur, bend left, "\zeta_A"] \arrow[urr, "\xi_A"] & & D^{B}_{\underline{A}} \arrow[ll, "D^{A}_{\underline{f}}"] \arrow[ur, "D^{f}_{\underline{B}}"'] &
\end{tikzcd}$$

Remark 3.3. This means that the (p,q)-end of D is the terminal object of the category of wedges of D, whose morphisms $h : (\alpha : \Delta_X \Longrightarrow\!\!\circ\, D) \to (\beta : \Delta_Y \Longrightarrow\!\!\circ\, D)$ are defined as the morphisms $h : X \to Y$ of $\mathcal{D}$ such that for every $A \in C_o$ one has $\beta_A \circ h = \alpha_A$:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\alpha_A"'] & & Y \arrow[dl, "\beta_A"] \\
 & D^{A}_{\underline{A}}. &
\end{tikzcd}$$

Remark 3.3 can be dualised to define (p,q)-*co*ends as initial (p,q)-cowedges. This is straightforward, and we leave it to the reader to spell out.

In the following proposition, we will make use of the (p,q)-diagonal functor $\Delta_{p,q}$ introduced in Definition 1.5, and duplicated in the following

Notation 3.4. We say that

- A functor $F : \mathcal{C}^{(p+r,q+s)} \longrightarrow \mathcal{D}$ is (r,s)-*dummy* if it factors through the canonical projection $\pi_{r,s} : \mathcal{C}^{(p+r,q+s)} \longrightarrow \mathcal{C}^{(p,q)}$ that cancels the last r contravariant components, and the last s covariant components.
- Given a functor $F : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ we define its (r,s)-*dummification* to be the composition $\eth^r_s F : \mathcal{C}^{(p+r,q+s)} \xrightarrow{\pi_{r,s}} \mathcal{C}^{(p,q)} \xrightarrow{F} \mathcal{D}$; this promotes every functor of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ to an (r,s)-dummy one.

It's immediate that every functor that is mute in *some* of its variables can be made into an (r,s)-dummy one by suitably reshuffling its arguments.

Proposition 3.5 (Properties of (p,q)-ends and (p,q)-coends)**.** Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor.

PE1) *Functoriality.* Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor. The assignments $D \mapsto {}_{(p,q)}\!\int_A D^{\underline{A}}_{\underline{A}}, {}^{(p,q)}\!\int^A D^{\underline{A}}_{\underline{A}}$ define functors

$$\begin{aligned} {}_{(p,q)}\!\int_{A\in\mathcal{C}} &: \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \to \mathcal{D}, \\ {}^{(p,q)}\!\int^{A\in\mathcal{C}} &: \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \to \mathcal{D} \end{aligned}$$

with domain the category of functors from $\mathcal{C}$ of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ to $\mathcal{D}$ and natural transformations between them.

PE2) *(p,q)-Wedges and (p,q)-diagonals.* For each $X \in \mathcal{C}_o$ we have natural bijections

$$\begin{aligned} \mathsf{Wd}^{(p,q)}_{(-)}(D) &\cong \mathsf{Wd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big), \\ \mathsf{CWd}^{(p,q)}_{(-)}(D) &\cong \mathsf{CWd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big). \end{aligned}$$

where $\Delta_{p,q}$ is the functor introduced in Definition 1.5.

PE3) *(p,q)-Ends as ordinary ends.* We have natural isomorphisms

$$\begin{aligned} {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \int_{A\in\mathcal{C}} \Delta^{(p,q)}_*(D)^A_A, \\ {}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \int^{A\in\mathcal{C}} \Delta^{(p,q)}_*(D)^A_A. \end{aligned}$$

where $\Delta_{p,q}$ is the functor introduced in Definition 1.5. In other words, the (p,q)-end functor factors as a composition

$$\mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \xrightarrow{\Delta^{(p,q)}_*} \mathsf{Cat}\big(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\big) \xrightarrow{\quad \int_A \quad} \mathcal{D},$$

and similarly so do (p,q)-coends.

PE4) *(p,q)-Ends as limits.* The (p,q)-end and (p,q)-coend of D fit respectively into an equaliser and into a coequaliser diagram

$$
{}_{(p,q)}\int_{A\in C} D^A_A \longrightarrow \prod_{A\in C_o} D^A_A \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \prod_{u\colon A\to B} D^A_B
$$

$$
\coprod_{u\colon A\to B} D^A_B \overset{\lambda'}{\underset{\rho'}{\rightrightarrows}} \coprod_{A\in C_o} D^A_A \longrightarrow {}^{(p,q)}\int^{A\in C} D^A_A
$$

for suitable maps $\lambda, \rho, \lambda', \rho'$, induced by the morphisms D^A_u, D^u_B.

PE5) *(p,q)-Ends as limits, again.* We have natural isomorphisms

$$
{}_{(p,q)}\int_{A\in C} D^A_A \cong \lim\Big(\mathsf{Tw}(C) \xrightarrow{\Sigma_{p,q}} C^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),
$$

$$
{}^{(p,q)}\int^{A\in C} D^A_A \cong \operatorname{colim}\Big(\mathsf{Tw}(C) \xrightarrow{\Sigma_{p,q}} C^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),
$$

where $\Sigma_{p,q}\colon \mathsf{Tw}(C) \to C^{(p,q)}$ is the composition $\Delta^{(p,q)} \circ \Sigma$, with Σ the usual forgetful functor from $\mathsf{Tw}(C)$ to $C^{\mathrm{op}} \times C$. Explicitly, $\Sigma^{(p,q)}$ is the functor

$$
\begin{aligned}
\mathsf{Tw}(C) &\longrightarrow C^{(p,q)},\\
\begin{bmatrix} A \\ f\downarrow \\ B \end{bmatrix} &\longmapsto (\boldsymbol{A}, \boldsymbol{B}),\\
\begin{bmatrix} A & \xrightarrow{f} & B \\ \phi\uparrow & & \downarrow\psi \\ C & \xrightarrow[g]{} & D \end{bmatrix} &\longmapsto (\boldsymbol{\phi}, \boldsymbol{\psi}).
\end{aligned}
$$

PE6) *(p,q)-Ends as limits, yet again.* There exists a category $\mathsf{Tw}^{(p,q)}(C)$ together with a universal fibration

$$
\Sigma\colon \mathsf{Tw}^{(p,q)}(C) \twoheadrightarrow C^{(p,q)}
$$

inducing natural isomorphisms

$$
{}_{(p,q)}\int_{A\in C} D^A_A \cong \lim\Big(\mathsf{Tw}^{(p,q)}(C) \xrightarrow{\Sigma} C^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),
$$

$$
{}^{(p,q)}\int^{A\in C} D^A_A \cong \operatorname{colim}\Big(\mathsf{Tw}^{(p,q)}(C) \xrightarrow{\Sigma} C^{(p,q)} \xrightarrow{D} \mathcal{D}\Big).
$$

PE7) *(p,q)-Ends as $(p+r,q+s)$-ends.* we have

$$
{}_{(p,q)}\int_{A\in C} D^A_A \cong {}_{(p+r,q+s)}\int_{A\in C} \eth^r_s(D)^A_A,
$$

$$
{}^{(p,q)}\int^{A\in C} D^A_A \cong {}^{(p+r,q+s)}\int^{A\in C} \eth^r_s(D)^A_A,
$$

where $\eth^r_s(-)$ is the (r,s)-dummification introduced in Notation 3.4.

PE8) *Commutativity of (p,q)-ends with homs.* We have natural isomorphisms

$$
\mathcal{D}\Big(-, {}_{(p,q)}\int_{A\in C} D^A_A\Big) \cong {}_{(p,q)}\int_{A\in C} \mathcal{D}\Big(-, D^A_A\Big)
$$

$$\mathcal{D}\left({}^{(p,q)}\!\!\int^{A\in C} D^A_A, -\right) \cong {}_{(q,p)}\!\int_{A\in C} \mathcal{D}\Big(D^A_A, -\Big).$$

Proof. We provide proofs of the above statements for (p, q)-ends only; the case of (p, q)-coends is dual.

Item PE1: Let $\alpha : D \Longrightarrow D'$ be a natural transformation and consider the composition

$$\mathsf{h}\Big(-, {}_{(p,q)}\!\textstyle\int_A D^A_A\Big) \xrightarrow{(\omega_D)_*} \mathsf{Wd}^{(p,q)}_X(D) \xrightarrow{\mathsf{Wd}^{(p,q)}_X(\alpha)} \mathsf{Wd}^{(p,q)}_X(D') \xrightarrow{(\omega_{D'})_*} \mathsf{h}\Big(-, {}_{(p,q)}\!\textstyle\int_A (D')^A_A\Big),$$

where we have used Proposition 2.7. This gives us a morphism between the representable functors associated to the (p, q)-ends of D and D'. The Yoneda lemma now yields a morphism

$${}_{(p,q)}\!\int_A D^A_A \to {}_{(p,q)}\!\int_A (D')^A_A.$$

between the (p, q)-ends. Since all constructions involved are functorial, it follows that (p, q)-ends preserve composition and identities, and hence define a functor.

Item PE2: This is the special case of Proposition 2.8 where $F = \Delta_{(-)}$.

Item PE3: We have

$$\mathsf{h}\left(-, {}_{(p,q)}\!\int_{A\in C} D^A_A\right) \cong \mathsf{Wd}^{(p,q)}_{(-)}(D) \cong \mathsf{Wd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big) \cong \mathsf{h}\left(-, \int_{A\in C} \Delta^{(p,q)}_*(D)^A_A\right),$$

from which the result follows from the Yoneda lemma.

Item PE4: This is again a combination of Item PE3 with the "products-and-equalisers" formula for ends.

Item PE5: This is just a combination of Item PE3 with the usual formula computing ends as limits of diagrams from the twisted arrow category.

Item PE6: This problem is studied in Section 5.3, with the statement of Item PE6 being proved in Proposition 5.18.

Item PE7: We have

$$\mathsf{h}\left(-, {}_{(p,q)}\!\int_{A\in C} D^A_A\right) \cong \mathsf{Wd}^{(p,q)}_{(-)}(D) \cong \mathsf{Wd}^{(p+r,q+s)}_{(-)}(\check{\eth}^r_s(D)) \cong \mathsf{h}\left(-, {}_{(p+r,q+s)}\!\int_{A\in C} \check{\eth}^r_s(D)^A_A\right),$$

from which the result follows from the Yoneda lemma.

Item PE8: This follows from Item PE3 and the fact that co/ends commute with homs. □

3.2. Adjoints and the Fubini rule

The scope of this section is to prove that (p, q)-ends are right adjoints (and, dually, that (p, q)-coends are left adjoints), and from this to derive a Fubini rule. Although the proofs are elementary, these properties of (p, q)-ends are more subtle to assess and require a certain amount of new terminology. Thus we separate them from the above list. Let's start with a simple definition:

Definition 3.6 (The $\hom_\Pi$ functor)**.** Let $p, q \geq 1$ be natural numbers; let's define a functor

$$\hom_{\Pi,p,q} : C^{(p,q)} \to \mathsf{Set}$$

by sending a pair of tuples $(\underline{A}, \underline{B})$ to the product $\prod^{p,q}_{i,j=1} \hom_C(A_i, B_j)$, namely to the iterated product $\prod^p_{i=1} \prod^q_{j=1}(A_i, B_j)$.

Remark 3.7. If C has finite products and finite coproducts, then we have a canonical factorisation

$$\begin{tikzcd} C^{(p,q)} \arrow[r, "\mathsf{M}_{p,q}"] \arrow[rr, bend right, "\hom_{\Pi,p,q}"'] & C^{\mathsf{op}} \times C \arrow[r, "\hom"] & \mathsf{Set}. \end{tikzcd}$$

where $\mathsf{M}_{p,q}$ is the functor of Notation 1.7.

Theorem 3.8. Let $G : \mathcal{C}^{(p,q)} \to \mathsf{Set}$ be a functor. There is an isomorphism, natural in G,

$$\mathrm{DiNat}^{(p,q)}(\mathrm{pt}, G) \cong \mathrm{Nat}(\hom_{\Pi,p,q}, G)$$

The proof of Theorem 3.8 requires several lemmas (Lemmas 3.9, 3.10 and 3.14), which we now discuss.

Lemma 3.9. Let $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$ be functors. We have a natural isomorphism

$$\mathrm{DiNat}(F, G) \cong \mathrm{Nat}(\mathsf{h}, [F, G]).$$

Proof. The proof is a formal derivation and mimics Example 4.6:

$$\begin{aligned}\mathrm{DiNat}(F, G) &\cong \int_{A\in\mathcal{C}} \hom_{\mathcal{D}}\left(F^A_A, G^A_A\right)\\ &\cong \int_{A,B\in\mathcal{C}} \left[\hom_{\mathcal{C}}(A,B), \hom_{\mathcal{D}}\left(F^B_A, G^A_B\right)\right]\\ &\cong \mathrm{Nat}\left(\hom_{\mathcal{C}}(-_1,-_2), \hom_{\mathcal{D}}\left(F^{-_2}_{-_1}, G^{-_1}_{-_2}\right)\right).\end{aligned}$$

□

Lemma 3.10. Let $G \colon \mathcal{C}^{(q,p)} \to \mathcal{D}$ be a functor. If $\mathcal{D}$ is cocomplete, then

$$\mathrm{DiNat}^{(p,q)}\left(\Delta_{\mathrm{pt}}, G\right) \cong \mathrm{Nat}\left(\mathsf{Lan}_{\Delta_{q,p}}\mathsf{h}, G\right).$$

Proof. We have

$$\begin{aligned}\mathrm{DiNat}^{(p,q)}\left(\Delta_{\mathrm{pt}}, G\right) &\cong \mathrm{DiNat}\left(\Delta^*_{p,q}\Delta_{\mathrm{pt}}, \Delta^*_{q,p}G\right),\\ &\cong \mathrm{Nat}\left(\mathsf{h}, \left[\Delta^*_{p,q}\Delta_{\mathrm{pt}}, \Delta^*_{q,p}G\right]\right).\\ &\cong \mathrm{Nat}\left(\mathsf{h}, \left[\Delta'_{\mathrm{pt}}, \Delta^*_{q,p}G\right]\right)\\ &\cong \mathrm{Nat}\left(\mathsf{h}, \Delta^*_{q,p}G\right)\\ &\cong \mathrm{Nat}\left(\mathsf{Lan}_{\Delta_{q,p}}\mathsf{h}, G\right).\end{aligned}$$

□

Remark 3.11 (Computing $\mathsf{Lan}_{\Delta_{q,p}}\mathsf{h}$)**.** We have

$$\begin{aligned}\mathsf{Lan}_{\Delta_{q,p}}\mathsf{h} &\cong \int^{A,B\in\mathcal{C}} \hom_{\mathcal{C}^{(q,p)}}\left(\Delta_{q,p}((A,B)); (-,-)\right) \odot \mathsf{h}^A_B\\ &\cong \int^{A,B\in\mathcal{C}} \hom_{\mathcal{C}^{(q,p)}}\left((\boldsymbol{A},\boldsymbol{B}); (-,-)\right) \odot \mathsf{h}^A_B\\ &\cong \int^{A\in\mathcal{C}} \hom_{\mathcal{C}^{(q,p)}}\left((\boldsymbol{A},\boldsymbol{A}); (-,-)\right)\\ &\overset{\mathrm{def}}{=} \int^{A\in\mathcal{C}} \mathsf{h}^{-_1}_A \times\cdots\times \mathsf{h}^{-_q}_A \times \mathsf{h}^A_{-_1} \times\cdots\times \mathsf{h}^A_{-_p},\end{aligned} \tag{3.12}$$

meaning the end of

$$(A,B) \mapsto \mathsf{h}^{-_1}_B \times\cdots\times \mathsf{h}^{-_q}_B \times \mathsf{h}^A_{-_1} \times\cdots\times \mathsf{h}^A_{-_p}. \tag{3.13}$$

Lemma 3.14. There is an isomorphism of functors

$$\mathsf{Lan}_{\Delta_{q,p}}\mathsf{h} \cong \hom_{\Pi,p,q}. \tag{3.15}$$

Proof. We shall prove the case $(p, q) = (2, 1)$, as the general case is analogous. Namely, we claim that

$$\begin{aligned}\int^{X\in C} \mathsf{h}^{-_1}_{X} \times \mathsf{h}^{X}_{-_2} \times \mathsf{h}^{X}_{-_3} &\overset{\text{(Remark 3.11)}}{\cong} \mathsf{Lan}_{\Delta_{1,2}}\mathsf{h} \\ &\cong \hom_{\Pi,2,1} \\ &\overset{\text{def}}{=} \mathsf{h}^{-_1}_{-_2} \times \mathsf{h}^{-_1}_{-_3}.\end{aligned}$$

Fix $A, B, C \in \mathrm{Obj}(C)$. We will show that the diagram

$$\coprod_{u\colon X\to Y} \mathsf{h}^{A}_{X} \times \mathsf{h}^{Y}_{B} \times \mathsf{h}^{Y}_{C} \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \coprod_{X\in C} \mathsf{h}^{A}_{X} \times \mathsf{h}^{X}_{B} \times \mathsf{h}^{X}_{C} \xrightarrow{\sigma} \mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{C},$$

where λ and ρ are the maps induced by the maps given by

$$\lambda\left(\left[\begin{smallmatrix}X\\ u\downarrow\\ Y\end{smallmatrix}\right];\left[\begin{smallmatrix}A\\ f\downarrow\\ X\end{smallmatrix}\right],\left[\begin{smallmatrix}Y\\ g\downarrow\\ B\end{smallmatrix}\right],\left[\begin{smallmatrix}Y\\ h\downarrow\\ C\end{smallmatrix}\right]\right) = (f, g\circ u, h\circ u),$$

$$\rho\left(\left[\begin{smallmatrix}X\\ u\downarrow\\ Y\end{smallmatrix}\right];\left[\begin{smallmatrix}A\\ f\downarrow\\ X\end{smallmatrix}\right],\left[\begin{smallmatrix}Y\\ g\downarrow\\ B\end{smallmatrix}\right],\left[\begin{smallmatrix}Y\\ h\downarrow\\ C\end{smallmatrix}\right]\right) = (u\circ f, g, h),$$

and σ is the map induced by the maps

$$\sigma_{X,X,X}\left(\left[\begin{smallmatrix}A\\ f\downarrow\\ X\end{smallmatrix}\right],\left[\begin{smallmatrix}X\\ g\downarrow\\ B\end{smallmatrix}\right],\left[\begin{smallmatrix}X\\ h\downarrow\\ C\end{smallmatrix}\right]\right) \overset{\text{def}}{=} (g\circ f, h\circ f),$$

is a coequaliser diagram. Firstly, note that σ indeed coequalises λ and ρ, as

$$\begin{aligned}\sigma(\lambda(u, f, g, h)) &= \sigma(f, g\circ u, h\circ u) \\ &= ((g\circ u)\circ f, (h\circ u)\circ f) \\ &= (g\circ(u\circ f), h\circ(u\circ f)) \\ &= \sigma(u\circ f, g, h) \\ &= \sigma(\rho(u, f, g, h)).\end{aligned}$$

Now, given another morphism

$$\coprod_{X\in C} \mathsf{h}^{A}_{X} \times \mathsf{h}^{X}_{B} \times \mathsf{h}^{X}_{C} \xrightarrow{\zeta} E,$$

coequalising (λ, ρ), i.e. such that

$$\zeta(f, g\circ u, h\circ u) = \zeta(u\circ f, g, h), \tag{3.16}$$

we claim that there exists a unique morphism $\bar{\zeta}\colon \mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{C} \overset{\exists!}{\dashrightarrow} E$ making the diagram

$$\begin{tikzcd}
\coprod\limits_{u\colon X\to Y} \mathsf{h}^{A}_{X} \times \mathsf{h}^{Y}_{B} \times \mathsf{h}^{Y}_{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & \coprod\limits_{X\in C} \mathsf{h}^{A}_{X} \times \mathsf{h}^{X}_{B} \times \mathsf{h}^{X}_{C} \arrow[r, "\sigma"] \arrow[dr, "\zeta"'] & \mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{C}, \arrow[d, dashed, "\bar{\zeta}"] \\
 & & E,
\end{tikzcd} \tag{3.17}$$

commute. Indeed:

TRP1) *Existence.* For each pair $\left(\left[\begin{smallmatrix}A\\ f\downarrow\\ B\end{smallmatrix}\right],\left[\begin{smallmatrix}A\\ g\downarrow\\ C\end{smallmatrix}\right]\right)$ in $\mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{C}$, we define[2]

$$\bar{\zeta}(f, g) \overset{\text{def}}{=} \zeta(\mathrm{id}_A, f, g).$$

[2]Note that in writing $\zeta(\mathrm{id}_A, f, g)$ we are applying ζ to the triple (id_A, f, g) in the component $\mathsf{h}^{A}_{A} \times \mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{C}$ of $\coprod_{X\in C} \mathsf{h}^{A}_{X} \times \mathsf{h}^{X}_{B} \times \mathsf{h}^{X}_{C}$.

TRP2) *Uniqueness.* Let $\widetilde{\zeta}\colon \mathsf{h}^A_B \times \mathsf{h}^A_C \to E$ be another morphism making Diagram 3.17 commute. Then, for each pair $\left(\left[\begin{smallmatrix} A \\ f\downarrow \\ B \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ g\downarrow \\ C \end{smallmatrix}\right]\right)$ in $\mathsf{h}^A_B \times \mathsf{h}^A_C$, we have

$$\begin{aligned}
\widetilde{\zeta}(f,g) &= \widetilde{\zeta}(f \circ \mathrm{id}_A, g \circ \mathrm{id}_A) \\
&\overset{\text{def}}{=} \widetilde{\zeta}(\sigma(\mathrm{id}_A, f, g)) \\
&= \zeta(\mathrm{id}_A, f, g) && \text{by the commutativity of Diagram 3.17,} \\
&\overset{\text{def}}{=} \overline{\zeta}(f,g).
\end{aligned}$$

TRP3) *Diagram 3.17 commutes.* For each triple $\left(\left[\begin{smallmatrix} A \\ f\downarrow \\ X \end{smallmatrix}\right], \left[\begin{smallmatrix} X \\ g\downarrow \\ B \end{smallmatrix}\right], \left[\begin{smallmatrix} X \\ h\downarrow \\ C \end{smallmatrix}\right]\right)$ in $\coprod_{X\in C} \mathsf{h}^A_X \times \mathsf{h}^X_B \times \mathsf{h}^X_C$, we have

$$\begin{aligned}
\overline{\zeta}(\sigma(f,g,h)) &= \overline{\zeta}(g \circ f, h \circ f) \\
&\overset{\text{def}}{=} \zeta(\mathrm{id}_A, g \circ f, h \circ f) \\
(\text{Equation (3.16)}) &= \zeta(\mathrm{id}_A \circ f, g, h) \\
&= \zeta(f,g,h).
\end{aligned}$$
□

Taken all together, Lemma 3.9, Lemma 3.10, and Lemma 3.14 yield Theorem 3.8.

Corollary 3.18 ((p,q)-co/ends as weighted co/limits)**.** Let $G : C^{(p,q)} \to \mathsf{Set}$ be a functor. We have functorial isomorphisms[3]

$$ {}_{(p,q)}\!\int_{A\in C} G^A_A \cong \lim^{\hom_{\Pi,p,q}} G, \qquad {}^{(p,q)}\!\int^{A\in C} G^A_A \cong \operatorname{colim}^{\hom_{\Pi,p,q}} G, $$

Proof. We have isomorphisms, natural in $A \in \mathcal{D}$

$$\begin{aligned}
\mathcal{D}\left(A, {}_{(p,q)}\!\int_A G^A_A\right) &\overset{\text{def}}{=} \mathrm{DiNat}^{(p,q)}\left(\mathrm{pt}, \mathcal{D}(A,G)\right) \\
\text{Theorem 3.8} &\cong \mathrm{Nat}\left(\hom_{\Pi,p,q}, \mathcal{D}(A,G)\right) \\
&\overset{\text{def}}{=} \mathcal{D}(A, \lim^{\hom_{\Pi,p,q}} G).
\end{aligned}$$

The result then follows from the Yoneda lemma. A dual argument yields the second identity. □

From Corollary 3.18, a general fact about weighted limits ([Lor15, Lemma 4.3.1]) yields

Corollary 3.19. There is an adjunction

$$\begin{tikzcd} \mathsf{Cat}(C^{(p,q)}, \mathcal{D}) \arrow[r, shift right, "{}_{(p,q)}\int_C"'] \arrow[r, phantom, "\perp"] & \mathcal{D}, \arrow[l, shift right, "\hom_{\Pi,p,q} \odot -"'] \end{tikzcd}$$

where the left adjoint $\hom_{\Pi,p,q} \odot -$ is defined by $D \mapsto \left((\underline{A}, \underline{B}) \mapsto \hom_{\Pi,p,q}(\underline{A}, \underline{B}) \odot D\right)$.

Dually, there is an adjunction

$$\begin{tikzcd} \mathsf{Cat}(C^{(p,q)}, \mathcal{D}) \arrow[r, shift left, "{}^{(p,q)}\int^C"] \arrow[r, phantom, "\perp"] & \mathcal{D}, \arrow[l, shift left, "\hom_{\Pi,p,q} \pitchfork -"] \end{tikzcd}$$

where the left adjoint $\hom_{\Pi,p,q} \odot -$ is defined by $D \mapsto \left((\underline{A}, \underline{B}) \mapsto \hom_{\Pi,p,q}(\underline{A}, \underline{B}) \odot D\right)$, and the right adjoint $\hom_{\Pi,p,q} \pitchfork -$ is defined by $D \mapsto \left((\underline{A}, \underline{B}) \mapsto \hom_{\Pi,p,q}(\underline{A}, \underline{B}) \pitchfork D\right)$.

[3]For $(p,q) = (1,1)$, this amounts to the well-known statement that the co/end of $T : C^{\mathrm{op}} \times C$ is the weighted co/limit of T by the hom functor $\hom_C(-,-) : C^{\mathrm{op}} \times C \to \mathsf{Set}$; see [Kel05, Section 3.10].

Lemma 3.20 (Shishido identity, first form)**.** The product-hom functor of Definition 3.6 satisfies the Shishido identity:[4]

$$\hom_{\Pi,p,q} \times \hom_{\Pi,r,s} \cong \hom_{\Pi,r,s} \times \hom_{\Pi,p,q} \cong \prod_{i=1}^{p+r} \prod_{j=1}^{q+s} \mathsf{h}^{(-_i,-_i)}_{(-_j,-_j)}.$$

Proof. The first isomorphism is clear. For the second one, let $(\underline{A}, \underline{B}) \in C^{(p,q)}$ and $\underline{A}', \underline{B}' \in C^{(r,s)}$; we then have natural isomorphisms

$$\begin{aligned}
\hom_{\Pi,p,q} \times \hom_{\Pi,r,s}((\underline{A}; \underline{A}'), (\underline{B}; \underline{B}')) &= \prod_{i,j=1}^{p,q} \hom_C (A_i, B_j) \times \prod_{h,k=1}^{r,s} \hom_C (A'_h, B'_k) \\
&\cong \prod_{i,j=1}^{p,q} \prod_{h,s}^{r,s} \hom_C (A_i, B_j) \times \hom_C (A'_h, B'_k) \\
&\cong \prod_{i,j=1}^{p+r,q+s} \hom_C ((\underline{A}; \underline{A}')_i, (\underline{B}; \underline{B}')_j) .
\end{aligned}$$

where the tuple $\underline{A}; \underline{A}'$ is the juxtaposition of $\underline{A}, \underline{A}'$. ◻

Theorem 3.21 (The Fubini Rule)**.** Let $D : \mathcal{A}^{(p,q)} \times \mathcal{B}^{(r,s)} \longrightarrow \mathcal{D}$ be a functor. Then

$$_{(p+r,q+s)}\int_{(A,B)} D^{A,B}_{A,B} \cong {}_{(p,q)}\int_A {}_{(r,s)}\int_B D^{A,B}_{A,B} \cong {}_{(r,s)}\int_B {}_{(p,q)}\int_A D^{A,B}_{A,B}, \tag{3.22}$$

$$^{(p+r,q+s)}\int^{(A,B)} D^{A,B}_{A,B} \cong {}^{(p,q)}\int^A {}^{(r,s)}\int^B D^{A,B}_{A,B} \cong {}^{(r,s)}\int^B {}^{(p,q)}\int^A D^{A,B}_{A,B} \tag{3.23}$$

as objects of $\mathcal{D}$, meaning that any of these expressions exist if and only if the others do, and, if so, they are are all canonically isomorphic.

Proof. To prove that three expressions in Equation (3.22) are isomorphic, it suffices to show their adjoints (Corollary 3.19)

$$\begin{gathered}
(\hom_{\Pi,p,q} \pitchfork \hom_{\Pi,r,s}) \pitchfork (-) \\
(\hom_{\Pi,r,s} \pitchfork \hom_{\Pi,p,q}) \pitchfork (-) \\
\left(\prod_{i=1}^{p+r} \prod_{j=1}^{q+s} \mathsf{h}^{(-_i,-_i)}_{(-_j,-_j)} \right) \pitchfork (-)
\end{gathered}$$

are isomorphic, since adjoints are unique. As $(A \pitchfork B) \pitchfork C \cong (A \times B) \pitchfork C$, this follows from Lemma 3.20.

A suitably dualised argument yields the result for higher arity coends. ◻

Remark 3.24 (Fubini does not reduce arity)**.** Note that p, q, r, s can't be broken further: given a functor G of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right]$, its (p, q)-end isn't in general expressible in terms of $(p - r, q - s)$-ends for suitable $r, s \geq 1$. This confirms the fact that iterated ends *are not* higher arity ends. Instead, higher arity ends are particular ends.

That is, Theorem 3.21 does not allow us to reduce the arity of a higher arity co/end when $\mathcal{A} = \mathcal{B}$:

$$_{(p,q)}\int_A {}_{(r,s)}\int_B D^{A,B}_{A,B} \cong {}_{(p+r,q+s)}\int_{(A,B)\in\mathcal{A}\times\mathcal{A}} D^{A,B}_{A,B} \ncong {}_{(p+r,q+s)}\int_{A\in C} D^{A}_{A}.$$

[4] Shishido Baiken is the name of a Japanese swordsman (his existence is attested in the *Nitenki* written in 1776, but the reliability of the text is currently object of debate). Baiken was a skilled master of *kusarigama-jutsu* and, according to the legend, lost a duel (and his life) with Miyamoto Musashi.

This is already apparent from the classical Fubini rule, where, given a functor $T\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \times \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \to \mathcal{D}$ with $\mathcal{C} = \mathcal{E}$, we have once again

$$\int_{(A,B)\in\mathcal{C}\times\mathcal{C}} T((A,B),(A,B)) \ncong \int_{A\in\mathcal{C}} T(A,A,A,A).$$

The main point in both cases is that we are "integrating" over a pair (A,B), and not over a single variable A.

From the point of view of adjoints, we have in (e.g.) the $(p,q) = (1,1)$ case

$$(-) \odot \left(\mathsf{h}^{-1}_{-3} \times \mathsf{h}^{-1}_{-4} \times \mathsf{h}^{-2}_{-3} \times \mathsf{h}^{-2}_{-4}\right) \dashv {}^{(2,2)}\int^{A\in\mathcal{C}} D^{A,A}_{A,A}$$

$$(-) \odot \underbrace{\mathsf{h}^{(-1,-2)}_{(-3,-4)}}_{\mathsf{h}^{-1}_{-3}\times\mathsf{h}^{-2}_{-4}} \dashv \int^{(A,B)\in\mathcal{C}\times\mathcal{C}} D^{(A,B)}_{(A,B)},$$

and of course

$$\mathsf{h}^{-1}_{-3} \times \mathsf{h}^{-1}_{-4} \times \mathsf{h}^{-2}_{-3} \times \mathsf{h}^{-2}_{-4} \neq \mathsf{h}^{(-1,-2)}_{(-3,-4)} = \mathsf{h}^{-1}_{-3} \times \mathsf{h}^{-2}_{-4},$$

so $\int^{A\in\mathcal{C}} D^{A,A}_{A,A}$ and $\int^{(A,B)\in\mathcal{C}\times\mathcal{C}} D^{(A,B)}_{(A,B)}$ are different as well.

4. Examples: a session of callisthenics

4.1. Examples arranged by dimension

The first roundup of examples is a series of sanity checks:

Example 4.1 ($(0,0)$-co/ends)**.** For the case where $p = q = 0$, we look at functors of the form $D\colon \mathsf{pt} \to \mathcal{D}$, where pt is the terminal category. It is evident that such a functor corresponds precisely to an object of $\mathcal{D}$, a $(0,0)$-wedge corresponds to the identity on that object, and the $(0,0)$-end of D is precisely that object.

Example 4.2 ($(1,0)$- and $(0,1)$-co/ends)**.** When $(p,q) = (1,0)$, we consider functors of the form $D\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$, and we see from the universal property of (p,q)-ends that the $(1,0)$-end of D is the limit of D. Similarly, the $(0,1)$-end of a functor $D\colon \mathcal{C} \to \mathcal{D}$ is again the limit of D.

In particular, starting with a functor $D\colon \mathcal{C} \to \mathcal{D}$ and passing to the opposite functor $D^{\mathsf{op}}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$, we get isomorphisms

$${}_{(0,1)}\int_{A\in\mathcal{C}} D = \lim(D),$$

$${}_{(1,0)}\int_{A\in\mathcal{C}} D^{\mathsf{op}} = \operatorname{colim}(D).$$

Example 4.3 ($(1,1)$-co/ends)**.** Let $(p,q) = (1,1)$ and consider a diagram in $\mathcal{D}$ of the form $D\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$. Again from the universal property of (p,q)-ends, we see that $(1,1)$-ends are nothing but ordinary ends. That is:

$${}_{(1,1)}\int_{A\in\mathcal{C}} D^A_A = \int_{A\in\mathcal{C}} D^A_A.$$

Furthermore, $(n,0)$-co/ends and $(0,n)$-co/ends are just suitable co/limits:

Example 4.4 ($(2,0)$-, $(0,2)$-co/ends; $(n,0)$- and $(0,n)$-co/ends)**.** Given a diagram $D\colon \mathcal{C}^2 \to \mathcal{D}$, we have

$${}_{(0,2)}\int_{A\in\mathcal{C}} D^{A,A} = \lim\left(D \circ \Delta_{\mathcal{C}}\right), \qquad {}^{(0,2)}\int^{A\in\mathcal{C}} D^{A,A} = \operatorname{colim}\left(D \circ \Delta_{\mathcal{C}}\right),$$

where $\Delta_C\colon C \to C \times C$ is the diagonal functor of C in the Cartesian monoidal structure of Cat.

A similar argument yields, for a diagram $D\colon C^n \to \mathcal{D}$:

$$ {}_{(0,n)}\!\int_{A\in C} D^A = \lim\left(D \circ \Delta_C^n\right), \qquad {}^{(0,n)}\!\int^{A\in C} D^A = \operatorname{colim}\left(D \circ \Delta_C^n\right). $$

We consider next the first nontrivial example:

Example 4.5 ($(2,1)$- and $(1,2)$-co/ends)**.** Given a functor $T : C^{-2} \times C \to \mathcal{D}$ let's flesh out what a $(2,1)$-wedge is: it consists of an object X endowed with maps

$$ \omega_A : X \longrightarrow T(AA;A) $$

with the property that, for every $f : A \to B$ in C, the square

$$ \begin{tikzcd} X \arrow[r, "\omega"] \arrow[d, "\omega"'] & T(AA;A) \arrow[d, "T(11f)"] \\ T(BB;B) \arrow[r, "T(ff1)"'] & T(AA;B) \end{tikzcd} $$

The $(2,1)$-end of T is the terminal object in the category $\mathsf{Wd}_{(2,1)}(T)$ of wedges for T.

As a particular example, let C be a Cartesian category. Let us consider the functor

$$ \begin{aligned} T = \hom(\_{}_1 \times \_{}_2, \_{}_3) : C^{\mathrm{op}} \times C^{\mathrm{op}} \times C &\longrightarrow \mathsf{Set} \\ (A,B;C) &\longmapsto C(A \times B, C) \end{aligned} $$

What is a $(2,1)$-wedge for T? It consists of a set X, and a family of functions $\omega_A : X \to C(A \times A, A)$ with the property that for each $f : A \to B$, the square

$$ \begin{tikzcd} X \arrow[r] \arrow[d] & C(A\times A, A) \arrow[d, "f_*"] \\ C(B\times B, B) \arrow[r, "(f\times f)^*"'] & C(A\times A, B) \end{tikzcd} $$

commutes. In other words, each $\omega_A(x)$ is a morphism $A \times A \to A$ in C with the property that each $f : A \to B$ is a "homomorphism" with respect to $\omega_A(x), \omega_B(x)$:

$$ \begin{tikzcd} A\times A \arrow[r, "\omega_A(x)"] \arrow[d, "f\times f"'] & A \arrow[d, "f"] \\ B\times B \arrow[r, "\omega_B(x)"'] & B \end{tikzcd} $$

This structure is easy to determine: let $b : 1 \to B$ be a point of A (e.g., let $C = \mathsf{Set}$). Then the commutativity of

$$ \begin{tikzcd} 1 \arrow[r, "\omega_A(x)"] \arrow[d, "f\times f"'] & 1 \arrow[d, "f"] \\ B\times B \arrow[r, "\omega_B(x)"'] & B \end{tikzcd} $$

tells that $\omega_B(x) : B \times B \to B$ is a section of the diagonal Δ_B (this means: $\omega_B(x)(b,b) = b$ for every $b \in B$). Moreover, the family $\omega_A : A \times A \to A$ is natural in A, i.e. it is a natural transformation

$$ \times \circ \Delta \xrightarrow{\ \omega\ } \mathrm{id} $$

that is a section of the natural transformation in the opposite direction, unit of the adjunction constant-product.

There are few such transformations. First, observe that the functor $\times \circ \Delta$ coincides with the functor $X \mapsto X^2$, so corresponds to the cotensoring with $2 = \{0, 1\}$ in an abstract category, and it is just the corepresentable presheaf on 2 in the category of sets. Similarly, the identity is the corepresentable over the point in the category of sets. All in all, in the category of sets

$$\begin{aligned} {}_{(2,1)}\!\int_A \hom(A \times A, A) &\cong [\mathsf{Set}, \mathsf{Set}](\mathsf{Set}(2, \_), \mathsf{Set}(1, \_)) \\ &\cong \mathsf{Set}(1, 2) \cong 2 \end{aligned}$$

by Yoneda.

Similarly, in a category $\mathcal{C}$ with Set-cotensors,

$${}_{(2,1)}\!\int_A \mathcal{C}(A \times A, A) \cong [\mathcal{C}, \mathsf{Set}]((2 \pitchfork \_), (1 \pitchfork \_))$$

where the functor $(n \pitchfork \_)$ coincides with the n-fold iterated product $X \mapsto X^n = X \times \cdots \times X$. A similar argument shows that ${}_{(n,1)}\!\int_A \mathcal{C}(A^n, A) \cong [\mathcal{C}, \mathsf{Set}]((n \pitchfork \_), (1 \pitchfork \_))$.

Example 4.6 (Dinatural transformations as a $(2, 2)$-end)**.** This example was first discovered in [DS70, Theorem 1]. We give an account of it in our language.

Let $F, G \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. Then

$$\begin{aligned} \mathrm{DiNat}(F, G) &\cong \int_{A \in \mathcal{C}} \hom_{\mathcal{D}}\left(F^A_A, G^A_A\right), \\ &\cong {}_{(2,2)}\!\int_{A \in \mathcal{C}} \hom_{\mathcal{D}}\left(F^A_A, G^A_A\right). \end{aligned}$$

Proof. The proof of the first isomorphism is divided in two steps:

DEP1) First, consider the functor

$$\hom_{\mathcal{D}}\left(F^{-_2}_{-_1}, G^{-_1}_{-_2}\right) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$

sending

(a) An object (A, X) of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to the set $\mathcal{D}\left(F^X_A, G^A_X\right)$;

(b) A morphism $\left(A \xrightarrow{f} B, X \xrightarrow{g} Y\right)$ of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to the map

$$\mathcal{D}\left(F^g_f, G^f_g\right) : \mathcal{D}\left(F^X_A, G^A_X\right) \to \mathcal{D}\left(F^Y_B, G^B_Y\right)$$

defined as the composition

$$\begin{tikzcd}[column sep=large] \mathcal{D}(F^X_A, G^A_X) \arrow[r, "\mathcal{D}(F^X_A, G^A_f)"'] \arrow[rrrr, bend left, "\mathcal{D}(F^g_f, G^f_g)"] & \mathcal{D}(F^X_A, G^A_Y) \arrow[r, "\mathcal{D}(F^f_A, G^A_Y)"'] & \mathcal{D}(F^Y_A, G^A_Y) \arrow[r, "\mathcal{D}(F^Y_A, G^f_Y)"'] & \mathcal{D}(F^Y_A, G^B_Y) \arrow[r, "\mathcal{D}(F^Y_f, G^B_Y)"'] & \mathcal{D}(F^Y_B, G^B_Y). \end{tikzcd}$$

By functoriality of homs, the assignment $(A, X) \mapsto \mathcal{D}(F^X_A, G^A_X)$ preserves identities and composition, defining therefore a functor.

DEP2) Second, we compute the end $\int_{A \in \mathcal{C}} \hom_{\mathcal{D}}\left(F^A_A, G^A_A\right)$; this is given by the equaliser of the pair of maps

$$\textstyle\prod_{A \in \mathcal{C}_o} \mathcal{D}(F^A_A, G^A_A) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{f : A \to B} \mathsf{Set}(\mathcal{C}(A, B), \mathcal{D}(F^B_A, G^A_B))$$

where λ and ρ are the morphisms induced by the universal property of the product by the morphisms

$$\lambda_{A,B}\colon \prod_A \mathcal{D}(F^A_A, G^A_A) \to \mathsf{Set}((C(A,B), \mathcal{D}(F^B_A, G^A_B)),$$
$$\rho_{A,B}\colon \prod_A \mathcal{D}(F^A_A, G^A_A) \to \mathsf{Set}((C(A,B), \mathcal{D}(F^B_A, G^A_B))$$

acting on elements as

$$\left(\alpha_A\colon F^A_A \to G^A_A\right) \mapsto \left(\left[\begin{smallmatrix} A \\ f\downarrow \\ B \end{smallmatrix}\right] \mapsto \left(G^{\mathrm{id}_A}_f \circ \alpha_A \circ F^f_{\mathrm{id}_A}\right)\right),$$
$$\left(\alpha_B\colon F^B_B \to G^B_B\right) \mapsto \left(\left[\begin{smallmatrix} A \\ f\downarrow \\ B \end{smallmatrix}\right] \mapsto \left(G^f_{\mathrm{id}_B} \circ \alpha_B \circ F^{\mathrm{id}_B}_f\right)\right),$$

and hence asking for λ and ρ to be equal is precisely the dinaturality condition for a family

$$\left\{\alpha_A\colon F^A_A \to G^A_A\right\}_{A\in C_o}$$

of morphisms of $\mathcal{D}$. As an element of the end $\int_{A\in C} \mathcal{D}\left(F^A_A, G^A_A\right)$ is precisely such a family equalising λ and ρ, the result follows.

As for the second isomorphism, we define a functor

$$\mathcal{D}(F_\uparrow, G_\downarrow)\colon C^{(2,2)} \to \mathsf{Set}$$

in a similar manner as we did above and then invoke Item PE3 of Proposition 3.5. The universal property of the $(2,2)$-end of $\mathcal{D}(F_\uparrow, G_\downarrow)$ is the same as the universal property of the equaliser defining $\mathrm{DiNat}(F,G)$. □

Generalising Example 4.6, we have the following.

Example 4.7 (((p,q)-Dinatural transformations as a (q,p)-end)**.** Let F and G be functors of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$, respectively. Then

$$\mathrm{DiNat}^{(p,q)}(F,G) \cong {}_{(q,p)}\!\int_{A\in C} \hom_{\mathcal{D}}\left(F^{\underline{A}}_{\underline{A}}, G^{\underline{A}}_{\underline{A}}\right),$$

where the "integrand" is the functor

$$\begin{aligned} C^{(q,p)} &\longrightarrow \mathcal{D} \\ (\underline{A}, \underline{B}) &\longmapsto \hom_{\mathcal{D}}\left(F^{B_1,\ldots,B_p}_{A_1,\ldots,A_q}, G^{A_1,\ldots,A_q}_{B_1,\ldots,B_p}\right). \end{aligned}$$

Proof. This is a combination of Proposition 2.8, Example 4.6, and Item PE3 of Proposition 3.5. □

4.2. Classes of higher arity coends

4.2.1. A glance at weighted co/ends

We now introduce a natural factory of examples for higher arity co/ends. In a nutshell, weighted co/ends are to co/ends as weighted co/limits are to co/limits.

Definition 4.8 (Weighted co/end)**.** Let C and $\mathcal{D}$ be $\mathcal{V}$-enriched categories and $D\colon C^{\mathsf{op}} \otimes_{\mathcal{V}} C \to \mathcal{D}$ a $\mathcal{V}$-functor, and $W : C^{\mathsf{op}} \times C \to \mathcal{V}$ a $\mathcal{V}$-presheaf.

WE1) The *end of D weighted by W* is, if it exists, the object $\int^W_{A\in C} D^A_A$ of $\mathcal{D}$ with the property that

$$\hom_{\mathcal{D}}\left(-, \int^W_{A\in C} D^A_A\right) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_C(-, D)).$$

WE2) The *coend of D weighted by W* is, if it exists, the object $\int^{A\in C}_W D^A_A$ of $\mathcal{D}$ with the property that

$$\hom_{\mathcal{D}}\left(\int^{A\in C}_W D^A_A, -\right) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_C(D, -)).$$

Example 4.9 (Weighted co/ends are $(2,2)$-co/ends)**.** A quick argument (to be discussed in future work [dLb]) gives $(2,2)$-co/end formulas for weighted co/ends:

$$\int_{A\in C}^{[W]} D_A^A \cong {}_{(2,2)}\int_{A\in C} W_A^A \pitchfork D_A^A,$$
$$\int_{[W]}^{A\in C} D_A^A \cong \int^{(2,2)\,A\in C} W_A^A \odot D_A^A.$$

Example 4.10 (Weighting Increases Arity)**.** Let $F, G\colon C \to \mathcal{D}$ and $W\colon C^{\mathsf{op}} \times C \to \mathcal{V}$ be $\mathcal{V}$-functors. In analogy with

$$\mathrm{Nat}_{\mathcal{V}}(F,G) \stackrel{\text{def}}{=} \int_{A\in C} \mathbf{hom}_{\mathcal{D}}(F_A, G_A),$$

we define the *object* $\mathsf{Nat}^{[W]}(F,G)$ *of natural transformations from F to G weighted by W* by

$$\mathsf{Nat}^{[W]}(F,G) \stackrel{\text{def}}{=} \int_{A\in C}^{[W]} \mathbf{hom}_{\mathcal{D}}(F_A, G_A). \tag{4.11}$$

Taking W to be mute in its contravariant variable, we can give a reformulation of the universal property of weighted limits:

$$\mathsf{h}\left(-, \lim^W(D)\right) \cong \mathsf{Nat}^{[W]}\left(\Delta_{(-)}, D\right).$$

Defining $\mathsf{DiNat}_{\mathcal{V}}^{[W]}(F,G)$ by a similar formula, we also obtain the following isomorphism in the case of weighted ends:

$$\mathsf{h}\left(-, \int_{A\in C}^{[W]} D_A^A\right) \cong \mathsf{DiNat}_{\mathcal{V}}^{[W]}\left(\Delta_{(-)}, D\right).$$

This naturally suggests a definition of "doubly-weighted ends":

$$\mathsf{h}\left(-, \int_{A\in C}^{[W_1,W_2]} D_A^A\right) \cong \mathsf{DiNat}_{\mathcal{V}}^{[W_1]}(W_2, D).$$

Repeating this process give you ends weighted by a collection of n functors $W_1, \dots, W_n$. These however, can be actually computed as $(n+1, n+1)$-ends ([dLb]):

$$\int_{A\in C}^{[W_1,\dots,W_n]} D_A^A \cong {}_{(n+1,n+1)}\int_{A\in C} \left((W_1)_A^A \times \cdots \times (W_n)_A^A\right) \odot D_A^A.$$

As such, we see that weighting an end increases its arity by $(1,1)$.

4.2.2. Weighted Kan extensions

Another source of examples comes from "weighing" left and right Kan extensions. While the most general such weight is a profunctor, having type $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, weights of type $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are specially interesting, as they give a more direct parallel with the classical theory of weighted co/limits (see Example 4.16).

For Definitions 4.12 and 4.14 below, recall from Equation (4.11) the definition of the object $\mathsf{Nat}^{[W]}(F,G)$ of weighted natural transformations.

Definition 4.12. The *left Kan extension of F along K weighted by W* is, if it exists, the $\mathcal{V}$-functor

$$\left(\mathsf{Lan}_K^{[W]}F\colon \mathcal{D} \to \mathcal{E}\right):\qquad
\begin{tikzcd}
 & & \mathcal{D} \arrow[d, dashed, "\mathsf{Lan}_K^{[W]}F"] \\
W \arrow[r, phantom, "\looparrowleft"] & C \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{E},
\end{tikzcd}$$

for which we have a $\mathcal{V}$-natural isomorphism

$$\mathsf{Nat}_{\mathcal{V}}\left(\mathsf{Lan}_K^{[W]}F, G\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(F, G \circ K\right), \tag{4.13}$$

natural in G.

One defines weighted right Kan extensions in a dual manner:

Definition 4.14. The *right Kan extension of F along K weighted by W* is, if it exists, the $\mathcal{V}$-functor

$$\left(\mathsf{Ran}_K^{[W]}F \colon \mathcal{D} \to \mathcal{E}\right) : \qquad \begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{Ran}_K^{[W]}F"] \\ W \arrow[r, phantom, "\looparrowleft"] & C \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift left=3, "\Downarrow"] & \mathcal{E}, \end{tikzcd}$$

for which we have a $\mathcal{V}$-natural isomorphism

$$\mathsf{Nat}_{\mathcal{V}}\left(G, \mathsf{Ran}_K^{[W]}F\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(G \circ K, F\right), \tag{4.15}$$

natural in G.

Example 4.16 (Weighted co/limits as weighted Kan extensions)**.** Let $D\colon C \to \mathcal{D}$ be a diagram on a category $\mathcal{D}$. Then we may canonically identify the left Kan extension of D along the terminal functor with its colimit:

$$\mathsf{Lan}_! D \cong \ulcorner\mathrm{colim}(D)\urcorner \qquad \begin{tikzcd} & \mathsf{pt} \arrow[d, dashed, "\ulcorner\mathrm{colim}(D)\urcorner"] \\ C \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift left=3, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

Similarly, given a weight $W\colon C^{\mathsf{op}} \to \mathsf{Set}$, we have

$$\mathsf{Lan}_!^{[W]} D \cong \ulcorner\mathrm{colim}^W(D)\urcorner \qquad \begin{tikzcd} & & \mathsf{pt} \arrow[d, dashed, "\ulcorner\mathrm{colim}^W(D)\urcorner"] \\ W \arrow[r, phantom, "\looparrowleft"] & C \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift left=3, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

One can also prove that the following formulas hold ([dLb]):

$$\mathsf{Lan}_K^{[W]}F \cong \int_{[W]}^{A\in C} \mathbf{hom}_C\left(K_A, -\right) \odot F_A \cong {}^{(2,2)}\!\!\int^{A\in C} \left(W_A^A \times \mathbf{hom}_C(K_A, -)\right) \odot F_A, \tag{4.17}$$

$$\mathsf{Ran}_K^{[W]}F \cong \int_{A\in C}^{[W]} \mathbf{hom}_C\left(-, K_A\right) \pitchfork F_A \cong {}_{(2,2)}\!\!\int_{A\in C} \left(W_A^A \times \mathbf{hom}_C(-, K_A)\right) \pitchfork F_A. \tag{4.18}$$

Equipped with these, we now proceed to compute a few weighted Kan extensions.

Example 4.19. Consider the functor $[0]^{\mathsf{op}} : \mathsf{pt}^{\mathsf{op}} \to \Delta^{\mathsf{op}}$; the left and right Kan extensions of a set $X_\bullet\colon \mathsf{pt} \to \mathsf{Set}$ along $[0]^{\mathsf{op}}$ are given by

$$\mathsf{Lan}_{[0]^{\mathsf{op}}}(X) \cong \underline{X}_\bullet$$

$$\mathsf{Ran}_{[0]^{\mathsf{op}}}(X) \cong \check{\mathrm{C}}_\bullet(X).$$

Now take a weight $W\colon \mathsf{pt}^{\mathsf{op}} \times \mathsf{pt} \to \mathsf{Set}$:

$$\begin{tikzcd} & & \Delta^{\mathsf{op}} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{[0]^{\mathsf{op}}} X"] \\ W \arrow[r, phantom, "\curvearrowright"] & \mathsf{pt}^{\mathsf{op}} \arrow[ur, "{[0]^{\mathsf{op}}}"] \arrow[r, "X"'] \arrow[ur, phantom, "\Uparrow" near end] & \mathsf{Set}. \end{tikzcd}$$

Then

$$\begin{aligned} \mathsf{Lan}^{[W]}_{[0]^{\mathsf{op}}}(X) &\cong \underline{W \times X_\bullet} \\ \mathsf{Ran}^{[W]}_{[0]^{\mathsf{op}}}(X) &\cong \check{\mathsf{C}}_\bullet(W \times X). \end{aligned}$$

Example 4.20. The above example has a more interesting counterpart, in which we consider the functor

$$\begin{aligned} t^{\mathsf{op}}\colon \Delta^{\mathsf{op}} &\longrightarrow \mathsf{pt}^{\mathsf{op}} \\ [n] &\longmapsto \star. \end{aligned}$$

The left and right Kan extensions of a simplicial set $X_\bullet\colon \Delta^{\mathsf{op}} \to \mathsf{Set}$ along t^{op} are given by

$$\begin{aligned} \mathsf{Lan}_{t^{\mathsf{op}}}(X_\bullet) &\cong \pi_0(X_\bullet) \\ \mathsf{Ran}_{t^{\mathsf{op}}}(X_\bullet) &\cong \mathsf{ev}_0(X_\bullet) \stackrel{\text{def}}{=} X_0. \end{aligned}$$

There is a great deal of flexibility in the choice of weight, as we may choose as such any cosimplicial space $W^\bullet_\bullet\colon \Delta^{\mathsf{op}} \times \Delta \to \mathsf{Set}$:

$$\begin{tikzcd} & & \mathsf{pt}^{\mathsf{op}} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{t^{\mathsf{op}}} X_\bullet"] \\ W \arrow[r, phantom, "\curvearrowright"] & \Delta^{\mathsf{op}} \arrow[ur, "t^{\mathsf{op}}"] \arrow[r, "X_\bullet"'] \arrow[ur, phantom, "\Uparrow" near end] & \mathsf{Set}. \end{tikzcd}$$

For instance, taking $W = \Delta^\bullet$ almost gives the geometric realisation of $X_\bullet$:

$$\mathsf{Lan}^{[\Delta^\bullet]}_{t^{\mathsf{op}}}(X_\bullet) \cong \int^{[n]\in\Delta} \Delta^n \times X_n,$$

with the caveat that the geometric realisation involves $|\Delta^n|$, rather than Δ^n itself. Dually, taking again $W = \Delta^\bullet$ but now for a cosimplicial object $X^\bullet\colon \Delta \to \mathsf{Set}$, we have

$$\mathsf{Ran}^{[\Delta^\bullet]}_{t}(X^\bullet) = \mathrm{Tot}(X_\bullet).$$

Example 4.21 (Stalks of a sheaf ([SGAIV, Paragraph 6.8 and Section 7.1])). Let $i_p\colon \{p\} \hookrightarrow X$ be the inclusion of a point into a topological space X. We get an induced functor

$$\begin{aligned} \mathsf{Op}(i_p)\colon \mathsf{Op}(X) &\longrightarrow \mathsf{Op}(\{p\}) \\ U &\longmapsto i_p^{-1}(U). \end{aligned}$$

Considering now left Kan extensions along the opposite of $\mathsf{Op}(i_p)$,

$$\begin{tikzcd} & \mathsf{Op}(\{p\})^{\mathsf{op}} \arrow[d, dashed, "\mathsf{Lan}_{\mathsf{Op}(i_p)^{\mathsf{op}}} \mathcal{F}"] \\ \mathsf{Op}(X)^{\mathsf{op}} \arrow[ur, "\mathsf{Op}(i_p)^{\mathsf{op}}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, "\Uparrow" near end] & \mathsf{Set}, \end{tikzcd}$$

we obtain a functor $\mathsf{Lan}_{\mathsf{Op}(i_p)^{\mathsf{op}}} \colon \mathsf{PSh}(X) \longrightarrow \mathsf{PSh}(\{p\})$, whose image at $\mathcal{F}$ is written $\lceil \mathcal{F}_p \rceil$ for simplicity. The restriction of this functor to $\mathsf{Shv}(X)$ can be identified with the stalk functor $(-)_p \colon \mathsf{Shv}(X) \to \mathsf{Set}$: we have $\mathsf{Op}(\{p\}) = \{\emptyset \hookrightarrow \{p\}\}$ and computing the images of $\emptyset$ and $\{p\}$ under $\lceil \mathcal{F}_p \rceil$ via the usual colimit formula for left Kan extensions gives

$$
\begin{aligned}
\lceil \mathcal{F}_p \rceil(\{p\}) &\cong \operatorname{colim}\left(\left(\mathsf{Op}(\lceil p \rceil) \downarrow \underline{\{p\}} \right)^{\mathsf{op}} \xrightarrow{\mathrm{pr}^{\mathsf{op}}} \mathsf{Op}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Set} \right), \\
&\cong \operatorname*{colim}_{U \ni p} (\mathcal{F}(U)), \\
&\cong \mathcal{F}_p \\
\lceil \mathcal{F}_p \rceil(\emptyset) &\cong \operatorname{colim}\left(\left(\mathsf{Op}(\lceil p \rceil) \downarrow \underline{\emptyset} \right)^{\mathsf{op}} \xrightarrow{\mathrm{pr}^{\mathsf{op}}} \mathsf{Op}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Set} \right), \\
&\cong \operatorname*{colim}_{U \hookrightarrow \emptyset} (\mathcal{F}(U)), \\
&\cong \mathcal{F}(\emptyset).
\end{aligned}
$$

(in case $\mathcal{F}$ is a sheaf, $\mathcal{F}(\emptyset)$ is the singleton set.) Consider the same situation, but now with a weight $W \colon \mathsf{Op}(X) \times \mathsf{Op}(X)^{\mathsf{op}} \to \mathsf{Set}$ (a "diagonal presheaf on X"; see Section 4.2.3 below):

$$
\begin{tikzcd}
 & & \mathsf{Op}(\{p\})^{\mathsf{op}} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{\mathsf{Op}(i_p)^{\mathsf{op}}}\mathcal{F}"] \\
W & \mathsf{Op}(X)^{\mathsf{op}} \arrow[ur, "\mathsf{Op}(i_p)^{\mathsf{op}}"] \arrow[r, "\mathcal{F}"'] \arrow[loop left] & \mathsf{Set}. \arrow[ul, phantom, "\Uparrow"]
\end{tikzcd}
$$

Using Equation (4.17), we may compute $\mathsf{Lan}^{[W]}_{\mathsf{Op}(i_p)^{\mathsf{op}}}\mathcal{F} \stackrel{\text{def}}{=} \lceil \mathcal{F}_p^{[W]} \rceil$ as the weighted coend

$$
\begin{aligned}
\lceil \mathcal{F}_p^{[W]} \rceil &\stackrel{\text{def}}{=} \int_{[W]}^{U \in \mathsf{Op}(X)} \hom_{\mathsf{Op}(X)^{\mathsf{op}}}\left(\mathsf{Op}(i_p^{\mathsf{op}})(U), - \right) \odot \mathcal{F}(U) \\
&\cong \int^{U \in \mathsf{Op}(X)} W_U^U \times \hom_{\mathsf{Op}(X)}\left(\chi_p(U), - \right) \times \mathcal{F}(U),
\end{aligned}
$$

where

$$
\chi_p(U) = \begin{cases} \emptyset & \text{if } p \notin U, \\ U & \text{otherwise.} \end{cases}
$$

For instance, taking W to be a sheaf $\mathcal{G}$ on X gives

$$
\mathcal{F}_p^{[\mathcal{G}]} \stackrel{\text{def}}{=} \lceil \mathcal{F}_p^{[\mathcal{G}]} \rceil(\{p\}) \cong \left(\mathcal{F} \times \mathcal{G} \right)_p.
$$

4.2.3. A glance at diagonality

In a nutshell, "diagonal" category theory arises when, instead of considering a natural transformation filling a higher-dimensional cell, we consider a *dinatural* one. Transformations that are more general than natural ones notoriously do not compose (see [Kel72b; Kel72a] and mostly [San19] for a modern account); yet, the category theory arising from this generalisation is interesting.

For the purposes of our exposition here, left/right Kan extensions are the most interesting categorical gadget to "diagonalise"; when this is done, they provide examples of higher arity co/ends.

Definition 4.22 (Diagonal left Kan extensions)**.** The *diagonal left Kan extension* of a functor $F\colon C^{\mathsf{op}} \times C \to \mathcal{D}$ along a functor $K\colon C^{\mathsf{op}} \times C \to \mathcal{D}$ is, if it exists the functor $\mathsf{DiLan}_K F\colon \mathcal{D} \to \mathcal{E}$ such that we have an isomorphism

$$\mathsf{Nat}(\mathsf{DiLan}_K F, G) \cong \mathsf{DiNat}(F, G \circ K), \qquad \begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\ C^{\mathsf{op}} \times C \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & \mathcal{E} \end{tikzcd}$$

natural in G.

Example 4.23. Standard examples of diagonal left Kan extensions are ends: Generalising the fact that the left Kan extension of a functor $D\colon C \to \mathcal{D}$ along the terminal functor $\boldsymbol{t}\colon C \twoheadrightarrow \mathsf{pt}$ can be identified with the colimit of $\mathcal{D}$, the diagonal left Kan extension of a functor $D\colon C^{\mathsf{op}} \times C \to \mathcal{D}$ along the terminal functor $\boldsymbol{t}\colon C^{\mathsf{op}} \times C \twoheadrightarrow \mathsf{pt}$ can be identified with the coend of $\mathcal{D}$.

$$\begin{tikzcd} & \mathsf{pt} \arrow[d, dashed, "\ulcorner \mathrm{colim}(D) \urcorner"] \\ C \arrow[ur, "\boldsymbol{t}"] \arrow[r, "D"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & \mathcal{D}, \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{pt} \arrow[d, dashed, "\ulcorner \int^A D^A_A \urcorner"] \\ C^{\mathsf{op}} \times C \arrow[ur, "\boldsymbol{t}"] \arrow[r, "D"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

Now, while ordinary Kan extensions can be computed via co/end formulas, diagonal Kan extensions admit $(2, 2)$-co/end formulas ([dLa]):

$$\mathsf{DiLan}_K F \cong {}^{(2,2)}\!\!\int^{A \in C} \mathcal{D}\left(K^A_A, -\right) \odot F^A_A, \tag{4.24}$$

$$\mathsf{DiRan}_K F \cong {}_{(2,2)}\!\!\int_{A \in C} \mathcal{D}\left(-, K^A_A\right) \pitchfork F^A_A, \tag{4.25}$$

where the pairing in Equation (4.24) is such that $\mathsf{DiLan}_K F$ is the coend of

$$(A, B) \mapsto \mathcal{D}\left(K^B_A, -\right) \odot F^A_B.$$

Alternatively, we may compute diagonal Kan extensions as hom-weighted Kan extensions ([dLb; dLa]):

$$\mathsf{DiLan}_K F \cong \int^{A,B \in C}_{[\hom_C(-,-)]} \mathcal{D}\left(K^B_A, -\right) \odot F^A_B,$$

$$\mathsf{DiRan}_K F \cong \int_{A,B \in C}^{[\hom_C(-,-)]} \mathcal{D}\left(-, K^B_A\right) \pitchfork F^A_B.$$

This is a generalisation, as per Example 4.23, of the fact that ends are hom-weighted limits. A forthcoming work [dLa] will address the topic of this remark in its entirety, studying the category theory arising from the notion of a weighted co/end (Definition 4.8).

Example 4.26. As a very special case, we note that diagonal left and right Kan extensions of identity functors are very important: with a universal property of the form

$$\mathsf{Nat}(\mathsf{DiLan}_{\mathrm{id}} F, G) \cong \mathsf{DiNat}^{(p,q)}(F, G),$$

$$\mathsf{Nat}(F, \mathsf{DiRan}_{\mathrm{id}} G) \cong \mathsf{DiNat}^{(p,q)}(F, G),$$

they allow us to study (p, q)-dinatural transformations from F to G in terms of ordinary natural transformations. These will play a fundamental role in Section 5.

Example 4.27. Let C be a closed monoidal category and $D\colon C^{\mathsf{op}} \times C \to \mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_{[-,-]}D$ and $\mathsf{DiRan}_{[-,-]}D$?

$$\begin{tikzcd} & C \arrow[d, dashed, "\mathsf{DiLan}_{[-,-]}D"] \\ C^{\mathsf{op}} \times C \arrow[ur, "{[-,-]}"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

$$\mathsf{DiLan}_{[-,-]}D \cong \int^{A\in C} \hom_C([A,A],-) \odot D^A_A.$$

Example 4.28. Let $D\colon C^{\mathsf{op}} \times C \to \mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_{\text{よ}}D$ and $\mathsf{DiRan}_{\text{よ}}D$?

$$\begin{tikzcd} & \mathsf{PSh}(C^{\mathsf{op}} \times C) \arrow[d, dashed, "\mathsf{DiLan}_{\text{よ}}D"] \\ C^{\mathsf{op}} \times C \arrow[ur, hook, "\text{よ}"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

$$\begin{aligned}
\mathsf{DiLan}_{\text{よ}}D &\cong \int^{A,B\in C}_{\hom_C(-,-)} \hom_{\mathsf{PSh}(C^{\mathsf{op}}\times C)}\left(\text{よ}^B_A,-\right) \odot F^A_B,\\
&\cong \int^{A\in C} \hom_{\mathsf{PSh}(C^{\mathsf{op}}\times C)}\left(\text{よ}^A_A,-\right) \odot F^A_A,\\
&\cong {}^{(2,2)}\!\!\int^{A\in C} \hom_{\mathsf{PSh}(C^{\mathsf{op}}\times C)}\left(\text{よ}^A_A,-\right) \odot F^A_A,\\
&\overset{\text{def}}{=} {}^{(2,2)}\!\!\int^{A\in C} \hom_{\mathsf{PSh}(C^{\mathsf{op}}\times C)}\left(\hom_{C^{\mathsf{op}}\times C}(-,(A,A)),-\right) \odot F^A_A,\\
&\overset{\text{def}}{=} {}^{(2,2)}\!\!\int^{A\in C} \hom_{\mathsf{PSh}(C^{\mathsf{op}}\times C)}\left(\mathsf{h}^A \times \mathsf{h}_A,-\right) \odot F^A_A.
\end{aligned}$$

In order to introduce the next example, we recall the following notation: we have an adjunction

$$(\boldsymbol{t} \dashv [0])\colon\ \begin{tikzcd} \mathsf{pt} \arrow[r, hook, shift right, "{[0]}"'] \arrow[r, phantom, "\perp"] & \Delta, \arrow[l, two heads, shift right, "\boldsymbol{t}"'] \end{tikzcd}$$

where

- $[0]\colon \mathsf{pt} \hookrightarrow \Delta$ is the functor choosing the terminal object;
- $\boldsymbol{t}\colon \Delta \twoheadrightarrow \mathsf{pt}$ is the terminal functor;

This induces a quadruple adjunction

$$\left(\pi_0 \dashv \underline{(-)}_{\bullet} \dashv \mathsf{ev}_0 \dashv \check{\mathsf{C}}_{\bullet}\right)\colon\ \begin{tikzcd} \mathsf{Set} \arrow[r, shift left=1.5] \arrow[r, shift right=4.5] & \mathsf{sSet} \arrow[l, shift right=4.5, "\pi_0"'] \arrow[l, shift left=1.5] \end{tikzcd}$$

(see [Gra80, §3]).

Example 4.29. Let $S^\bullet_\bullet \colon \Delta^{\mathsf{op}} \times \Delta \to \mathsf{Set}$ be a cosimplicial space. What is $\mathsf{DiLan}_{t^{\mathsf{op}}\times t}(S^\bullet_\bullet)$?

$$\begin{tikzcd}
 & \mathsf{pt}^{\mathsf{op}} \times \mathsf{pt} \arrow[d, dashed, "\mathsf{DiLan}_{t^{\mathsf{op}}\times t}(S^\bullet_\bullet)"] \\
\Delta^{\mathsf{op}} \times \Delta \arrow[ur, "t^{\mathsf{op}}\times t"] \arrow[r, "S^\bullet_\bullet"'] & \mathsf{Set}.
\end{tikzcd}$$

It is just the end of $S^\bullet_\bullet$ (btw do you know what this is?):

$$\begin{aligned}
\mathsf{DiLan}_{t^{\mathsf{op}}\times t}(S^\bullet_\bullet) &\cong \int^{[n]\in\Delta} \hom_{\mathsf{pt}}(\star,\star) \odot S^n_n, \\
&\cong \int^{[n]\in\Delta} S^n_n.
\end{aligned}$$

Similarly, given a set $X \colon \mathsf{pt}^{\mathsf{op}} \times \mathsf{pt} \to \mathsf{Set}$, we have

$$\begin{tikzcd}
 & \Delta^{\mathsf{op}} \times \Delta \arrow[d, dashed, "\mathsf{DiLan}_{[0]^{\mathsf{op}}\times[0]}(X)"] \\
\mathsf{pt}^{\mathsf{op}} \times \mathsf{pt} \arrow[ur, hook, "{[0]^{\mathsf{op}}\times[0]}"] \arrow[r, "X"'] & \mathsf{Set}.
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{DiLan}_{[0]^{\mathsf{op}}\times[0]}(X) &\cong \int^{\star\in\mathsf{pt}} \hom_{\Delta^{\mathsf{op}}\times\Delta}(([0],[0]),(-_1,-_2)) \odot X, \\
&\cong \hom_{\Delta^{\mathsf{op}}\times\Delta}(([0],[0]),(-_1,-_2)) \odot X \\
&\cong \hom_{\Delta}([0],-_2) \odot X \\
&\cong \Delta^{-_2}[0] \odot X.
\end{aligned}$$

Similarly, let $X^\bullet_\bullet \colon \Delta^{\mathsf{op}} \times \Delta \to \mathsf{Set}$ be a cosimplicial space again. What is $\mathsf{DiLan}_\Delta(X^\bullet_\bullet)$?

$$\begin{tikzcd}
 & \mathsf{Set} \arrow[d, dashed, "\mathsf{DiLan}_\Delta(X^\bullet_\bullet)"] \\
\Delta^{\mathsf{op}} \times \Delta \arrow[ur, "{\Delta^{-_2}[-_1]}"] \arrow[r, "X^\bullet_\bullet"'] & \mathsf{Set}.
\end{tikzcd}$$

$$\mathsf{DiLan}_\Delta(X^\bullet_\bullet) \cong \int^{[n]\in\Delta} \mathsf{Set}(\Delta^n[n],-) \odot X^n_n.$$

4.2.4. Weighted diagonal Kan extensions

In the same spirit, one can define weighted diagonal Kan extensions, mixing the two perspectives in Definitions 4.8 and 4.22, and considering now the diagram

$$\begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\ W \arrow[r, phantom, "\curvearrowright"] & C^{\mathsf{op}} \times C \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & \mathcal{E}, \end{tikzcd}$$

just to discover that these are actually computed as $(4,4)$-co/ends:

$$\mathsf{DiLan}_K^{[W]} F \cong {}^{(4,4)}\!\!\int^{A\in C} \left(W_{A,A}^{A,A} \times \mathbf{hom}_C\left(K_A^A, -\right)\right) \odot F_A^A,$$

$$\mathsf{DiRan}_K^{[W]} F \cong {}_{(4,4)}\!\!\int_{A\in C} \left(W_{A,A}^{A,A} \times \mathbf{hom}_C\left(-, K_A^A\right)\right) \pitchfork F_A^A.$$

At this point, the reader shall be convinced that the list of examples is virtually endless. We defer a thorough study of the topic to separate works [dLa; dLb].

4.2.5. Daydreaming About Operads

Day convolution was introduced by B. Day in [Day70a; Day70b], in order to classify monoidal structures on the category $\mathsf{PSh}(C)$ of presheaves on C. Day proved that $\mathsf{PSh}(C)$ can be turned into a monoidal category in as many ways as C can be turned into a pseudomonoid in the bicategory Prof of profunctors.[5]

We now propose a generalisation of this framework based on higher arity coends: let $(C, \otimes, 1)$ be a monoidal category, and let $\mathcal{K} \stackrel{\text{def}}{=} \mathsf{PSh}(C)$. Higher arity Day convolution is defined as a family of functors $\circledast_n : \mathcal{K}^n \to \mathcal{K}$:

Definition 4.30. The **Day** (n,n)**-convolution** of an n-tuple of presheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ is the presheaf

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) \colon C^{\mathsf{op}} \to \mathsf{Set}$$

defined at $A \in C_o$ as the (n,n)-coend

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) \stackrel{\text{def}}{=} A \mapsto {}^{(n,n)}\!\!\int^{A\in C} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times C\left(-, A^{\otimes n}\right),$$

where $A^{\otimes n}$ is shorthand for the n-fold tensor product of A with itself.

Example 4.31 (Day convolution operad)**.** The **Day convolution operad associated to** $(C, \otimes, 1)$ is the free symmetric operad Day whose set of generating operations (see [Fre17, Section 1.2.5]) is given by $\{\mathrm{id}, \circledast_2, \circledast_3, \dots, \circledast_n, \dots\}$.

Remark 4.32 (Unwinding Example 4.31)**.** We spell out in detail the first four sets of n-ary operations of Day:

$$\mathsf{Day}_1 = \{\mathrm{id}\}$$
$$\mathsf{Day}_2 = \{\circledast_2(-,-)\}$$

[5] More formally, let $S : \mathsf{Cat} \to \mathsf{Cat}$ be the 2-monad of pseudomonoids; let $\tilde{S} : \mathsf{Prof} \to \mathsf{Prof}$ be the lifting of S to the bicategory of profunctors (i.e. to the Kleisli bicategory of the presheaf construction PSh); then, given an object C of Cat, there is a bijection between pseudo-S-algebra structures on $\mathsf{PSh}(C)$ and pseudo-$\tilde{S}$-algebras on C, as an object of Prof.

$$\begin{aligned}\mathsf{Day}_3 &= \{\circledast_3(-,-,-), \circledast_2(\circledast_2(-,-),-), \circledast_2(-,\circledast_2(-,-))\}\\ \mathsf{Day}_4 &= \{\circledast_4(-,-,-,-), \circledast_2(\circledast_3(-,-,-),-), \circledast_2(-,\circledast_3(-,-,-)), \circledast_2(\circledast_2(-,-),\circledast_2(-,-)),\\ &\qquad \circledast_3(\circledast_2(-,-),-,-), \circledast_3(-,\circledast_2(-,-),-), \circledast_3(-,-,\circledast_2(-,-))\}\end{aligned}$$

All in all, the set Day_n can be succinctly described as

$$\mathsf{Day}_n = \{\circledast_n\} \cup \sum_{p+q=n} \mathsf{Day}_p \times \mathsf{Day}_q$$

The operadic composition of Day is now defined via 'grafting' in the usual way:

$$\begin{aligned}\mathsf{Day}_n \times \mathsf{Day}_{k_1} \times \cdots \times \mathsf{Day}_{k_n} &\longrightarrow \mathsf{Day}_{\sum k_i}\\ (\theta;\theta_1,\dots,\theta_k) &\longmapsto \theta(\theta_1(-_1,\dots,-_{k_1}),\dots,\theta_k(-_1,\dots,-_{k_n})).\end{aligned}$$

5. Kusarigamas and twisted arrow categories

The aim of this section is to introduce and study a fundamental computational tool that will endow higher arity coends with a fairly rich calculus. Generalising a construction of Street–Dubuc introduced in [DS70, Theorem 2], we introduce in Definition 5.1 functors

$$\begin{aligned}\daleth^{p,q} &: \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}) \to \mathsf{Cat}(\mathcal{C}^{(q,p)},\mathcal{D}),\\ \Gamma^{q,p} &: \mathsf{Cat}(\mathcal{C}^{(q,p)},\mathcal{D}) \to \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}),\end{aligned}$$

dubbed *co/kusarigamas*, which generalise the product-hom functor of Definition 3.6, in the sense that

$$\hom_{\Pi,p,q}(\underline{A},\underline{B}) \cong \daleth^{p,q}(\mathrm{pt})^{\underline{A}}_{\underline{B}},$$

where pt is the terminal functor. These constructions allow us to pass from dinaturality to naturality and underpin a number of results in the theory of higher arity co/ends, such as the construction of higher arity twisted arrow categories.

Overall, the entire structure of the section concentrates on studying the properties of the functors $\Gamma^{q,p}(G), \daleth^{p,q}(F)$, regarded as

- Universal objects among (p,q)-dinatural transformations, through which all other (p,q)-dinaturals factor (Definition 5.1 and Remark 5.3):
$$\mathrm{DiNat}^{(p,q)}(F,G) \cong \mathrm{Nat}\left(F,\Gamma^{p,q}(G)\right) \cong \mathrm{Nat}\left(\daleth^{p,q}(F),G\right);$$
- Functors that can be inductively defined through suitable Kan extensions (Item PK5), starting from the case $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$:
$$\daleth^{p,q}(F) \cong \mathsf{Lan}_{\Delta_{q,p}}\left(\daleth(\Delta^*_{p,q}(F))\right), \qquad \Gamma^{p,q}(G) \cong \mathsf{Ran}_{\Delta_{p,q}}\left(\Gamma(\Delta^*_{q,p}(G))\right).$$
- "Twisted versions" of F and G, which may be computed by use of a similar formula as the one computing co/ends as co/limits via the twisted arrow category (Section 5.4).

Finally, the paramount property of the co/kusarigama functors is that given a category $\mathcal{C}$, the category of elements of $\daleth^{p,q}(\mathrm{pt})$, where $\mathrm{pt} : \mathcal{C}^{(p,q)} \longrightarrow \mathsf{Set}$ is the terminal presheaf, is the universal fibration needed to build a higher-arity version of the *twisted arrow category* (i.e., the category of elements of $\hom_{\mathcal{C}}$): we study this construction in Section 5.3. This makes it possible to express the (p,q)-co/end of a diagram $G : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ as a co/limit over the (p,q)-twisted arrow category of $\mathcal{C}$.

5.1. Co/kusarigamas: basic definitions

Let C and $\mathcal{D}$ be categories.

Definition 5.1. Let F and G be functor from C to $\mathcal{D}$ of types $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]$.

CK1) The *kusarigama*[6] of G is, if it exists, the object

$$\Gamma^{q,p}(G)\colon C^{(p,q)} \to \mathcal{D}$$

of $\mathsf{Cat}(C^{(p,q)}, \mathcal{D})$ representing the functor

$$\mathrm{DiNat}^{(p,q)}(-, G)\colon \mathsf{Cat}(C^{(p,q)}, \mathcal{D}) \to \mathsf{Set}.$$

CK2) The *cokusarigama* of F is, if it exists, the object

$$⅃^{p,q}(F)\colon C^{(q,p)} \to \mathcal{D}$$

of $\mathsf{Cat}(C^{(p,q)}, \mathcal{D})$ corepresenting the functor

$$\mathrm{DiNat}^{(p,q)}(F, -)\colon \mathsf{Cat}(C^{(q,p)}, \mathcal{D}) \to \mathsf{Set}.$$

Remark 5.2. Thus, co/kusarigamas are defined by the following relations:

$$\mathrm{Nat}\left(⅃^{p,q}(F), -\right) \cong \mathrm{DiNat}^{(p,q)}(F, -),$$
$$\mathrm{Nat}\left(-, \Gamma^{p,q}(G)\right) \cong \mathrm{DiNat}^{(p,q)}(-, G).$$

It is crucial to focus on the exact way in which the types of F, G, and of $⅃^{p,q}(F), \Gamma^{p,q}(G)$ interchange: asking that F, G be of type of types $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]$ is the only possible choice for the three objects $\mathrm{Nat}(⅃^{p,q}(F), G)$, $\mathrm{Nat}(F, \Gamma^{p,q}(G))$ and $\mathrm{DiNat}^{(p,q)}(F, G)$ to exist, according to our Definition 2.1.

This means that $\Gamma^{p,q}, ⅃^{p,q}$ are candidates to be functors

$$⅃^{p,q} : \mathsf{Cat}(C^{(p,q)}, \mathcal{D}) \to \mathsf{Cat}(C^{(q,p)}, \mathcal{D}) \qquad \Gamma^{p,q} : \mathsf{Cat}(C^{(q,p)}, \mathcal{D}) \to \mathsf{Cat}(C^{(p,q)}, \mathcal{D})$$

Among many other properties, we prove in Proposition 5.9 that these correspondences are indeed functors.

Remark 5.3 (Unwinding Definition 5.1)**.** The co/representability conditions defining co/kusarigamas unwind as the following universal properties:

UK1) The *cokusarigama* of a functor $F : C^{(p,q)} \to \mathcal{D}$ is, if it exists, the pair $(⅃^{p,q}(F), \eta)$ with

$$⅃^{q,p}(F) : C^{(q,p)} \to \mathcal{D}$$

a functor of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$, and

$$\eta : F \Longrightarrow\!\!\!\circ\; ⅃^{p,q}(F)$$

a (p, q)-dinatural transformation satisfying the following universal property:

[6] A *kusarigama* (鎖鎌) is a Japanese compound weapon made of a sickle (*kama*) and a blunt weight (*fundo*) attached to the opposite ends of a chain (*kusari*). The weight was used to disarm the opponent by entangling their sword in the chain, or as a single weapon; disarmed or damaged the opponent, the sickle was then used to deliver the final, fatal strike. Kusarigamas were probably adapted from an old farming tool, and first adopted by Koga ninjas as a fast, compact weapon; its use then spread to tactic-oriented esoteric weaponry schools like Shinkage-ryū and Suiō-ryū. See [Ino98] for more information.

For us, the chain consists of the tuple of hom-functors, at the end of which the "weight" G is attached:

$$\Gamma^{q,p}(G) \overset{\text{def}}{=} \int_{A\in C} \mathsf{h}_A \times \cdots \times \mathsf{h}_A \times \mathsf{h}^A \times \cdots \times \mathsf{h}^A \pitchfork G^A_A.$$

(★) Given a (p, q)-dinatural transformation $\theta : F \Longrightarrow\!\circ\, G$, there exists a unique natural transformation $⅃^{p,q}(F) \Longrightarrow G$ making the diagram

$$\begin{tikzcd}
⅃^{p,q}(F) \arrow[dr, Rightarrow, "\exists!"] & \\
F \arrow[u, Rightarrow, "\eta"] \arrow[r, Rightarrow, "\theta"'] & G
\end{tikzcd}$$

commute.

UK2) The *kusarigama* of a functor $G : C^{(q,p)} \to \mathcal{D}$ is, if it exists, the pair $(\Gamma^{q,p}(G), \epsilon)$ with

$$\Gamma^{q,p}(G) : C^{(p,q)} \to \mathcal{D}$$

a functor of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, and

$$\epsilon : \Gamma^{q,p}(G) \Longrightarrow\!\circ\, G$$

a (p, q)-dinatural transformation satisfying the following universal property:

(★) Given a (p, q)-dinatural transformation $\theta : F \Longrightarrow\!\circ\, G$, there exists a unique natural transformation $F \Longrightarrow \Gamma^{q,p}(G)$ making the diagram

$$\begin{tikzcd}
& \Gamma^{q,p}(G) \arrow[d, Rightarrow, "\epsilon"] \\
F \arrow[ur, Rightarrow, "\exists!"] \arrow[r, Rightarrow, "\theta"'] & G
\end{tikzcd}$$

commute.

Notation 5.4. Given tuples $\boldsymbol{A}, \boldsymbol{C} \in C^{-p} = (C^p)^{\mathrm{op}}, \boldsymbol{B}, \boldsymbol{D} \in C^q$ we make use of the notation

$$\hom_{C^{(p,q)}}((\boldsymbol{A}, \boldsymbol{B}), (\boldsymbol{C}, \boldsymbol{D})) \overset{\text{def}}{=} \mathsf{h}^{(\boldsymbol{A},\boldsymbol{B})}_{(\boldsymbol{C},\boldsymbol{D})},$$

as well as of the equalities

$$\mathsf{h}^{(\boldsymbol{A},\boldsymbol{B})}_{(\boldsymbol{C},\boldsymbol{D})} \overset{\text{def}}{=} \mathsf{h}^{\boldsymbol{C}}_{\boldsymbol{A}} \times \mathsf{h}^{\boldsymbol{B}}_{\boldsymbol{D}} = \mathsf{h}^{C_1}_{A_1} \times \cdots \times \mathsf{h}^{C_p}_{A_p} \times \mathsf{h}^{B_1}_{D_1} \times \cdots \times \mathsf{h}^{B_q}_{D_q}.$$

Construction 5.5 (Constructing cokusarigamas)**.** Suppose that $\mathcal{D}$ is cocomplete. Then

$$^{(p,q)}\!\!\int^{\boldsymbol{A}\in C} \left(\mathsf{h}^{-}_{\boldsymbol{A}_q} \times \mathsf{h}^{\boldsymbol{A}_p}_{-}\right) \odot F^{\boldsymbol{A}_p}_{\boldsymbol{A}_q}$$

meaning the (p, q)-coend of

$$\begin{aligned}
C^{(p,q)} &\longrightarrow \mathsf{Cat}(C^{(q,p)}, \mathcal{D}) \\
(\underline{A}, \underline{B}) &\longmapsto \hom_{C^{(q,p)}}\left((\underline{B}, \underline{A}); (-,-)\right) \odot F^{\underline{A}}_{\underline{B}},
\end{aligned}$$

satisfies the universal property in Item CK2.

Proof. The proof is merely a formal manipulation:

$$\mathrm{DiNat}^{(p,q)}(F, G) \cong {}_{(q,p)}\!\!\int_{\boldsymbol{X}\in C} \hom_{\mathcal{D}}\left(F^{\boldsymbol{X}}_{\boldsymbol{X}}, G^{\boldsymbol{X}}_{\boldsymbol{X}}\right)$$

$$
\begin{aligned}
&\cong {}_{(q,p)}\!\int_{X\in C} \hom_{\mathcal{D}}\left(F^{X}_{X}, \int_{\underline{A},\underline{B}\in C} \left(\mathsf{h}^{\underline{A}}_{X}\times \mathsf{h}^{X}_{\underline{B}}\right)\pitchfork G^{\underline{A}}_{\underline{B}}\right)\\
&\cong {}_{(q,p)}\!\int_{X\in C}\int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(F^{X}_{X}, \left(\mathsf{h}^{\underline{A}}_{X}\times \mathsf{h}^{X}_{\underline{B}}\right)\pitchfork G^{\underline{A}}_{\underline{B}}\right)\\
&\cong {}_{(q,p)}\!\int_{X\in C}\int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(\left(\mathsf{h}^{\underline{A}}_{X}\times \mathsf{h}^{X}_{\underline{B}}\right)\odot F^{X}_{X}, G^{\underline{A}}_{\underline{B}}\right)\\
&\cong \int_{\underline{A},\underline{B}\in C}{}_{(q,p)}\!\int_{X\in C} \hom_{\mathcal{D}}\left(\left(\mathsf{h}^{\underline{A}}_{X}\times \mathsf{h}^{X}_{\underline{B}}\right)\odot F^{X}_{X}, G^{\underline{A}}_{\underline{B}}\right)\\
&\cong \int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left({}^{(p,q)}\!\int^{X\in C}\left(\mathsf{h}^{\underline{A}}_{X}\times \mathsf{h}^{X}_{\underline{B}}\right)\odot F^{X}_{X}, G^{\underline{A}}_{\underline{B}}\right)\\
&\overset{\text{def}}{=} \int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(\daleth^{p,q}(F)^{\underline{A}}_{\underline{B}}, G^{\underline{A}}_{\underline{B}}\right)\\
&\cong \mathrm{Nat}\left(\daleth^{p,q}(F), G\right).
\end{aligned}
$$

□

Construction 5.6 (Constructing Kusarigamas)**.** Suppose that $\mathcal{D}$ is complete. Then

$$
{}_{(q,p)}\!\int_{A\in C}\left(\mathsf{h}^{A_p}_{-}\times \mathsf{h}^{-}_{A_q}\right)\pitchfork G^{A_q}_{A_p},
$$

meaning the (q,p)-coend of

$$
\begin{aligned}
C^{(q,p)} &\longrightarrow \mathsf{Cat}(C^{(p,q)}, \mathcal{D})\\
(\underline{A},\underline{B}) &\longmapsto \hom_{C^{(q,p)}}\left((\underline{A},\underline{B});(-,-)\right)\pitchfork G^{\underline{A}}_{\underline{B}},
\end{aligned}
\tag{5.7}
$$

satisfies the universal property in Item CK2.

Proof. While this is dual to Construction 5.5, we register its derivation below for the sake of completeness.

$$
\begin{aligned}
\mathrm{DiNat}^{(p,q)}(F,G) &\cong {}_{(q,p)}\!\int_{X\in C} \hom_{\mathcal{D}}\left(F^{Y}_{X}, G^{X}_{Y}\right)\\
&\cong {}_{(q,p)}\!\int_{X\in C} \hom_{\mathcal{D}}\left(\int^{\underline{A},\underline{B}\in C}\left(\mathsf{h}^{Y}_{\underline{A}}\times \mathsf{h}^{\underline{B}}_{X}\right)\odot F^{\underline{A}}_{\underline{B}}, G^{X}_{Y}\right)\\
&\cong {}_{(q,p)}\!\int_{X\in C}\int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(\left(\mathsf{h}^{Y}_{\underline{A}}\times \mathsf{h}^{\underline{B}}_{X}\right)\odot F^{\underline{A}}_{\underline{B}}, G^{X}_{Y}\right)\\
&\cong {}_{(q,p)}\!\int_{X\in C}\int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(F^{\underline{A}}_{\underline{B}}, \left(\mathsf{h}^{Y}_{\underline{A}}\times \mathsf{h}^{\underline{B}}_{X}\right)\pitchfork G^{X}_{Y}\right)\\
&\cong \int_{\underline{A},\underline{B}\in C}{}_{(q,p)}\!\int_{X\in C} \hom_{\mathcal{D}}\left(F^{\underline{A}}_{\underline{B}}, \left(\mathsf{h}^{Y}_{\underline{A}}\times \mathsf{h}^{\underline{B}}_{X}\right)\pitchfork G^{X}_{Y}\right)\\
&\cong \int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(F^{\underline{A}}_{\underline{B}}, {}_{(q,p)}\!\int_{X\in C}\left(\mathsf{h}^{Y}_{\underline{A}}\times \mathsf{h}^{\underline{B}}_{X}\right)\pitchfork G^{X}_{Y}\right)\\
&\overset{\text{def}}{=} \int_{\underline{A},\underline{B}\in C} \hom_{\mathcal{D}}\left(F^{\underline{A}}_{\underline{B}}, \Gamma^{p,q}(G)^{\underline{A}}_{\underline{B}}\right)\\
&\cong \mathrm{Nat}(F, \Gamma^{p,q}(G)).
\end{aligned}
$$

□

Notation 5.8 ($(1,1)$-kusarigamas)**.** For the sake of brevity, we often write $\Gamma(D)$ and ⅃(D) for $\Gamma^{1,1}(D)$ and ⅃$^{1,1}(D)$, respectively.

Proposition 5.9 (Properties of co/kusarigamas)**.** Let $D, F, G : \mathcal{C}^{(p,q)} \rightrightarrows \mathcal{D}$ be functors, where $\mathcal{D}$ is a bicomplete category.

PK1) *Functoriality.* The assignments $D \mapsto \Gamma(D)$, ⅃(D) define functors

$$⅃^{p,q} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \to \mathsf{Cat}\big(\mathcal{C}^{(q,p)}, \mathcal{D}\big),$$
$$\Gamma^{p,q} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \to \mathsf{Cat}\big(\mathcal{C}^{(q,p)}, \mathcal{D}\big).$$

PK2) *Adjointness.* We have an adjunction

$$\begin{tikzcd} \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \arrow[r, shift left, "⅃^{p,q}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}\left(\mathcal{C}^{(q,p)}, \mathcal{D}\right) . \arrow[l, shift left, "\Gamma^{q,p}"] \end{tikzcd}$$

PK3) *Commutativity with homs.* Let $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$ be a functor, and let us consider the functors

$$\mathcal{D}(F,1) : \mathcal{D} \to \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathsf{Set}), D \mapsto \left((\underline{A}, \underline{B}) \mapsto \mathcal{D}\left(F^{\underline{A}}_{\underline{B}}, D\right)\right),$$
$$\mathcal{D}(1,F) : \mathcal{D}^{\mathsf{op}} \to \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathsf{Set}), D \mapsto \left((\underline{A}, \underline{B}) \mapsto \mathcal{D}\left(D, F^{\underline{A}}_{\underline{B}}\right)\right),$$

then the diagrams

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\mathcal{D}(⅃^{p,q}(F),1)"'] \arrow[dr, "\mathcal{D}(F,1)"] & \\ \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathsf{Set}) & & \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathsf{Set}) \arrow[ll, "\Gamma^{q,p}"] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{D} \arrow[dl, "\mathcal{D}(1,\Gamma^{p,q}(F))"'] \arrow[dr, "\mathcal{D}(1,F)"] & \\ \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathsf{Set}) & & \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathsf{Set}) \arrow[ll, "\Gamma^{p,q}"] \end{tikzcd}$$

commute:

$$\mathcal{D}(⅃^{p,q}(F), 1) \cong \Gamma^{q,p}(\mathcal{D}(F,1)) \qquad \mathcal{D}(1, \Gamma^{p,q}(F)) \cong \Gamma^{p,q}(\mathcal{D}(1,F)).$$

PK4) *Limits of kusarigamas.* Let $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$ be a functor; we have functorial isomorphisms

$$_{(p,q)}\!\int_{A\in\mathcal{C}} F^{\boldsymbol{A}}_{\boldsymbol{A}} \cong \lim\left(\Gamma^{p,q}(F)\right), \qquad {}^{(p,q)}\!\!\int^{A\in\mathcal{C}} F^{\boldsymbol{A}}_{\boldsymbol{A}} \cong \operatorname{colim}\left(⅃^{q,p}(F)\right).$$

PK5) *Higher arity co/kusarigamas from* $(1,1)$*-co/kusarigamas.* The cokusarigama

$$⅃^{p,q}(F) \colon \mathcal{C}^{(q,p)} \to \mathcal{D}$$

of a functor $F \colon \mathcal{C}^{(p,q)} \to \mathcal{D}$ is the left Kan extension of the $(1,1)$-cokusarigama of $\Delta^*_{p,q}(F)$ along $\Delta_{q,p}$:

$$⅃^{p,q}(F) = \mathsf{Lan}_{\Delta_{q,p}}\left(⅃\big(\Delta^*_{p,q}(F)\big)\right) \qquad \begin{tikzcd} & \mathcal{C}^{(q,p)} \arrow[d, dashed, "⅃^{p,q}(F)"] \\ \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[ur, "\Delta_{q,p}"] \arrow[r, "⅃\left(\Delta^*_{p,q}(F)\right)"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

Moreover, if $\mathcal{C}$ has finite products and finite coproducts, then $⅃^{p,q}(-)$ factors as

$$\mathsf{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right) \xrightarrow{\Delta^*_{p,q}} \mathsf{Cat}\left(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right) \xrightarrow{⅃} \mathsf{Cat}\left(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right) \xrightarrow{\left(\mathsf{W}^{q,p}_{i,j}\right)^*} \mathsf{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right).$$

Dually, the kusarigama

$$\Gamma^{q,p}(G)\colon \mathcal{C}^{(p,q)} \to \mathcal{D}$$

of $G\colon \mathcal{C}^{(q,p)} \to \mathcal{D}$ is the right Kan extension of the $(1,1)$-kusarigama of $\Delta^*_{q,p}(G)$ along $\Delta_{p,q}$:

$$\Gamma^{q,p}(G) = \mathsf{Ran}_{\Delta_{p,q}}\left(\Gamma\left(\Delta^*_{q,p}(G)\right)\right) \qquad \begin{tikzcd} & \mathcal{C}^{(p,q)} \arrow[d, dashed, "\Gamma^{q,p}(G)"] \\ \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[ur, "\Delta_{p,q}"] \arrow[r, swap, "\Gamma\left(\Delta^*_{q,p}(G)\right)"] \arrow[ur, phantom, shift right=4, "\Swarrow"] & \mathcal{D}. \end{tikzcd}$$

Moreover, if $\mathcal{C}$ has finite products and finite coproducts, then $\Gamma^{q,p}(-)$ factors as

$$\mathsf{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right) \xrightarrow{\Delta^*_{q,p}} \mathsf{Cat}\left(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right) \xrightarrow{\Gamma} \mathsf{Cat}\left(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right) \xrightarrow{(\mathsf{M}^{p,q})^*} \mathsf{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right).$$

In fact, the adjunction yielding $\text{⅃}^{p,q} \dashv \Gamma^{p,q}$ can be extended as in the following diagram of adjunctions:

$$\begin{tikzcd}[column sep=huge] \left[\mathcal{C}^{(p,q)}, \mathcal{D}\right] \arrow[r, shift left=12, "\mathsf{Lan}_{\mathsf{W}_{p,q}}"] \arrow[r, phantom, shift left=9, "\perp"] \arrow[r, shift left=6, leftarrow, "\mathsf{W}^*_{p,q}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, "\Delta^*_{p,q}" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, leftarrow, "\mathsf{M}^*_{p,q}" description] \arrow[r, phantom, shift right=9, "\perp"] \arrow[r, shift right=12, swap, "\mathsf{Ran}_{\mathsf{M}_{p,q}}"] & \left[\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right] \arrow[r, shift left=2, "\text{⅃}"] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, leftarrow, swap, "\Gamma"] & \left[\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}\right] \arrow[r, shift left=12, leftarrow, "\mathsf{Lan}_{\mathsf{W}_{q,p}}"] \arrow[r, phantom, shift left=9, "\perp"] \arrow[r, shift left=6, "\mathsf{W}^*_{q,p}" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, leftarrow, "\Delta^*_{q,p}" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\mathsf{M}^*_{q,p}" description] \arrow[r, phantom, shift right=9, "\perp"] \arrow[r, shift right=12, leftarrow, swap, "\mathsf{Ran}_{\mathsf{M}_{q,p}}"] & \left[\mathcal{C}^{(q,p)}, \mathcal{D}\right]; \end{tikzcd}$$

see Corollary 1.10.

Proof. We often prove the statements for cokusarigamas only, as the ones for kusarigamas follow by an easy dualisation.

Item PK1: This follows from [Mac98, Theorems IX.7.2 and IX.7.3].

Item PK2: This follows straight from the definition of co/kusarigamas.

Item PK3: For the first statement, we have

$$\begin{aligned} \mathcal{D}\left(1, {}^{(q,p)}\!\!\int_{A\in\mathcal{C}} \mathsf{h}^{(A,\ldots,A)} \pitchfork F^A_A\right) &\cong {}^{(q,p)}\!\!\int_{A\in\mathcal{C}} \mathcal{D}\left(-, \mathsf{h}^{(A,\ldots,A)} \pitchfork F^A_A\right) \\ &\cong {}^{(q,p)}\!\!\int_{A\in\mathcal{C}} \mathsf{h}^{(A,\ldots,A)} \pitchfork \mathcal{D}\left(-, F^A_A\right) \\ &\overset{\text{def}}{=} \Gamma^{p,q}(\mathcal{D}(1,F)). \end{aligned}$$

Item PK4: We just prove the first statement, the other being a straightforward dualisation. We have

$$\begin{aligned} \mathcal{D}\left(-, {}^{(p,q)}\!\!\int_A D^A_A\right) &\overset{\text{def}}{=} \mathrm{DiNat}^{(p,q)}\left(\Delta_{\mathrm{pt}}, h_D\right) \\ &\cong \mathrm{Nat}\left(\Delta_{\mathrm{pt}}, \Gamma^{p,q}(h_D)\right) && \text{by Remark 5.2,} \\ &\cong \mathrm{Nat}\left(\Delta_{\mathrm{pt}}, h_{\Gamma^{p,q}(D)}\right) && \text{by Item PK3,} \\ &\overset{\text{def}}{=} h_{\lim(\Gamma^{p,q}(D))}. \end{aligned}$$

The result then follows from the Yoneda lemma.

Item PK5: We have

$$\begin{aligned}
\mathrm{Nat}\left(\mathsf{Lan}_{\Delta_{q,p}} \daleth(\Delta^*_{p,q}(F)), G\right) &\stackrel{\text{def}}{=} \mathrm{Nat}\left(\daleth(\Delta^*_{p,q}(F)), \Delta^*_{q,p}(G)\right) \\
&\cong \mathrm{DiNat}\left(\Delta^*_{p,q}(F), \Delta^*_{q,p}(G)\right) && \text{by Remark 5.2,} \\
&\cong \mathrm{DiNat}^{(p,q)}(F, G) && \text{by Proposition 2.8,} \\
&\cong \mathrm{Nat}\left(\daleth^{p,q}(F), G\right) && \text{by Remark 5.2 again.}
\end{aligned}$$

The stated factorisation follows from the isomorphism $\mathsf{Lan}_{\Delta_{q,p}} \cong \left(\mathsf{W}^{q,p}_{i,j}\right)^*$ of Corollary 1.10. □

5.2. Examples of co/kusarigamas

Example 5.10 (Cokusarigamas of hom functors)**.** The computation given in the proof of Lemma 3.14 generalises to show that, given $(\underline{A}, \underline{B}) \in C_o^{(p,q)}$, the cokusarigama of the functor $\mathsf{h}_{(\underline{A},\underline{B})} \colon C^{(q,p)} \to \mathsf{Set}$, which may be written as

$$\begin{aligned}
\hom_{C^{(p,q)}}((-,-);(\underline{A},\underline{B})) &\stackrel{\text{def}}{=} \hom_{C^p}(\underline{A},-) \times \hom_{C^q}(-,\underline{B}) \\
&\stackrel{\text{def}}{=} \mathsf{h}^{A_1}_{-_{q+1}} \times \cdots \times \mathsf{h}^{A_p}_{-_{q+p}} \times \mathsf{h}^{-_1}_{B_1} \times \cdots \times \mathsf{h}^{-_q}_{B_q},
\end{aligned}$$

is given by

$$\begin{aligned}
\daleth^{q,p}(\mathsf{h}_{(\underline{A},\underline{B})}) &\stackrel{\text{def}}{=} \int^{X \in C} \mathsf{h}^X_{-_{p+1}} \times \cdots \times \mathsf{h}^X_{-_{p+q}} \times \mathsf{h}^{-_1}_X \times \cdots \times \mathsf{h}^{-_p}_X \times \mathsf{h}^{A_1}_X \times \cdots \times \mathsf{h}^{A_p}_X \times \mathsf{h}^X_{B_1} \times \cdots \times \mathsf{h}^X_{B_q} \\
&\cong \left(\mathsf{h}^{A_1}_{B_1} \times \cdots \times \mathsf{h}^{A_1}_{B_q} \times \cdots \times \mathsf{h}^{A_p}_{B_1} \times \cdots \times \mathsf{h}^{A_p}_{B_q}\right) \\
&\times \left(\mathsf{h}^{A_1}_{-_{p+1}} \times \cdots \times \mathsf{h}^{A_1}_{-_{p+q}} \times \cdots \times \mathsf{h}^{A_p}_{-_{p+1}} \times \cdots \times \mathsf{h}^{A_p}_{-_{p+q}}\right) \\
&\times \left(\mathsf{h}^{-_1}_{B_1} \times \cdots \times \mathsf{h}^{-_1}_{B_q} \times \cdots \times \mathsf{h}^{-_p}_{B_1} \times \cdots \times \mathsf{h}^{-_p}_{B_q}\right) \\
&\times \left(\mathsf{h}^{-_1}_{-_{p+1}} \times \cdots \times \mathsf{h}^{-_1}_{-_{p+q}} \times \cdots \times \mathsf{h}^{-_p}_{-_{p+1}} \times \cdots \times \mathsf{h}^{-_p}_{-_{p+q}}\right)
\end{aligned}$$

Example 5.11 (Co/kusarigamas of constant functors)**.** Let E be a set and let's equally denote $E : C^{(p,q)} \to \mathsf{Set}$ the constant functor on E; assume C has finite products and coproducts; then, we can compute the kusarigama of E as the integral

$$\begin{aligned}
\Gamma(\underline{E}) &\cong \int_A \left(C[\mathbf{Y}|A] \times C[A|\mathbf{X}]\right) \pitchfork E \\
&\cong \int_A \mathsf{Set}\Big(C(A, \textstyle\prod X_i), \mathsf{Set}\big(C(\textstyle\coprod Y_j, A), E\big)\Big) \\
&\cong \mathsf{Cat}(C^{(p,q)}, \mathsf{Set})\Big(C(-, \textstyle\prod X_i), \mathsf{Set}\big(C(\textstyle\coprod Y_j, -), E\big)\Big) \\
&\cong \mathsf{Set}\big(C(Y, X), E\big),
\end{aligned}$$

where $X \stackrel{\text{def}}{=} \prod X_i$, $Y \stackrel{\text{def}}{=} \coprod Y_j$.

In particular, when $\mathcal{D} = \mathsf{Set}$:

$$\Gamma(\mathrm{pt}) = \int_{A \in C} \left[\mathsf{h}_A \times \mathsf{h}^A, \mathrm{pt}\right] \cong \mathrm{pt}.$$

This is in accordance with the fact that dinatural transformations to Δ_{pt} coincide with natural transformations to Δ_{pt}.

Dually,

$$
\begin{aligned}
⅃(E)^{Y}_{X} &= \int^{A} (\mathsf{h}^{A})^{p} \times (\mathsf{h}_{A})^{q} \times E \\
&\cong \int^{A} C[Y|A] \times C[A|\mathbf{X}] \times E \\
&\cong \Big(\int^{A} C(Y,A) \times C(A,X) \Big) \times E \\
&\cong C\big(Y, X\big) \times E, \qquad (5.12)
\end{aligned}
$$

where $X \stackrel{\text{def}}{=} \prod X_i$ and $Y \stackrel{\text{def}}{=} \coprod Y_j$.

Example 5.13 (The co/kusarigama of the identity functor)**.** Let C be a complete and cocomplete category (so that the co/ends in Construction 5.5 and Construction 5.6 exist).

We want to compute the co/kusarigama of the identity functor $\mathrm{id}_{C^{(p,q)}} : C^{(p,q)} \to C^{(p,q)}$. By virtue of the universal property of the product category $C^{(p,q)}$, it is then enough to determine the functor

$$C^{(p,q)} \xrightarrow{⅃^{p,q}(\mathrm{id})} C^{(q,p)} \xrightarrow{\pi_j} C^{\pm}$$

where the functor π_j projects to the factor C for $1 \le j \le q$, and to C^{op} for $p+1 \le j \le q+p$.

In case $(p,q) = (2,1)$ one sees that for objects (X_1, X_2, Y) the diagram

$$
\begin{tikzcd}
\coprod_{f:B\to A} C(X_1,B) \times C(X_2,B) \times C(A,Y) \odot B \arrow[d, shift right, "\beta"'] \arrow[d, shift left, "\alpha"] & \\
\coprod_{A\in C} C(X_1,A) \times C(X_2,A) \times C(A,Y) \odot A \arrow[d, "c"] & \left(\begin{bmatrix} X_1 \\ u\downarrow \\ A \end{bmatrix}, \begin{bmatrix} X_2 \\ v\downarrow \\ A \end{bmatrix}, \begin{bmatrix} A \\ w\downarrow \\ Y \end{bmatrix}, a \right) \arrow[d, mapsto] \\
C(X_1,Y) \times C(X_2,Y) \odot Y & (wu, wv, wa)
\end{tikzcd}
$$

commutes and in fact that it is a coequaliser: every other $\zeta : \coprod_{A\in C} C(X_1,A) \times C(X_2,A) \times C(A,Y) \odot A \to E$ coequalising the pair (α,β) must factor through $C(X_1,Y) \times C(X_2,Y) \odot Y$ with a uniquely determined map.

A standard argument, carried over the general case, to find the coequaliser defining the end and coend in Construction 5.5 and Construction 5.6 now yields

$$⅃^{p,q}(\mathrm{id})(\underline{X},\underline{Y}) = \mathrm{hom}_{\Pi,p,q}(\underline{Y},\underline{X}) \odot (\underline{Y},\underline{X}), \qquad \Gamma^{p,q}(\mathrm{id})(\underline{X},\underline{Y}) = \mathrm{hom}_{\Pi,p,q}(\underline{X},\underline{Y}) \pitchfork (\underline{Y},\underline{X}).$$

Remark 5.14. The previous argument hides a technical point. It holds by virtue of the following fact: if two categories $\mathcal{A}, \mathcal{B}$ are co/tensored over Set, then so is their product $\mathcal{A} \times \mathcal{B}$, with the component-wise action of a functor $\odot : \mathsf{Set} \times \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B}$.

A similar result does *not* hold for a generic base of enrichment.

5.3. Higher arity twisted arrow categories

Classically, it is possible to compute the co/end of a diagram $D\colon C^{\mathrm{op}} \times C \longrightarrow \mathcal{D}$ as the co/limit of D over the *twisted arrow category* $\mathsf{Tw}(C)$ of C, i.e. over the category of elements of the hom functor of C. The purpose of this section is to formulate and prove an analogous description for higher arity co/ends.

In this section, we abbreviate $⅃^{p,q}(\Delta_{\mathrm{pt}})$ as $⅃^{p,q}(\mathrm{pt})$.

Definition 5.15. The *(p,q)-twisted arrow category of C* is the category $\mathsf{Tw}^{(p,q)}(C)$ defined as the category of elements of $⅃^{p,q}(\mathrm{pt})$:

$$\begin{tikzcd}
\mathsf{Tw}^{(p,q)}(C) \arrow[r, two heads, "\Sigma_{(p,q)}"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & C^{(p,q)} \arrow[d, "\daleth^{p,q}(\mathrm{pt})"] \\
\mathsf{pt} \arrow[r, hook, "\ulcorner \mathrm{pt} \urcorner"'] & \mathsf{Set}.
\end{tikzcd}$$

Remark 5.16 (Unwinding Definition 5.15)**.** By the calculation in the proof of Lemma 3.14, we have $\daleth^{p,q}(\mathrm{pt}) \cong \hom_{\Pi,p,q}$. As a result, we see that $\mathsf{Tw}^{(p,q)}(C)$ may be described as the category whose

KCC1) Objects are collections $\{f_{ij} \colon A_i \longrightarrow B_j\}$ of morphisms of $\mathcal{D}$ with $0 \leq i \leq p$ and $0 \leq j \leq q$;
KCC2) Morphisms are collections of factorisations of the codomain through the domain, of the form

$$\begin{tikzcd}
A_i \arrow[r, "f"] & B_j \arrow[d, "\psi_j"] \\
A'_i \arrow[u, "\phi_i"] \arrow[r, "g"'] & B'_j,
\end{tikzcd}$$

one for each $0 \leq i \leq p$ and each $0 \leq j \leq q$.

Lemma 5.17. Let $D \colon C^{(p,q)} \longrightarrow \mathcal{D}$ be a diagram. We have natural isomorphisms

$$^{(p,q)}\!\!\int^{A \in C} D^A_A \cong \operatorname{colim}^{\daleth^{p,q}(\mathrm{pt})}(D) \qquad {}_{(p,q)}\!\!\int_{A \in C} D^A_A \cong \lim{}^{\daleth^{p,q}(\mathrm{pt})}(D),$$

generalising the well-known isomorphisms

$$\int^{A \in C} D^A_A \cong \operatorname{colim}^{\hom_C}(D) \qquad \int_{A \in C} D^A_A \cong \lim{}^{\hom_C}(D),$$

valid for $(p,q) = (1,1)$.

Proof. We have

$$\begin{aligned}
\mathsf{h}\left(-, {}_{(p,q)}\!\!\int_{A \in C} D^A_A\right) &\cong \mathrm{DiNat}(\Delta_{\mathrm{pt}}, \mathsf{h}_D) \\
&\cong \mathrm{Nat}(\daleth^{p,q}(\mathrm{pt}), \mathsf{h}_D) \\
&\cong h_{\lim^{\daleth^{p,q}(\mathrm{pt})}(D)}.
\end{aligned}$$

The proof of the remaining isomorphism is formally dual to the above one. □

Proposition 5.18 ((p,q)-ends as limits, yet again)**.** Let $D : C^{(p,q)} \longrightarrow \mathcal{D}$ be a functor. We have isomorphisms

$$\begin{aligned}
{}_{(p,q)}\!\!\int_{A \in C} D^{\underline{A}}_{\underline{A}} &\cong \lim\left(\mathsf{Tw}^{(p,q)}(C) \xrightarrow{\Sigma_{(p,q)}} C^{(p,q)} \xrightarrow{D} \mathcal{D}\right), \\
^{(p,q)}\!\!\int^{A \in C} D^{\underline{A}}_{\underline{A}} &\cong \operatorname{colim}\left(\mathsf{Tw}^{(p,q)}(C^{\mathsf{op}})^{\mathsf{op}} \xrightarrow{\Sigma_{(p,q)}} C^{(p,q)} \xrightarrow{D} \mathcal{D}\right).
\end{aligned}$$

Proof. This result follows from Lemma 5.17 and the classical description of weighted colimits as conical ones [Kel05, Section 3.4, Equation 3.33]. □

5.4. Twisted arrow categories associated to cokusarigamas

In this short section, we give a co/comma category formula for computing co/kusarigamas. These generalise the construction in Section 5.3 and work for arbitrary (p, q). However, these turn out to be too complicated for $p, q \geq 2$ as to be practically useful[7], so we restrict our attention to the case $(p, q) = (1, 1)$ below. Let $F \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor and fix $A, B \in \mathcal{C}_o$.

Definition 5.19. The *twisted arrow category of* $\mathcal{C}$ *for* $(1, 1)$*-cokusarigamas at* (A, B) is the category $\mathsf{Tw}^{A,B}_{\lrcorner}(\mathcal{C})$ defined as the category of elements of $\mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{-_2} \times \mathsf{h}^{-_1}_{B} \times \mathsf{h}^{-_1}_{-_2}$.

Remark 5.20. Concretely, $\mathsf{Tw}^{A,B}_{\lrcorner}(\mathcal{C})$ may be described as the category whose

KCC1) Objects are squares of the form

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & B \\ Y & A \arrow[l, "\psi"] \arrow[u, "g"'] \end{tikzcd}$$

with $X, Y \in \mathcal{C}_o$ and $f, g, \phi, \psi \in \mathrm{Mor}(\mathcal{C})$;

KCC2) Morphisms are twisted commutative cubes

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dd, "f"'] & & B & \\ & X' \arrow[ul, "\xi_1"] \arrow[rr, "\phi'"] \arrow[dd, "f'"] & & B \arrow[ul, "\xi_3"'] \\ Y \arrow[dr, "\xi_2"'] & & A \arrow[ll, "\psi"'] \arrow[uu, "g"] \arrow[dr, "\xi_4"] & \\ & Y' & & A \arrow[ll, "\psi'"] \arrow[uu, "g'"'] \end{tikzcd}$$

Remark 5.21 ($\mathsf{Tw}^{A,B}_{\lrcorner}(\mathcal{C})$ as a generalisation of the twisted arrow category)**.** The twisted arrow category of $\mathcal{C}$ naturally fits inside $\mathsf{Tw}^{A,B}_{\lrcorner}(\mathcal{C})$:

$$\begin{tikzcd} W \arrow[r, "f"] & X \arrow[d, "q"] \\ Y \arrow[u, "p"] \arrow[r, "g"'] & Z \end{tikzcd} \rightsquigarrow \begin{tikzcd} W \arrow[rr] \arrow[dd, "f"'] & & \bullet & \\ & Y \arrow[ul, "p"'] \arrow[rr] \arrow[dd, "g"] & & \bullet \arrow[ul] \\ X \arrow[dr, "q"'] & & \bullet \arrow[ll] \arrow[uu] \arrow[dr] & \\ & Z & & \bullet \arrow[ll] \arrow[uu] \end{tikzcd}.$$

This comes from the identity $\lrcorner(\mathrm{pt}) \cong \hom$.

[7]Similarly to how a morphism of $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ turned out to involve pq arrows of $\mathcal{C}$, unravelling the construction given in this section for arbitrary (p, q) gives a category $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ whose morphisms now consist of $4pq$ morphisms of $\mathcal{C}$. Additionally, *each of these* points now in a different directions (i.e. they cannot anymore be arranged as morphisms in product categories). Together, these two points make $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ unusable in practice when p and q are too large. As a compromise, we work out the case $(p, q) = (1, 1)$, which is both the simplest case as well as the most useful one.

Proposition 5.22 (Co/kusarigamas as limits)**.** Given a functor $D\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$, we have isomorphisms

$$⅃(D)^{A}_{B} \cong \operatorname{colim}\left(\mathsf{Tw}^{A,B}_{⅃}(\mathcal{C}) \xrightarrow{\mathrm{pr}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{D} \mathcal{D}\right),$$

$$\Gamma(D)^{A}_{B} \cong \lim\left(\mathsf{Tw}^{A,B}_{⅃}(\mathcal{C}) \xrightarrow{\mathrm{pr}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{D} \mathcal{D}\right).$$

Proof. Firstly, observe that we may compute $⅃(D)$ as the following weighted coend:

$$\begin{aligned}\int^{X\in\mathcal{C}}_{[\mathsf{h}^{A}_{-_2} \times \mathsf{h}^{-_1}_{B}]} D^{X}_{X} &\cong \int^{X\in\mathcal{C}} \left(\mathsf{h}^{A}_{X} \times \mathsf{h}^{X}_{B}\right) \odot D^{X}_{X} \\ &\cong ⅃(D)^{A}_{B}.\end{aligned}$$

Now, weighted coends corepresent functors of the form $\mathrm{DiNat}(W, \mathsf{h}^{D})$, but since $\mathrm{DiNat}(W, \mathsf{h}^{D}) \cong \mathrm{Nat}(⅃(W), \mathsf{h}^{D})$, we see that the above weighted coend is the weighted colimit of D by $⅃(\mathsf{h}^{A}_{-_2} \times \mathsf{h}^{-_1}_{B})$. From the computation in the proof of Lemma 3.14, we have $⅃(\mathsf{h}^{A}_{-_2} \times \mathsf{h}^{-_1}_{B}) \cong \mathsf{h}^{A}_{B} \times \mathsf{h}^{A}_{-_2} \times \mathsf{h}^{-_1}_{B} \times \mathsf{h}^{-_1}_{-_2}$. The result then follows from the classical description of weighted colimits as conical ones [Kel05, Section 3.4, Equation 3.33].

The second formula is proved in a dual fashion. □

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