Logic in 2D, Metalogic in 3D: The Language of Category Theory

Christian Williams

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https://sites.google.com/view/logic-in-color

The Language of Category Theory

Category theory is known as a language of mathematics.

Applied CT: developing a language for all kinds of science.

My thesis proposes that

category theory is **the language of thinking**.

Categories form a *bifibrant double category*,
which can be seen as a *logic*: a system of “thoughts of a world”.

We define the 3D language of all bifibrant double categories,
which can be seen as *metalogic*: “thinking about thinking”.

Logics

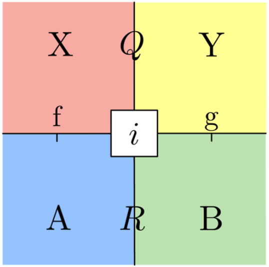
The fundamental notions of category theory

type and process, relation and transformation
composition and identity, adjunction and representation

are systematized in the language of a (bi)*fibrant double category*, also known as proarrow equipment, or framed bicategory. [3]

We understand this structure as a **logic**.

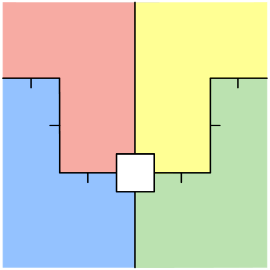
fib. dbl. cat.	dim.	**logic**
object	0	type
tight arrow	V	process
loose arrow	H	relation
square	2	inference



Logics

The “bifibrance” of a double category is the *action* of processes on relations, pushing forward or pulling backwards in “time”.

This property-like structure is *essential*,
both for the expressiveness of a logic,
and the coherence and expressiveness of metalogic.



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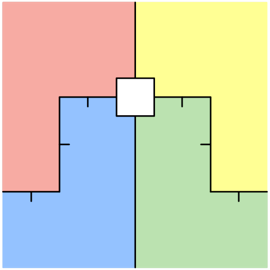
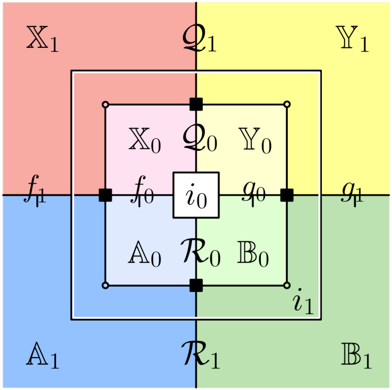


substitution

Metalogic

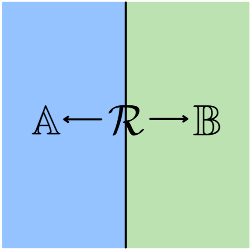
Logics form a three-dimensional multiverse
(FDCs form a fibrant triple category w/o interchange)



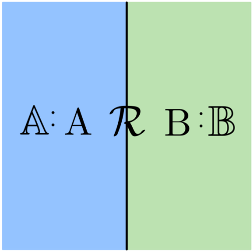
which we can explore in both *imagery* and *syntax*.

Color syntax

Imagery is *dual* to syntax; so they unite in **color syntax**: a string diagram is a general concept, and *substituting* syntax determines a specific instance of the concept.



span category
$\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$

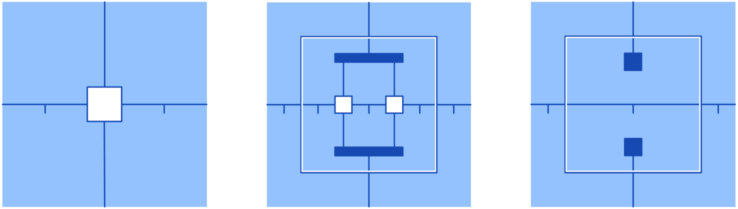


fiber category
$\mathcal{R}(\mathrm{A}, \mathrm{B}) : \mathrm{Cat}$

"total" $\simeq$ "fiberwise" is the basis of dependent category theory.

The metalogic of logics

The key to metalogic is to see a logic as like a category:
a category is a matrix with composition and identity;
a logic is a *matrix category* with composition and identity.



logic composition identity

"Matrix category" is a short name for *two-sided bifibration*,
so a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a *bifibrant double category*.

The metalogic of logics

Matrix categories are *exponentiable*, so metalogic is *higher-order*: the **co/descent calculus** is the higher co/end calculus. [2]

$$\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R} \otimes \mathcal{S}, \mathcal{T})$$

$$\begin{aligned}
&= \vec{\Pi} A, C && \mathbb{C}\mathrm{at}((\mathcal{R} \otimes \mathcal{S})(A, C), \mathcal{T}(A, C)) \\
&= \vec{\Pi} A, C && \mathbb{C}\mathrm{at}(\vec{\Sigma} B \;\; \mathcal{R}(A, B) \times \mathcal{S}(B, C), \mathcal{T}(A, C)) \\
&\simeq \vec{\Pi} A, C \;\; \vec{\Pi} B && \mathbb{C}\mathrm{at}(\mathcal{R}(A, B) \times \mathcal{S}(B, C), \mathcal{T}(A, C)) \\
&\simeq \vec{\Pi} A, B, C && \mathbb{C}\mathrm{at}(\mathcal{S}(B, C), [\mathcal{R}(A, B) \to \mathcal{T}(A, C)]) \\
&\simeq \vec{\Pi} B, C && \mathbb{C}\mathrm{at}(\mathcal{S}(B, C), \vec{\Pi} C \;\; [\mathcal{R}(A, B) \to \mathcal{T}(A, C)])
\end{aligned}$$

$$= \quad \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{S}, [\mathcal{R} \to \mathcal{T}])$$

A system for double weighted co/limits, and much more.

Outline

We develop the underlying structure of a logic;
then metalogic is the monad completion of this structure.

- A *span of categories* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$
  $\simeq$ a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$.
- The *weave double category* is the equational logic of $\mathbb{A}$.

$$\langle \mathbb{A} \rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$$

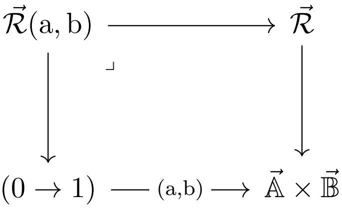
- A *matrix category* is a bimodule of weave double categories.

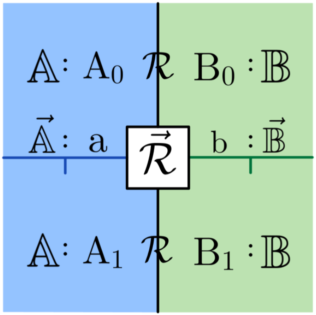
A *logic*, or *bifibrant double category*,
is a matrix category $\underline{\mathbb{C}} \leftarrow \mathbb{C} \rightarrow \underline{\mathbb{C}}$ with comp. and identity.

Spans of categories

The basic data of a logic is a span of categories:
relations and inferences, over pairs of types and processes.

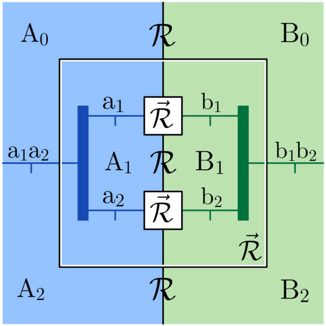
A span of categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ $\sim$ a matrix of categories: [4]
a **displayed category** is a normal lax functor $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$.





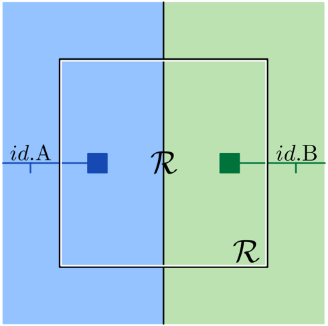
$\vec{\mathcal{R}}(a, b) : \mathcal{R}(A_0, B_0) \mid \mathcal{R}(A_1, B_1)$

Spans of categories



composition

$\vec{\mathcal{R}}(a_1, b_1) \circ \vec{\mathcal{R}}(a_2, b_2) \Rightarrow \vec{\mathcal{R}}(a_1 a_2, b_1 b_2)$

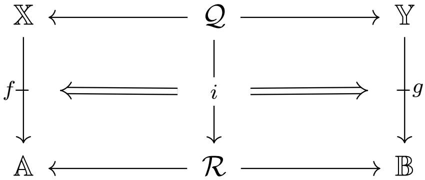


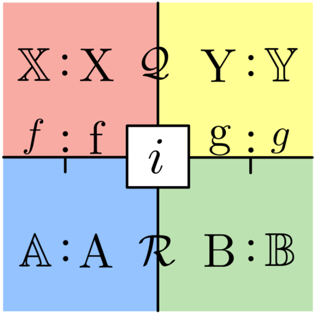
identity

$\mathcal{R}(A, B) \Rightarrow \vec{\mathcal{R}}(id.A, id.B)$

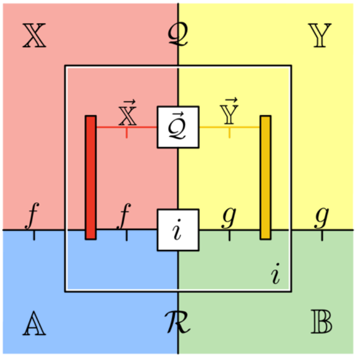
Spans of profunctors

This idea generalizes to spans of profunctors $f \leftarrow i \rightarrow g$.
A **displayed profunctor** is a map $i(f, g) : \mathrm{Prof}$ which forms a bimodule of lax functors $\mathcal{Q}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{R}(\mathbb{A}, \mathbb{B})$.



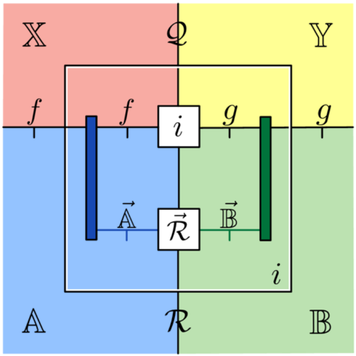


$i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \mid \mathcal{R}(\mathrm{A}, \mathrm{B})$



precomposition

$\vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ \imath(\mathrm{f},\mathrm{g}) \Rightarrow \imath(\mathrm{xf},\mathrm{yg})$



postcomposition

$\imath(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow \imath(\mathrm{fa},\mathrm{gb})$

Equivalence: spans are matrices

Inverse image is functorial, defining “displayed functors” and “displayed transformations”.

Theorem

The double category of span categories is equivalent to the double category of displayed categories.

	SpanCat	$\simeq$	DisCat
0	*span cat.* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$	$\sim$	*dis. cat.* $\mathcal{R}(\mathrm{A}, \mathrm{B}) : \mathrm{Cat}$
V	*span fun.* $[\![\mathcal{R}]\!] : \mathcal{R}_0 \rightarrow \mathcal{R}_1$	$\sim$	*dis. fun.* $[\![\mathcal{R}]\!] : \mathcal{R}_0(\mathrm{A}_0, \mathrm{B}_0) \rightarrow \mathcal{R}_1([\![\mathrm{A}_0]\!], [\![\mathrm{B}_0]\!])$
H	*span prof.* $f \leftarrow i \rightarrow g$	$\sim$	*dis. prof.* $i(f, g) : \mathrm{Prof}$
2	*span trans.* $[\![i]\!] : i_0 \rightarrow i_1$	$\sim$	*dis. trans.* $[\![i]\!] : i_0(\mathrm{f}_0, \mathrm{g}_0) \Rightarrow i_1([\![\mathrm{f}_0]\!], [\![\mathrm{g}_0]\!])$

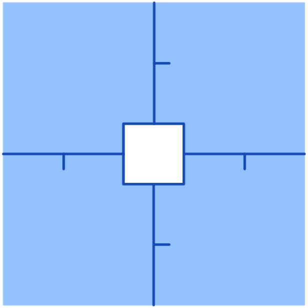
Arrow double categories

If $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is to be *relations* from $\mathbb{A}$ to $\mathbb{B}$, then relations should *vary* over processes in $\mathbb{A}$ and $\mathbb{B}$.

The **arrow double category** $\overrightarrow{\mathbb{A}}$ is that of commuting squares.

$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{\mathrm{a}}_0^1} & \mathrm{A}_0^1 \\ {\scriptstyle \mathrm{a}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_1^0 & \xrightarrow{\hat{\mathrm{a}}_1^1} & \mathrm{A}_1^1 \end{array}$$

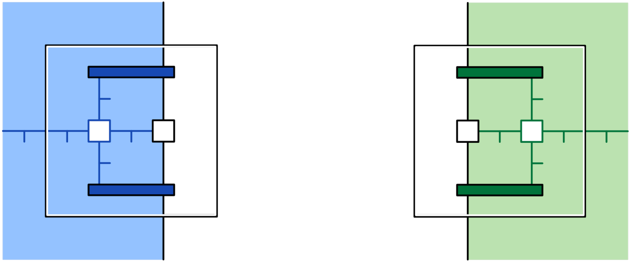
$$(\mathrm{a}_0, \mathrm{a}_1) : \overrightarrow{\mathbb{A}}(\hat{\mathrm{a}}_0^1 \rightarrow \hat{\mathrm{a}}_1^1)$$



Now $\mathbb{A} \leftarrow \overrightarrow{\mathbb{A}} \rightarrow \mathbb{A}$ and $\mathbb{B} \leftarrow \overrightarrow{\mathbb{B}} \rightarrow \mathbb{B}$ can *act* on $\mathcal{R}$.

Fibered and opfibered categories

A **fibered category** over $\mathbb{A}$ is a left $\overrightarrow{\mathbb{A}}$-module. [5]
An **opfibered category** over $\mathbb{B}$ is a right $\overrightarrow{\mathbb{B}}$-module.



substitution

$\odot : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1) \to \mathcal{R}(\mathrm{A}_0)$

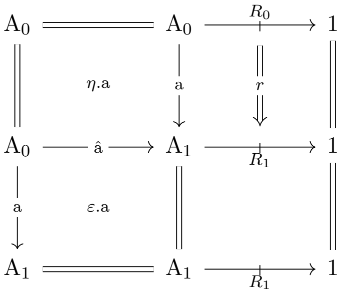
image

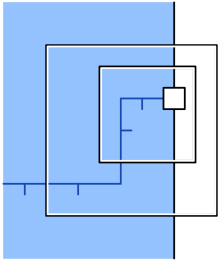
$\odot : \mathcal{R}(\mathrm{B}_0) \times \overrightarrow{\mathbb{B}}(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{B}_1)$

These are often denoted $\mathrm{a}^* R$ “pullback” and $\mathrm{b}_! R$ “pushforward”.

Fibered and opfibered categories

In a fibered category $\mathcal{R}$ over $\mathbb{A}$, a morphism $r : R_0 \to R_1$ over $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1)$ is equivalent to $\eta.\mathrm{a} \circ r : R_0 \to \hat{\mathrm{a}} \odot R_1$ over $\mathrm{id}.\mathrm{A}_0$, by factoring through the **cartesian** morphism $\varepsilon.\mathrm{a} \circ \mathrm{id}.R_1$.



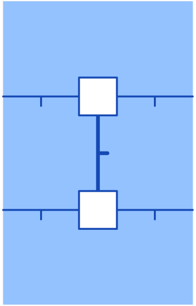


This gives a contravariant representation of morphisms over a.

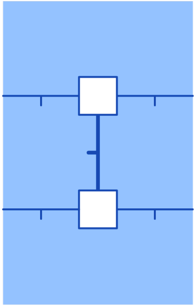
$$\vec{\mathcal{R}}(\mathrm{a})(R_0, R_1) \cong \mathcal{R}(R_0, \mathrm{a} \odot R_1)$$

Weave double category

Yet an arrow double category is not a *logic*.
There is a limitation to the equational reasoning of $\overrightarrow{\mathbb{A}}$.



$(\mathrm{a}_0, \mathrm{a}_1 \cdot \mathrm{a}_2) = (\mathrm{a}_0 \cdot \mathrm{a}_1, \mathrm{a}_2)$

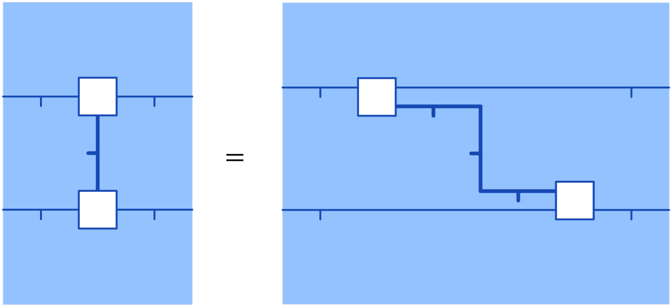


$(\mathrm{a}_0 \cdot \mathrm{a}_1, \mathrm{a}_2) = (\mathrm{a}_0, \mathrm{a}_1 \cdot \mathrm{a}_2)$

Composable pairs are only defined *up to associativity*.

Weave double category

The latter cannot be expressed in the arrow double category.



So, we define the *weave double category*:
the union of the arrow double category $\overrightarrow{\mathbb{A}}$ with its opposite $\overleftarrow{\mathbb{A}}$.

Weave double category

Let $\mathbb{A}$ be a category, with arrow double category $\overrightarrow{\mathbb{A}}$.
The **op-arrow double category** $\overleftarrow{\mathbb{A}}$ is the horizontal opposite.

$$\overleftarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \equiv \overrightarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$$

Denote an **arrow** $\hat{\mathrm{a}} : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1)$, and an **op-arrow** $\check{\mathrm{a}} : \overleftarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$.

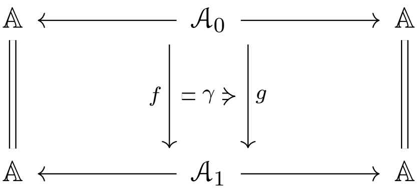
$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{\mathrm{a}}_1^0} & \mathrm{A}_1^0 \\ \downarrow{\scriptstyle \mathrm{a}_0} & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_0^1 & \xrightarrow[\hat{\mathrm{a}}_1^1]{} & \mathrm{A}_1^1 \end{array} \qquad \begin{array}{ccc} \mathrm{A}_0^0 & \xleftarrow{\check{\mathrm{a}}_1^0} & \mathrm{A}_1^0 \\ \downarrow{\scriptstyle \mathrm{a}_0} & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_0^1 & \xleftarrow[\check{\mathrm{a}}_1^1]{} & \mathrm{A}_1^1 \end{array}$$

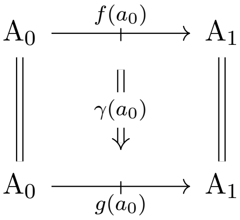
We use $\bar{\mathrm{a}}$ for objects of $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$.

Weave double category

Define $\mathrm{Dbl}_{\mathbb{A}}$ be the 2-category of double categories on $\mathbb{A}$, double functors over $\mathrm{id}.\mathbb{A}$, and identity-component transformations.

Given double categories $\mathcal{A}_0$ and $\mathcal{A}_1$ on $\mathbb{A}$, and double functors $f, g : \mathcal{A}_0 \to \mathcal{A}_1$ over $\mathrm{id}.\mathbb{A}$, an icon $\gamma : f \Rightarrow g$ gives for each $a_0 : \mathcal{A}_0$ a 2-morphism $\gamma(a_0) : f(a_0) \Rightarrow g(a_0)$, subject to naturality.





Weave double category

Let $\mathbb{A}$ be a category. The **weave double category** $\langle\mathbb{A}\rangle$ is the coproduct of the arrow and op-arrow double categories in $\mathrm{Dbl}_{\mathbb{A}}$.

$$\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$$

$\langle\mathbb{A}\rangle$ is generated by squares of $\overrightarrow{\mathbb{A}}$, opsquares of $\overleftarrow{\mathbb{A}}$, and isomorphisms of identity arrows and op-arrows.

$$\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$$

Theorem

$\langle\mathbb{A}\rangle$ *is a logic.*

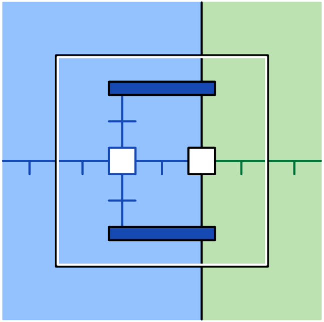
Theorem

$\langle\mathbb{A}\rangle$*-modules are bifibered categories over* $\mathbb{A}$*.*

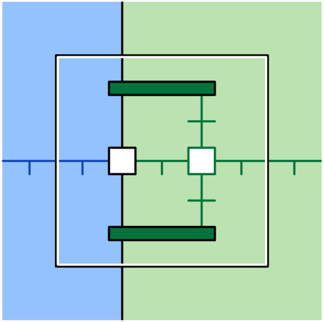
Matrix categories

Let $\mathbb{A}, \mathbb{B}$ be categories, with weave double categories $\langle\mathbb{A}\rangle, \langle\mathbb{B}\rangle$.

A **matrix category** or **two-sided bifibration** $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ which is a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$.



$\odot_{\mathbb{A}} : \langle\mathbb{A}\rangle(A_0, A_1) \times \mathcal{R}(A_1, B) \rightarrow \mathcal{R}(A_0, B)$

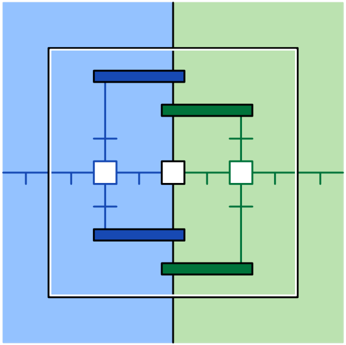


$\odot_{\mathbb{B}} : \mathcal{R}(A, B_0) \times \langle\mathbb{B}\rangle(B_0, B_1) \rightarrow \mathcal{R}(A, B_1)$

Matrix categories

The actions of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ on $\mathcal{R}$ are associative and unital up to coherent isomorphism.

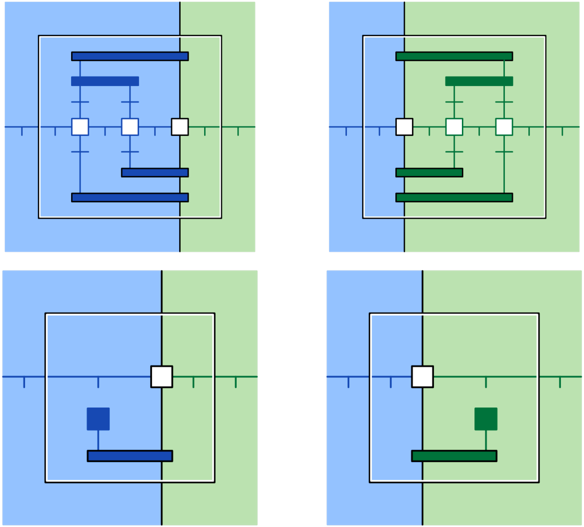
$$\begin{array}{ccc}
\langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle & \xrightarrow{\langle\mathbb{A}\rangle * \odot_{\mathbb{B}}} & \langle\mathbb{A}\rangle * \mathcal{R} \\
\downarrow{\scriptstyle \odot_{\mathbb{A}} * \langle\mathbb{B}\rangle} & \Downarrow{\scriptstyle \alpha_{\mathcal{R}}} & \downarrow{\scriptstyle \odot_{\mathbb{A}}} \\
\mathcal{R} * \langle\mathbb{B}\rangle & \xrightarrow{\odot_{\mathbb{B}}} & \mathcal{R}
\end{array}$$



center associator

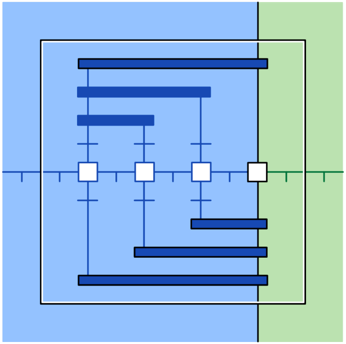
$$\alpha_{\mathcal{R}} : \ \bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) \cong (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$$

Matrix categories

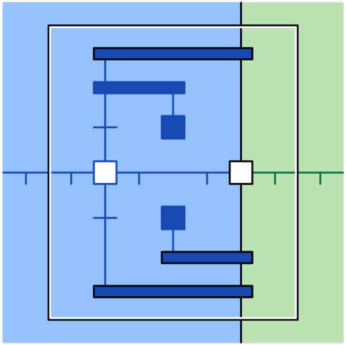


Matrix categories

The coherence means that reassociating a composite is well-defined, and reassociating a unit is well-defined.



$$(\langle \bar{a}_k \rangle \circ \langle \bar{a}_\ell \rangle \circ \langle \bar{a}_m \rangle) \odot R$$
$$\Rrightarrow \langle \bar{a}_k \rangle \odot (\langle \bar{a}_\ell \rangle \odot (\langle \bar{a}_m \rangle \odot R))$$



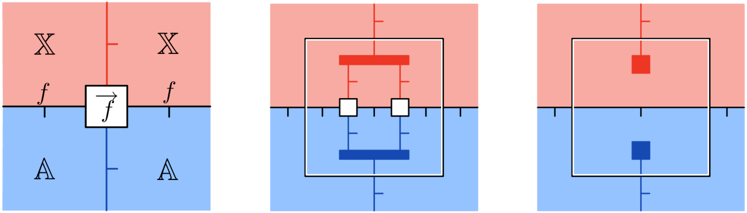
$$(\langle \bar{a}_k \rangle \circ \mathrm{id.A}) \odot R$$
$$\Rrightarrow \langle \bar{a}_k \rangle \odot (\mathrm{id.A} \odot R)$$

Matrix profunctors

We now define relations of matrix categories.

Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor; then the **arrow profunctor** of arrow categories $\overrightarrow{f} : \overrightarrow{\mathbb{X}} \mid \overrightarrow{\mathbb{A}}$ consists of commutative squares; its projections form a span profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$.

$$\overrightarrow{f}(\hat{\mathrm{x}}, \hat{\mathrm{a}}) = \{(\mathrm{f}_0 : f(\mathrm{X}_0, \mathrm{A}_0), \mathrm{f}_1 : f(\mathrm{X}_1, \mathrm{A}_1)) \mid \mathrm{a} \cdot \mathrm{f}_0 = \mathrm{f}_1 \cdot \mathrm{x}\}$$



This forms a *vertical profunctor* of arrow double categories.

Matrix profunctors

Dually, the **op-arrow profunctor** of f is the profunctor of op-arrow categories $\overleftarrow{f} : \overleftarrow{\mathbb{X}} \mid \overleftarrow{\mathbb{A}}$.

$$\overleftarrow{f}(\check{\mathrm{x}}, \check{\mathrm{a}}) = \{\mathrm{f}_0 : f(\mathrm{X}_0, \mathrm{A}_0), \mathrm{f}_1 : f(\mathrm{X}_1, \mathrm{A}_1) \mid \mathrm{x} \cdot \mathrm{f}_0 = \mathrm{f}_1 \cdot \mathrm{a}\}$$

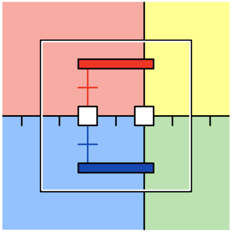
The **weave vertical profunctor** of weave double categories $\langle f \rangle : \langle \mathbb{X} \rangle \mid \langle \mathbb{A} \rangle$ is the coproduct of $\overrightarrow{f}$ and $\overleftarrow{f}$ in the category of vertical profunctors over f.

Just like the weave double category, this is generated from squares and opsquares in f, plus the actions of $\langle \mathbb{X} \rangle$ and $\langle \mathbb{A} \rangle$, subject to naturality of the isomorphisms $\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$.

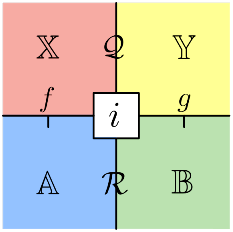
Matrix profunctors

Let $\mathcal{Q}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{R}(\mathbb{A}, \mathbb{B})$ be matrix categories.
Let $f : \mathbb{X} \mid \mathbb{A}$ and $g : \mathbb{Y} \mid \mathbb{B}$ be profunctors,
with weave profunctors $f \leftarrow \langle f \rangle \rightarrow f$ and $g \leftarrow \langle g \rangle \rightarrow g$.

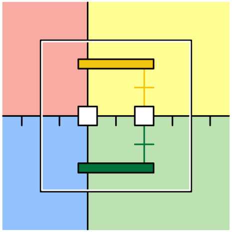
A **matrix profunctor** $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a span profunctor which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$, coherent with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$.



$\odot_f : \langle f \rangle * i \rightarrow i$



$i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$

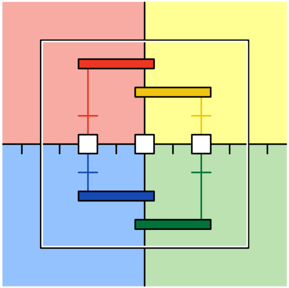


$\odot_g : i * \langle g \rangle \rightarrow i$

Matrix profunctors

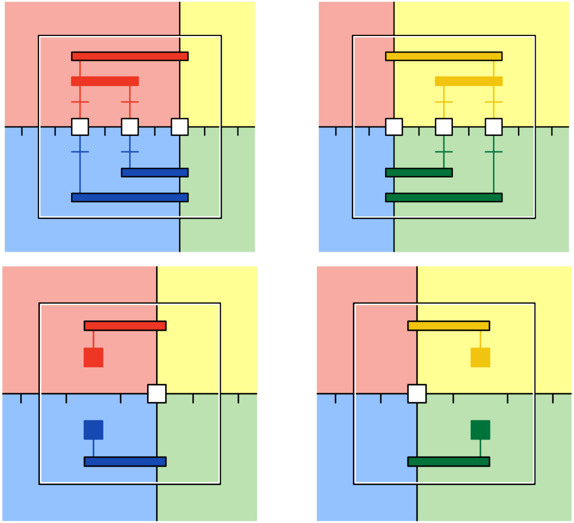
The matrix profunctor $i(f, g)$ is a relation of matrix categories $\mathcal{Q}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{R}(\mathbb{A}, \mathbb{B})$, so it coheres with associators and unitors.

$$\begin{array}{ccc} \bar{\mathrm{x}} \odot (Q \odot \bar{\mathrm{y}}) & \xrightarrow{\alpha_{\mathcal{Q}}} & (\bar{\mathrm{x}} \odot Q) \odot \bar{\mathrm{y}} \\ \Big\downarrow {\scriptstyle \mathrm{f}\odot(i\odot\mathrm{g})} & & \Big\downarrow {\scriptstyle (\mathrm{f}\odot i)\odot\mathrm{g}} \\ \bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) & \xrightarrow[\alpha_{\mathcal{R}}]{} & (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}} \end{array}$$



associator coherence

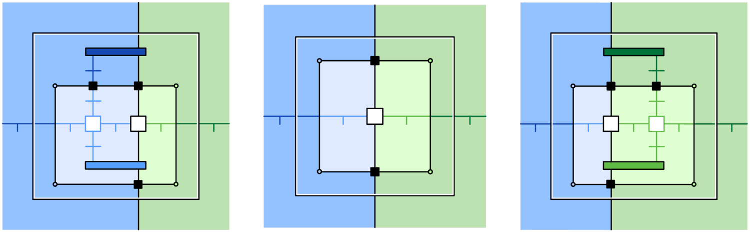
Matrix profunctors



Matrix functors and transformations

Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $\mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ and $\mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix categories.

A **matrix functor** $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ is a morphism of bimodules, preserving composition and identity up to coherent isos.



left join

$[\![\langle \bar{a}_k \rangle]\!] \odot_1 [\![R]\!] \;\cong\; [\![\langle \bar{a}_k \rangle \odot_0 R]\!]$

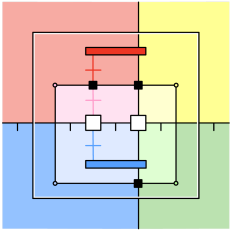
right join

$[\![R]\!] \odot_1 [\![\langle \bar{b}_\ell \rangle]\!] \cong [\![R \odot_0 \langle \bar{b}_\ell \rangle]\!]$

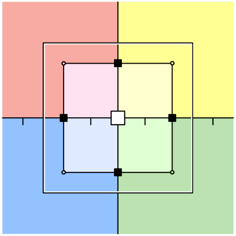
Matrix functors and transformations

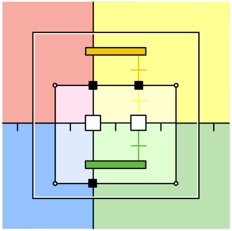
Let $[\![\mathcal{Q}]\!](\mathbb{X}, \mathbb{Y})$ and $[\![\mathcal{R}]\!](\mathbb{A}, \mathbb{B})$ be matrix functors, and let $i_0(f_0, g_0) : \mathcal{Q}_0 \mid \mathcal{R}_0$ and $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be matrix profunctors.

A **matrix transformation** $[\![i]\!] : i_0 \to i_1$ is a span transformation which coheres with the left and right joins of $[\![\mathcal{Q}]\!]$ and $[\![\mathcal{R}]\!]$.



$[\![\mathrm{x}]\!] \odot [\![Q]\!] \rightrightarrows [\![\mathrm{a} \odot R]\!]$



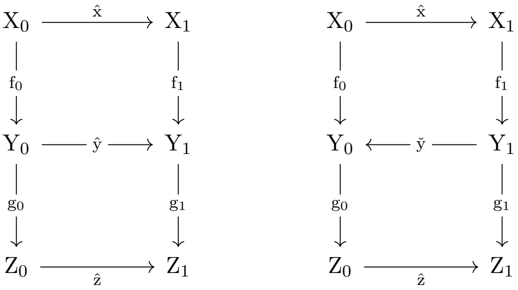


$[\![Q]\!] \odot [\![\mathrm{y}]\!] \rightrightarrows [\![R \odot \mathrm{b}]\!]$

Sequential composition

We now see how matrix categories and functors, matrix profunctors and transformations form a *logic*.

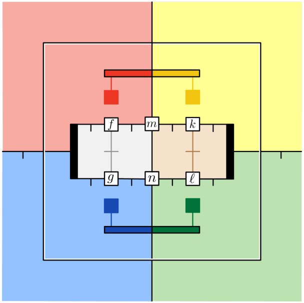
How do we compose matrix profunctors? By using *weaves*.

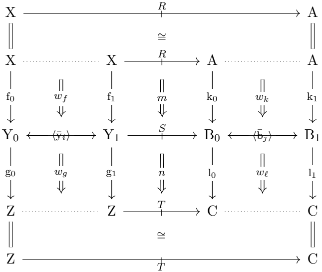


Both squares of $\langle f \circ g \rangle$ can be expressed in $\langle f \rangle \circ \langle g \rangle$ — so an action by $\langle f \rangle$ and one by $\langle g \rangle$ defines an action by $\langle f \circ g \rangle$.

Sequential composition

So, we ensure the actions are *well-defined* on the *identities*, associativity zig-zags in $\langle f \circ g \rangle$ and $\langle k \circ \ell \rangle$: so to compose $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$, we quotient $m \circ n$ by their actions.





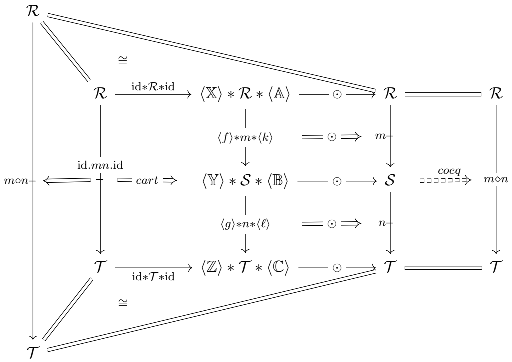
$$[S.(m,n)] \;\equiv\; [u_{\mathcal{R}} \cdot (\langle \bar{y}_i \rangle \odot S \odot \langle \bar{b}_j \rangle).(w_f \odot m \odot w_k,\; w_g \odot n \odot w_\ell) \cdot u_{\mathcal{T}}^{-1}]$$

Sequential composition

Let $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ be matrix profunctors. The **sequential composite**

$$(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$$

is the following coequalizer.



The logic of matrix categories

Theorem

Matrix categories form a logic.

Proof.

As sequential composition of matrix profunctors is defined by coequalizer, it is canonically functorial. The associator and unitors are inherited from $\mathrm{Span}\mathbb{C}\mathrm{at}$, because the coequalizer is orthogonal to span profunctor composition.

Hence $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a double category. Moreover it is a logic: substitution of matrix functors in matrix profunctors is exactly analogous to that of functors in profunctors, in $\mathbb{C}\mathrm{at}$. □

The logic of matrix categories

A **double fibration** [1] is a category in the 2-category of fibered categories, fibered functors, and fibered transformations.

$$\begin{array}{ccccc} \mathrm{MatCat} & \longleftarrow & \mathrm{MatProf} & \longrightarrow & \mathrm{MatCat} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cat} \times \mathrm{Cat} & \longleftarrow & \mathrm{Prof} \times \mathrm{Prof} & \longrightarrow & \mathrm{Cat} \times \mathrm{Cat} \end{array}$$

Theorem

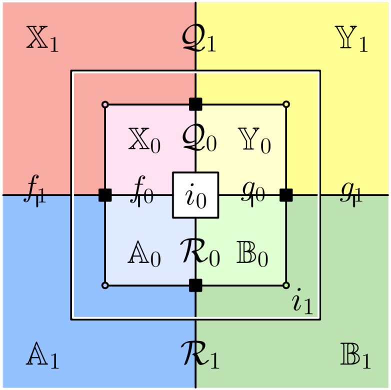
Matrix categories are fibered over pairs of categories.

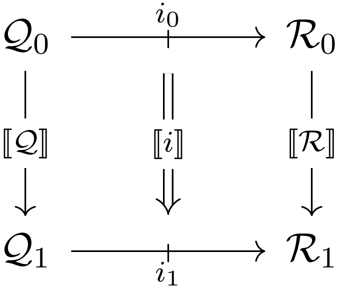
Proof.

Substitution of functors in matrix categories, and transformations in matrix profunctors, is defined by pullback. Matrix profunctor composition preserves substitution. □

The logic of matrix categories

This is the logic of matrix categories, over pairs of categories.





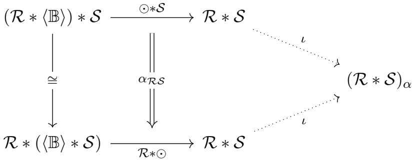
$\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$

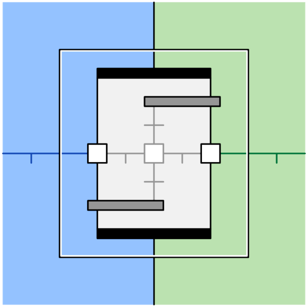
Now, we define *parallel composition* of matrix categories.

Parallel composition

Now, we define composition of matrix categories.

Let $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,\|\, \mathbb{C}$ be matrix categories.
The **parallel composite** $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,\|\, \mathbb{C}$ is constructed as follows.
On $\mathbb{A} \leftarrow \mathcal{R} * \mathcal{S} \rightarrow \mathbb{C}$ we form the *iso-coinserter* of actions by $\langle \mathbb{B} \rangle$.

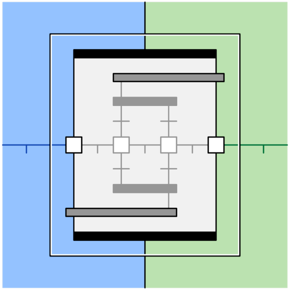


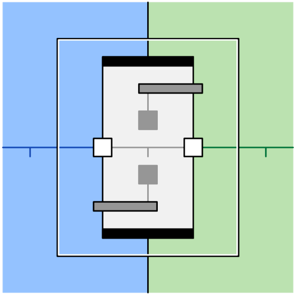


This adjoins an associator $\alpha_{\mathcal{R}\mathcal{S}} : \mathrm{B}_0.(R, \bar{\mathrm{b}} \odot S) \;\cong\; \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, S)$.

Parallel composition

On the associator, two equations are imposed by *coequifier*, for reassociating a composite and a unit.





associator coherence

$(R, \bar{\mathrm{b}}_1 \odot (\bar{\mathrm{b}}_2 \odot S))$
$\Rrightarrow ((R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2), S)$

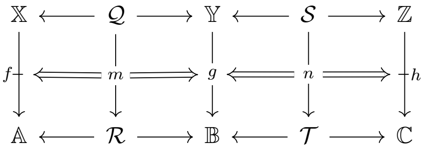
unitor coherence

$(R, \overline{\mathrm{id}}.\mathrm{B} \odot S)$
$\Rrightarrow (R \odot \overline{\mathrm{id}}.\mathrm{B}, S)$

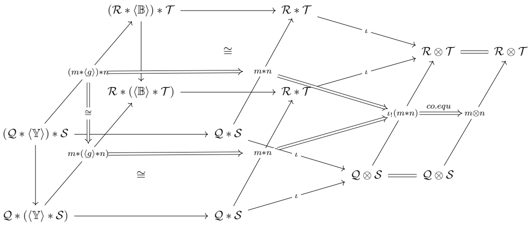
Hence $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \,||\, \mathbb{C}$ is a *codescent object*. [5]

Parallel composition

Let $m(f,g)$ and $n(g,h)$ be matrix profunctors.



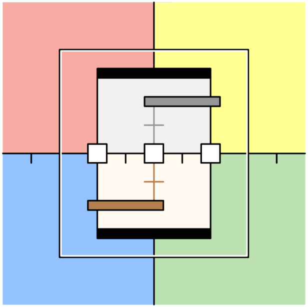
The **parallel composite** matrix profunctor $m \otimes n : \mathcal{Q} \otimes \mathcal{S} \mid \mathcal{R} \otimes \mathcal{T}$ is the following coequalizer.



Parallel composition

So the elements of $(m \otimes n)(f, h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X}, \mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A}, \mathbb{C})$ are composites of: morphisms $\mathrm{y}.(q, s)$, associators $\alpha_{\mathcal{Q}\mathcal{S}}$, elements $\mathrm{g}.(m, n)$, associators $\alpha_{\mathcal{R}\mathcal{T}}$, and morphisms $\mathrm{b}.(r, t)$, such that for any $[\mathrm{g}_0, \mathrm{g}_1] : \langle g \rangle (\bar{\mathrm{y}}, \bar{\mathrm{b}})$ and $m : m(\mathrm{f}, \mathrm{g}_0)$, $n : n(\mathrm{g}_1, \mathrm{h})$ the following commutes.

$$\begin{array}{ccc}
\mathrm{Y}_0.(Q, \bar{\mathrm{y}} \odot S) & \xrightarrow{\alpha_{\mathcal{Q}\mathcal{S}}} & \mathrm{Y}_1.(Q \odot \bar{\mathrm{y}}, S) \\
\Big\downarrow {\scriptstyle \mathrm{g}_0.(m,[\mathrm{g}_0,\mathrm{g}_1]\odot n)} & & \Big\downarrow {\scriptstyle \mathrm{g}_1.(m\odot[\mathrm{g}_0,\mathrm{g}_1],n)} \\
\mathrm{B}_0.(R, \bar{\mathrm{b}} \odot T) & \xrightarrow[\alpha_{\mathcal{R}\mathcal{T}}]{} & \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, T)
\end{array}$$

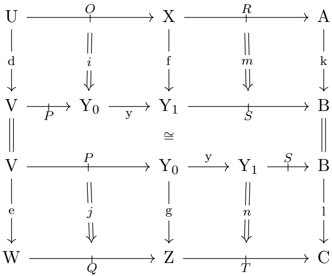


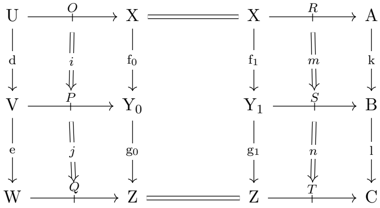
Parallel composition

Parallel composition does *not* preserve sequential composition.

$$(i \otimes m) \diamond (j \otimes n) \qquad \not\leftrightarrow \qquad (i \diamond j) \otimes (m \diamond n)$$

Parallel composition *creates* an associator element, while sequential composition *equates* elements.





The metalogic of matrix categories

A **metalogic** is a logic $\mathbb{C}$ and a fibered logic $\mathbb{C} \leftarrow \mathbb{M} \rightarrow \mathbb{C}$ which forms an *intramonad* in $\mathrm{Span}(\mathrm{Span}\mathbb{C}\mathrm{at})$: analogous to an intermonad in an intercategory, but vertically 1-weak, horizontally 2-weak, and no interchange.

Theorem

Matrix categories form a metalogic.

$$
\begin{array}{ccccc}
C & \longleftarrow & MC & \longrightarrow & C \\
\uparrow & & \uparrow & & \uparrow \\
P & \longleftarrow & MP & \longrightarrow & P \\
\downarrow & & \downarrow & & \downarrow \\
C & \longleftarrow & MC & \longrightarrow & C
\end{array}
$$

This is a “bifibrant triple category” without interchange.

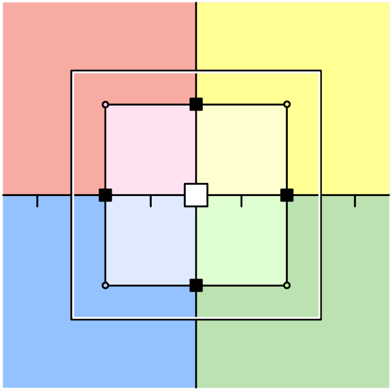
The metalogic of logics

A **logic** is a pseudomonad in $\mathbb{M}\mathrm{at}\mathbb{C}\mathrm{at}$.

Theorem

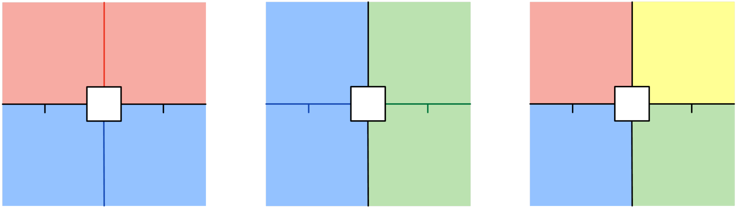
Logics form a metalogic.

$$\begin{array}{ccccc} L & \longleftarrow & R & \longrightarrow & L \\ \uparrow & & \uparrow & & \uparrow \\ P & \longleftarrow & I & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ L & \longleftarrow & R & \longrightarrow & L \end{array}$$



The metalogic of logics

There are two kinds of relations between logics.
a *vertical* profunctor consists of *processes* between logics, and
a *horizontal* profunctor consists of *relations* between logics.

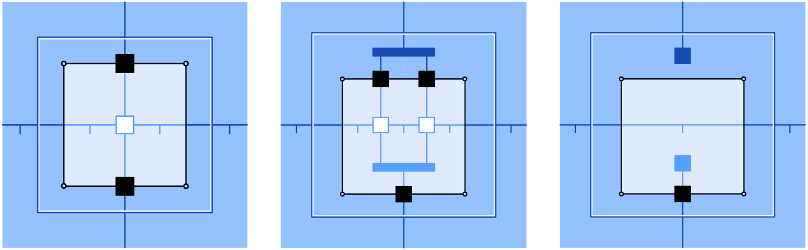


meta process (v-prof.) — meta relation (h-prof.) — meta inference (d-prof.)

Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.

The metalogic of logics

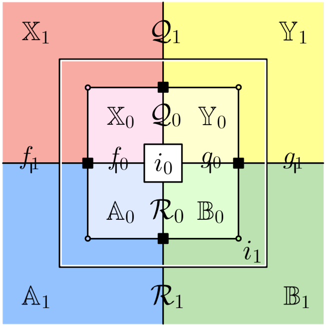
Logics have two kinds of relation, and one kind of function: a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to $\mathbb{A}_1$, preserving relation composition and unit up to coherent iso.



This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition and unit.

The metalogic of logics

All together, logics form a metalogic.



A cube is a double transformation, the fully general notion of what is known as a modification.

The metalogic of logics

The metalanguage is extremely powerful;
there are just three basic “limitations” or complexities:

1. *No interchange.* Parallel (horizontal) composition is neither lax nor colax with respect to sequential (vertical) composition of double profunctors.
2. *No vertical collage.* In general there is no collage of a vertical profunctor, because its elements do not act on the relations of the bifibrant double categories.
3. *No vertical closure.* Neither $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ nor $bf.\mathrm{Dbl}\mathbb{P}\mathrm{rof}$ are closed logics.

Yet $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is *horizontally* closed: lifts and extensions are derived just as in the co/end calculus, giving formulae for double weighted co/limits.

Prospectus

The language extends to virtual equipments, and moreover their poly- generalization, by specifying any “shape” of 2-cell as a matrix profunctors, equipped with multi- or poly- composition.

The pseudomonad construction generalizes lax or colax double functors; but this complicates the co/descent calculus. It is likely best to use pseudo double functors, and encode co/laxity.

As of now, I do not know any aspect of category theory which is beyond the scope of this metalanguage. There is a huge research program of unification, just waiting for people to explore.

Thank you.

References

- Geoffrey Cruttwell, Michael Lambert, Dorette Pronk, and Martin Szyld.
  Double fibrations, 2022.
- Fosco Loregian.
  (Co)end Calculus.
  Cambridge University Press, 2021.
- Michael Shulman.
  Framed bicategories and monoidal fibrations.
  Theory and Applications of Categories, 20:650–738, 2008.
- Ross Street.
  Powerful functors, 2001.
- Tamara von Glehn.
  Polynomials, fibrations, and distributive laws.
  Theory and Applications of Categories, 33:1111–1144, 2018.