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Type Theory for (∞,∞)-Categories, and Decomposition Spaces

Thomas Jan Mikhail

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Type Theory for (∞, ∞)-Categories, and Decomposition Spaces

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Abstract

This thesis has two chapters, one on `CaTT`, a type theory for (∞,∞)-categories, and one on decomposition spaces, which is joint work with Joachim Kock.

The first chapter contains two parts. The first part begins with the fact that the dimension of an operation is equal to that of the underlying pasting diagram being composed, whereas the dimension of a coherence is strictly larger than that of the underlying pasting diagram. Based on this observation we propose a new set of rules describing (∞,∞)-categories, in which the free-variable side conditions of the original rules in `CaTT` are replaced by a dimension side condition. The new rules have the advantage of being more geometric. The main result is then that the new rules and the original rules are mutually admissible. Building up to the main result are a number of technical results, which are of independent interest. A key result states that the free variables of a term in a pasting context form themselves a pasting context.

In the second part we introduce and study a purely syntactic notion of lax cones and (∞,∞)-limits on finite computads in `CaTT`. In particular, we define a new theory $\mathtt{CaTT}_{\mathtt{lim}}$ extending `CaTT`, which describes (∞,∞)-categories with lax limits for finite computads. Conveniently, finite computads are precisely the contexts in `CaTT`. We define a cone over a context to be a context, which is obtained by induction over the list of variables of the underlying context. In the case where the underlying context is globular we give an explicit description of the cone and conjecture that an analogous description continues to hold also for general contexts. We use the cone to control the types of the term constructors for the universal cone. The implementation of the universal property follows a similar line of ideas. Starting with a cone as a context, a set of context extension rules produce a context with the shape of a transfor between cones, i.e. a higher morphism between cones. As in the case of cones, we use this context as a template to control the types of the term constructor required for universal property.

The second chapter on decomposition spaces relates to a theorem of Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer, which states an equivalence between 2-Segal spaces and certain augmented stable doubleSegal spaces, and the work of Carlier, who introduced the notion of bicomodule configuration. We establish more general equivalences, involving simplicial maps of 2-Segal spaces and abacus bicomodule configurations, extending results of Carlier. The BOORS equivalence is recovered from the special case of the identity map. One main ingredient is an analysis of the relationship between the BOORS and Carlier notions of augmentation, hitherto considered unrelated.

Preface

A finished product rarely retains any traces of the story behind it and everything that lead up to it. In this rather lengthy preface I tell parts of this story, with an attempt at adhering to a chronological structure, as an excuse to recount important events and influences leading up to this thesis. Embedded in it are also the acknowledgements I wish to express.

In a certain sense, this thesis grew out of a desire to understand ∞-groupoids and thereby the foundations of higher category theory. It is a long-term goal of my research to develop an $(\infty, 1)$-analogue of Lawvere's Elementary Theory of the Category of Sets (ETCS) [53] which I call the Elementary Theory of the $(\infty, 1)$-Category of ∞-Groupoids (∞-ETCG). As part of the desiderata, this theory should include an axiomatization of what we might call an elementary $(\infty, 1)$-topos and the underlying shape should be globular. Ultimately, as in the case of Lawvere's ETCS for set theory, this language is meant to serve as a synthetic foundation, intended to reflect the usage and the intuition of homotopy theorists.

The search for ∞-ETCG was not a part of my work since the beginning of my PhD. The original project with which I had applied for the PhD position and grant was motivated by perturbative quantum field theory. Without getting into any details, the question asked was whether Zimmerman's forest formula [83, 82] can be lifted to the abstract setting of decomposition spaces [33]. Given my background in physics, this project made sense. The desire for an ∞-groupoid native foundational language was mentioned by Joachim in one of our early conversations. The idea fascinated me. After some inquiring about the nature and ambition level of such a project, and after discussing it for some time, it was agreed that I could try.

On the practical level, this project meant having a mountain of things to read, as the project required learning and digesting not only higher category theory but also type theory and logic and the way in which they all interact. As luck has it, and without really knowing beforehand, a big part of what became relevant to look into had already been on my reading list.

The first and obvious place to start learning higher category theory was Lurie's monumental *Higher Topos Theory* [57]. Fortunately I did not have to navigate through this rather intimidating book by myself. Wilson Forero, at that time also a PhD student of Joachim organized and led a reading group on quasicategories. He was already an experienced user of higher category theory and proposed the reading group to help familiarize other group member with the basics. Sadly, and with the pandemic to blame, the reading group had to take place online. Still, everybody welcomed the initiative and appreciated the group-feeling which was partly restored by it. I take the opportunity here to thank Alex, Guille and Wilson, fellow PhD students who became friends to whom I owe many happy memories and upon whom I could always rely.

Having some basic higher category theory under the belt, I next turned my attention towards logic, starting with Makkai's First-Order Logic with Dependent Sorts (FOLDS) [59]. I was impressed by the philosophical thoughts out of which this work had grown. Despite eventually not playing a role in the thesis, it helped solidify my understanding of logic and widened my horizon. It also may have helped the transition into the type-theoretic way of thinking, in which I completely immersed myself immediately after FOLDS. Homotopy type theory had already established itself as synthetic foundational language describing ∞-groupoids, and the *HoTT book* [77], which ended up being an absolute pleasure to read, was an obvious entry in my reading list. The synthetic approach helped grasp part of the essence of ∞-groupoids, and the corresponding way of thinking still governs the way I understand ∞-groupoids until today.

The following semester, the winter semester of 2021/22, I took up teaching. Wolfgang (Wolf) Pitsch, who later became my tutor (which is distinct from being a supervisor) had a free slot in the tutorials for his course on manifolds. This was particularly exciting for me, as Wolf was experimenting with the concept of flipped classrooms. Although this course totally consumed my time, it was a great chance to brush up on some differential geometry and (something I admit with some embarrassment) to practice my Spanish. A big thank you goes to all the students in that class who so patiently dealt with my language deficiencies.

During the same semester I was also introduced to the topology group of the University of Barcelona (UB), which was regularly running seminars on higher category theory and its foundations. After having been put in touch with Carles Casacuberta by Joachim, I began attending the seminar. The many (extended) trips of mine did not permit me to be present as much as I would have like to. Nevertheless, this seminar has been formative in my pursuit of understanding higher category theory, and I have had the fortunate to have participated enough to have collected a number of fond memories over the years. I thank everyone at the UB topology group for taking me in and making me feel part of the group.

By this time, conferences had restarted taking place. The first conference as a PhD student I attended was the Logic and Higher Structures at the CIRM in Marseille, organized by Dimitri Ara, Thierry Coquand and Samuel Mimram in February 2022. It was the complete experience for me: a place to be inspired academically and a place to connect with people and make friends. It was here that I first met Victor Iwaniack who would become a good friend and to whom I owe many good memories from all the shared conferences we attended. It was also here that I was first introduced to the type theory `CaTT` (due to Eric Finster and Samuel Mimram [30]), thanks to a talk by Thibaut Benjamin. At the time it just seemed like a curiosity, being able to axiomatize the complex structure (∞, ∞)-categories in such a simple manner, and I had no idea how central of a role this would later play in my thesis.

It was also around this time that I had started seeing some results in my work. One of the early developments was the generalization of Martin-Löf's identity type to a "fiber type". This came about as part of my investigations of and experiments in homotopy type theory in the search for a more elementary-topos-like reformulation. My attempts at replacing the identity type with some "pullback type" had failed, but the efforts were successful with the special case of the fiber type. At the same time I developed an interest in categorical semantics and specifically in understanding better the relation between HoTT and ∞-topoi. The understanding gathered from reading Lambek and Scott's *Introduction to Higher-Order Categorical Logic* [52] and Johnstone's *Elephant* [47] was compiled into a set of expository notes. Somewhat ironically, only after I was close to being done, I discovered Michael Shulman's draft *Categorical Logic from the Categorical Point of View* [74]. This contained the conceptually clearest exposition of the relation between categories and type theories I had seen, placing the theory in a uniform framework by emphasizing the adjunction between the syntactic category functor and the internal language functor. The downside was that this made my notes obsolete. The upside was that Mike's draft cleared up a lot of fog and helped me make sense of the bigger picture.

The project received a major boost as soon as I had visited Michael (Mike) Shulman in the US the following winter semester of 2022/23. It was truly a privilege spending three months in San Diego and I can't thank Mike enough for his guidance and for his kindness, his patience and his generosity with his time. It was during one of our conversations where it was suggested to me to look into it `CaTT`, which had faded in my memory. Being also based on globular shapes, `CaTT` became a natural setting and a fertile testing ground for my subsequent investigations.

Experiments with the rules of `CaTT` quickly lead to an alternative description of (∞, ∞)-categories, in which the side condition on free variables has been replaced by a side condition on the dimension of the terms involved. Parts of this had been worked out during my stay in San Diego, but large parts resisted falling into place up until a few months after. This project forms Section 2 of the chapter Type Theory for (∞, ∞)-Categories.

The next goal became defining the required structure in `CaTT` which would allow spelling out the axioms of ∞-ETCG. Here, limits seemed to be an appropriate place to start. The idea was to implement a rule which generates all the required higher cells, just as one might have done intuitively, if one were to spell out the universal property of, say, a terminal object. Since the structures involved (e.g. cones and morphisms thereof) were already of the appropriate generality, defining fully lax (∞,∞)-limits seemed to be only a few steps away. Thus, even though it was always the restriction to $(\infty,1)$-categories which ultimately was the goal, it seemed worthwhile tackling the problem at this higher level of generality, obtaining $(\infty,1)$-limits as as a special case. This work has been made available on arXiv [64] and forms Section 3 in the chapter Type Theory for (∞,∞)-Categories.

The road which finally lead me to the current rendition of the definition of limits in `CaTT` took up a large part of the academic year of 2023/24 and was far from straightforward. The description of cones as it appears now already appeared in some crude form in the summer of 2023. After this discovery, most of the time was invested in providing a construction of such cones. Even though some minor progress was made in that direction, the issue quickly became intractable due to the complex nature of (∞,∞)-categories. As a matter of fact, some of my ideas turned out to be misguided, as was pointed out to me by Thibaut, whom I had been in contact with thanks to Samuel Mimram. And so, leaving some work unfinished I eventually abandoned these efforts and adopted a different point of view. Following the suggestion of Thibaut in turning the description of cones into type theoretic rules is what eventually lead me to the final and distilled definition present in this work. My gratitude extends also to Thibaut, all conversations with whom have been a big help.

In the meanwhile, a number of other projects had emerged. Early 2023, I restarted becoming more involved in decomposition space theory. Decomposition spaces had never really ceded to be present in our conversations due to a natural interest of mine in Joachim's work. They offered themselves as a fitting place to gain more experience working with higher categories model independently. Discussions with Joachim naturally lead to a project centered around some apparent connections between the so-called BOORS equivalence [16, 15] and the work of Carlier. It culminated with the establishment of more general equivalences involving Carlier's bicomodule configurations [21], from which the original BOORS equivalence can be recovered, and resulted in a joint paper which has been submitted to the journal Algebraic & Geometric Topology and is available on the arXiv [51]. The paper is included in this thesis in chapter Decomposition Spaces.

In the summer of 2023, while visiting Lukas Barth, a fellow mathematician and close friend from our days back in Heidelberg, we decided to study Gödel's incompleteness theorem. We decided to attack it from a categorical point of view, due to Lawvere [54], thinking that this would present the most conceptual clarity. The fact that Lawvere's fixed point theorem, which lies at the heart of the categorical approach to Gödel's theorem, captures the essence of many other paradoxes in logic made this approach even more attractive. The time invested in this beautiful subject lead to some ideas on how this may be applied also to Gödel's second

incompleteness theorem (which unfortunately never came to fruition) as well as some expository notes which I hope to make available some day.

With the project on Lawvere's fixed point theorem put on hold, Lukas and I restarted discussions, but this time about a new project he had been involved in. This project grew out of the desire of Lukas' group members at the MPI in Leipzig to understand the mathematics behind UMAP [63], a recent and successful algorithm for dimension reduction in data visualization. Somewhat surprisingly, UMAP was inspired by a categorical construction, namely a nerve-realization adjunction between so-called fuzzy simplicial sets and certain generalized metric spaces. This is how Lukas and eventually I got involved in this project. Most of the work had been already done by Lukas. Nevertheless, there was still sufficient room for improvement and streamlining, because of which I was asked to join. This project eventually became a paper, which has been submitted to the journal Applied Categorical Structures and is available on the arXiv [4].

There is one important development that has not been mentioned yet in this recount. Thanks to Joachim's move to Copenhagen I was fortunate enough to visit Copenhagen multiple times. The topology group in Copenhagen is truly special in its vibrancy and the sheer amount of activities. Every visit would leave me inspired and as I increasingly felt more as part of the group with each visit, Copenhagen became more and more a second academic home.

There are many more who have given me their time and feedback, many of which haven't made it into the storyline traced out above. Among them, and to all of whom I extend my gratitude, are Dimitri Ara, Simon Henry, Felix Loubaton, Samuel Mimram and Chaitanya Leena Subramaniam who was (and still is at the time of writing) a postdoc of Mike during my visit.

Dimitri Ara and Eric Finster have kindly accepted to be the external examiners and Carles Casacuberta, Natàlia Castellana and Samuel Mimram have kindly accepted to be the members of the defense committee. I am very grateful to all of them, especially for their understanding when there were delays in the submission of the thesis. My gratitude extends also to Carlos Broto, Joana Cirici and Javier Gutiérrez, for kindly accepting to be substitutes for the defense committee.

I am also thankful to everyone at the department of mathematics at the UAB for creating such a pleasant environment. The revival of the group seminar in the last year of my thesis had a truly positive effect and restored some of the energy that had been dampened by the pandemic. Although the overlap of my interests with that of the rest of the group is limited, I still benefitted greatly from the seminar. I am also happy that this seminar also gave me the chance to share more about my work in a number of talks outside of the main line of the seminar. I am particularly grateful to Wolfgang (Wolf) Pitsch and Natàlia Castellana. Wolf agreed to become my tutor after the departure of Joachim, and always made sure I don't lose sight of the end goal. Natàlia, on the other hand, always reminded me of her availability if ever I was in need of something, and that despite her overwhelmingly busy schedule. I also thank Natàlia for initiating a reading group soon after the arrival of her PhD student Gabriel Martinez, a friend with whom it always was an absolute pleasure to talk about maths.

It is hard to overestimate the amount of support and encouragement I have received from my family, Mom, Dad and Linda, and friends throughout my PhD. I can't thank my parents enough for backing me up in every choice I made and for providing me with a certain stability I could always fall back to if necessary. Living and working away from family is a result of following what I love doing the most – something which hasn't always been easy, but which has been made easier thanks to their unconditional support. Since my move to Germany for my masters degree, my aunt and godmother Steffi also played an increasingly more important role in my life. For this alone I am very thankful. Without her I would have never come this far. There are two more names I need to add here, Lukas and Laura, as they have been a big part of my life since I have met them and who have helped shape me into the person I am today. Having them be part of my life is something for which I will be forever grateful.

This thesis would also not have been possible without the the financial support provided by the PhD scholarship I was awarded by the Government of Catalunya. The three-month research stay in San Diego, on the other hand, would not have been possible, without the scholarship I was awarded by the Ferran Sunyer i Balaguer Foundation, especially when one takes into consideration that San Diego is not the cheapest place. I thank all at the University of San Diego – especially at the Office for International Students and Scholars – who have made this trip possible. All my visits to Copenhagen have been supported by Joachim's grant and all other trips with which I attended conferences and workshops were supported by the grant held by the topology group at the UAB.

As I sit here and reflect upon the past years, I am filled with gratitude for everything this PhD has given me: the privileged life devoted to study and exploration surrounded by inspiring people. It's been a rollercoaster of a ride - one for which I will be forever grateful.

It only seems fitting to end this preface by expressing my deepest gratitude to my supervisor Joachim for his constant support and encouragement at every step, for his patience and his inspiring endurance in our extended meetings (without the need of a single sip of coffee), for reading all documents I've produced with such attention to detail, returning them with valuable advice, and for his invaluable guidance, all within a nurturing environment in which I felt free to explore.

Contents

Introduction

1 Background and Historical Remarks

1.1 Type Theory and the Foundations of Mathematics

Origin of type theory. Type theory can be traced back to Bertrand Russell [71] and his work in the early twentieth century (see also [68]). The purpose of the introduction of types was to conceive a language allowing the formulation of a consistent foundation for mathematics, which is free of the paradox plaguing naive set theory that Russell had exposed. Since its inception, type theory has undergone extensive development and currently plays an essential role in the foundations of logic, computer science and higher category theory. As a discipline, it has come to be understood as the study of a family of languages belonging to symbolic logic sharing the fact that terms are typed.

λ-Calculus. One important milestone in the development of type theory is formulation of λ-calculus, formalizing the notion of function abstraction and function application. It was developed by Church [24], as part of his work on the foundations of mathematics. Despite its simple appearance, λ-calculus is Turing complete and as such serves as a model of computation. On the other hand, typed λ-calculus, also due to Church [23], serves as the basis for functional programming languages and forms a fragment of Martin-Löf type theory.

Curry–Howard Correspondence. When interpreted as a programming language, type theory views programs as terms. The type of a program is then, broadly speaking, understood as parametrizing the input and output of the program. In logic, on the other hand, types may be interpreted as propositions, the proofs of which are given by the corresponding terms. Since the information of the various differing proofs is retained, one also speaks of proof relevance. This sets up a relation between computer science and mathematical logic, which is captured by the slogans: "propositions as types" or equivalently "proofs as programs." This is the content of the Curry-Howard correspondence [45], first observed by Curry [26] (see also Girard [37] for a modern account). Under this correspondence, for example, the function application rule in λ-calculus is matched up with the modus ponens. More generally, if we allow for simply typed λ-calculus to contain additional structure (such as products), then, under the correspondence this can be identified with propositional intuitionistic logic.

Categorical Semantics. The notions of function abstraction and function application have also been formalized in the language of categories where they are captured by the axioms of cartesian closed categories. Lambek showed [52], that cartesian closed categories and simply typed λ-calculus are two sides of the same coin (see also [47]). As a matter of fact, this relationship extends to other structured categories and corresponding type theories, relating, for example, toposes and higher order intuitionistic type theory. This upgrades the Curry–Howard correspondence to a three-way correspondence between computer science, mathematical logic and category theory. In its cleanest form, the relationship between categories and type theories is expressed in terms of an adjunction, in which the two adjoints are known as the syntactic category functor and the internal language functor. The categorical structures involved are said to provide the semantics. In these terms, higher-order intuitionistic type theory (to be thought of as a theory of sets) is the internal language of toposes, and toposes provide semantics for higher-order intuitionistic type theory (see [52, 47]).

Martin-Löf and dependent types. In 1972 Martin-Löf [61] introduced his eponymous type theory (see also [62]). This type theory has the additional feature of allowing its types to depend on terms, hence we speak of dependent types. Under the "propositions as types" philosophy it has enough structure to support an interpretation of first-order intuitionistic logic. But perhaps most importantly, it contains the identity type, which internalizes the notion of identity. Following the "propositions as types" philosophy, the identity type is meant to be thought of as the proposition asking whether a given pair of terms of a fixed type are equal. Technically it is defined as a relation inductively generated by reflexivity. Properties, such as symmetry and transitivity, are derived as a consequence.

Excursion into Synthetic vs. Analytic Theories. The word "synthetic" originally comes from the ancient greek verb "συντίθημι" which can be translated as "put together". The word "analytic", on the other hand, stems from the ancient greek verb "αναλύω" which can be translated as "to resolve into its elements". Very broadly the distinction between synthetic and analytic has been used to describe contrasting approaches in reasoning: the synthetic method appeals to a set of first principles, and starting with these derives the solution, whereas the analytic method deconstructs the proposition to arrive at a solution. We do not intend to go through the differing usages of the the words synthetic and analytic in philosophy and mathematics throughout time here. Rather, we shall restrict ourselves to a usage in mathematics which is exemplified by the distinction between synthetic geometry and analytic geometry. Analytic geometry begins with the notion of euclidean space which serves as a substrate on top of which the theory lives. This is a characteristic feature of analytic theories. Geometric objects are then carved out of the underlying euclidean space and their properties inherited from those of the embedding euclidean space. Synthetic geometry, on the other hand, gives a direct axiomatization of the properties of geometric objects without an appeal to some ambient space. More generally, for us a synthetic theory will mean an axiomatic theory in which the objects of interest are primitive and in which the properties thereof are encoded directly into the axioms of the theory.

1.2 Higher Category Theory

Higher Category Theory in the 21st centure. With the advent of higher category theory, the 21st century is casting new light on abstract homotopy theory. Initiated by Grothendieck and his school, we are essentially witnessing a shift from set-based to ∞-groupoid-based mathematics (see [75] for a historical overview). With ∞-groupoids corresponding to homotopy types the language of ∞-categories has been accepted as the legitimate language of homotopy theory, fulfilling a dream previously addressed by the ad hoc language of model categories. The theory of ∞-categories has matured thanks to the pioneering work of Joyal and Lurie among others and has found numerous applications in areas such as geometry, mathematical physics, arithmetic geometry and combinatorics.

∞-Groupoids. A question lying at the center of algebraic topology is the classification of spaces with the help of algebraic invariants. To each space one can associate its fundamental groupoid, the objects of which are the points and the morphisms of which are homotopy classes of paths. The fact that this is actually a groupoid expresses the fact that paths can be traversed in reverse. As an algebraic invariant, the fundamental groupoid is fine enough to distinguish all homotopy 1-types. In fact, groupoids provide a classification of all homotopy 1-types. In order to "detect" the higher structure of n-types for larger n we must refrain from quotienting out the homotopies and incorporate them into the structure. The first step sees the fundamental groupoid giving way to the fundamental 2-groupoid, in which the (homotopy classes) of homotopies between paths are built in as 2-dimensional morphisms. Since homotopies can be traversed in reverse, the fundamental 2-groupoid is indeed a 2-groupoid. What this buys us is the improved result that 2-groupoids classify all 2-types. To capture all homotopy types, it becomes necessary to go all the way to infinity, by which we arrive at ∞-groupoids. This is the content of the homotopy hypothesis: homotopy types are the same thing ∞-groupoids. It is for this reason that ∞-groupoids also go by the name spaces. In total we think of an ∞-groupoid as a category-like structure with morphisms of every dimension, all of which are invertible. Crucially, the algebraic identities satisfied by ∞-groupoids are weak in the sense that they hold only up to homotopy. For 1-types and 2-types a strict structure suffices, but already at dimension 3 counter examples exist, showing that strict 3-groupoids do not model all 3-types [75].

From ∞-Groupoids to $(\infty, 1)$-Categories. The natural notion of morphism between ∞-groupoids is that of a functor (or ∞-functor). All higher notions of morphisms, such as natural transformations also make sense, but as these refer to morphisms in the target ∞-groupoid, these will necessarily be invertible. As a result the collection ∞-groupoids forms an $(\infty, 1)$-category, an infinite-dimensional structure with morphisms of any dimension, in which all but the 1-dimensional morphisms are necessarily invertible (explaining the prefix in the name). This is analogous to how sets form a 1-category. As a matter of fact, the analogy is strong: putting aside a number of pitfalls, a large part of the theory of 1-categories generalizes to $(\infty, 1)$-categories (see Lurie [58, 57]).

Models of $(\infty, 1)$-Categories. Throughout the years, many definitions have been proposed as models for $(\infty, 1)$-categories (see [14, 55]). These models fall into two categories: the geometric definitions and the algebraic definitions. In geometric definitions, it is just the mere existence of composites that is guaranteed. Definitions of this flavor include quasicategories, complete Segal spaces, Segal categories and Θ_1-spaces. Algebraic models, on the other hand, come equipped with actual operations providing the composites and the higher coherences. Models of this kind are the definition of Batanin–Leinster, that of Trimble and that of Grothendieck–Maltsiniotis. Models can also be organized according to the underlying shape involved in the higher structure. Quasicategories, complete Segal spaces and Segal categories, for example, are simplicial, whereas Θ_1-spaces, the Batanin–Leinster definition and that of Grothendieck–Maltsiniotis is globular. All models mentioned here constitute analytic theories of higher categories, as these require an underlying theory of sets (except for Segal spaces, which rely on an underlying notion of space). An important endeavour has been establishing equivalence between the different models, an endeavour which has been carried out successfully to a large extent. The most prominent model in the literature so far has been that of quasicategories, owing to the work of Joyal and Lurie [48, 58, 57].

(Complete) Segal Spaces. As was first observed by Grothendieck [38], 1-Categories can be characterized as those simplicial sets satisfying a certain condition which came to be known as the Segal condition [72]. In one of its equivalent formulations, the Segal condition says that certain simplicial identities form pullbacks. Intuitively, in a Segal simplicial set the 0-simplices form the objects and an n-simplex is simply a sequence of n composable 1-simplices. The Segal condition can be translated in a straightforward manner to the setting of higher categories. Replacing sets with spaces, the strict pullback condition with a homotopy pullback and finally the condition of being an isomorphism with that of being an equivalence of spaces converts the original definition into its ∞-analogue. A simplicial space satisfying this condition is what is known as a Segal space. However, contrary to the case of 1-categories, this does not yet provide a model for $(\infty, 1)$-categories, as subtleties arise in the ∞-world. This is because the collection of objects forms a space, so that objects already comes equipped with a notion of equivalence, which a priori is unrelated to the canonical notion of equivalence defined on the collection 1-morphisms. A Segal space is said to be complete if these two notions of equivalence are identified (see [65] for an introduction). This completeness condition is due to Rezk [67] and indeed, complete Segal spaces form a model of $(\infty, 1)$-categories.

From $(\infty, 1)$-Categories to (∞, ∞)-Categories. There is a notion of functor between $(\infty, 1)$-categories with which $(\infty, 1)$-categories assemble into an $(\infty, 1)$-category. But since the 1-dimensional morphisms in $(\infty, 1)$-categories are not necessarily invertible, there also exists a meaningful notion of noninvertible natural transformations. As a result, $(\infty, 1)$-categories form an $(\infty, 2)$-category, an infinite-dimensional structure in which all morphisms are invertible, except those of dimension less than or equal to 2. This is analogous to how 1-categories form a 2-category. The pattern is now evident and it becomes natural to consider (∞, n)-categories for arbitrary $n \geq 0$ (where the case $n = 0$ reduces to ∞-groupoids). In fact at this point it makes sense to consider the most general such structure, namely (∞, ∞)-categories, which may contain noninvertible morphism of any dimension. Higher categories of this level of generality have become increasingly more relevant and have found applications in the study of topological quantum field theories.

Grothendieck–Maltsiniotis (∞, ∞)-Categories. Remarkably, the work of Grothendieck already contained a definition of ∞-groupoids [39] (see also Maltsiniotis [60]). This was taken up by Maltsiniotis, who extended this to a full-blown definition of an (∞, ∞)-categories. This definition constitutes an algebraic model, the underlying shape of which is globular. At the base of this definition lies a category, called the coherator, which is defined in an inductive manner and serves as an indexing category. Its objects are pasting diagrams (the diagrams which can be composed) and its morphisms include all higher-categorical operations and coherences. The Grothendieck–Maltsiniotis (∞, ∞)-categories are then defined as presheaves on the coheretor preserving certain colimits. Ara [1] and Bourke [19] showed that Grothendieck–Maltsiniotis (∞, ∞)-categories are equivalent to those of Batanin and Leinster. Notably, the homotopy hypothesis with respect to the ∞-groupoids à la Grothendieck is still an open problem, although progress has been made thanks to the work of Henry [41] and Henry–Lanari [42].

1.3 Foundations: Higher Categories and Type Theory

Homotopy type theory. In the recent years a type theory emerged as a synthetic theory axiomatizing the $(\infty, 1)$-category of ∞-groupoids. Homotopy type theory, as it became known (or more specifically Book HoTT, after the book which popularized it), is given by Martin-Löf type theory with the addition of the univalence axiom, due to Voevodsky, and higher inductive types. From this perspective, types are interpreted as spaces and the identity type plays the role of the path space between two given points. Early signs of this homotopic interpretation of Martin-Löf type theory were already present in the work of Hoffman and Streicher [43], who showed that Martin-Löf type theory supports an interpretation in the category of groupoids. In fact, their work [44] already included a shadow of the univalence axiom under the name universe extensionality. The interpretation as ∞-groupoids was later solidified with the construction of a model in simplicial sets by Voevodsky (see Kapulkin–Lumbsdaine [49]) and the work of Awodey–Warren [3, 80], who constructed models in Quillen model categories. Building on this, Shulman [73] showed that homotopy type theory can be interpreted in any $(\infty, 1)$-topos. From the ∞-topos point of view, univalence corresponds to the existence object classifiers. As a consequence of Shulman's work, all results proven in homotopy type theory automatically hold in any $(\infty, 1)$-topos. This can be understood as a generalization of the fact that 1-toposes provide semantics for intuitionistic higher-order type theory.

The type theory `CaTT`. In his thesis Brunerie [20] developed a type-theoretic version of the definition of ∞-groupoids with the aim of showing that the types in homotopy type theory possess this structure. The type theory is remarkable in its simplicity, having (apart from the variable rule) only a single term constructor rule generating all the operations and coherences of ∞-groupoids in an inductive manner. Inspired by this, Finster and Mimram constructed a type-theoretic definition of (∞, ∞)-categories called `CaTT` [30]. It forms a synthetic definition of (∞, ∞)-categories based on globular shapes. Together with Benjamin, Finster and Mimram showed that its models are precisely the Grothendieck–Maltsiniotis (∞, ∞)-categories [10].

1.4 Applications: Higher Category and Combinatorics

Decomposition Spaces/2-Segal Spaces. Decomposition spaces are certain simplicial spaces capturing and generalizing the properties of combinatorial structures. They were introduced by Gálvez, Kock, and Tonks [33] with the purpose of providing the most general setting for the incidence coalgebra construction, originally introduced by Rota [69] for posets. The main insight of decomposition spaces is that combinatorial structures in many cases cannot composed, but rather decomposed into substructures. In terms of properties of simplicial objects, this means that they do not satisfy the Segal condition, but instead a weaker property, also phrased in terms of certain pullback conditions. Independently of Gálvez, Kock and Tonks, Dyckerhoff and Kapranov had introduced the notion 2-Segal space [28] as part of their work in representation, geometry and homological algebra. It was later understood that decomposition spaces and 2-Segal spaces are one and the same. Nevertheless, the definitions have distinct flavors and the theory developed by Gálvez–Kock–Tonks and that by Dyckerhoff–Kapranov were initially rather orthogonal to each other.

Möbius Inversion and Bicomodules. An important motivation for Gálvez, Kock and Tonks was Möbius inversion. The classical Möbius inversion was introduced in combinatorics as a statement about arithmetic functions. It was abstracted by Rota to the setting of posets (satisfying an appropriate finiteness condition), the classical case being recovered by the poset of natural numbers and divisibility. A first goal of the theory of decomposition spaces was to upgrade the classical theory of incidence algebras and Möbius inversion from posets to decomposition spaces [32]. Carlier [21] took an important step in this direction with a generalization of Rota's formula for the Möbius functions of two posets related by a Galois connection [69]. He generalized Rota's formula to adjunctions of ∞-categories, and went further to the situation of certain correspondences between decomposition spaces, which were shown to induce bicomodule configurations, namely suitably augmented stable double Segal spaces designed to have incidence bicomodules. Carlier established a Möbius inversion principle for certain Möbius bicomodule configurations, which reduces to Rota's formula in the case of a Galois connection.

The BOORS equivalence. An important source of decomposition spaces is provided by Waldhausen's $S_\bullet$-construction [79]. This example plays a central role in the work of Dyckerhoff and Kapranov, who show that the Waldhausen construction applied to any proto-exact $(\infty, 1)$-category forms a 2-Segal space (see also [33]). Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [16], [15] (BOORS) made the surprising discovery that *every* 2-Segal space arises as an $S_\bullet$-construction, if just the $S_\bullet$-construction is extended to more general inputs. They identified certain augmented stable double Segal spaces as the appropriate input to produce (all) 2-Segal spaces, and showed that this generalized $S_\bullet$-construction is part of an equivalence. The BOORS theorem is quite remarkable, as it gives a completely new perspective on 2-Segal spaces, and at the same time gives new insight on the $S_\bullet$-construction (see [17]), by staging it in a setting where it has an inverse. It is striking that this inverse is another well-appreciated construction: the inverse takes a 2-Segal space to its total decalage, suitably augmented.

Waldhausen's original $S_\bullet$-construction dates back to 1983, while the total decalage construction is credited to Illusie [46] (1972). For an overview of how various Waldhausen constructions relate in this perspective, see [17]; for an introduction to the BOORS equivalence, see [70].

2 Technical Background and Summary of Results

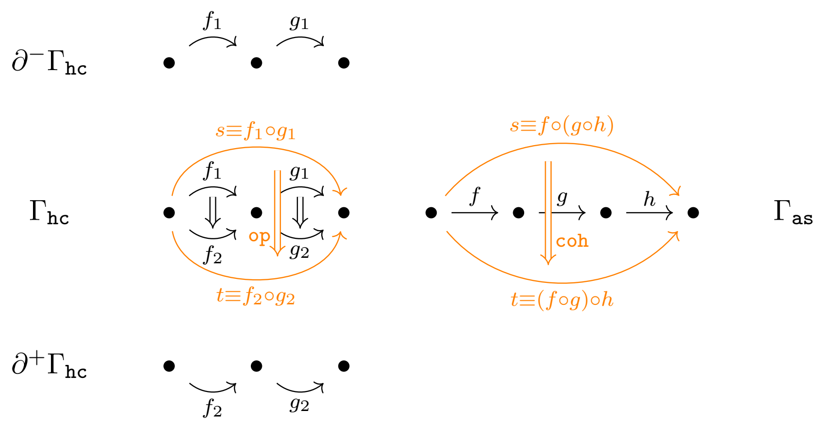
This thesis has two parts, one on `CaTT`, type theory for (∞, ∞)-categories, and one on decomposition spaces. The second part is joint work with Joachim Kock, and a corresponding paper *Abacus bicomodule configurations and the Bergner-Osorno-Ozornova-Rovelli-Scheimbauer equivalence* has been posted to the arXiv as [51] and submitted to the journal Algebraic & Geometric Topology. The two authors contributed equally to that work. The first part is entirely my own work. It contains two subparts: the first is about a set of alternative rules for `CaTT` and the second about limits in `CaTT`. The work on limits has been posted to the arXiv under the title *A Type-Theoretic Definition of Lax* (∞, ∞)*-Limits*, [64].

Note to the reader: This section has an overlap with the local introductions present in the main body of the thesis. Duplications are indicated with pointers in the margin.

2.1 Type Theory for (∞, ∞)-Categories: Technical Background

`CaTT` is a dependent type theory with two type constructors. The first, functioning as the base case, introduces the type `Ob` which one may think of as the type of objects. The second one takes two terms $s, t : A$ as input and produces the type $s \to_A t$ which is understood as the type of morphisms from $s \to_A t$. Starting with the terms of `Ob` and applying the second type constructor iteratively, one can access the type of all higher dimensional morphisms.

In addition to the type constructors, `CaTT` also contains two term constructors. The first term constructor generates all (∞, ∞)-categorical operations, which includes all binary operations, and the second term constructor generates all coherences, interpolating between the different ways of composing cells. These include the unit, the associator, the unit laws and so on. The diagrams that can be composed in an (∞, ∞)-category are called pasting schemes (or pasting diagrams). In `CaTT`, it is contexts which take up the role of diagrams and we will use these words interchangeably. A context is a list of variables $x_1 : A_1, \ldots, x_n : A_n$, such that the type A_i can be constructed using the variables of $x_1 : A_1, \ldots, x_{i-1} : A_{i-1}$. The context which have the shape of pasting diagrams are called pasting context and are generated by certain rules. The operations and coherence are then built in pasting contexts. The following two diagrams depict horizontal composition of 2-dimensional cells as an example of an operation (center left) and the associator as an example of a coherence (center right)



The underlying pasting context $\Gamma_{\texttt{op}}, \Gamma_{\texttt{coh}}$ for the operation and the coherence respectively are depicted in black. In the associator on the right we notice that both s (for source) and t (for target) make use of all variables of the pasting context. In `CaTT`, this condition is expressed by $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma_{\texttt{as}}) = \mathrm{FV}(t : A)$. In the case of the horizontal composition, on the other hand, we have $\mathrm{FV}(s : A) = \mathrm{FV}(\partial^- \Gamma_{\texttt{hc}})$ and $\mathrm{FV}(t : A) = \mathrm{FV}(\partial^+ \Gamma_{hz})$, where $\partial^- \Gamma_{\texttt{hc}}$ and $\partial^+ \Gamma_{\texttt{hc}}$ are the so-called source and target diagrams associated to $\Gamma_{\texttt{hc}}$. These observations turn out to be the defining features of operations and coherences. The two rules can then be stated to some approximation as follows

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \to t}{\Gamma \vdash \texttt{op}_{\Gamma, s \to t} : s \to t} \text{ (OP)} \qquad \begin{array}{c} \mathrm{FV}(\partial^- \Gamma) = \mathrm{FV}(s) \\ \mathrm{FV}(\partial^+ \Gamma) = \mathrm{FV}(t) \end{array}$$

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \to t}{\Gamma \vdash \texttt{coh}_{\Gamma, s \to t} : s \to t} \text{ (COH)} \qquad \begin{array}{c} \mathrm{FV}(\Gamma) = \mathrm{FV}(t) \\ \mathrm{FV}(\Gamma) = \mathrm{FV}(s) \end{array}$$

Now, in the above two examples, we see that the horizontal composition has the same dimension as that of the underlying pasting scheme while the dimension of the associator is strictly larger and this pattern persists when going through further examples. Indeed, we have the following lemma:

Lemma (Benjamin [9] Lemma 76 (see Lemma 2.2), Lemma 2.1)**.** *In* `CaTT` *we have that* $\dim(\texttt{op}_{\Gamma, s \to t}) = \dim(\Gamma)$ *and* $\dim(\texttt{coh}_{\Gamma, s \to t}) > \dim(\Gamma)$*.*

2.2 Type Theory for (∞, ∞)-Categories: Summary of Results

In view of the above lemma I propose the following new rules.

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \to t}{\Gamma \vdash \texttt{op}_{\Gamma, s \to t} : s \to t} \text{ (OP')} \qquad \dim(\Gamma) = \dim(s \to t)$$

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \to t}{\Gamma \vdash \mathtt{coh}_{\Gamma, s \to t} : s \to t} \text{ (COH')} \qquad \dim(\Gamma) < \dim(s \to t)$$

Consider a type theory `CaTT'` which is just like `CaTT` but with (OP) and (COH) replaced by (OP') and (COH'). The new rules have a less syntactic and more geometric flavor. We may ask how this new relates to the original and whether the new type theory still describes (∞, ∞)-categories. The main result about this type theory is then the following:

Theorem (Theorems 2.20, 2.18, 2.28, and 2.27)**.** *The rules (OP') and (COH') are admissible in* `CaTT`*. and the rules* (OP) *and* (COH) *are admissible in* `CaTT'`

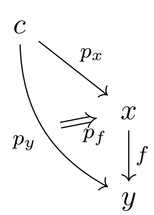
Leading up to this theorem are a number of technical lemmas, the key result of which states that the intersection of two subpasting diagrams of a pasting diagram is again a pasting diagram.

Note that a proof of equivalence of models of `CaTT` and `CaTT'` requires more work. The results here may be considered a first step towards establishing such an equivalence.

♣ Next we turn to limits in `CaTT`. The goal is to spell out rules which make `CaTT` into a type theory for (∞, ∞)-categories with lax (∞, ∞)-limits. To implement limits in `CaTT` we rely on the fact that contexts can be thought of as diagrams. More precisely, contexts are finite computads, thanks to a result by Benjamin, Markakis and Sarti [13]. This limitation is brought upon us by the finiteness of type theory's contexts. It may be possible to translate and extend these ideas to other frameworks, such as that of Dean, Finster, Markakis, Reutter and Vicary [27], in which one is liberated from this restriction.

♣ The remaining subsection has a duplicate in Section 3 (∞, ∞)-Limits

We define the cone over a context Γ to be another context K. As a low dimensional example, consider the cone over $\Gamma \equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y$, given by



$$K \equiv \Gamma,\ c : \mathtt{Ob},\ p_x : c \to x,\ p_y : c \to y,\ p_f : p_y \to p_x \star_0^1 f$$

The definition of cones is based on the observation that the types of the projections exhibit a certain pattern. Take, for example, the variable $f : x \to y$. First of all, the source of p_f is built out of the projections associated to the target of f, namely p_y. Second, the target of p_f is built out of the projections associated to the source of f, namely p_x as well as f itself. Finally, the variable f appears in a certain linear way. We collect all these properties into a set of conditions and use these to spell out the following definition.

Definition (Definition **??**). *The derivable judgments* K `cone` (Γ, c) *are generated by the rules*

$$\frac{}{c : \mathtt{Ob} \quad \mathtt{cone} \quad (\emptyset, c)}$$

$$\frac{\Gamma, c : \mathtt{Ob}, \Pi \quad \mathtt{cone} \quad (\Gamma, c) \qquad \Gamma, x : X, c : \mathtt{Ob}, \Pi \vdash s \to_A t}{\Gamma, x : X, c : \mathtt{Ob}, \Pi, p_x : s \to_A t \quad \mathtt{cone} \quad ((\Gamma, x : X), c)} \qquad \textit{(+ side conditions)}$$

If K `cone` (Γ, c) *is derivable we say that* K *is a cone over* Γ *with apex* c*.*

The rules exploit the inductive definition of contexts. For the empty context the cone is simply an apex. This is the first rule. In the second rule we begin with a cone K over a diagram Γ as well as a context extension $\Gamma, x : X$. The side conditions ensure that the type $s \to_A t$ is of the appropriate form so as to be the type of a projection corresponding to the appended variable $x : X$. Given this, the rule extends the original cone over Γ to a cone over $\Gamma, x : X$.

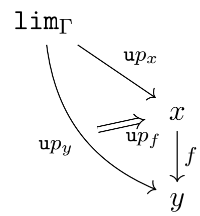
The definition of the cone involves a choice of orientation for the higher cells. The orientation chosen has the benefit of exhibiting a certain uniformity. All other choices can be obtained by making suitable adjustments to the rules. In addition to that, reversing the orientation of all 1-dimensional cells turns the definition into one for colimits.

If Γ is a globular diagram, meaning that all terms involved are variables, we show that there exists a context K such that K is the cone over Γ. In particular we construct a cone over a globular diagram Γ with an explicit description of the type of the projection of a variable $(x : A) \in \Gamma$:

$$\begin{aligned} &c \to x, && \text{if } \dim(x) = 0 \\ &p_{\tau(x)} \to p_{\sigma(x)} \overset{d}{\underset{d-1}{*}} \left(1^d_{p_{\sigma^2(x)}} \overset{d}{\underset{d-2}{*}} \cdots \left(1^d_{p_{\sigma^d(x)}} \overset{d}{\underset{0}{*}} x\right)\right), && \text{if } \dim(x) > 0. \end{aligned} \tag{1.1}$$

where $d = \dim(x)$. We then examine two classes of examples of two dimensional non-globular diagrams, the first one containing sequences of composable 1-dimensional morphisms and the second a sequence of composable 2-dimensional morphisms. Considering both examples in the strict case we show that there exists a cone with a similar description to that in the globular case. The argument relies on the fact that, given a term in the diagram $t : A$, the projections associated to the free variables of t may be composed in a way to obtain a certain term p_t the type of which is described by a formula analogous to equation 1.1. Restricting ourselves to the strict case does not spoil the argument, as the coherences are absorbed by the terms p_t. Motivated by these examples we conjecture the existence of such terms for all $t : A$ in a diagram. Using these we construct a cone with an explicit description for arbitrary diagrams.

Given a cone K over a diagram Γ, we can build the universal cone with the help of term-constructor rules, which produce a term for the apex and for each projection. As a collection these terms organize themselves into a context morphism $\Gamma \vdash \texttt{ucone} : K$. Diagrammatically we may depict this as



Implementing the universal property amounts to asking the functor of (∞, ∞)-categories given by postcomposition with the universal cone, schematically depicted by

$$\texttt{cone}_* : \{\text{terms of } c \to \texttt{lim}_\Gamma\} \longrightarrow \{\text{cones over } \Gamma \text{ with apex } c\} \tag{1.2}$$

to an equivalence. Here, the domain is the (∞, ∞)-category of terms of the type $c \to \texttt{lim}_\Gamma$ and the codomain is given by the (∞, ∞)-category of cones over Γ with apex c. We refer to the n-dimensional cells of the codomain as $(n+1)$-transfors.[1] We define an equivalence of (∞, ∞)-categories to be a functor which is (essentially) surjective on all higher hom-(∞, ∞)-categories. To ensure that the functor in equation 1.2 is an equivalence we first spell out a set of rules which, in a manner similar to those generating cones, produce out of a given context a new context of the shape of a higher transfor between cones on that context. As in the case of the universal cone, we use this context as a template, to control the types of the terms we need to build with term constructor rules. In its first application, given a arbitrary cone over a given diagram, our constructions will produce a cone morphism (i.e. a modification) from the given cone to the universal cone.

2.3 Decomposition Spaces: Technical Background

Bisimplicial Spaces feature prominently both in the work of Carlier and in that of BOORS. Carlier introduced the notion of bicomodule configuration, which is a certain suitably augmented stable double Segal space, owing its name to the fact that it induce a bicomodule. BOORS also work with augmented stable double Segal spaces, but here the adjective augmented. We now explain the commonalities and the differences between the two bisimplicial space structures.

[1] More generally, an n-dimensional cell in a functor category is called an n-transfor, this terminology being coined by Crans [25].

♣ **Stability and augmentations.** The notion of stable double Segal space is central to both the BOORS equivalence and Carlier's theory of bicomodules, but with very different motivations. A double Segal space is called stable if certain vertical-horizontal bisimplicial identities are pullback squares. For BOORS, the purpose was to capture certain bipullback features of stable or proto-exact ∞-categories as used in the classical $S_\bullet$-construction; for Carlier the purpose was to encode the bicomodule condition of a left and a right coaction. (Carlier [21] learned about the stability condition from BOORS [16], but reformulated it in a way suitable for ∞-categories.)

♣ The remainin subsection has a duplicate in Section

However, the notions of augmentation used by BOORS and Carlier are very different. For Carlier, an augmentation of a bisimplicial space B consists of an augmentation (in the usual sense of simplicial spaces) of each row and each column, assembling into two extra new simplicial spaces: one forming an augmentation column $B_{\bullet,-1}$ and the other forming an augmentation row $B_{-1,\bullet}$; these play the role of the coalgebras that the bicomodule is over. A bicomodule configuration in the sense of Carlier thus has the shape

$$\begin{array}{ccccccccc}
 & & B_{-1,0} & \leftrightarrows & B_{-1,1} & \leftrightarrows & B_{-1,2} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & \\
B_{0,-1} & \longleftarrow & B_{0,0} & \leftrightarrows & B_{0,1} & \leftrightarrows & B_{0,2} & \cdots \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & \\
B_{1,-1} & \longleftarrow & B_{1,0} & \leftrightarrows & B_{1,1} & \leftrightarrows & B_{1,2} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array} \tag{1.3}$$

Technically, this is the shape of a presheaf on the category $\mathbb{\Delta}_{/[1]}$. It should still obey a number of axioms: stability, double Segalness, and some exactness conditions imposed on the augmentations.

In contrast, what for BOORS is called an augmentation of a bisimplicial space B is the addition of a single extra space B_{-1} with a morphism *to* B_{00}, subject to some pullback conditions. This space parametrizes the objects which play the role of the zero object in a proto-exact ∞-category. A BOORS-augmented bisimplicial space thus has the shape

$$\begin{array}{cccccccc} B_{-1} & & & & & & & \\ & \searrow & & & & & & \\ & & B_{0,0} & \leftrightarrows & B_{0,1} & \leftrightarrows & B_{0,2} & \cdots \\ & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & \\ & & B_{1,0} & \leftrightarrows & B_{1,1} & \leftrightarrows & B_{1,2} & \cdots \\ & & \vdots & & \vdots & & \vdots & \end{array}$$

— it is a presheaf on a certain category Σ.

In the present work (outside of this introduction), in order to avoid confusion regarding the word "augmentation", we will instead refer to the map $B_{-1} \to B_{0,0}$ as a *pointing*, and call the pullback conditions the *pointing axioms*. For all such larger diagram shapes extending bisimplicial spaces, we call the nonnegatively-indexed part the *bulk*.

Augmentations of the total decalage. Before we embark on a deeper comparison between the two notions of augmentation, let us note how the total decalage exemplifies both. For X a simplicial space, the *total decalage* $\mathrm{Tot}(X)$ is the bisimplicial space obtained by pulling back X along the ordinal sum functor $\mathbb{\Delta} \times \mathbb{\Delta} \to \mathbb{\Delta}$ (see for example [76]). It has the upper decalage $\mathrm{Dec}_\top(X)$ as its zeroth column and the lower decalage $\mathrm{Dec}_\bot(X)$ as its zeroth row, so a picture of $\mathrm{Tot}(X)$ starts like this (suppressing degeneracy maps):

$$\begin{array}{cccc} X_1 & \overset{d_2}{\underset{d_1}{\leftleftarrows}} & X_2 & \cdots \\ d_0 \uparrow\uparrow d_1 & & d_0 \uparrow\uparrow d_1 & \\ X_2 & \overset{d_3}{\underset{d_2}{\leftleftarrows}} & X_3 & \cdots \\ \vdots & & \vdots & \end{array}$$

(It also contains the edgewise subdivision $\mathrm{sd}(X)$ as its diagonal. This has recently turned out to be of relevance for 2-Segal spaces: Bergner et al. [18] show that X is 2-Segal if and only if $\mathrm{sd}(X)$ is 1-Segal, and Hackney–Kock [40] show that a simplicial map $F : X \to Y$ is culf if and only if $\mathrm{sd}(F)$ is a right fibration.)

The Tot construction can be refined to provide either a BOORS augmentation (i.e. pointing) or a row-and-column augmentation à la Carlier. In the BOORS case this is just to take the degeneracy map $s_0 : X_0 \to X_1$ as pointing. The row-and-column-augmented Tot instead simply puts the original simplicial space X both in the augmentation column and in the augmentation row, where they fit in by simplicial operators.

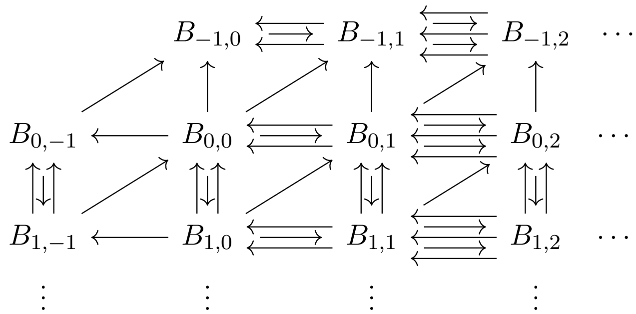
The BOORS version is $\mathrm{Tot} = \mathsf{p}^*$, where $\mathsf{p} : \Sigma \to \mathbb{\Delta}$ extends the ordinal-sum functor. This is important because the BOORS equivalence is established in [15] as a restriction of the adjunction $\mathsf{p}^* \dashv \mathsf{p}_*$, with p_* being interpreted as a generalized Waldhausen construction. The similarly defined functor $\mathbb{\Delta}_{/[1]} \to \mathbb{\Delta}$, the pullback along which is the row-and-column-augmented Tot, has not been studied from a similar viewpoint, as far as we are aware. Nevertheless, the Tot example does play a role in Carlier's theory: it is the result of applying his constructions to the identity correspondence or the identity functor of an ∞-category. This observation is the starting point of our paper.

2.4 Decomposition Spaces: Summary of Results

We have two main contributions (as well as the theory building up to these results): one is that we upgrade some of Carlier's constructions to equivalences, by identifying the conditions allowing one to "go back". Our equivalences go between certain simplicial maps and certain abacus bicomodule configurations. Secondly we explain the relationship between the two notions of augmentation, so as to allow the BOORS equivalence to be derived from more general equivalences involving simplicial maps and bicomodule configurations.

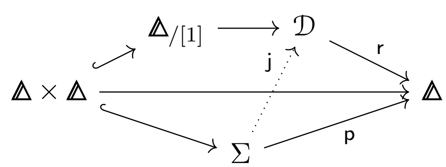
Simplicial infrastructure. The paper is primarily about bisimplicial spaces. However, we do need some groundwork which is simplicial rather than bisimplicial, culminating with a proposition (2.17) comparing the BOORS augmentation axioms with certain coalgebras for the lower-decalage comonad. This is a generalization of the observation of Garner–Kock–Weber [36] that Dec-coalgebra structure on a 1-category expresses a local-initial-objects structure (or local-terminal-objects structure, which plays a key role in the theory of operadic categories of Batanin and Markl [8].)

Abacus maps. A prominent role is played throughout by the so-called *abacus maps*: these are a family of diagonal maps $B_{i+1,j} \to B_{i,j+1}$ in a row-and-column-augmented bisimplicial space B satisfying a number of relations. Such an object thus looks like this:



It is a presheaf on a suitable index category $\mathcal{D}$, which we study in some detail. As part of the description of $\mathcal{D}$, the data of abacus maps is shown to be equivalent to having horizontal splittings (i.e. $\mathrm{Dec}_\perp$-coalgebra structures) on all bulk rows. Abacus maps appear already in Carlier's first paper [21] and were named and axiomatized in his second paper [22].

The shapes mentioned so far fit into the diagram



(The subtle functor j is explained further on.)

Equivalences in the relative setting. On the presheaf level, the abacus maps encode a simplicial map from the augmentation column to the augmentation row. Conversely, $\mathcal{D}$-presheaves arise naturally from general simplicial maps (that is, presheaves on $\mathbb{\Delta} \times \Delta^1$), by way of the functor q_* in the adjunction

$$\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) \underset{\mathsf{q}^*}{\overset{\mathsf{q}_*}{\rightleftarrows}} \boldsymbol{Pr}(\mathcal{D}),$$

given by right Kan extension along a functor $\mathsf{q} : \mathbb{\Delta} \times \Delta^1 \to \mathcal{D}$. In Theorem 4.6 we identity a certain condition $(\star)$ on $\mathcal{D}$-presheaves, required for the adjunction to restrict to an equivalence

$$\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) \simeq \boldsymbol{Pr}^\star(\mathcal{D}).$$

This has the flavor of the BOORS equivalence, but in a relative settings, and with more elaborate categories in place of $\mathbb{\Delta}$ and Σ. From here we gradually impose more conditions on the two sides of the equivalence to arrive at objects of special interest. We introduce the notion of relatively upper 2-Segal simplicial maps between 2-Segal spaces, identified as precisely those simplicial maps which correspond to Carlier's bicomodule configurations (with abacus maps satisfying ($\star$)). In particular, the adjunction $\mathsf{q}^* \dashv \mathsf{q}_*$ restricts to an equivalence (Theorem 4.14)

$$\boldsymbol{Pr}^{\mathrm{up}\,2\text{-Seg}}(\mathbb{\Delta} \times \Delta^1) \simeq \boldsymbol{ABC}^{\star},$$

where $\boldsymbol{ABC}^{\star}$ stands for the full subcategory of $\boldsymbol{Pr}^{\star}(\mathcal{D})$ spanned by abacus bicomodule configurations. By establishing an equivalence, this improves upon a result of Carlier [22], who showed how to construct an abacus bicomodule configuration from any functor of ∞-categories. (Note that between ∞-categories, any functor is relatively upper 2-Segal.) Finally, restricting further, to the full subcategory of $\boldsymbol{Pr}(\mathbb{\Delta})$ spanned by the 2-Segal spaces (interpreted as equivalences of 2-Segal spaces), and to the full subcategory of bicomodule configurations with invertible abacus maps, we arrive at an equivalence (Theorem 4.19)

$$\boldsymbol{Pr}^{2\text{-Seg}}(\mathbb{\Delta}) \simeq \boldsymbol{ABC}^{\simeq},$$

which we interpret as a BOORS-type equivalence but with Σ-presheaves replaced by $\mathcal{D}$-presheaves. In fact, the functor $\boldsymbol{Pr}^{2\text{-Seg}}(\mathbb{\Delta}) \to \boldsymbol{ABC}^{\simeq}$ is equivalent to the total decalage, version r^*.

Relating the two notions of augmentation. The results above prompt a closer analysis of the relationship between the two notions of augmentation, leading finally to a derivation of the BOORS equivalence from equivalences involving bicomodules.

The comparison of the two notions of augmentation is mediated by a functor $\mathsf{j} : \Sigma \to \mathcal{D}$, which maps $[-1]$ to $[0, -1]$. In outline, the translation from a Σ-presheaf satisfying the BOORS axioms to a bicomodule configuration (with invertible abacus maps) then runs as follows. First, with the help of Proposition 2.17, we reinterpret the BOORS axioms on augmentation (pointing) as providing horizontal bottom splitting (more precisely, $\mathrm{Dec}_\bot$-coalgebra structure) on the zeroth row, and by stability on all rows. Similarly, the pointing also induces vertical top splittings (more precisely, $\mathrm{Dec}_\top$-coalgebra structure) on all columns. Next, we can take geometric realization of all rows and columns to produce the Carlier augmentations. A descent argument in Proposition 3.6 involving stability ensures that the augmentations satisfy Carlier's bicomodule axioms (2-Segal augmentation column and row and culf augmentation maps). Furthermore, since both the splittings were induced from the single pointing $B_{-1} \to B_{0,0}$, both the augmentation row and the augmentation column must have B_{-1} in degree zero. As a matter of fact, the induced abacus maps are forced to be invertible, so that altogether, starting with a Σ-presheaf satisfying the BOORS axioms we have produced a bicomodule configuration with invertible abacus maps. Moreover, when restricted along j, this bicomodule configuration recovers the Σ-presheaf we started with.

A similar reasoning shows that the restriction of a bicomodule configuration with invertible abacus maps along j yields a Σ-presheaf satisfying the BOORS axioms. All told, j^* restricts to an equivalence (Theorem 5.9)

$$\boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma) \xrightarrow{\simeq} \boldsymbol{ABC}^{\simeq}$$

between Σ-presheaves satisfying the BOORS axioms and bicomodule configurations with invertible abacus maps. This equivalence, together with those described in paragraph Equivalences in the relative setting can be organized into the following diagram

$$\begin{array}{ccccc}
\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) & \underset{\mathsf{q}^*}{\overset{\mathsf{q}_*}{\rightleftarrows}} & \boldsymbol{Pr}(\mathcal{D}) & \xrightarrow{\mathsf{j}^*} & \boldsymbol{Pr}(\Sigma) \\
\| & & \uparrow & & \uparrow \\
\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) & \xrightarrow[4.6]{\simeq} & \boldsymbol{Pr}^{\star}(\mathcal{D}) & & \\
\uparrow & & \uparrow & & \\
\boldsymbol{Pr}^{\mathrm{up\ 2\text{-}Seg}}(\mathbb{\Delta} \times \Delta^1) & \xrightarrow[4.14]{\simeq} & \boldsymbol{ABC}^{\star} & & \\
\uparrow & & \uparrow & & \\
\boldsymbol{Pr}^{\mathrm{2\text{-}Seg}}(\mathbb{\Delta}) & \xrightarrow[4.19]{\simeq} & \boldsymbol{ABC}^{\simeq} & \xrightarrow[5.9]{\simeq} & \boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma)
\end{array} \tag{1.4}$$

where all the vertical inclusions are full. The composite of the bottom two horizontal functors turns out to be precisely the total decalage p^*, whereby we recover the original BOORS equivalence (Corollary 5.10). The final result of the paper (Theorem 5.12) states a more refined equivalence, which does not fit into the diagram above. It goes between Σ-presheaves that satisfy "half" of the BOORS axioms and certain $\mathcal{D}$-presheaves but without the augmentation row.

Type Theory for (∞, ∞)-Categories

Contents

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1 The Type Theory `CaTT`

1.1 Type Theory

We begin with a short and informal introduction to type theory. In type theory there are four basic entities:

types, *terms*, *contexts*, and *context morphisms.*

These can be put together into valid expressions which are called *judgments*. There are four judgments (in `CaTT`), which we give here along with their interpretation:

$\Gamma \vdash$	Γ is a well-formed context
$\Gamma \vdash A$	*A is a well-formed type in the context* Γ
$\Gamma \vdash t : A$	*t is a well-formed term of type A* in Γ
$\Delta \vdash \gamma : \Gamma$	*γ is a well-formed context morphism from* Γ to Δ.

Intuitively we may think of types as containers, the elements of which are terms. Among all terms of a given type we have the variables, which appear in the context.

A well-formed context Γ is a finite list of variables with their corresponding type $(x_i : A_i)_{1\leq i\leq n}$, such that A_{i+1} is a well-formed type in the context $(x_k : A_k)_{1\leq k\leq i}$. Context can be thought of as the data we use to build types and terms. As a simple example, consider the judgment $(x_i : A_i)_{1\leq i\leq n} \vdash x_i : A_i$, which says we can always built the variable x_i, if it was already available in the context. The variables appearing in the construction of a certain type or term are called the *free variables* of the given type or term.

A well-formed context morphism $\Delta \vdash \gamma : \Gamma$, where $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n}$, consists of a list of terms $\gamma_1, \ldots, \gamma_n$ in context Δ. The type of the term γ_i is determined by the type A_i in a specified way. All together, the context morphism $\Delta \vdash \gamma : \Gamma$ provides us with a way to substitute a list of terms $\gamma_1, \ldots, \gamma_n$ into the free variables of a type or a term in a well-formed way.

A given type theory also specifies a list of *rules*. A rule has the form

$$\frac{J_1 \quad \cdots \quad J_n}{J}$$

where $J_1, \ldots, J_n, J$ are judgments. The judgments $J_1, \ldots, J_n$ are said to be the hypothesis and the judgment J is the conclusion. Rules can be composed to give *derivation trees*

$$\dfrac{\dfrac{J_1' \quad \cdots \quad J_m'}{J_1} \qquad \cdots \qquad \dfrac{\dfrac{J_1''' \quad \cdots \quad J_l'''}{J_1''} \qquad \cdots \quad J_m''}{J_n}}{J}$$

The root is called the *conclusion* of the derivation (in the above example J) and the leaves are the *hypothesis* (in the above example $J_1', \ldots, J_m', J_1''', \ldots, J_l''', \ldots$). A judgment is said to be *derivable* if it is the conclusion of a derivation tree with an empty set of leaves.

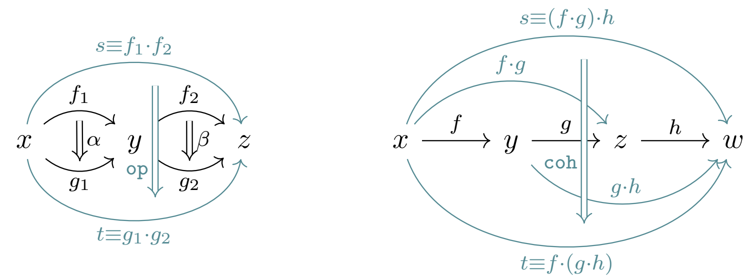
1.2 The Rules

The type theory `CaTT`, developed by Mimram and Finster [30] describes a single (∞, ∞)-category. Its models are precisely the (∞, ∞)-categories in the sense of Grothendieck and Maltsiniotis, as shown by Benjamin, Finster and Mimram [10]. The type theory `CaTT` has two type constructors

$$\frac{}{\Gamma \vdash \mathtt{Ob}} \qquad\qquad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \to_A t}$$

Here `Ob` is the type of all objects, while $s \to_A t$ may be thought of as a "directed hom-type", containing all morphisms from s to t. Starting with `Ob`, the second rule allows us to iteratively build all higher hom-types. To reduce clutter we will often omit the subscript A in $s \to_A t$.

In a (weak) higher category there are two types of cells one needs to produce as part of the axioms, the operations and the coherences. Operations are non-invertible and include precisely all the compositions while the coherences are invertible and include cells such as the units, the associators and the interchange laws. As an example, consider the horizontal composition of two 2-dimensional cells (left) and the associator (right):



(1.5)

The collection of solid arrows form the pasting diagram. These are the diagrams which may be composed via the operations. The remaining arrows in blue are built using the data of the underlying pasting diagram. In each case we are ultimately constructing a 2-dimensional cell $s \to t$. In the case of the horizontal composition $\texttt{op} : f_1 \cdot f_2 \to g_1 \cdot g_2$, the source only makes use of variables f_1 and f_2 which define a sub-pasting-diagram called the source and denoted by $\partial^- \Gamma$. Similarly, the target of $\texttt{op}$ only makes use of g_1 and g_2 which define a sub-pasting-diagram called the target diagram and denoted by $\partial^+ \Gamma$. In the case of the associator $\texttt{coh} : (f \cdot g) \cdot h \to f \cdot (g \cdot h)$, both the source and the target make use of the whole diagram defined by f, g and h. These observations are the defining features of operations and coherences respectively.

In the type theory $\texttt{CaTT}$ the underlying pasting diagram is encoded as a context Γ. A judgment of the form $\Gamma \vdash_{\text{ps}}$ asserts that Γ as a diagram has the shape of a pasting diagram. $\texttt{CaTT}$ contains two rules (OP) and (COH), producing the operations and coherences respectively. Each of these two rules is accompanied by a side condition referencing the free variables of the terms involved, which expresses precisely the intuition explained in the previous paragraph. The rules read

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \to_A t}{\Gamma \vdash \texttt{op}_{\Gamma, s \to_A t} : s \to_A t} \qquad \begin{array}{c} \text{FV}(s : A) = \text{FV}(\partial^- \Gamma) \\ \text{FV}(t : A) = \text{FV}(\partial^+ \Gamma) \end{array}$$

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \to_A t}{\Gamma \vdash \texttt{coh}_{\Gamma, s \to_A t} : s \to_A t} \qquad \begin{array}{c} \text{FV}(s : A) = \text{FV}(\Gamma) \\ \text{FV}(t : A) = \text{FV}(\Gamma) \end{array}$$

It is now just a matter of building in a cut into the rules to make the cut rule admissible. Intuitively we may think of this as allowing us to compose arbitrary terms, not just the variables of the given pasting diagram. In the next section we will give the full list of all rules with all the required details.

Definition 1.1 ($\texttt{CaTT}$ Rules)**.** $\texttt{CaTT}$ *is defined by the rules:*

Rules for types:

$$\frac{\Gamma \vdash}{\Gamma \vdash \texttt{Ob}}\ (\text{Ob}) \qquad\qquad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \to_A t}\ (\to)$$

Rules for terms:

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\text{VAR})$$

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{op}_{\Gamma, s \to_A t}[\gamma] : s[\gamma] \to_{A[\gamma]} t[\gamma]}\ (\text{OP}) \qquad \begin{array}{c}\text{FV}(\partial^-\Gamma) = \text{FV}(s : A)\\ \text{FV}(\partial^+\Gamma) = \text{FV}(t : A)\end{array}$$

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \to_A t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{coh}_{\Gamma, s \to_A t}[\gamma] : s[\gamma] \to_{A[\gamma]} t[\gamma]}\ (\text{COH}) \qquad \begin{array}{c}\text{FV}(\Gamma) = \text{FV}(t : A)\\ \text{FV}(\Gamma) = \text{FV}(s : A)\end{array}$$

Rules for contexts:

$$\frac{}{\emptyset \vdash}\ (\text{EC}) \qquad\qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}\ (\text{CE})$$

Rules for substitutions:

$$\frac{\Delta \vdash}{\Delta \vdash \langle\,\rangle : \emptyset}\ (\text{ES}) \qquad\qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, t \rangle : (\Gamma, x : A)}\ (\text{SE})$$

Rules for ps-contexts:

$$\frac{}{x : \texttt{Ob} \vdash_{\text{ps}} x : \texttt{Ob}}\ (\text{PSS}) \qquad\qquad \frac{\Gamma \vdash_{\text{ps}} x : A}{\Gamma, y : A, f : x \to_A y \vdash_{\text{ps}} f : x \to_A y}\ (\text{PSE})$$

$$\frac{\Gamma \vdash_{\text{ps}} f : x \to_A y}{\Gamma \vdash_{\text{ps}} y : A}\ (\text{PSD}) \qquad\qquad \frac{\Gamma \vdash_{\text{ps}} x : \texttt{Ob}}{\Gamma \vdash_{\text{ps}}}\ (\text{PS})$$

Given a context morphism $\Delta \vdash \gamma : \Gamma$ and a variable $x \in \text{FV}(\Gamma)$ we may denote by γ_x the term in γ corresponding to x.

To make sense of the rules we need to define the free variables on all the constructors, substitution on the constructors, the dimension of contexts and finally the source and the target of a ps-context.

Definition 1.2 (Free Variables)**.** *The set of free variables are inductively defined* *as follows:*

$$\mathrm{FV}(\mathtt{Ob}) := \emptyset \qquad \mathrm{FV}(\emptyset) := \emptyset$$
$$\mathrm{FV}(s \to_A t) := \mathrm{FV}(s) \cup \mathrm{FV}(t) \cup \mathrm{FV}(A) \qquad \mathrm{FV}(\Gamma, x : A) := \mathrm{FV}(\Gamma) \cup \{x\}$$

$$\mathrm{FV}(x) := \{x\} \qquad \mathrm{FV}(\langle\,\rangle) := \emptyset$$
$$\mathrm{FV}(\mathtt{op}_{\Gamma, s \to_A t}[\gamma]) := \mathrm{FV}(\gamma) \qquad \mathrm{FV}(\langle \gamma, t\rangle) := \mathrm{FV}(\gamma) \cup \mathrm{FV}(t)$$
$$\mathrm{FV}(\mathtt{coh}_{\Gamma, s \to_A t}[\gamma]) := \mathrm{FV}(\gamma)$$

As a shorthand we also write $\mathrm{FV}(t : A) = \mathrm{FV}(t) \cup \mathrm{FV}(A)$ for a term t of type A. Moreover, if a context Γ is of the form Γ', Γ'', then we define $\mathrm{FV}(\Gamma'') = \mathrm{FV}(\Gamma) \backslash \mathrm{FV}(\Gamma')$. The cardinality of $\mathrm{FV}(\Gamma'')$ will be denoted by $|\Gamma''|$.

For technical reasons it is convenient to make substitution an admissible rule rather than an explicit one. This requires defining substitution on the constructors and building in just enough substitution into the term constructors.

Definition 1.3 (Substitution)**.** *For the operation of substitution, we define*

$$\mathtt{Ob}[\gamma] :\equiv \mathtt{Ob} \qquad \langle\,\rangle \circ \gamma :\equiv \langle\,\rangle$$
$$(s \underset{A}{\to} t)[\gamma] :\equiv s[\gamma] \underset{A[\gamma]}{\to} t[\gamma] \qquad \langle \theta,\, t\rangle \circ \gamma :\equiv \langle \theta \circ \gamma,\, t[\gamma]\rangle$$

$$x_i[\gamma] :\equiv \gamma_i$$
$$\mathtt{op}_{\Gamma, s \to t}[\gamma][\delta] :\equiv \mathtt{op}_{\Gamma, s \to t}[\gamma \circ \delta]$$
$$\mathtt{coh}_{\Gamma, s \to t}[\gamma][\delta] :\equiv \mathtt{coh}_{\Gamma, s \to t}[\gamma \circ \delta]$$

Moreover, if $(x_i : A_i) \in \Gamma$ *and* $\Delta \vdash \gamma : \Gamma$*, then we define* $A_i[\gamma] :\equiv A_i[\gamma^{i-1}]$.

The rules also rely on the definition of the source and target of a ps-context. This definition makes use of the dimension of contexts which we define first.

Definition 1.4 (Dimension)**.** *The dimension of a type is defined inductively by:*

$$\dim(\mathtt{Ob}) := 0$$
$$\dim(s \to_A t) := \dim(A) + 1$$

Moreover, given a context $\Gamma \equiv (x_i : A_i)_{1 \le i \le n}$ *we define* $\dim(\Gamma) := \max_i\{\dim(A_i)\}$ *and given a term* $\Gamma \vdash t : A$ *we define* $\dim(t) := \dim(A)$.

Given a term $\Gamma \vdash t : A$ with $\dim(t) > 0$, then A is necessarily the result of an application of $(\to \mathfrak{I})$, and we write $A \equiv \sigma(t) \to_{\partial A} \tau(t)$. We will also write $\partial^- t$ for $\sigma(t)$ and $\partial^+ t$ for $\tau(t)$. Moreover, if t is a variable in Γ, say x_i, then we write $x_{\partial^- i}$ for $\partial^- x_i$ and similarly for the target.

Definition 1.5 (Codimension)**.** *Let $t : A$ be a term and let $x_i \in \mathrm{FV}(t)$. The codimension of x_i in t is defined to be* $\mathrm{codim}_t(x_i) := \dim(t) - \dim(x_i)$*.*

Definition 1.6 (Context Source and Target)**.** *Let $\Gamma \vdash_{\mathrm{ps}}$ be a ps-context and let $k \in \mathbb{N}$ be a natural number. We define k-source by $\partial_k^-(x : \mathtt{Ob}) :\equiv x : \mathtt{Ob}$ and*

$$\partial_k^-(\Gamma, y : A, f : x \to y) :\equiv \begin{cases} \partial_k^-(\Gamma) & \text{if } k \leq \dim(A) \\ \partial_k^-(\Gamma),\ y : A,\ f : x \to y & \text{else} \end{cases}$$

and the k-th target by $\partial_k^+(x : \mathtt{Ob}) :\equiv x : \mathtt{Ob}$ and

$$\partial_k^+(\Gamma,\ y : A,\ f : x \to y) :\equiv \begin{cases} \partial_k^+(\Gamma) & \text{if } k < \dim(A) \\ \mathrm{drop}\big(\partial_k^+(\Gamma)\big),\ y : A & \text{if } k = \dim(A) \\ \partial_k^+(\Gamma),\ y : A,\ f : x \to y & \text{else} \end{cases}$$

Here $\mathrm{drop}(\Gamma)$ *is given by Γ with the last element removed. If $\dim(\Gamma) > 0$, then the source and target of Γ are defined to be $\partial^-\Gamma :\equiv \partial^-_{\dim(\Gamma)-1}\Gamma$ and $\partial^+\Gamma :\equiv \partial^+_{\dim(\Gamma)-1}\Gamma$.*

If we forget the rules (OP) and (COH) along with all the required rules (such as those for the ps-contexts and their boundaries), we are left with a type theory called `GSet`, which has been studied by Benjamin, Finster and Mimram in [10]. The models of this type theory are precisely globular sets. A term, type or context in `CaTT` is said to be globular, if it is derivable in the subtype theory `GSet`. This amounts to saying that all of the terms appearing in its construction are variables. Similarly a term, type or context is said to be categorical if it is constructed purely using the rules of `CaTT`.

1.3 Admissible Structural Rules and General Features

When spelling out type theoretic rules containing, say, a term $\Gamma \vdash t : A$, it is not necessary to include also the judgment $\Gamma \vdash A$, affirming that A is derivable in Γ. This is because we can deduce the latter from the former. Other similar properties also hold, and we collect all of them in the next lemma. All properties discussed in this subsection are standard and are already mentioned in Benjamin, Finster, Mimram [10] and Benjamin's thesis [10].

Definition 1.7 (Admissibility)**.** *Let $\mathcal{T}$ be some type theory. A rule*

$$\frac{J_1 \qquad \cdots \qquad J_n}{J}$$

where $J_1, \ldots, J_n, J$ are some judgments in $\mathcal{T}$ is said to be admissible in $\mathcal{T}$, if given a derivation of $J_1, \ldots, J_n$ in $\mathcal{T}$ we can derive J in $\mathcal{T}$.

Proposition 1.8. *The following rules are admissible in* `CaTT`

$$\frac{\Gamma \vdash A}{\Gamma \vdash} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash A} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Gamma \vdash}$$

Another important feature (common to all cut-free type theories) is that all derivable judgments are well-behaved in terms of free variable structure.

Proposition 1.9. *The following hold in* `CaTT`*:*

1. *For every derivable type* $\Gamma \vdash A$*, we have* $\mathrm{FV}(A) \subset \mathrm{FV}(\Gamma)$*;*
2. *For every derivable term* $\Gamma \vdash t : A$*, we have* $\mathrm{FV}(t) \subset \mathrm{FV}(\Gamma)$*;*
3. *For every derivable context morphism* $\Delta \vdash \gamma : \Gamma$*, we have* $\mathrm{FV}(\gamma) \subset \mathrm{FV}(\Delta)$

Substitution is admissible and it satisfies all the required properties to ensure that the syntactic category is actually a category (in fact a category with families).

Proposition 1.10. *The following rules are admissible in* `CaTT`

(i) *For types:* $$\frac{\Gamma \vdash A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]} \, \mathcal{S}_{types} \qquad \textit{and} \qquad A[\gamma][\delta] \equiv A[\gamma \circ \delta].$$

(ii) *For terms:* $$\frac{\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash t[\gamma] : A[\gamma]} \, \mathcal{S}_{types} \qquad \textit{and} \qquad t[\gamma][\delta] \equiv t[\gamma \circ \delta].$$

(iii) *For contexts:* $$\frac{\Gamma \vdash \theta : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \theta \circ \gamma : \Theta} \, \mathcal{S}_{cont.} \qquad \textit{and} \qquad (\theta \circ \gamma) \circ \delta \equiv \theta \circ (\gamma \circ \delta).$$

Substitution interacts with the free variables as expected.

Lemma 1.11. *Given a type* $\Gamma \vdash A$ *and a term* $\Gamma \vdash t : A$ *in context* Γ *as well as a context morphism* $\Delta \vdash \gamma : \Gamma$*, the following equations hold in* `CaTT`

$$\mathrm{FV}(A[\gamma]) = \bigcup_{x_i \in \mathrm{FV}(A)} \mathrm{FV}(\gamma_i), \qquad \mathrm{FV}(t[\gamma]) = \bigcup_{x_i \in \mathrm{FV}(t)} \mathrm{FV}(\gamma_i).$$

The next lemma is a useful tool for proving equalities between context morphisms.

Lemma 1.12. *Let* $\Delta \vdash \gamma : \Gamma$ *and* $\Delta \vdash \gamma' : \Gamma$ *be two context morphisms. If* $x_i[\gamma] \equiv x_i[\gamma']$ *for all* $1 \leq i \leq n$*, then* $\gamma \equiv \gamma'$*.*

Lemma 1.13. *The following rule is admissible in* `CaTT`

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathtt{id}_\Gamma : \Gamma} \, \mathcal{U} \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash \pi_{\Gamma,A} : \Gamma} \, \mathcal{P}$$

Moreover,

1. *For types* $\Gamma \vdash A$
$$A[\mathtt{id}] \equiv A$$
2. *For terms* $\Gamma \vdash t : A$
$$t[\mathtt{id}] \equiv t$$
3. *For context morphisms* $\Delta \vdash \gamma : \Gamma$
$$\mathtt{id}_\Gamma \circ \gamma \equiv \gamma \qquad\qquad \gamma \circ \mathtt{id}_\Delta \equiv \gamma.$$
as well as $\pi_{\Gamma,A} \circ \gamma \equiv \gamma^{n-1}$.

Finally, weakening and exchange are also admissible.

Proposition 1.14. *The weakening and exchange rules are admissible in* CaTT

(i) For types:
$$\frac{\Gamma \vdash A \qquad \Gamma, x_{n+1} : A_{n+1} \vdash}{\Gamma, x_{n+1} : A_{n+1} \vdash A_w}\ \mathcal{W}_{types}$$
Moreover, $A_w[\langle \gamma, \gamma_{n+1} \rangle] \equiv A[\gamma]$ *and in fact* $A_w \equiv A$.

(ii) For terms:
$$\frac{\Gamma \vdash t : A \qquad \Gamma, x_{n+1} : A_{n+1} \vdash}{\Gamma, x_{n+1} : A_{n+1} \vdash t_w : A_w}\ \mathcal{W}_{terms}$$
Moreover, $t_w[\langle \gamma, \gamma_{n+1} \rangle] \equiv t[\gamma]$ *and in fact* $t_w \equiv t$.

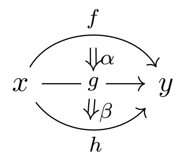
(iii) For context morphisms:
$$\frac{\Gamma \vdash \theta : \Theta \qquad \Gamma, x_{n+1} : A_{n+1} \vdash}{\Gamma, x_{n+1} : A_{n+1} \vdash \theta_w : \Theta}\ \mathcal{W}_{cont.}$$
Moreover, $\theta_w \circ \langle \gamma, \gamma_{n+1} \rangle \equiv \theta \circ \gamma$ *and in fact* $\gamma_w \equiv \gamma$.

Proposition 1.15. *The following rules hold in* CaTT
$$\frac{\Gamma,\, x : A,\, y : B,\, \Delta \vdash \qquad (x : A) \notin \mathrm{FV}(B)}{\Gamma,\, y : B,\, x : A,\, \Delta \vdash}\ \mathcal{E}$$
Moreover, we can perform this exchange in any derivable rule.

1.4 Operations and Identities

In a higher category, there are various ways of composing higher dimensional cells. Here we make explicit the d canonical binary compositions with which d-dimensional cells may be composed. The i-th binary composition is defined by having an $(i-1)$-dimensional locus of composition.

For 2-dimensional cells we have vertical and horizontal composition. The relevant diagram for vertical composition is given by



As a ps-context it can be obtained by the derivation tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : \texttt{Ob} \vdash_{\text{ps}} x : \texttt{Ob}}\ (\text{PSS})}{x : \texttt{Ob},\ y : \texttt{Ob},\ f : x \to y \vdash_{\text{ps}} f : x \to y}\ (\text{PSE})}{x : \texttt{Ob},\ y : \texttt{Ob},\ f : x \to y,\ g : x \to y,\ \alpha : f \to g \vdash_{\text{ps}} \alpha : f \to g}\ (\text{PSE})}{x : \texttt{Ob},\ y : \texttt{Ob},\ f : x \to y,\ g : x \to y,\ \alpha : f \to g \vdash_{\text{ps}} g : x \to y}\ (\text{PSD})}{x : \texttt{Ob},\ y : \texttt{Ob},\ f : x \to y,\ g : x \to y,\ \alpha : f \to g,\ h : x \to y,\ \beta : g \to h \vdash_{\text{ps}} \beta : g \to h}\ (\text{PSE})$$

and we define the context in the last judgment to be $O^2_1(\alpha, \beta)$, that is we set

$$O^2_1(\alpha, \beta) :\equiv x : \texttt{Ob}, y : \texttt{Ob}, f : x \to y, g : x \to y, \alpha : f \to g, h : x \to y, \beta : g \to h.$$

The source and boundary ps-contexts are given by

$$\partial^- O^2_1(\alpha, \beta) \equiv x : \texttt{Ob}, y : \texttt{Ob}, f : x \to y \equiv: D^1(f)$$

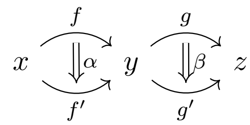
$$\partial^+ O^2_1(\alpha, \beta) \equiv x : \texttt{Ob}, y : \texttt{Ob}, h : x \to y \equiv: D^1(h).$$

With these we may apply the (OP) rule and obtain the vertical composition

$$\frac{O^2_1(\alpha, \beta) \vdash_{\text{ps}} \qquad D^1(f) \vdash f : x \to z \qquad D^1(h) \vdash h : x \to z \qquad \Delta \vdash \gamma : O^2_1(\alpha, \beta)}{\Delta \vdash \gamma_\alpha \overset{2}{\underset{1}{*}} \gamma_\beta : \gamma_f \to \gamma_h}\ (\text{OP})$$

where we have defined $\gamma_\alpha \overset{2}{\underset{1}{*}} \gamma_\beta :\equiv \texttt{op}_{O^2_1(\alpha,\beta)}[\gamma]$. In particular we get the composite $\alpha \overset{2}{\underset{1}{*}} \beta : f \to h$ using the identity context morphism.

Similarly we can build horizontal composition by considering the diagram



As a ps-context it is given by

$$\begin{aligned} O_0^2(\alpha, \beta) :\equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y, f' : x \to y, \alpha : f \to f', \\ z : \mathtt{Ob}, g : y \to z, g' : y \to z, \beta : g \to g' \end{aligned}$$

in which we may construct the horizontal composite $\alpha \overset{2}{\underset{0}{*}} \beta : f \overset{1}{\underset{0}{*}} g \to f' \overset{1}{\underset{0}{*}} g'$, where $\overset{1}{\underset{0}{*}}$ denotes the composition of 1-cells.

We now construct all binary composites more systematically. The d-dimensional globe as a context can be defined using the rules for ps-context. First of all, by (PSS) we have $x : \mathtt{Ob} \vdash_{\text{ps}} x : \mathtt{Ob}$ and we define $D^0(x : \mathtt{Ob}) :\equiv x : \mathtt{Ob}$. By construction, the judgment $D^0(x : \mathtt{Ob}) \vdash_{\text{ps}} x : \mathtt{Ob}$ is derivable and $\dim(x) = 0$. Applying inductively the rule (PSE) we get

$$\frac{D^d(x : A) \vdash_{\text{ps}} x : A}{D^d(x : A), y : A, f : x \to_A y \vdash_{\text{ps}} f : x \to_A y}$$

and we define $D^{d+1}(f : x \to y) :\equiv D^d(x : A), y : A, f : x \to_A y$. Given a d-dimensional globe $D^d(x : A)$, by an inductive argument one can show that $\dim(x) = d$ as well as $\text{FV}(D^d(x : A)) = \text{FV}(x : A)$. Often we will suppress the type and simply write $D^d(x)$ instead of $D^d(x : A)$.

Next we define a collection of ps-contexts, the gluings of which will give us the binary compositions. Let $n \in \mathbb{N}$ be some natural number. Applying (PSD) and then (PSE) again to $D^{n+1}(f : x \to_A y) \vdash_{\text{ps}} f : x \to_A y$ we get a context

$$\overbrace{D^{n+1}(f : x \to_A y),\ z : A,\ g : y \to z}^{O_n^{n+1}(f : x \to_A y,\, g : y \to_A z)\ :\equiv} \vdash_{\text{ps}} g : y \to_A z$$

Again for brevity we may also write $O_n^{n+1}(f, g)$. These contexts will allow us to define sequential composition (or vertical composition in the 2-dimensional case). As a sequence of rules this judgment is represented by (PSS)(PSE)$^{d+1}$(PSD)(PSE).

For the contexts of the remaining compositions we take the ps-context which as a sequence of rules is given by (PSS)(PSE)^d(PSD)$^{d-n}$(PSE)$^{d-n}$. The judgment obtained is of the form

$$\overbrace{D^d(\alpha : X), \Delta, \beta : Y}^{O_n^d(\alpha : X,\, \beta : Y)\ \equiv} \vdash_{\text{ps}} \beta : Y$$

where $d > n$, Δ is a string of variables containing the target of β and variables thereof.

Now, a calculation shows that

$$\begin{aligned} \partial^- O_n^{n+1}(f : x \to_A y, g : y \to_A z) &\equiv D^n(x : A) \\ \partial^+ O_n^{n+1}(f : x \to_A y, g : y \to_A z) &\equiv D^n(z : A) \end{aligned} \tag{1.6}$$

as well as

$$\begin{aligned} \partial^- O_n^{d+1}(\phi : \alpha \to_X \alpha', \psi : \beta \to_Y \beta') &\equiv O_n^d(\alpha : X, \beta : Y) \\ \partial^+ O_n^{d+1}(\phi : \alpha \to_X \alpha', \psi : \beta \to_Y \beta') &\equiv O_n^d(\alpha' : X, \beta' : Y) \end{aligned} \tag{1.7}$$

We use these ps-context to construct all binary operations $\underset{n}{\overset{d}{*}}$. By equation (1.6) and with the abbreviation $O_n^{n+1}(f, g)$ for $O_n^{n+1}(f : x \to_A y, g : y \to_A z)$ we have

$$\frac{O_n^{n+1}(f,g) \vdash_{\text{ps}} \qquad D^n(x : A) \vdash x : A \qquad D^n(z : A) \vdash z : A \qquad \Delta \vdash \gamma : O_n^{n+1}(f,g)}{\Delta \vdash \gamma_f \underset{n}{\overset{n+1}{*}} \gamma_g : \gamma_x \to_{A[\gamma]} \gamma_z} \text{ (OP)}$$

where we have defined $\gamma_f \underset{n}{\overset{n+1}{*}} \gamma_g :\equiv \mathtt{op}_{O_n^{n+1}(f,g)}[\gamma]$.

The remaining binary compositions are built by induction. To start consider the ps-context $O_n^{d+1}(\phi : \alpha \to \alpha', \psi : \beta \to \beta')$. By the inductive hypothesis, the terms $O_n^d(\alpha, \beta) \vdash \alpha \underset{n}{\overset{d}{*}} \beta : T$ and $O_n^d(\alpha', \beta') \vdash \alpha' \underset{n}{\overset{d}{*}} \beta' : T$ are derivable and parallel. This allows us to apply the (OP) rule in the following way:

$$\frac{O_n^{d+1}(\phi,\psi) \vdash_{\text{ps}} \qquad O_n^d(\alpha,\beta) \vdash \alpha \underset{n}{\overset{d}{*}} \beta : T \qquad O_n^d(\alpha',\beta') \vdash \alpha' \underset{n}{\overset{d}{*}} \beta' : T \qquad \Delta \vdash \gamma : O_n^{d+1}(\phi,\psi)}{\Delta \vdash \phi \underset{n}{\overset{d+1}{*}} \psi : \alpha \underset{n}{\overset{d}{*}} \beta \to \alpha' \underset{n}{\overset{d}{*}} \beta'} \text{ (OP)}$$

where we have used equation (1.7) and where we defined $\phi \underset{n}{\overset{d+1}{*}} \psi \equiv \mathtt{op}_{O_n^{d+1}(\phi,\psi)}[\gamma]$.

A context morphisms $\Delta \vdash \gamma : O_n^d(\phi, \psi)$ contains the same information as two d-dimensional terms $\Delta \vdash u : s \to t$ and $\Delta \vdash v : s' \to t'$ such that $\mathrm{cod}^{d-n}(u) \equiv \mathrm{dom}^{d-n}(v)$. Thus, for any two such terms we may derive their composite $u \underset{n}{\overset{d}{*}} v$, which will also use the notation $u \cdot v$ when $d = n+1$. Given a substitution $\Theta \vdash \delta : \Delta$ we have $(u \underset{n}{\overset{d}{*}} v)[\delta] \equiv u[\delta] \underset{n}{\overset{d}{*}} v[\delta]$.

Consider the globe context $D^d(x : A)$. Given any term $\Delta \vdash t : T$ let us set $1_t^0 :\equiv t$. Applying the (COH) rule inductively then produces all higher identities

$$\frac{D^d(x:A) \vdash_{\text{ps}} \qquad D^d(x:A) \vdash 1^n_x \to 1^n_x \qquad \Delta \vdash \gamma : D^d(x:A)}{\Delta \vdash 1^{n+1}_{\gamma_x} : 1^n_{\gamma_x} \to 1^n_{\gamma_x}} \text{(COH)}$$

where we have defined $1^{n+1}_{\gamma_x} :\equiv \mathtt{coh}_{D^d(x:A)}[\gamma]$. In the case of $n = 1$ we also write 1_{γ_x} for $1^1_{\gamma_x}$. Since a term $\Delta \vdash t : T$ contains the same information as a context morphism $\Delta \vdash \hat{t} : D^d(x : A)$ with $\hat{t}_x \equiv t$ (see Benjamin–Finster–Mimram [10] Lemma 74), we can construct the identity $1^{n+1}_t : 1^n_t \to 1^n_t$ for any d-dimensional term $\Delta \vdash t : T$. Given a substitution $\Theta \vdash \delta : \Delta$ we have $1^n_t[\delta] \equiv 1^n_{t[\delta]}$.

1.5 Ps-Contexts

Ps-contexts are globular contexts, i.e. they contain only variables and no operations nor coherences. As diagrams, ps-contexts are precisely those which can be glued (or composed) using the (OP) rule in `CaTT`. In this section we study their properties. The results of this section will be used in Section 2.

Lemma 1.16. *Let $\Gamma \vdash_{\text{ps}}$ be a ps-context. If $\Gamma \equiv (x_i : A_i)_{1 \leq n \leq n} \equiv \Gamma^{n-2}, y : A, f : x \to y$, then $x \equiv (\partial^+)^p(x_{n-2})$ for some $p \in \mathbb{N}$.*

Proof. Both statements can be shown by inducting over all the rules for ps-context. □

Lemma 1.17 (Finster–Mimram [30] Lemma 34, Proposition 41)**.** *The following statements hold in* `CaTT`

(i) If $\Gamma \vdash_{\text{ps}}$ is derivable, then so is $\Gamma \vdash$

(ii) If $\Gamma \vdash_{\text{ps}}$ is derivable, then so are $\partial^-\Gamma \vdash_{\text{ps}}$ and $\partial^+\Gamma \vdash_{\text{ps}}$.

Lemma 1.18. *Let $\Gamma \vdash_{\text{ps}}$ be a ps-context such that $\dim(\Gamma) > 0$. Then $\text{FV}(\partial^-\Gamma) \neq \text{FV}(\partial^+\Gamma)$.*

Proof. If $\dim(\Gamma) = d > 0$, then Γ contains a d-dimensional morphism. When constructing ps-contexts, new terms always come in pairs of the form $y : A,\ f : x \to y$. Any d-dimensional morphism cannot be on the left, because Γ would otherwise contain a $(d+1)$-dimensional morphism, leading to a contradiction. Let f be the d-dimensional morphism introduced last in the construction of Γ so that we may write

$$\Gamma \equiv \Delta,\ y : A,\ f : x \to y,\ \Delta'$$

where $\Delta \vdash_{\text{ps}}$ and Δ' is a list of terms (which is not necessarily a context!) not containing any d-dimensional morphisms. Then

$$\begin{aligned}\partial^-\Gamma &\equiv \partial^-(\Delta,\ y : A,\ f : x \to y,\ \Delta') \\ &\equiv \partial^-(\Delta,\ y : A,\ f : x \to y) \\ &\equiv \partial^-\Delta\end{aligned}$$

while

$$\begin{aligned}\partial^+\Gamma &\equiv \partial^+(\Delta,\ y : A,\ f : x \to y,\ \Delta') \\ &\equiv \partial^+(\Delta,\ y : A,\ f : x \to y) \\ &\equiv \text{drop}\big(\partial^+\Delta\big),\ y : A.\end{aligned}$$

In particular $(y : A) \in \partial^+\Gamma$ while $(y : A) \notin \partial^-\Gamma$ so that $\text{FV}(\partial^-\Gamma) \neq \text{FV}(\partial^+\Gamma)$. □

1.5.1 Paths and the structure of Ps-Contexts

Ps-contexts have a very rigid structure. Our main tool for studying this structure will be sequences of terms forming paths, used extensively by Benjamin [9]. Using these we can, for example, show that in a certain sense, ps-contexts have no "holes" (see Lemma 1.25). Some of the following lemmas already appeared in Benjamin [9] – we indicate this in the margin. Nevertheless, for completeness sake, we have included (new) proofs, which we hope are of value.

Definition 1.19 (Parallel Terms)**.** *Terms $s, t : A$ of the same type are said to be parallel. We will also use the notation $s \parallel_A t$ and even drop the subscript as in $s \parallel t$ if no confusion can occur.*

Remark 1.20. Note that being parallel is an equivalence relation.

Definition 1.21 (Paths)**.** *Let Γ be a context and let $x, y \in \text{FV}(\Gamma)$ be two variables such that $x \parallel y$. A path $\overline{f} : x \rightsquigarrow y$ in some subset $S \subset \text{FV}(\Gamma)$ is an ordered list of variables $\{z_a\}_{0 \leq a \leq l}$ in S where $z_0 = x$ and $z_l = y$ together with an ordered list of variables $\{f_a\}_{1 \leq a \leq l}$ in S such that $(f_a : z_{a-1} \to z_a) \in \Gamma$. Visually*

$$x \xrightarrow{f_1} z_1 \xrightarrow{f_2} \dots \xrightarrow{f_{l-1}} z_{l-1} \xrightarrow{f_l} y.$$

If $S = \text{FV}(\Gamma)$, then we say $\overline{f}$ is a path in Γ. The ordered sets $\texttt{nodes}(\overline{f}) := \{z_a\}_{0 \leq a \leq l}$ and $\texttt{comp}(\overline{f}) := \{f_a\}_{1 \leq a \leq l}$ are called the nodes and the components of $\overline{f}$ respectively. For convenience we will also denote the path $\overline{f}$ by $(f_1, \dots, f_l)$. The length $|\overline{f}|$ of a path $\overline{f}$ is defined to be the cardinality of the set of components, that is $|\overline{f}| := |\texttt{comp}(\overline{f})| = l$. The free variables of the path $\overline{f}$ are defined to be the set $\text{FV}(\overline{f}) := \bigcup_{1 \leq a \leq l} \text{FV}(f_a : z_{a-1} \to z_a)$ and its dimension is defined to be that of its components, i.e. $\dim(\overline{f}) := \dim(f_a)$ for some $1 \leq a \leq l$.

Remark 1.22. Every variable has a path to itself, namely the empty path which has length zero.

Definition 1.23 (Concatenation of Paths)**.** *Let* Γ *be a context, and let* $\overline{f} : x \rightsquigarrow y$ *and* $\overline{g} : y \rightsquigarrow z$ *be two paths. Then* $\overline{f} * \overline{g} : x \rightsquigarrow z$ *is the path* $(f_1, \dots, f_p, g_1, \dots, g_q)$ *obtained by concatenating the two lists* $\overline{f} = (f_1, \dots, f_p)$ *and* $\overline{g} = (g_1, \dots, g_q)$*.*

Remark 1.24. By construction $\texttt{nodes}(\overline{f} * \overline{g}) = \texttt{nodes}(\overline{f}) \cup \texttt{nodes}(\overline{g})$ and $\texttt{comp}(\overline{f} * \overline{g}) = \texttt{comp}(\overline{f}) \cup \texttt{comp}(\overline{g})$ ordered appropriately and $|\overline{f} * \overline{g}| = |\overline{f}| + |\overline{g}|$.

Lemma 1.25. *Let* $\Gamma \vdash_{\text{ps}}$ *be a ps-context and let* $x_{i_1}, x_{i_2} \in \text{FV}(\Gamma)$ *be two parallel variables, i.e.* $x_{i_1} \parallel x_{i_2}$ *and such that* $i_1 \leq i_2$*. Then*

(i) (♣) *there exists a path* $\overline{p} : x_{i_1} \rightsquigarrow x_{i_2}$ *in* Γ *;* (♣) Benjamin [9] Lemma 94

(ii) for all paths $\overline{q} : x_{i_1} \rightsquigarrow x_{i_2}$*, we have* $\texttt{nodes}(\overline{q}) = \texttt{nodes}(\overline{p})$*.*

Proof. We prove the lemma by induction on ps-context for Γ. If $\Gamma \equiv (x : \texttt{Ob})$, then the statement is trivially true: we necessarily have $x_{i_1} \equiv x_{i_2} \equiv: x$ and there is a unique path $x \rightsquigarrow x$ namely the empty path. Assume now that $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n} \equiv \Gamma^{n-2}, y : A, f : x \to y$ where $(x : A) \in \Gamma^{n-2}$ and that the statement holds for Γ^{n-2}. There are four cases to be considered.

(1) $x_{i_1}, x_{i_2} \in \text{FV}(\Gamma^{n-2})$

Then, by the inductive hypothesis, there exists a path $x_{i_1} \rightsquigarrow x_{i_2}$ in Γ^{n-2} and therefore also in Γ. The second statement, however, is not automatic, because by induction the statement is only true inside Γ^{n-2} and the presence of new variables in Γ might spoil this.

First of all, any path $\overline{q} : x_{i_1} \rightsquigarrow x_{i_2}$ in Γ cannot involve f, neither as a node, nor as a component. This is because if f were a node, then there would have to exist a component of the form $q_a : f \to z_a$ or $q_{a-1} : z_{a-1} \to f$ for some nodes z_a, z_{a-1} in the path. But, since q_a and q_{a-1} depend on f, they must appear after f in the context Γ, contradicting the fact that f is the last variable in Γ. If $f : x \to y$ were a component, this would mean that $y \in \texttt{nodes}(\overline{q})$, which would allow us to extract from $\overline{q}$ a path $y \rightsquigarrow x_{i_2}$. This new path must be nonempty since $y \notin \text{FV}(\Gamma^{n-2})$. In particular there exists a component $q_a : y \to z_a$ in the path. Since the type of q_a depends on y it would have to appear after y does in Γ, which again results in a contradiction. This paragraph thus shows that f cannot be involved in $\overline{q}$.

Next we investigate the effects of having y involved in a path $\overline{q} : x_{i_1} \rightsquigarrow x_{i_2}$. If y is a node, then we derive a contradiction, as in the above paragraph. If y is a component, then, since x and y are parallel, i.e. $x \parallel y$, we can replace y with x and obtain a new path with all its variables contained in Γ^{n-2}. In total, we have shown that any path $\overline{q} : x_{i_1} \rightsquigarrow x_{i_2}$ can be modified to obtain a path $x_{i_1} \rightsquigarrow x_{i_2}$ with the same nodes which lies in Γ^{n-2}, which allows us to use the inductive hypothesis.

(2) $x_{i_1} \in \mathrm{FV}(\Gamma^{n-2})$ and $x_{i_2} \equiv y$

The last component of any path $\overline{p} : x_{i_1} \rightsquigarrow y$ must be of the form $p_a : z_{a-1} \to y$ for some node z_a. But any such variable must appear after y in Γ and the only option is f. So every path $\overline{p} : x_{i_1} \to y$ must end with $f : x \to y$.

But $x \in \mathrm{FV}(\Gamma^{n-2})$ and by the previous paragraph there exists a path $\overline{r} : x_{i_1} \rightsquigarrow x$ and all other such paths have the same nodes. We then get a path $x_{i_1} \rightsquigarrow y$ by concatenating $\overline{r}$ with f. Moreover, by the inductive hypothesis, every other path $\overline{q} : x_{i_1} \to y$ has the same nodes as $\overline{r} * f$.

(3) $x_{i_1} \in \mathrm{FV}(\Gamma^{n-2})$ and $x_{i_2} \equiv f$

For any variable $x_{i_1} \in \mathrm{FV}(\Gamma^{n-2})$ we necessarily have $x_{i_1} \nparallel f$. This is because the type $x \to y$ of f depends on y, so that x_{i_1} would have to appear after y in Γ, contradicting the fact that it lives in Γ^{n-2}. So the statement is vacuously true.

(4) $x_{i_1} \equiv y$ and $x_{i_2} \equiv f$

Then $x_{i_1} \nparallel x_{i_2}$ and the statement is vacuously true. □

There are a number of rather intuitive and very useful consequences we can extract from the above lemma.

Corollary 1.26. *Let Γ be a ps-context and let $x, y, z \in \mathrm{FV}(\Gamma)$ be two variables such that $x \parallel y \parallel z$. Then*

(i) if $x \not\equiv y$ and if there exists a path $x \rightsquigarrow y$ then there exist no path $y \rightsquigarrow x$;

(ii) all paths $x \rightsquigarrow y$ have the same length;

(iii) (♣) if $\dim(x) = \dim(\Gamma) - 1$, then there exists at most one path $x \rightsquigarrow y$; (♣) Benjamin [9] Lemma 95

(iv) if $\dim(x) = \dim(\Gamma)$, then $x \equiv y$;

(v) if there exists a path $x \rightsquigarrow y$, then it is nonempty if and only if $x \not\equiv y$;

(vi) if there exist paths $\overline{p} : z \rightsquigarrow x$ and $\overline{q} : z \rightsquigarrow y$ then if $|\overline{p}| \leq |\overline{q}|$, then $x \in$ `nodes`$(\overline{q})$. If $|\overline{p}| = |\overline{q}|$, then $x \equiv y$;

(vii) if there exist paths $\overline{p} : x \rightsquigarrow z$ and $\overline{q} : y \rightsquigarrow z$ then if $|\overline{p}| \leq |\overline{q}|$, then $x \in$ `nodes`$(\overline{q})$. If $|\overline{p}| = |\overline{q}|$, then $x \equiv y$.

Proof. *(i)* Assume the contrary, namely that there exists a path $x \rightsquigarrow y$ and a path $y \rightsquigarrow x$. Since $x \not\equiv y$ these paths are nonempty. By concatenation we get a nonempty path $x \rightsquigarrow x$. But we already have a path $x \rightsquigarrow x$, namely the empty path. Since one path contains y and the other does not, this contradicts Lemma 1.25, according to which all paths between the same variables have the same nodes.

(ii) This follows directly from the fact that any two paths have the same family of nodes, see Lemma 1.25.

(iii) Assume we have two paths $\overline{f}, \overline{g} : x \rightsquigarrow y$ where $\dim(x) = \dim(y) = \dim(\Gamma) - 1$. Then, by Lemma 1.25, these paths have the same families of nodes $\{z_a\}_{1 \leq a \leq p}$. Let $f_a : z_{a-1} \to z_a$ and $g_a : z_{a-1} \to z_a$ be the components between z_{a-1} and z_a of $\overline{f}$ and $\overline{g}$ respectively. Then $f_a \parallel g_a$ and by Lemma 1.25 there exists a path between f_a and g_a. But the components of such a path must then be of dimension $\dim(\Gamma) + 1$. Since the components of the path must themselves be variables of Γ, a nonempty path would thus lead to a contradiction. Thus we necessarily have $f_a \equiv g_a$. Since the index a was arbitrary, it follows that $\texttt{comp}(\overline{f}) = \texttt{comp}(\overline{g})$ and therefore $\overline{f} = \overline{g}$.

(iv) By Lemma 1.25 there exists a path between x and y. But any nonempty path would involve a variable of dimension $\dim(\Gamma) + 1$, contradicting the fact that this variable must live in Γ. Thus, any path between x and y must be empty, i.e. $x \equiv y$.

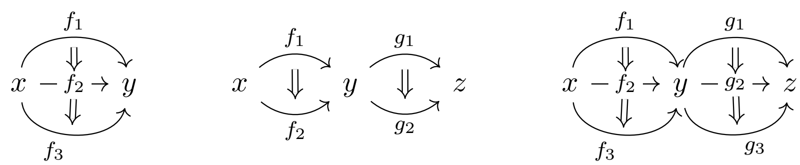
(v) Assume there exists a path $\overline{p} : x \rightsquigarrow y$. If $\overline{p}$ is nonempty, then if $x \equiv y$ we would also have the empty path, contradicting item (*(ii)*). If, on the other hand $x \not\equiv y$, then trivially $\overline{p}$ must be nonempty.

(vi) Assume we are given two paths $\overline{p} : z \rightsquigarrow x$ and $\overline{q} : z \rightsquigarrow y$. Then, by Lemma 1.25, since $x \parallel y$, there exists a path $\overline{f} : x \rightsquigarrow y$ or a path $\overline{g} : y \rightsquigarrow x$. If the $\overline{g}$ exists, then by concatenation we have a path $\overline{q} * \overline{g} : z \rightsquigarrow x$ and by part *(ii)* $|\overline{q} * \overline{g}| = |\overline{p}|$. If in addition we assume $|\overline{p}| \leq |\overline{q}|$, it must be the case that $\overline{g}$ is empty so that $x \equiv y$ and $x \in \texttt{nodes}(\overline{q})$.

If, on the other hand $\overline{f} : x \rightsquigarrow y$ exists, by concatenation, we get a path $\overline{p} * \overline{f} : z \rightsquigarrow y$. Thus, by Lemma 1.25, $\texttt{nodes}(\overline{p} * \overline{f}) = \texttt{nodes}(\overline{q})$. In particular $x \in \texttt{nodes}(\overline{q})$. If $|\overline{p}| = |\overline{q}|$, then we necessarily must have $|\overline{f}| = 0$ so that $x \equiv y$.

(vii) The proof of this is analogous to that of part *(vi)*. □

For our next result, consider the ps-context for composition of 2-morphisms, horizontal composition of 2-morphisms and a combination thereof



In all three cases one sees that each variable of dimension $\dim(\Gamma)-1$ is connected to a parallel variable in $\partial^-\Gamma$ and one in $\partial^+\Gamma$ via a path. This is a general phenomenon in ps-contexts independent of the dimension, as we now show.

Lemma 1.27. *Let $\Gamma \vdash_{\mathrm{ps}}$ be a ps-context and let us write $\Gamma \equiv (x_i : A_i)_{1\le i\le n}$ as usual. Then, if $x_k \in \mathrm{FV}(\Gamma)$ is such that $\dim(x_k) = \dim(\Gamma) - 1$, then*

1. *there exists a $x_{k^-} \in \mathrm{FV}(\partial^-\Gamma)$ with $\dim(x_{k^-}) = \dim(\Gamma) - 1$ and a path $x_{k^-} \rightsquigarrow x_k$ in Γ;*
2. *there exists a $x_{k^+} \in \mathrm{FV}(\partial^+\Gamma)$ with $\dim(x_{k^+}) = \dim(\Gamma) - 1$ and a path $x_k \rightsquigarrow x_{k^+}$ in Γ.*

Proof. We perform and induction on the ps-context Γ. If $\Gamma \equiv x : \mathtt{Ob}$ the statement is trivially true since $\partial^-(x : \mathtt{Ob}) \equiv (x : \mathtt{Ob}) \equiv \partial^+(x : \mathtt{Ob})$ so that the empty path does the job. So assume we are given a ps-context of the form $\Gamma, y : A, f : x \to y$. The argument differs according to the dimension of f in relation to that of Γ.

(i) $\dim(\Gamma) < \dim(f)$. Since $\dim(\Gamma, y : A, f : x \to y) = \dim(f)$, the variable x_k cannot be f. So there are two cases, either $x_k \equiv y$ or $x_k \in \mathrm{FV}(\Gamma)$.

The source boundary is given by $\partial^-(\Gamma, y : A, f : x \to y) = \Gamma$. If $x_k \equiv y$, since $x \in \mathrm{FV}(\Gamma)$ we can take the path $(f) : x \rightsquigarrow y$. If $x_k \in \mathrm{FV}(\Gamma)$ then we may simply take the empty path.

On the other hand $\partial^+(\Gamma, y : A, f : x \to y) \equiv \mathrm{drop}(\Gamma), y : A$. But, since $\dim(f) > \dim(\Gamma)$, we must have $\Gamma \equiv \Gamma^{n-1}, x : A$ so that the target boundary is computed by $\Gamma^{n-1}, y : A$. Again we have two cases. If $x_k \equiv y$, then we take the empty path. If $x_k \in \mathrm{FV}(\Gamma)$, then if $x_k \equiv x$ we take the path $(f) : x \rightsquigarrow y$. Otherwise we take the empty path.

(ii) $\dim(\Gamma) = \dim(f)$. As before there are two cases, either $x_k \equiv y$ or $x_k \in \mathrm{FV}(\Gamma)$.

The source boundary is given by $\partial^-(\Gamma, y : A, f : x \to y) \equiv \partial^-(\Gamma)$. If $x_k \equiv y$, then, by induction, there exists a variable $x_{k^-} \in \mathrm{FV}(\partial^-\Gamma)$ and a path $x_{k^-} \rightsquigarrow x$, which when concatenated with $(f) : x \rightsquigarrow y$ yields the desired data. If, on the other hand $x_k \in \mathrm{FV}(\Gamma)$, then we simply take the path given by induction, since $\partial^-(\Gamma)$ is contained in $\partial^-(\Gamma, y : A, f : x \to y)$.

As for the target boundary we have $\partial^+(\Gamma, y : A, f : x \to y) \equiv \mathrm{drop}(\partial^+\Gamma), y : A$. If $x_k \equiv y$, then we may simply take the empty path. For the other case notice first that Γ cannot be 0-dimensional since $\dim(\Gamma) = \dim(f)$. If we write $\Gamma \equiv \Gamma^{n-2}, w : B, g : v \to w$ then there are two possibilities, either $\dim(g) = \dim(\Gamma)$ or $\dim(g) = \dim(\Gamma) - 1$. Analyzing both cases shows that $\partial^+\Gamma$ ends with the variable x, which is therefore discarded in $\mathrm{drop}(\partial^+\Gamma)$. Now, if $x_k \in \mathrm{FV}(\Gamma)$, by induction there exists a $x_{k^+} \in \mathrm{FV}(\partial^+\Gamma)$ and a path $x_k \rightsquigarrow x_{k^+}$. If $x_{k^+} \not\equiv x$, then we are done. Otherwise, we concatenate this path with $(f) : x \rightsquigarrow y$.

(iii) $\dim(\Gamma) > \dim(f)$. In this case either $x_k \equiv f$ or $x_k \in \mathrm{FV}(\Gamma)$. If $x_k \equiv f$, then we may simply take the empty path on f. If, on the other hand $x_k \in \mathrm{FV}(\Gamma)$, we take the path in $\partial^\pm$ given to us by induction, since $\partial^\pm(\Gamma)$ is contained in $\partial^\pm(\Gamma, y : A, f : x \to y)$. □

Lemma 1.28. *Let $\Gamma \vdash_{\mathrm{ps}}$ be a ps-context. Then*

(i) $\mathrm{FV}(\Gamma)|_{\dim(\Gamma)-2} \subset \mathrm{FV}(\partial^-\Gamma)$*;*

(ii) $\mathrm{FV}(\Gamma)|_{\dim(\Gamma)-2} \subset \mathrm{FV}(\partial^+\Gamma)$.

(where the restriction to variables of dimension $\dim(\Gamma)-1$ *is empty if* $\dim(\Gamma) < 2$*) and if* $\dim(\Gamma) > 0$*, then*

(iii) $\mathrm{FV}(\partial^-\Gamma) \subset \mathrm{FV}(\Gamma)|_{\dim(\Gamma)-1}$*;*

(iv) $\mathrm{FV}(\partial^+\Gamma) \subset \mathrm{FV}(\Gamma)|_{\dim(\Gamma)-1}$.

Proof. We prove the statement by induction. If $\Gamma \equiv x : \mathtt{Ob}$, then the statement is trivially true. So assume now that Γ is of dimension greater than 0.

Consider the source boundary $\partial^-\Gamma$ of a ps-context Γ. Let us call a consecutive pair of variables $y : A, f : x \to y$ in Γ a ps-pair, if it is a pair introduced by the rule (PSE). The formula for the source boundary is such that it kills precisely those ps-pairs for which $\dim(f) = \dim(\Gamma)$. In particular, in the process of applying ∂^- we loose all top dimensional variables and all other variables also removed are of codimension 1 with respect to the dimension of the ps-context. This implies the results regarding the source boundary.

As for the target boundary, assume Γ is of the form $\Gamma', y : A, f : x \to y$. Let $n := \dim(\Gamma)$. Then if $\dim(f) < \dim(\Gamma)$, the ps-pair survives the application of the target boundary and remains part of $\partial^+_{n-1}\Gamma$. If, on the other hand, $\dim(f) = \dim(\Gamma)$, then $\partial^+\Gamma \equiv \mathrm{drop}(\partial^+\Gamma'), y : A$. We now perform a nested induction on Γ' and claim that ∂^+_{n-1} always ends with the variable $x : \mathtt{Ob}$. Indeed, the base case is given by $\Gamma' \equiv x : \mathtt{Ob}$. If, on the other hand, $\Gamma' \equiv \Gamma'', w : B, g : v \to w$, then we either have $\dim(g) = \dim(\Gamma)$ and $w \equiv x$ or $\dim(g) = \dim(\Gamma) - 1$ and $g \equiv x$. In either case, $\partial_n^+\Gamma'$ will end with the variable x. The application of ∂_{n-}^+ to Γ will again kill all top (i.e. all n-dimensional) variables and all other variables killed are of dimension $n-1$. □

1.5.2 Subcontexts, Unions and Intersections

Definition 1.29. *Given a context $\Gamma \equiv (x_i : A_i)_{1\leq i\leq n}$ a subcontext of Γ is a context $\Delta \equiv (x_{i_1} : A_{i_1}, \ldots, x_{i_m} : A_{i_m})$ where $i_1 < \cdots < i_m$. We denote this by $\Delta \subset \Gamma$.*

If $\Delta \subset \Gamma$, then there is a canonical context morphism $\Gamma \vdash \pi_\Delta : \Delta$, such that $(\pi_\Delta)_j \equiv x_{i_j}$.

Lemma 1.30. *Let $\Gamma \vdash_{\mathrm{ps}}$ be a ps-context. Then the ordered sublist $\Gamma|_d$ of variables of Γ containing all variables with dimension at most d is a subcontext.*

Lemma 1.31. *Given a context Γ and two subcontexts $\Delta_1, \Delta_2 \subset \Gamma$ then*

(i) The union $\mathrm{FV}(\Delta_1) \cup \mathrm{FV}(\Delta_2)$ assemble into a subcontext $\Delta_1 \cup \Delta_2 \subset \Gamma$.

(ii) The intersection $\mathrm{FV}(\Delta_1) \cap \mathrm{FV}(\Delta_2)$ assemble into a subcontext $\Delta_1 \cap \Delta_2 \subset \Gamma$.

(iii) Assume $\Gamma, \Delta_1, \Delta_2$ are ps-contexts. If $\Delta_1 \cap \Delta_2$ is a ps-context up to reordering of variables, then so is $\Delta_1 \cup \Delta_2$.

Before we give the proof of Lemma 1.31 we make some remarks and introduce the required tools.

Remark 1.32. Technically, Lemma 1.31*(iii)* only shows that $\Delta_1 \cup \Delta_2$ is a ps-context up to reordering of the variables. We make the following conjecture:

Conjecture 1.33. *Given a ps-context Γ and a subcontext $\Gamma' \subset \Gamma$ (which has the induced order), if Γ' is a ps-context up to reordering, then it is a ps-context with the specified order.*

With this conjecture, $\Delta_1 \cup \Delta_2$ becomes a ps-context. Lemma 2.15 implicitly makes use of this conjecture. Nevertheless, the results of this chapter do not depend on the conjecture, as we could otherwise simply keep track of the fact that a given context may need to be reordered to be an actual ps-context. We will not mention this subtlety again in the remainder.

In the proof of Lemma 1.31 we will make use of a characterization of ps-context (up to reordering of context), due to Finster–Mimram [30]. It was originally spelled out in terms of globular sets, but for us it is enough to only work the variables of a context. In the remainder of this section we will temporarily denote the source of an $(n+1)$-dimensional variable x by $\sigma(x)$ and its target by $\tau(x)$. More generally, for some $0 \leq i \leq n$ its i-dimensional source will be denoted by $\sigma_i(x)$, that is, $\sigma_i(x)$ is obtained by applying the source map appropriately many times until we reach dimension i. Similarly, the i-dimensional target of x will be denoted by $\tau_i(x)$. If $i = \dim(x)$, then $\sigma_i(x) = x = \tau_i(x)$.

Definition 1.34 (Finster–Mimram [30] Definition 21)**.** *Given a globular context Γ, we define the relation $\lhd$ on $\mathrm{FV}(\Gamma)$ to be the transitive closure of the relation generated by*

$$\sigma(x) \lhd x \lhd \tau(x). \tag{1.8}$$

for all $x \in \mathrm{FV}(\Gamma)$ such that $\dim(x) > 0$.

Theorem 1.35 (Finster–Mimram [30])**.** *Ps-contexts are precisely those globular contexts Γ in which the relation $\lhd$ is a total order, i.e.*

$$x \lhd y \text{ or } y \lhd x \qquad \text{iff} \qquad x \neq y$$

for all $x, y \in \mathrm{FV}(\Gamma)$.

If for some variables $x, y \in \mathrm{FV}(\Gamma)$ we have $x \lhd y$ or $y \lhd x$, then we say x and y are related. We will also make use of the following result:

Lemma 1.36 (Finster–Mimram [30])**.** *Let Γ be a contexts, let $\Delta \subset \Gamma$ be a subcontext and let x, y be two variables in Δ. If $x \lhd y$ holds in Δ, then $x \lhd y$ also holds in Γ.*

The converse of Lemma 1.36 does not necessarily hold. Take for example Γ to be the 2-globe with 2-cell $\alpha : f \to g$ and let $\Delta \subset \Gamma$ be its 1-dimensional boundary, comprised of two 1-globes $f, g : x \to y$. Then, since Γ contains a 2-cell $\alpha : f \to g$ we have $f \lhd \alpha \lhd g$ and in particular $f \lhd g$. In Δ, however, f and g are not related.

Remark 1.37. Let Γ be a ps-context.

(i) Let Δ_1, Δ_2 be two subcontexts of Γ containing the variables $x, z \in \mathrm{FV}(\Delta_1)$ and $y, z \in \mathrm{FV}(\Delta_2)$. Assume that we have a path between x and z in Δ_1 and a path between y and z in Δ_2. Then there exists a path between x and y in $\Delta_1 \cup \Delta_2$. This follows from Lemma 1.26*(vi)*,*(vii)*.

(ii) Let $\Delta \subset \Gamma$ be a subcontext, which is not necessarily a ps-context. Assume we are given two variables $f_x : s_x \to t_x$ and $f_y : s_y \to t_y$ in Δ such that there exists a path between t_x and s_y also in Δ. Then, one of the following mutually exclusive conditions will hold in Δ:

- $t_x \lhd s_y$
- $t_x \equiv s_y$
- $s_x \equiv s_y$ and $t_x \equiv t_y$
- $t_y \lhd s_x$.

This follows from Lemma 1.26*(vi)*,*(vii)* and *(ii)*.

Proof of Lemma 1.31. Parts *(i)* and *(ii)* can be proven by induction over contexts.

For part *(iii)*, we may equivalently show that if $\Gamma, \Delta_1, \Delta_2$ and $\Delta_1 \cap \Delta_2$ are ps-contexts, if $\Delta_1 \cap \Delta_2$ is not a ps-context context, then we get a contradiction.

So assume there exist two nonequal variables $x, y \in \mathrm{FV}(\Delta_1 \cup \Delta_2)$ which are unrelated. There are three cases, depending on whether both variables, only one, or none of the two are 0-dimensional.

Consider first the case where both x, y are 0-dimensional. Since $\Delta_1 \cap \Delta_2$ is a ps-context, it is nonempty and it contains in particular a 0-dimensional variable z. Since Δ_1 is a ps-context, we have a path between x and z in Δ_1 and similarly we have a path between y and z in Δ_2. By Remark 1.37(i) we get a path between x and y in $\Delta_1 \cup \Delta_2$. But this implies that x and y are related in $\Delta_1 \cup \Delta_2$ leading to a contradiction.

Consider now the case where both x, y are not 0-dimensional. The case where only one of the two is 0-dimensional uses similar ideas but is easier, so we skip it.

If x, y are both in Δ_1, then, since Δ_1 is a ps-context, x and y must be related in Δ_1. Since Δ_1 is a subcontext of $\Delta_1 \cup \Delta_2$ they must also be related in $\Delta_1 \cup \Delta_2$ by Lemma 1.36, leading to a contradiction. Similarly x and y cannot both be in Δ_2. By symmetry it remains to consider the case where $x \in \mathrm{FV}(\Delta_1)$ and $y \in \mathrm{FV}(\Delta_2)$.

Claim 1: There exists an $n \in \mathbb{N}$, such that $\sigma_n(x) \equiv \sigma_n(y)$ and $\tau_n(x) \equiv \tau_n(y)$.

In fact, we show that $\sigma_0(x) \equiv \sigma_0(y)$ and $\tau_0(x) \equiv \tau_0(y)$. Notice first that we have two terms $\sigma_1(x) : \sigma_0(x) \to \tau_0(x)$ and $\sigma_1(y) : \sigma_0(y) \to \tau_0(y)$ in $\Delta_1 \cup \Delta_2$. With an argument as in the case where x and y are 0-dimensional, there exists a path between $\tau_0(x)$ and $\sigma_0(y)$ in $\Delta_1 \cup \Delta_2$. These are the conditions in Remark 1.37(ii), with $\Delta_1 \cup \Delta_2$ playing the role of the subcontext in question. According to the remark there are four cases. The first case implies that $\tau_0(x) \triangleleft \sigma_0(y)$ in $\Delta_1 \cup \Delta_2$. But then

$$x \triangleleft \tau_0(x) \triangleleft \sigma_0(y) \triangleleft y,$$

contradicting the fact that x and y are unrelated in $\Delta_1 \cup \Delta_2$. The second and fourth cases of the remark are similar. This leaves us with the third case, which says $\sigma_0(x) \equiv \sigma_0(y)$ and $\tau_0(x) \equiv \tau_0(y)$, proving claim 1 with $n = 0$.

Now, let $n \in \mathbb{N}$ be the maximal natural number such that $\sigma_n(x) \equiv \sigma_n(y)$ and $\tau_n(x) \equiv \tau_n(y)$. Note that in particular $\sigma_n(x), \tau_n(x) \in \mathrm{FV}(\Delta_1 \cap \Delta_2)$.

Claim 2: $\sigma_n(x)$ and $\tau_n(x)$ are unrelated in $\Delta_1 \cap \Delta_2$.

First of all, we can't have $\tau_n(x) \triangleleft \sigma_n(x)$ in $\Delta_1 \cap \Delta_2$. For, if this were the case, then this must also hold in Δ_1 by Lemma 1.36, contradicting the fact that $\sigma_n(x) \triangleleft x \triangleleft \tau_n(x)$ in Δ_1 and that Δ_1 is a ps-context.

Assume now that $\sigma_n(x) \triangleleft \tau_n(x)$ in $\Delta_1 \cap \Delta_2$. Since $\Delta_1 \cap \Delta_2$ is a ps-context there exists a path between $\sigma_n(x)$ and $\tau_n(x)$ in $\Delta_1 \cap \Delta_2$. But we already have such a path in Δ_1, namely the path $\sigma_n(x) \rightsquigarrow \tau_n(x)$ of length 1, with component $\sigma_{n+1}(x)$. Thus, the path in $\Delta_1 \cap \Delta_2$ must also be of the same orientation and of length 1 by Lemma 1.26*(ii)*. Let $w : \sigma_n(x) \to \tau_n(x)$ be the component of that path. By construction, $\tau_{n+1}(x), w, \sigma_{n+1}(y)$ are parallel. Since Δ_1 is a ps-context there is a path between $\tau_{n+1}(x)$ and w in Δ_1 and similarly there exists a path between $\sigma_{n+1}(y)$ and w in Δ_2. Thus, by Remark 1.37(i) there is a path between $\tau_{n+1}(x)$ and $\sigma_{n+1}(y)$ in $\Delta_1 \cup \Delta_2$.

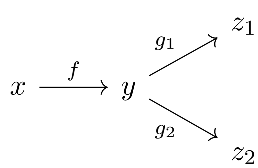
There are again three cases to consider, depending on the dimension of x and y. If both are $(n+1)$-dimensional, then we have a path between x and y in $\Delta_1 \cup \Delta_2$, contradicting the fact that x and y are unrelated in $\Delta_1 \cup \Delta_2$. Next we consider the case where both are of dimension strictly greater than $n+1$. The case where only one of the two is $(n+1)$-dimensional is similar but easier so we skip it. So if both x and y are of dimension greater than $n+1$, we can consider the variables $\sigma_{n+2}(y) : \sigma_{n+1}(x) \to \tau_{n+1}(x)$ and $\sigma_{n+2}(y) : \sigma_{n+1}(y) \to \tau_{n+1}(y)$, where we additionally have a path between $\tau_{n+1}(x)$ and $\sigma_{n+1}(y)$. This puts us again in the situation of Remark 1.37(ii). The first, second and fourth cases of the remark all contradict the fact that x and y are unrelated in $\Delta_1 \cup \Delta_2$. The first case for example says $\tau_{n+1}(x) \triangleleft \sigma_{n+1}(y)$ which gives

$$x \triangleleft \tau_{n+1}(x) \triangleleft \sigma_{n+1}(y) \triangleleft y.$$

The third case, which says $\sigma_{n+1}(x) \equiv \sigma_{n+1}(y)$ and $\tau_{n+1}(x) \equiv \tau_{n+1}(y)$ contradicts the maximality of n with this property. This proves claim 2.

But claim 2 contradicts the fact that $\Delta_1 \cap \Delta_2$ is a ps-contexts, which completes the proof. □

Remark 1.38. Lemma 1.31*(iii)* requires Γ to be a ps-context. As a counter example consider the context Γ given by the diagram



where Δ_i is spanned by f and g_i and $\Delta_1 \cap \Delta_2$ is spanned by f. The dependence on Γ implicitly appears in the proof Lemma 1.31*(iii)* via Remark 1.37.

1.5.3 Boundary Variables

As one might intuitively expect, given a ps-context Γ and a variable $x_k \in \partial^+\Gamma$ of dimension $\dim(\Gamma)-1$, there are no paths pointing outwards of Γ starting from x_k. Other cells of different dimension exhibit a similar behavior and we call any such variable a boundary variable.

Definition 1.39. *Let $\Gamma \vdash_{\text{ps}}$ be a ps-context. A variable $x_k \in \mathrm{FV}(\Gamma)$ is said to be a right boundary variable if the empty path $x_k \rightsquigarrow x_k$ is the unique path in Γ starting at x_k. We may also write $\Gamma \blacktriangleright x_k$.*

Lemma 1.40. *Let $\Gamma \equiv \Gamma^{n-2},\ y : A,\ f : x \to y$ be a ps-context. Then*

(i) $\Gamma \blacktriangleright f$

(ii) $\Gamma \blacktriangleright y$

(iii) $(f) : x \rightsquigarrow y$ *is the unique nonempty path in* Γ *starting at* x.

(iv) $\Gamma^{n-2} \blacktriangleright x$ *and* $\Gamma \not\blacktriangleright x$.

(v) If $x_i \in \mathrm{FV}(\Gamma^{n-2})$ *then,* $\Gamma \blacktriangleright x_i$ *implies* $\Gamma^{n-2} \blacktriangleright x_i$.

Proof. For the proof, let us write the context in the form $\Gamma \equiv \Gamma^{n-2},\ x_{n-1} : A_{n-1},\ x_n : x_{\partial^- n} \to x_{n-1}$.

(i) Any nonempty path starting at x_n necessarily involves a variable $f : x_n \to z$ for some node z. The type of f thus depends on x_n. This variable would therefore have to be introduced after x_n in Γ, contradicting the fact that x_n is the last variable. Thus, there cannot exist any nonempty paths starting at x_n.

(ii) Similar to part *(i)*.

(iii) Consider the variable $x_{\partial^- n} \in \mathrm{FV}(\Gamma)$ and let $\overline{q} : x_{\partial^- n} \rightsquigarrow x$ be a nonempty path in Γ starting at $x_{\partial^- n}$. Since we also have the path $(x_n) : x_{\partial^- n} \rightsquigarrow x_{n-1}$ with $|(x_n)| = 1 \leq |\overline{q}|$, it follows from Lemma 1.26*(vi)* that $x_{n-1} \in \texttt{nodes}(\overline{q})$. If $x \not\equiv x_{n-1}$, then $\texttt{comp}(\overline{q})$ would contain a term $q_a : x_{n-1} \to w$ for some node $w \in \texttt{nodes}(\overline{q})$. But any such term would have to appear after $(x_{n-1} : A_{n-1}) \in \Gamma$ leading to a contradiction (by inspecting Γ). So $x \equiv x_{n-1}$ and $|\overline{q}| = |(x_n)| = 1$ by Lemma 1.26*(ii)*. Since any term $x_{\partial^- n} \to x_{n-1}$ must appear after $(x_{n-1} : A_{n-1}) \in \Gamma$, the only option for which is x_n we must have $\overline{q} = (x_n)$.

(iv) By part *(iii)*, $(x_n) : x_{\partial^- n} \rightsquigarrow x_{n-1}$ is the unique nonempty path in Γ starting at $x_{\partial^- n}$. Thus, since $x_n \notin \Gamma^{n-2}$, the empty path $() : x_{\partial^- n} \rightsquigarrow x_{\partial^- n}$ is the unique path in Γ^{n-2} starting at $x_{\partial^- n}$.

The fact that $\Gamma^{n-2} \not\blacktriangleright x_{\partial^- n}$ follows from the fact that there exists a nonempty path $(x_n) : x_{\partial^- n} \rightsquigarrow x_{n-1}$ by part *(iii)*.

(v) This follows directly from the fact that we have less available paths in Γ^{n-2} than in Γ. □

2 Alternative Rules for `CaTT`

In this section, we propose alternative rules for `CaTT` and prove that the new and old rules are mutually admissible. In these new rules, all the judgments remain the same, but the free variable side conditions are replaced by conditions on the dimensions. As a result the new rules have a more geometric flavor compared to the original ones. Note that a proof of equivalence of models of the resulting theory and `CaTT` requires more work. The results here may be considered a first step towards establishing such an equivalence. We return to this discussion at the end of this section.

2.1 The Rules (OP') and (COH')

2.1.1 Two Lemmas and the Intuition behind (OP') and (COH')

Intuitively, if a variable is used in a term, then the dimension of that term must be at least that of the variable. This is captured by the following lemma.

Lemma 2.1. *The following two statements hold in* `CaTT`

(i) Given a judgment $\Gamma \vdash A$ *we have*

$$x_i \in \mathrm{FV}(A) \implies \dim(A_i) < \dim(A)$$

(ii) Given a judgment $\Gamma \vdash t : A$ *we have*

$$x_i \in \mathrm{FV}(t) \implies \dim(x_i) \leq \dim(t)$$

Proof. We prove all three statements simultaneously. The proof is obtained by performing a mutual induction on the rules for types and terms. We start with the rules for producing types.

(Ob) Consider the rule

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathtt{Ob}}\ (\mathrm{Ob})$$

Since $\mathrm{FV}(\mathtt{Ob}) = \emptyset$ there is nothing to check and the statement is vacuously true.

(→) Consider the rule

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \underset{A}{\to} t}\ (\to)$$

By definition $\mathrm{FV}(s \underset{A}{\to} t) = \mathrm{FV}(s) \cup \mathrm{FV}(t) \cup \mathrm{FV}(A)$. So let $x_i \in \mathrm{FV}(s \underset{A}{\to} t)$. If $x_i \in \mathrm{FV}(s)$ or $x_i \in \mathrm{FV}(t)$, then by the inductive hypothesis $\dim(x_i) \leq \dim(s)$ or $\dim(x_i) \leq \dim(t)$. Thus

$$\dim(A_i) = \dim(x_i) < \dim(s \underset{A}{\to} t). \tag{1.9}$$

Otherwise, if $x_i \in \mathrm{FV}(A)$ then $\dim(A_i) < \dim(A)$ by the inductive hypothesis and $\dim(A_i) < \dim(A) < \dim(s \underset{A}{\to} t)$.

Next we go through the rules for terms.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\mathrm{VAR})$$

By definition $\mathrm{FV}(x) = \{x\}$. So $x_i \in \mathrm{FV}(x)$ implies that $A_i \equiv A$ and $x_i \equiv x : A$, by which we trivially have $\dim(A_i) \leq \dim(A)$ or $\dim(x_i) \leq \dim(x)$.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]}\ (\mathrm{OP})$$

where $\mathrm{FV}(\partial^-\Gamma) = \mathrm{FV}(s : A)$ and $\mathrm{FV}(\partial^+\Gamma) = \mathrm{FV}(t : A)$. By definition $\mathrm{FV}(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma]) = \mathrm{FV}(\gamma)$.

First of all, $\dim(\partial^-\Gamma) \leq \dim(A)$. To see this let $x_j \in \mathrm{FV}(\partial^-\Gamma)$. Then either $x_j \in \mathrm{FV}(s)$ or $x_j \in \mathrm{FV}(A)$, and in both cases $\dim(A_j) \leq \dim(A)$ by the inductive hypothesis for types and terms. Since this is true for all $x_j \in \mathrm{FV}(\partial^-\Gamma)$, it follows that $\dim(\partial^-\Gamma) \leq \dim(A)$.

Now let $y_j \in \mathrm{FV}(\mathtt{op}_{\Gamma, s \to t}[\gamma]) = \mathrm{FV}(\gamma)$. By the definition of $\mathrm{FV}(\gamma)$ this means $y_j \in \mathrm{FV}(\gamma_i)$ for some $1 \leq i \leq n$. Putting everything together we have

$$\begin{aligned}
\dim(y_j) &\leq \dim(\gamma_i), && \text{by inductive hypothesis for terms} \\
&= \dim(x_i) \\
&\leq \dim(\Gamma) \\
&= \dim(\partial^- \Gamma) + 1 \\
&\leq \dim(A) + 1, \\
&= \dim(s \underset{A}{\to} t) \\
&= \dim\big(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]\big) \\
&= \dim(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma]).
\end{aligned}$$

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \ (\mathrm{COH})$$

where $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma) = \mathrm{FV}(t : A)$. In particular $\mathrm{FV}(s \underset{A}{\to} t) = \mathrm{FV}(\Gamma)$.

Let $y_j \in \mathrm{FV}(\mathtt{coh}_{\Gamma, s \to t}[\gamma])$. Then, since by definition $\mathrm{FV}(\mathtt{coh}_{\Gamma, s \to t}[\gamma]) = \mathrm{FV}(\gamma)$, there exists an $1 \leq i \leq n$ such that $y_j \in \mathrm{FV}(\gamma_i)$. So by the inductive hypothesis for terms we already know that $\dim(y_j) \leq \dim(\gamma_i) = \dim(x_i)$.

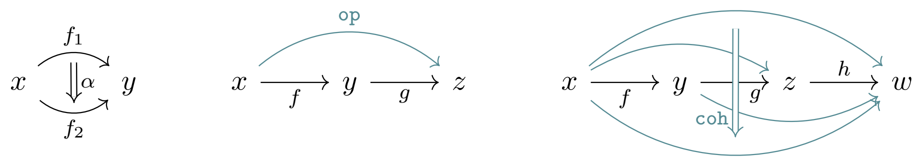
Consider now the corresponding variable $x_i \in \mathrm{FV}(\Gamma)$. Since, $\mathrm{FV}(\Gamma) = \mathrm{FV}(s \to t)$, it follows that $x_i \in \mathrm{FV}(s \to t)$ so that $\dim(A_i) < \dim(s \to t)$ by the inductive hypothesis. Putting everything together we have

$$\begin{aligned}
\dim(y_j) &\leq \dim(x_i) \\
&= \dim(A_i) \\
&< \dim(s \underset{A}{\to} t) \\
&= \dim(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]) \\
&= \dim(\mathtt{coh}_{\Gamma, s \underset{A}{\to} t}).
\end{aligned}$$

We see that in this case we even get a strict inequality, as one may have expected intuitively. □

For our next lemma, notice that for all the composition operations in our palette of standard examples (e.g. composition of 1-morphisms, vertical and horizontal composition of 2-morphisms), the constructed morphism always has the same dimension as the starting diagram. This turns out to be a general feature of the (OP) rule, as we prove in the next lemma.

In order to prove this statement about the dimension of operator terms, we must pair it with a statement about terms. Recall that CaTT gives us three rules for producing terms $t : A$, (VAR), (OP) and (COH) and consider the following representative examples



where the underlying context (in fact ps-context) is drawn in black. For the sake of simplicity we are ignoring here the inbuilt substitution in the rules. By inspection, we see that for (VAR) and (OP), we have $\mathrm{FV}(t)\backslash\mathrm{FV}(A) \neq \emptyset$. More precisely $\mathrm{FV}(\alpha)\backslash\mathrm{FV}(f_1 \to f_2) = \{\alpha\}$ and $\mathrm{FV}(\mathtt{op})\backslash\mathrm{FV}(x \to y) = \{y, f, h\}$. In both cases $\mathrm{FV}(t)\backslash\mathrm{FV}(A)$ contains a cell of dimension equal to $\dim(t)$. So we might expect that whenever $\mathrm{FV}(t)\backslash\mathrm{FV}(A) \neq \emptyset$, then $\mathrm{FV}(t)$ contains a variable of codimension 0. For the associator, on the other hand, $\mathrm{FV}(\mathtt{coh})\backslash\mathrm{FV}\big(h\circ(g\circ f) \to (h\circ g)\circ f\big) = \emptyset$ and $\mathrm{FV}(h\circ(g\circ f)) = \mathrm{FV}((h\circ g)\circ f)$. So we might expect that whenever $\mathrm{FV}(t)\backslash\mathrm{FV}(A) = \emptyset$, then $\dim(A) > 0$ and $\mathrm{FV}(\partial^- t : \partial A) = \mathrm{FV}(\partial^+ t : \partial A)$. If we include substitutions the case for (OP) becomes more subtle. The set $\mathrm{FV}(\mathtt{op}[\gamma])\backslash\mathrm{FV}(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma])$ may or may not be empty. The general intuition conveyed in this paragraph, however, is correct, as confirmed by the following lemma.

Lemma 2.2. *The following statements hold in* CaTT

(i) (♣) *Given a type in context* $\Gamma \vdash A$ *satisfying* (OP), *then* $\dim(\Gamma) = \dim(A)$. (♣) Benjamin [9] Lemma 76

(ii) Given a term $\Gamma \vdash t : A$

$$\mathrm{FV}(t)\backslash\mathrm{FV}(A) \neq \emptyset \quad \Longrightarrow \quad \Big(\exists\, x_i \in \mathrm{FV}(t) \quad \textit{such that} \quad \mathrm{codim}_t(x_i) = 0\Big)$$

and

$$\mathrm{FV}(t)\backslash\mathrm{FV}(A) = \emptyset \quad \Longrightarrow \quad \mathrm{FV}(\partial^- t : \partial A) = \mathrm{FV}(\partial^+ t : \partial A).$$

Remark 2.3. Note that in the second statement, the terms $\partial^- t$ and $\partial^+ t$ always exist. This is because the only terms of the type Ob are variables so that $x :$ Ob could never satisfy the hypothesis of the second statement. Thus the statement addresses only terms of types with dimension at least 1, so that indeed the terms $\partial^- t$ and $\partial^+ t$ exist.

Proof of Lemma 2.2. We prove all statements simultaneously by inducting over all the rules for types, terms and contexts in CaTT.

(Ob) Consider the rule

$$\frac{\Gamma \vdash}{\Gamma \vdash \texttt{Ob}} \text{ (Ob)}$$

Since $\Gamma \vdash \texttt{Ob}$ fails to satisfy the hypothesis (OP), the statement is trivially true.

($\to$) Consider the rule

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \underset{A}{\to} t} \text{ } (\to)$$

where by hypothesis

$$\Gamma \vdash_{\text{ps}}, \qquad \text{FV}(s : A) = \text{FV}(\partial^- \Gamma) \qquad \text{FV}(\partial^+ \Gamma) = \text{FV}(t : A).$$

By Lemma 1.18 $\text{FV}(\partial^- \Gamma) \neq \text{FV}(\partial^+ \Gamma)$. Thus, by hypothesis we either have $\text{FV}(t : A) \backslash \text{FV}(s : A) \neq \emptyset$ or $\text{FV}(s : A) \backslash \text{FV}(t : A) \neq \emptyset$. The argument is similar in both cases, so let us assume that $\text{FV}(t : A) \backslash \text{FV}(s : A) \neq \emptyset$ which implies that $\text{FV}(t) \backslash \text{FV}(A) \neq \emptyset$. Thus, by the inductive hypothesis for terms, there exists a $x_i \in \text{FV}(t) \subset \text{FV}(\partial^+ \Gamma)$ such that $\text{codim}_t(x_i) = 0$, or equivalently such that $\dim(A_i) = \dim(A)$. This means $\dim(\partial^+ \Gamma) \geq \dim(A)$. Since $\text{FV}(\partial^+ \Gamma) = \text{FV}(t : A)$, Lemma 2.1 shows that $\dim(\partial^+ \Gamma) \leq \dim(A)$. Putting these two together gives $\dim(\partial^+ \Gamma) = \dim(A)$ so that

$$\dim(\Gamma) = (\partial^+ \Gamma) + 1 = \dim(A) + 1 = \dim(s \underset{A}{\to} t)$$

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{ (VAR)}$$

where by definition $\text{FV}(x) = \{x\}$. Then $\text{FV}(x) \backslash \text{FV}(A) = \{x\}$ since $x \notin \text{FV}(A)$ (otherwise Lemma 2.1 would produce the contradiction $\dim(A) < \dim(A)$). Thus the statement holds in this case.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \partial^- \Gamma \vdash s : A \qquad \partial^+ \Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (OP)}$$

where $\dim(\Gamma) > 0$ and $\text{FV}(s : A) = \text{FV}(\partial^- \Gamma) \neq \text{FV}(\partial^+ \Gamma) = \text{FV}(t : A)$ (see Lemma 1.18).

Now, by definition, $\text{FV}(\texttt{op}[\gamma]) = \text{FV}(\gamma)$. On the other hand, $\text{FV}(s \underset{A}{\to} t) =$

$\mathrm{FV}(s) \cup \mathrm{FV}(t) \cup (A) = \mathrm{FV}(\partial^- \Gamma) \cup \mathrm{FV}(\partial^+ \Gamma) = \mathrm{FV}(\partial \Gamma)$. Thus, by Lemma 1.11

$$\mathrm{FV}(s[\gamma] \underset{A[\gamma]}{\rightarrow} t[\gamma]) = \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i) = \mathrm{FV}(\partial \gamma).$$

For this rule, we either have $\mathrm{FV}(\mathtt{op}[\gamma]) \backslash \mathrm{FV}(s[\gamma] \underset{A[\gamma]}{\rightarrow} t[\gamma]) \neq \emptyset$, or we have $\mathrm{FV}(\mathtt{op}[\gamma]) \backslash \mathrm{FV}(s[\gamma] \underset{A[\gamma]}{\rightarrow} t[\gamma]) = \emptyset$ and we must consider both cases.

Assume first that $\mathrm{FV}(\mathtt{op}[\gamma]) \backslash \mathrm{FV}(t[\gamma] \rightarrow s[\gamma]) \neq \emptyset$, i.e. that

$$\mathrm{FV}(\gamma) \Big\backslash \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i) \neq \emptyset.$$

We must show that there exists a $y_j \in \mathrm{FV}(\gamma)$ of dimension $\dim(y_j) = \dim(\mathtt{op}_{\Gamma, s \rightarrow t}[\gamma])$. Instead of this, however, we shall prove the contrapositive: assuming that $\mathtt{op}_{\Gamma, s \rightarrow t}[\gamma]$ has no free variables of codimension 0 we have

$$\mathrm{FV}(\gamma) \Big\backslash \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i) = \emptyset.$$

So let $y_j \in \mathrm{FV}(\gamma)$, which means $y_j \in \mathrm{FV}(\gamma_k)$ for some $x_k \in \mathrm{FV}(\Gamma)$. Our task is to show that $y_j \in \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i)$. There are three cases to consider.

If $\dim(x_k) \leq \dim(\Gamma) - 2$, then $x_k \in \partial \Gamma$, since by Lemma 1.28, $\Gamma|_{\dim(\Gamma)-2} \subset \partial^{\pm} \Gamma$. Thus we have $y_j \in \mathrm{FV}(\gamma_k) \subset \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i)$. If $\dim(x_k) = \dim(\Gamma) - 1$, then by Lemma 1.27 there exists a $x_{k^-} \in \mathrm{FV}(\partial \Gamma)$ and a path $\overline{p} : x_{k^-} \rightsquigarrow x_k$. Let $p_a : x_{a-1} \rightarrow x_a$ be a component in $\overline{p}$. By substitution we get a term $p_a[\gamma] : \gamma_{a-1} \rightarrow \gamma_a$. But by assumption $\mathtt{op}_{\Gamma, s \rightarrow t}[\gamma]$ contains no variables of codimension 0, i.e. $\mathrm{FV}(\gamma)$ contains no variables of dimension $\dim(\Gamma)$, where we've used the inductive hypothesis for types. So $p_a[\gamma]$ contains no variable of codimension 0, so that by the inductive hypothesis we must have $\mathrm{FV}(\gamma_{a-1} : T) = \mathrm{FV}(\gamma_a : T)$. Applying this argument to all components of $\overline{p}$ we conclude that $\mathrm{FV}(\gamma_{k^-} : T) = \mathrm{FV}(\gamma_k : T)$. Thus, again $y_j \in \mathrm{FV}(\gamma_{k^-}) \subset \bigcup_{x_i \in \mathrm{FV}(\partial \Gamma)} \mathrm{FV}(\gamma_i)$. Finally, if $\dim(x_k) = \dim(\Gamma)$, then $\dim(x_k) > 0$ and so we may write $x_k : x_{\partial^- k} \rightarrow x_{\partial^+ k}$. Since we are assuming that $\mathrm{FV}(\gamma)$ contains no variables of dimension $\dim(\Gamma)$, it follows by the inductive hypothesis that $\mathrm{FV}(\gamma_k) \subset \mathrm{FV}(\gamma_{\partial^- k} \rightarrow \gamma_{\partial^+ k})$. In particular, y_j is contained in some $\gamma_{k'}$ for some $x_{k'}$ of dimension less than or equal to $\dim(\Gamma) - 1$, and so this case reduces to the previous two and we are done.

Assume now that $\mathrm{FV}(\mathtt{op}[\gamma]) \backslash \mathrm{FV}(s[\gamma] \underset{A[\gamma]}{\rightarrow} t[\gamma]) = \emptyset$. We must show that $\mathrm{FV}(s[\gamma] : A[\gamma]) = \mathrm{FV}(t[\gamma] : A[\gamma])$, i.e. that

$$\bigcup_{x_i \in \mathrm{FV}(\partial^- \Gamma)} \mathrm{FV}(\gamma_i) = \bigcup_{x_i \in \mathrm{FV}(\partial^+ \Gamma)} \mathrm{FV}(\gamma_i)$$

We will show that the left-hand side is contained in the right-hand side. The other direction is given by a symmetric argument. So consider a variable $y_j \in \bigcup_{x_i \in \mathrm{FV}(\partial^- \Gamma)} \mathrm{FV}(\gamma_i)$. This means $y_j \in \mathrm{FV}(\gamma_k)$ for some $x_k \in \mathrm{FV}(s : A)$. If $\dim(x_k) \leq \dim(\Gamma) - 2$, then since $\Gamma|_{\dim(\Gamma)-2} \subset \partial^{\pm}\Gamma$ it follows that $x_k \in \mathrm{FV}(\partial^+\Gamma)$, so that $y_j \in \mathrm{FV}(\gamma_k) \subset \bigcup_{x_i \in \mathrm{FV}(\partial^+ \Gamma)} \mathrm{FV}(\gamma_i)$. If $\dim(x_k) = \dim(\Gamma) - 1$, then with an argument similar to that in the previous paragraph we can find a x_{k^+} and a path $x_k \rightsquigarrow x_{k^+}$ such that $x_{k^+} \in \mathrm{FV}(\partial^+\Gamma)$ where $\mathrm{FV}(\gamma_k : T) = \mathrm{FV}(\gamma_{k^+} : T)$. So again $y_j \in \mathrm{FV}(\gamma_{k^+}) \subset \bigcup_{x_i \in \mathrm{FV}(\partial^+ \Gamma)} \mathrm{FV}(\gamma_i)$.

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \ (\mathrm{COH})$$

On the one hand $\mathrm{FV}(\mathtt{coh}[\gamma]) = \mathrm{FV}(\gamma)$ by definition. On the other hand, by assumption, $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma) = \mathrm{FV}(t : A)$ so that $\mathrm{FV}(s \to t) = \mathrm{FV}(s : A) \cup \mathrm{FV}(t : A) = \mathrm{FV}(\Gamma)$ implying that $\mathrm{FV}(s[\gamma] \to t[\gamma]) = \mathrm{FV}(\gamma)$. Thus $\mathrm{FV}(\mathtt{coh}[\gamma]) \backslash \mathrm{FV}(s[\gamma] \to t[\gamma]) = \emptyset$ and indeed, by assumption again, $\mathrm{FV}(s[\gamma] : A[\gamma]) = \mathrm{FV}(t[\gamma] : A[\gamma])$. □

2.1.2 The Rules (OP') and (COH')

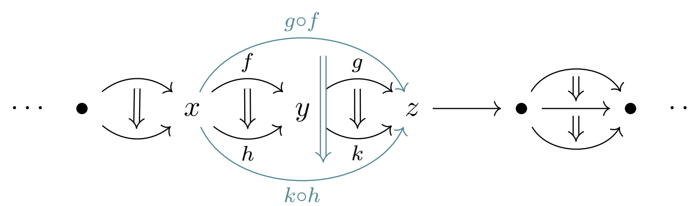
Lemma 2.2 says that $\dim(\mathtt{op}_{\Gamma, s \to t}[\gamma]) = \dim(\Gamma)$. Lemma 2.1, on the other hand, implies that $\dim(\mathtt{coh}_{\Gamma, s \to t}[\gamma]) > \dim(\Gamma)$. Inspired by this, we propose the following rules as alternatives for (OP) and (COH):

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}'_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \ (\mathrm{OP'}) \qquad \dim(\Gamma) = \dim(s \underset{A}{\to} t)$$

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}'_{\Gamma, s \to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \ (\mathrm{COH'}) \qquad \dim(\Gamma) < \dim(s \underset{A}{\to} t)$$

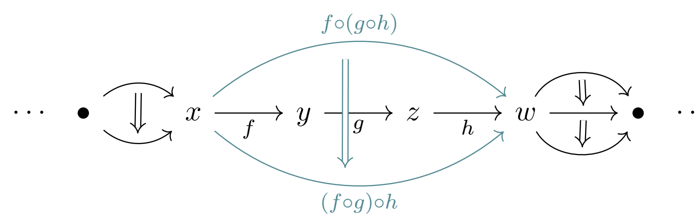
Notation: Let us denote by `CaTT'` the type theory `GSeTT` together with (OP') and (COH')

Compared to the original rules, the new ones are more geometric and less syntactic. On the other side, the new rules contain redundancies. The free variable side condition in the original rules ensures that the underlying pasting scheme is the minimal pasting scheme for the construction under consideration. This feature is lost in the new rules. In particular, the new rules are local, in the sense that they allow the construction of operations and coherences which only make use of a part of the underlying pasting scheme. The following example shows how we may compose horizontally two 2-cells using the (OP') rule inside a larger diagram directly.



where the black diagram is the underlying pasting scheme (of dimension 2) and the blue arrows are all constructed on top of it. Of course, this locality can also be achieved with the original rules, with the inbuilt cut in the rules.

On a conceptual level, one feature of the new rules is the blurring of operations and coherences. In particular, the (OP) rule now also constructs certain coherences. The following example shows how we may build the associator in a 2-dimensional pasting scheme.



Here again the underlying pasting context is depicted in black, and all arrows built in this context using the rules are colored blue.

2.1.3 Admissible Structural Rules and General Features

The free variables of $\mathtt{op}'_{\Gamma,s\to t}[\gamma]$ and $\mathtt{coh}'_{\Gamma,s\to t}[\gamma]$ are defined just as for $\mathtt{op}_{\Gamma,s\to t}[\gamma]$ and $\mathtt{coh}_{\Gamma,s\to t}[\gamma]$. As a consequence, all the results of Subsection 1.3 continue to hold in `CaTT'` with all proofs carrying over directly.

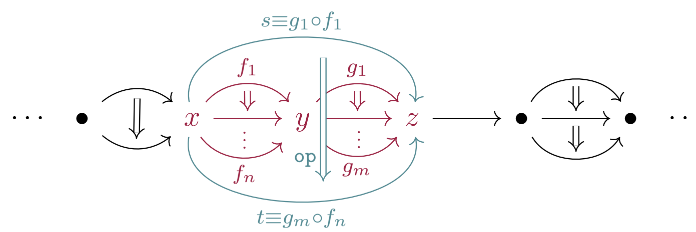
Now, in the original rules of `CaTT`, the side conditions ensure that the ps-context Γ under consideration is the minimal ps-context in which the corresponding operation or coherence may take place. In our case however, the dimension side condition does not ensure the minimality of the ps-context under consideration. As a result, there may be variables which are not relevant to the operation or coherence cell being constructed. To compare `CaTT` with `CaTT'` we first single out the relevant portion of the variables, defined below. To distinguish the relevant free variables of a term t from all free variables from the free variables, we use the notation $\overline{\mathrm{FV}}(t)$. The notion of relevant free variables will be the only notion of free variables used in Subsection **??**, where we study `CaTT'`, and where we will drop the adjective 'relevant' but keep the distinguished notation as a reminder.

When interpreting the rules (OP) and (COH) of `CaTT` in `CaTT'`, we will refer to the relevant free variables in the side conditions, but as a matter of fact, in this case these agree with the free variables.

We define the relevant free variables on all constructors. For contexts, types, the variable and context morphisms, the definition of the relevant free variables agrees with that of the free variables. For the coherence we define

$$\overline{\mathrm{FV}}(\mathtt{coh}'_{\Gamma,A}[\gamma]) := \overline{\mathrm{FV}}(A[\gamma]). \tag{1.10}$$

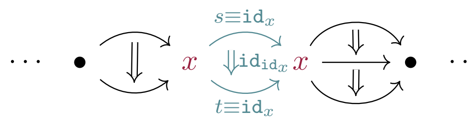
As for (OP') let us first consider the case without a context morphism, or equivalently with the identity context morphism $\Gamma \vdash \mathtt{id}_\Gamma : \Gamma$. So we are working with the term $\mathtt{op}'_{\Gamma,s\to t}[\mathtt{id}_\Gamma] : s \underset{A}{\to} t$. Since the new rules allow us to build cells locally, we must single out only the relevant variables and assign them to the operation. In the following diagram we have colored the morphisms which we construct using the operation rule in blue, as before. The underlying pasting scheme is colored black and red, where the red cells form what should be the relevant free variables of the operation.



Here $\overline{\mathrm{FV}}(A) = \{x, z\}$ while $\overline{\mathrm{FV}}(s : A) = \{x, y, z, f_1, g_1\}$ and $\overline{\mathrm{FV}}(t : A) = \{x, y, z, f_n, g_m\}$. To define $\overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\to t}[\mathtt{id}_\Gamma])$ we may thus begin with what is already given to us, namely $\overline{\mathrm{FV}}(s \underset{A}{\to} t)$. The remaining variables are contained in paths of dimension 2 from the free variables of s of dimension 1 to those of t. Said differently, they are contained in the paths $\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}$ where $\dim(\overline{p}) = \dim(\mathtt{op}'_{\Gamma,s\to t}[\mathtt{id}_\Gamma])$ and $\mathrm{codim}_s(x_{i_s}) = 0$ and $\mathrm{codim}_t(x_{i_t}) = 0$. We may therefore define

$$\overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\underset{A}{\to}t}[\mathtt{id}_\Gamma]) := \overline{\mathrm{FV}}\Big(s \underset{A}{\to} t\Big) \cup \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t} \\ \mathrm{codim}_s(x_{i_s})=0 \\ \mathrm{codim}_t(x_{i_t})=0}} \overline{\mathrm{FV}}(\overline{p}).$$

Remark 2.4. In the given example, notice that all the variables in $\overline{\mathrm{FV}}(s \underset{A}{\to} t)$ are already contained in the relevant free variables of all the paths. There are however cases in which this is not true, which is why we are forced to include this set in the definition. Consider for example the following diagram.



Here again the underlying pasting scheme is colored in black and red, with the red variables forming the free variables of the constructed path. The morphisms constructed on in the context are colored in blue. We have separated the domain and codomain x of $\mathtt{id}_x$ only to make the presentation more readable. We have thus constructed 1-dimensional cells $s, t : A$ in the 1-dimensional source and target of the pasting scheme, so that we may apply (OP') to inhabit the type $s \to t$. But $s, t : A$ contain no variables of codimension 0, so there can be no paths of the form given in the above definition. This should make clear the necessity of including $\overline{\mathrm{FV}}(s \to t)$ in the free variables of $\mathtt{op}'_{\Gamma, s\to t}[\mathtt{id}_\Gamma]$.

Once we include the substitution there are (at least) two canonical choices we could take for the second set containing the free variables of the paths. We could either build in the substitution in $s : A$ and $t : A$ and consider paths $\overline{q} : y_{j_s} \rightsquigarrow y_{j_t}$ with $\mathrm{codim}_{s[\gamma]}(y_{j_s}) = 0$ and $\mathrm{codim}_{t[\gamma]}(y_{j_t}) = 0$, or we could stick with paths in Γ and apply the substitution to the variables of the path. It turns out that the latter is more natural - it makes the proof of Proposition 1.11 straightforward, as one may have hoped for.[2] Before we proceed with the actual definition we introduce a shorthand to reduce the notational clutter.

Definition 2.5. *Let $\overline{p}$ be a path in a context Γ of length l, with components* $\mathtt{comp}(\overline{p}) = \{p_a\}_{1\le a\le l}$ *where $p_a : T_a$. Given a context morphism $\Delta \vdash \gamma : \Gamma$ we define the relevant free variables of $\overline{p}$ with $\Delta \vdash \gamma : \Gamma$ substituted to be the set*

$$\overline{\mathrm{FV}}(\overline{p}[\gamma]) := \bigcup_{1\le a\le l} \overline{\mathrm{FV}}(p_a[\gamma] : T_{p_a}[\gamma]).$$

Remark 2.6. Note that $\overline{p}[\gamma]$ is not a path again although one may intuitively expect this to form a collection of parallel paths.

In conclusion, we make the following definitions for the relevant free variables of the terms of `CaTT'`.

Definition 2.7. *The free variables of operations and coherences in* `CaTT'` *are*

[2] The former definition simplifies the admissibility proofs of (OP) and (COH) in `CaTT'`, but we find it more natural to make Lemma 1.11 trivial and work for the admissibility rather than the opposite.

defined as follows:

$$\overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\underset{A}{\to}t}[\gamma]) := \overline{\mathrm{FV}}\Big(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]\Big) \cup \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \mathrm{codim}_s(x_{i_s})=0\\ \mathrm{codim}_t(x_{i_t})=0}} \overline{\mathrm{FV}}(\overline{p}[\gamma])$$

$$\overline{\mathrm{FV}}(\mathtt{coh}'_{\Gamma,A}[\gamma]) := \overline{\mathrm{FV}}(A[\gamma]).$$

Proposition 1.9 and Lemma 1.11 in Subsection 1.3 continue to hold for the relevant free variables. As a matter of fact, parts of the inductive proof carry over and we only need to provide proofs for the new term constructors. We give here the proof for Lemma 1.11.

Proof of Lemma 1.11 using relevant free variables in `CaTT'`. It remains to consider the cases for the rules (OP') and (COH').

(OP') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \quad \partial^-\Gamma \vdash s : A \quad \partial^+\Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}'_{\Gamma,t\underset{A}{\to}s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]}\ (\mathrm{OP})$$

where $\dim(\Gamma) = \dim(s \underset{A}{\to} t)$. We compute

$$\begin{aligned}
\overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\to t}[\gamma][\delta]) &= \overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\to t}[\gamma\circ\delta]), && \text{by Definition 1.3}\\
&= \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \mathrm{codim}_s(x_{i_s})=0\\ \mathrm{codim}_t(x_{i_t})=0}} \overline{\mathrm{FV}}(\overline{p}[\gamma\circ\delta]), && \text{by Definition 2.7}\\
&= \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \cdots}} \bigcup_{a} \overline{\mathrm{FV}}(p_a[\gamma\circ\delta] : T_{p_a}[\gamma\circ\delta]), && \text{by Definition 2.5}\\
&= \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \cdots}} \bigcup_{a} \overline{\mathrm{FV}}(p_a[\gamma][\delta] : T_{p_a}[\gamma][\delta]), && \text{by Proposition 1.10}\\
&= \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \cdots}} \bigcup_{a} \bigcup_{y_j\in\overline{\mathrm{FV}}(p_a[\gamma]:T_{p_a}[\gamma])} \overline{\mathrm{FV}}(\delta_j) && \text{inductive hypothesis}\\
&= \bigcup_{\substack{\overline{p}\,:\,x_{i_s}\rightsquigarrow x_{i_t}\\ \cdots}} \bigcup_{y_j\in\overline{\mathrm{FV}}(\overline{p}[\gamma])} \overline{\mathrm{FV}}(\delta_j)\\
&= \bigcup_{y_j\in\overline{\mathrm{FV}}(\mathtt{op}'_{\Gamma,s\to t}[\gamma])} \overline{\mathrm{FV}}(\delta_j).
\end{aligned}$$

(COH') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}'_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (COH)}$$

where $\dim(\Gamma) < \dim(s \underset{A}{\to} t)$. We compute

$$\begin{aligned}
\overline{\mathrm{FV}}(\mathtt{coh}'_{\Gamma,s\to t}[\gamma][\delta]) &= \overline{\mathrm{FV}}(\mathtt{coh}'_{\Gamma,s\to t}[\gamma \circ \delta]) \\
&= \overline{\mathrm{FV}}\Big((s \underset{A}{\to} t)[\gamma \circ \delta]\Big) \\
&= \bigcup_{y_i \in \overline{\mathrm{FV}}((s \underset{A}{\to} t)[\gamma])} \overline{\mathrm{FV}}(\delta_j) \qquad \text{by the inductive hypothesis} \\
&= \bigcup_{y_i \in \overline{\mathrm{FV}}\left(\mathtt{coh}'_{\Gamma,s\to t}[\gamma]\right)} \overline{\mathrm{FV}}(\delta_j).
\end{aligned}$$

□

2.2 (OP') and (COH') are Admissible in CaTT

In this section we show that (OP') and (COH') are both admissible in CaTT. But before we get to the crucial lemmas which imply the admissibility of (OP') and (COH') in CaTT, we must first develop some tools.

2.2.1 Minimal Contexts and Free Variables

The strategy for proving the admissibility of (OP') and (COH') in CaTT is by reducing the ps-context in the judgment $\Gamma \vdash s \to t$ to an appropriate minimal ps-context still containing all the free variables of $s \to t$, but satisfying the free variable side conditions of (OP) and (COH). Crucially this minimal context will also turn out to be a ps-context again. We start by proving the existence of what we call the minimal context.

Lemma 2.8. *The following statements hold in* CaTT

(i) Given a derivable type $\Gamma \vdash A$, there exists a subcontext $\Gamma_A \subset \Gamma$ such that

- *$\Gamma_A \vdash A$ is derivable;*
- *$\mathrm{FV}(\Gamma_A) = \mathrm{FV}(A)$.*

(ii) Given a derivable term $\Gamma \vdash t : A$, there exists a subcontext $\Gamma_t \subset \Gamma$ such that

- *$\Gamma_t \vdash t : A$ is derivable;*
- *$\mathrm{FV}(\Gamma_t) = \mathrm{FV}(t : A)$.*

(iii) Given a substitutions $\Delta \vdash \gamma : \Gamma$*, there is a subcontext* Δ_γ *such that*

- $\Delta_\gamma \vdash \gamma : \Gamma$ *is derivable;*
- $\mathrm{FV}(\Delta_\gamma) = \mathrm{FV}(\gamma)$.

In each case, we call the constructed subcontext the minimal subcontext.

Proof. The proof is given by a mutual induction on all three statements.

(Ob) Consider the rule

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathtt{Ob}} \text{ (Ob)}$$

By definition $\mathrm{FV}(\mathtt{Ob}) = \emptyset$. Trivially $\emptyset \subset \Gamma$ is a subcontext and $\emptyset \vdash \mathtt{Ob}$ is a valid judgment by ($\mathtt{Ob}$).

($\to$) Consider the rule

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \underset{A}{\to} t} \ (\to)$$

By induction we assume that we are given two subcontexts $\Gamma_s \subset \Gamma$ and $\Gamma_t \subset \Gamma$ such that $\mathrm{FV}(\Gamma_s) = \mathrm{FV}(s : A)$, $\mathrm{FV}(\Gamma_t) = \mathrm{FV}(t : A)$ and $\Gamma_s \vdash s : A$ and $\Gamma_t \vdash t : A$. By Lemma 1.31, $\Gamma_s \cup \Gamma_t \subset \Gamma$ is again a subcontext such that $\mathrm{FV}(\Gamma_s \cup \Gamma_t) = \mathrm{FV}(\Gamma_s) \cup \mathrm{FV}(\Gamma_t)$ and we take this to be $\Gamma_{s \underset{A}{\to} t}$. By weakening we get $\Gamma_{s \underset{A}{\to} t} \vdash s : A$ and $\Gamma_{s \underset{A}{\to} t} \vdash t : A$ and applying the rule ($\to$) we get $\Gamma_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{ (VAR)}$$

By the inductive hypothesis the variables $\mathrm{FV}(A)$ forms a subcontext $\Gamma_A \subset \Gamma$ such that $\Gamma_A \vdash A$. We may therefore form the context $\Gamma_A,\, x : A$ which we take as our definition for Γ_x. By definition $\mathrm{FV}(\Gamma_A, x : A) = \mathrm{FV}(x : A)$ and by the variable rule (VAR) we get $\Gamma_A,\, x : A \vdash x : A$.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (OP)}$$

By definition $\mathrm{FV}(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma]) = \mathrm{FV}(\gamma)$. By the inductive hypothesis there is a subcontext $\Delta_\gamma \subset \Delta$ such that $\mathrm{FV}(\Delta_\gamma) = \mathrm{FV}(\gamma)$ and $\Delta_\gamma \vdash \gamma : \Gamma$. Applying the rule (OP) using $\Delta_\gamma \vdash \gamma : \Gamma$ yields $\Delta_\gamma \vdash \mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma]$, and we choose $\Delta_{\mathtt{op}_{\Gamma,s\to t}[\gamma]}$ to be Δ_γ.

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma,t\to s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (COH)}$$

This case is the same as that for (OP) and we may choose $\Delta_{\mathtt{coh}_{\Gamma,s\to t}[\gamma]} \equiv \Delta_\gamma$.

(ES) Consider the rule

$$\frac{\Delta \vdash}{\Delta \vdash \langle\,\rangle : \emptyset} \text{ (ES)}$$

By definition $\mathrm{FV}(\langle\,\rangle) = \emptyset$. Trivially $\emptyset \subset \Delta$ is a subcontext and as a special case of the rule (ES) we obtain

$$\frac{\emptyset \vdash}{\emptyset \vdash \langle\,\rangle : \emptyset} \text{ (ES)}$$

where the context $\emptyset$ satisfies the desired property $\mathrm{FV}(\emptyset) = \mathrm{FV}(\langle\,\rangle)$.

(SE) Consider the rule

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, t\rangle : (\Gamma, x : A)} \text{ (SE)}$$

By the inductive hypothesis there exists a subcontext $\Delta_\gamma \subset \Delta$ such that $\mathrm{FV}(\Delta_\gamma) = \mathrm{FV}(\gamma)$ and $\Delta_\gamma \vdash \gamma : \Gamma$. The inductive hypothesis also says that we have a context Δ_t such that $\mathrm{FV}(\Delta_t) = \mathrm{FV}(t : A[\gamma])$. Both contexts Δ_γ and Δ_t are subcontexts of Δ so we may form their union and get the subcontext $\Delta_{\langle\gamma,t\rangle} :\equiv \Delta_\gamma \cup \Delta_t \subset \Delta$. By construction $\mathrm{FV}(\Delta_{\langle\gamma,t\rangle}) = \mathrm{FV}(\langle\gamma, t\rangle)$ where in the last line we have used the fact that $\mathrm{FV}(A[\gamma]) \subset \mathrm{FV}(\gamma)$ (see Lemma 1.10). By weakening we get $\Delta_{\langle\gamma,t\rangle} \vdash \gamma : \Gamma$ and $\Delta_{\langle\gamma,t\rangle} \vdash t : A[\gamma]$ and applying the rule (SE) we get

$$\frac{\Delta_{\langle\gamma,t\rangle} \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta_{\langle\gamma,t\rangle} \vdash t : A[\gamma]}{\Delta_{\langle\gamma,t\rangle} \vdash \langle \gamma, t\rangle : (\Gamma, x : A)} \text{ (SE)}$$

□

The next lemma says that the free variables of types are, in some sense, downwards closed. A similar statement holds for terms and for contexts and as usual, all three statements need to be proven simultaneously.

Lemma 2.9. *The following statements hold in* `CaTT`

(i) *Given a type* $\Gamma \vdash A$

$$x_i \in \mathrm{FV}(A) \implies \mathrm{FV}(A_i) \subset \mathrm{FV}(A)$$

(ii) *Given a term* $\Gamma \vdash t : A$

$$x_i \in \mathrm{FV}(t) \implies \mathrm{FV}(A_i) \subset \mathrm{FV}(t : A)$$

(iii) *Given a substitutions* $\Delta \vdash \gamma : \Gamma$

$$y_j \in \mathrm{FV}(\gamma) \implies \mathrm{FV}(B_j) \subset \mathrm{FV}(\gamma)$$

where $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n}$, $\Delta \equiv (y_j : B_j)_{1 \leq j \leq m}$ *and* $\gamma \equiv \langle \gamma_i \rangle_{1 \leq i \leq n}$.

Remark 2.10. The statement for terms cannot be improved to $x_i \in \mathrm{FV}(t) \implies \mathrm{FV}(A_i) \subset \mathrm{FV}(t)$ nor to $x_i \in \mathrm{FV}(t) \implies \mathrm{FV}(A_i) \subset \mathrm{FV}(t)$. A simple counterexample for both is given by $\Gamma \vdash p : x \underset{\mathrm{Ob}}{\to} y$ where p, x and y are all variables in Γ. The reason why for substitutions we need not consider the type of the components of γ is that the variables thereof are already contained in $\mathrm{FV}(\gamma)$.

Proof of Lemma 2.9. The proofs of each of the three statements are very similar, so we only prove the statement for types.

(i) Let $\Gamma \vdash A$ be a derivable type and consider the minimal subcontext $\Gamma_A \subset \Gamma$, as constructed in Lemma 2.8. Let $x_i \in \mathrm{FV}(A)$ be a free variable. Then $x_i \in \mathrm{FV}(\Gamma_A)$, i.e. $(x_i : A_i) \in \Gamma_A$. Using the variable rule (VAR) and weakening we may thus construct the judgment $\Gamma_A \vdash x_i : A_i$. By Proposition 1.8 we have $\Gamma_A \vdash A_i$ and by Proposition 1.9 we get $\mathrm{FV}(A_i) \subset \mathrm{FV}(\Gamma_A) = \mathrm{FV}(A)$. □

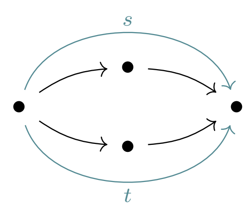
2.2.2 The Free Variables of Terms

We are aiming at proving that the minimal context of a term in a ps-context is again a ps-context. This requires developing some tools first. In particular, before showing that the free variables of terms (i.e. their minimal context) form a ps-context, we show that they satisfy a number of pleasant and intuitive properties.

To understand the following lemma, consider a term $\Gamma \vdash t : A$. By Lemma 2.1, if $x_i \in \mathrm{FV}(t : A)$ is of codimension 0, i.e. $\dim(x_i) = \dim(t)$, then $x_i \in \mathrm{FV}(t) \backslash \mathrm{FV}(A)$. On the other hand, by Lemma 2.2, if $\mathrm{FV}(t) \backslash \mathrm{FV}(A) \neq \emptyset$, then there exists a variable $x_i \in \mathrm{FV}(t)$ which is of codimension 0. But there is a priori no reason for all variables in $\mathrm{FV}(t) \backslash \mathrm{FV}(A)$ to be of codimension 0. In fact, it is not true, since the simple composition of 1-morphisms (or the vertical composition of 2-morphisms) already provides us with a counter example. What does however hold in this example, and turns out to be true in general, is that if $x_i \in \mathrm{FV}(t) \setminus \mathrm{FV}(A)$, then $\dim(x_i) = \dim(t)$ or $\dim(x_i) = \dim(t) - 1$. This statement, which we prove in its contrapositive form below, can be considered a consequence of Lemma 1.28, according to which $\mathrm{FV}(\Gamma|_{\dim(\Gamma)-2}) \subset \mathrm{FV}(\partial^- \Gamma) \cap \mathrm{FV}(\partial^+ \Gamma)$.

Lemma 2.11. *Let $\Gamma \vdash t : A$ be a term in a globular context Γ. Then, if $x_k \in \mathrm{FV}(t : A)$ and $\dim(x_k) \leq \dim(t) - 2$, then $\dim(A) > 0$ and $x_k \in \mathrm{FV}(\partial^- t : \partial A) \cap \mathrm{FV}(\partial^+ t : \partial A)$. In particular, $x_k \in \mathrm{FV}(A)$.*

Remark 2.12. Note that the assumption that Γ is globular cannot be relaxed. This is because of the variable case in the inductive proof, for which the following picture depicts a counterexample:



Here the black parts forms the context Γ and s, t are the composites of the top and bottom pairs of morphisms. In this situation we may derive the judgment $\Gamma, x : s \to t \vdash x : s \to t$, where s and t do not share all 0-dimensional cells. To see why coherences also spoil this property consider the example $x : \mathtt{Ob}, f : x \to x, \alpha : f \to 1_x \vdash \alpha : f \to 1_x$.

Proof. First, if $\dim(A) = 0$, then the only terms of A are variables and the set $\mathrm{FV}(t : A)$ contains a single element, namely the variable t. This explains why we must have $\dim(A) > 0$ and why it makes sense to consider the terms $\partial^- t,\ \partial^+ t : \partial A$. So we may write $A \equiv (\partial^- t \underset{\partial A}{\to} \partial^+ t)$.

The remainder of the proof is performed by an induction on terms.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{ (VAR)}$$

As already explained, $x : A$ must be of the form $x : \partial^- x \underset{\partial A}{\to} \partial^+ x$. Since Γ is a globular context, it follows that $\partial^- x : \partial A$ and $\partial^+ x : \partial A$ are again variables in Γ of dimension $\dim(x) - 1$. So any free variables $x_k \in \mathrm{FV}(x : \partial^- x \underset{\partial A}{\to} \partial^+ x)$ with $\dim(x_k) \leq \dim(x) - 2$ must necessarily lie in ∂A. Thus $x_k \in \mathrm{FV}(\partial^- x : \partial A) \cap \mathrm{FV}(\partial^+ x : \partial A)$.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \quad \partial^{-}\Gamma \vdash s : A \quad \partial^{+}\Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (OP)}$$

We will only explicitly prove that for $y_j \in \mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma])$ we have $y_j \in \mathrm{FV}(s[\gamma] : A[\gamma])$. The argument for $t[\gamma] : A[\gamma]$ is similar.

By hypothesis we have $\mathrm{FV}(s : A) = \mathrm{FV}(\partial^{-}\Gamma)$. Thus

$$\mathrm{FV}\big(s[\gamma] : A[\gamma]\big) = \bigcup_{x_i : \mathrm{FV}(\partial^{-}\Gamma)} \mathrm{FV}(\gamma_i)$$

(see Lemma 1.11). On the other hand, by definition $\mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma]) = \mathrm{FV}(\gamma)$. Thus $\mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]) = \mathrm{FV}(\gamma)$.

So let $y_j \in \mathrm{FV}(\gamma)$ be such that $\dim(y_j) \leq \dim(\mathtt{op}[\gamma]) - 2 = \dim(\Gamma) - 2$, where in the last equation we have used Lemma 2.2. Then $y_j \in \mathrm{FV}(\gamma_k)$ for some $x_k \in \mathrm{FV}(\Gamma)$. If we can show that there exists a $x_{k*} \in \mathrm{FV}(\partial^{-}\Gamma)$ such that $y_j \in \mathrm{FV}(\gamma_{k*})$, then

$$y_j \in \mathrm{FV}(\gamma_{k*}) \subset \bigcup_{x_i : \mathrm{FV}(\partial^{-}\Gamma)} \mathrm{FV}(\gamma_i) = \mathrm{FV}\big(s[\gamma] : A[\gamma]\big)$$

and we would be done. So let us now show that such a x_{k*} exists.

First, Lemma 2.1 implies that $\dim(y_j) \leq \dim(x_k) \leq \dim(\Gamma)$ where $\dim(y_j) \leq \dim(\Gamma) - 2$. There are three cases to consider. If $\dim(x_k) \leq \dim(\Gamma) - 2$, then, since $\Gamma|_{\dim(\Gamma)-2} \subset \partial^{-}\Gamma$ by Lemma 1.28 we may simply choose x_{k*} to be x_k.

If $\dim(x_k) = \dim(\Gamma) - 1$, then, by Lemma 1.27, there exists a $x_{k^-} \in \mathrm{FV}(\partial^{-}\Gamma) \subset \mathrm{FV}(\Gamma)$ and a path $\overline{p} : x_{k^-} \rightsquigarrow x_k$ in Γ. Assume now, that we are given some component of $\overline{p}$, say $x_a : x_{a-1} \to x_a$ (implying that $\dim(x_a) = \dim(\Gamma)$) and that we have already shown $y_j \in \mathrm{FV}(\gamma_a)$. Then $y_j \in \mathrm{FV}(\gamma_a : \gamma_{a-1} \to \gamma_a)$. Since $\dim(y_j) \leq \dim(\Gamma) - 2$ and $\dim(\gamma_a) = \dim(\Gamma)$, by the inductive hypothesis $y_j \in \mathrm{FV}(\gamma_{a-1})$. Applying this argument inductively on each component of the path $\overline{p}$, we conclude that $y_j \in \mathrm{FV}(\gamma_{k^-})$ and we may choose x_{k*} to be x_{k^-}.

Finally, if $\dim(x_k) = \dim(\Gamma)$, then since $\partial^{\pm}\Gamma$ only makes sense if $\dim(\Gamma) > 0$ we must have, $\dim(x_k) > 0$. So we can write $x_k : A_k$ in the form $x_k : x_{\partial^- k} \underset{\partial A_k}{\to} x_{\partial^+ k}$ where, since $\Gamma \vdash_{\mathrm{ps}}$, the terms $x_{\partial^- k}, x_{\partial^+ k} \in \mathrm{FV}(\Gamma)$. Since $\dim(y_j) \leq \dim(\Gamma) - 2$ and $\dim(x_k) = \dim(\Gamma)$, by the inductive hypothesis $y_j \in \mathrm{FV}(\gamma_{\partial^- k} : \partial A_k[\gamma])$. Thus, this case reduces to the previous case with y_j and $x_{\partial^- k}$.

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\rightarrow} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, s \underset{A}{\rightarrow} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\rightarrow} t[\gamma]} \text{ (COH)}$$

By hypothesis, $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma) = \mathrm{FV}(t : A)$. Thus $\mathrm{FV}(s[\gamma] : A[\gamma]) = \mathrm{FV}(\gamma) = \mathrm{FV}(t[\gamma] : A[\gamma])$. Since by definition we also have $\mathrm{FV}(\mathtt{coh}_{\gamma, s \rightarrow t}[\gamma]) = \mathrm{FV}(\gamma)$, the statement is trivially true. $\square$

The following lemma, in some sense, goes together with Lemma 1.25 for ps-contexts. It says that codimension 1 variables of a term $u : s \rightarrow t$ can always be connected via a path to a variable of the same dimension in s and in t.

Lemma 2.13 (Benjamin [9] Lemma 98)**.** *Let $\Gamma \vdash t : A$ be a term in a globular context Γ. Given a term $x_k \in \mathrm{FV}(t : A)$ such that $\dim(x_k) = \dim(t) - 1$, then*

(i) $\dim(A) > 0$;

(ii) there exists a $x_{k^-} \in \mathrm{FV}(\partial^- t)$ and a path $x_{k^-} \rightsquigarrow x_k$ in $\mathrm{FV}(t : A)$;

(iii) there exists a $x_{k^+} \in \mathrm{FV}(\partial^+ t)$ and a path $x_k \rightsquigarrow x_{k^+}$ in $\mathrm{FV}(t : A)$.

Proof. *(i)* The only terms of the type `Ob` are variables. Thus, the terms of type `Ob` cannot satisfy the premise of the lemma and we necessarily have $\dim(A) > 0$. So we may write $A \equiv \partial^- t \underset{\partial A}{\rightarrow} \partial^+ t$.

(ii) We prove the statement by induction on terms.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{ (VAR)}$$

Since $x : A$ is a variable and since Γ is, by assumption, a globular context, it follows that $(\partial^- x : \partial A), (\partial^+ x : \partial A) \in \Gamma$. Here, $\dim(\partial^- x) = \dim(\partial^+ x) = \dim(x) - 1$. By Lemma 2.1 $\mathrm{FV}(\partial A)$ cannot contain any variables of dimension $\dim(x)$ since $\dim(\partial A) = \dim(A) - 1$. Thus the only variables of dimension $\dim(x) - 1$ in $\mathrm{FV}(x : A)$ are $\partial^- x$ and $\partial^+ x$.

For the variable $\partial^- x$ just choose $\partial^- x$ itself and the empty path, while for $\partial^+ x$ choose again $\partial^- x$ and the path $(x) : \partial^- x \rightsquigarrow \partial^+ x$. Trivially the path (x) has free variables $\mathrm{FV}\big((x)\big) = \mathrm{FV}(x : A) \subset \mathrm{FV}(x : A)$.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \quad \partial^-\Gamma \vdash s : A \quad \partial^+\Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(OP)}$$

where $\mathrm{FV}(t : A) = \mathrm{FV}(\partial^-\Gamma)$ so that

$$\mathrm{FV}(t[\gamma] : A[\gamma]) = \bigcup_{x_i \in \partial^-\Gamma} \mathrm{FV}(\gamma_i)$$

(see Lemma 1.11). Recall that by definition $\mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma]) = \mathrm{FV}(\gamma)$ and by Lemma 2.2 $\dim(\mathtt{op}_{\Gamma,s\to t}[\gamma]) = \dim(\Gamma)$.

Let $y_j \in \mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$ be a variable of dimension $\dim(y_j) = \dim(\mathtt{op}_{\Gamma,s\to t}[\gamma]) - 1 = \dim(\Gamma) - 1$. Thus $y_j \in \mathrm{FV}(\gamma_k)$ for some $x_k \in \mathrm{FV}(\Gamma)$ and by Lemma 2.1 we necessarily have $\dim(x_k) = \dim(\Gamma)$ or $\dim(x_k) = \dim(\Gamma) - 1$.

Consider first the case $\dim(x_k) = \dim(\Gamma) - 1$. Then, by Lemma 1.25 there is a $x_{k^-} \in \mathrm{FV}(\partial^-\Gamma)$ and a path $\overline{p} : x_{k^-} \rightsquigarrow x_k$. Let $p_l : x_{l-1} \to x_k$ be the last component of $\overline{p}$. Then, remembering that $y_j \in \mathrm{FV}(\gamma_k)$, by the inductive hypothesis, there exists a $y_{j_l} \in \mathrm{FV}(\gamma_{l-1})$ and a path $y_{j_l} \rightsquigarrow y_k$ in $\mathrm{FV}(\gamma)$. Repeating this process for all components successively and by concatenating all the resulting paths yields a $y_{j^-} \in \mathrm{FV}(\gamma_{k^-})$ and a path $y_{j^-} \rightsquigarrow y_j$ in $\mathrm{FV}(\gamma)$. Since $x_{k^-} \in \partial^-\Gamma$, we have

$$y_{j^-} \in \mathrm{FV}(\gamma_{k^-}) \subset \bigcup_{x_i \in \partial^-\Gamma} \mathrm{FV}(\gamma_i) = \mathrm{FV}(t[\gamma] : A[\gamma]).$$

In fact, since $\dim(y_{j^-}) = \dim(t[\gamma])$, by Lemma 2.1 we must have $y_{j^-} \in \mathrm{FV}(t[\gamma])$. And since the path $y_{j^-} \rightsquigarrow y_j$ is in $\mathrm{FV}(\gamma) = \mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$ it has all the desired properties.

Consider now the case where $\dim(x_k) = \dim(\Gamma)$ which implies that $\dim(x_k) > 0$. Thus we may write $x_k : x_{\partial^- k} \to x_{\partial^+ k}$ (since $\Gamma \vdash_{\mathrm{ps}}$ is a ps-context). Since $y_j \in \mathrm{FV}(\gamma_k : \gamma_{\partial^- k} \to \gamma_{\partial^+ k})$, by the inductive hypothesis, there exists a $y_{j^-} \in \mathrm{FV}(\gamma_{\partial^- k})$ and a path $y_{j^-} \rightsquigarrow y_j$ in $\mathrm{FV}(\gamma)$. Since $\dim(x_{\partial^- k}) = \dim(\Gamma) - 1$, the previous case applied to $y_{j^-} \in \mathrm{FV}(\gamma_{\partial^- k})$ tells us that there exists a $y_{j^=} \in \mathrm{FV}(t[\gamma])$ and a path $y_{j^=} \rightsquigarrow y_{j^-}$ in $\mathrm{FV}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. By concatenation we get the path $y_{j^=} \rightsquigarrow y_j$ which has all the desired properties.

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \quad \Gamma \vdash s \underset{A}{\to} t \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, t \to s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(COH)}$$

where by assumption $\mathrm{FV}(t : A) = \mathrm{FV}(\Gamma) = \mathrm{FV}(s : A)$. Since by definition $\mathrm{FV}(\mathtt{coh}_{\Gamma,s\to t}[\gamma]) = \mathrm{FV}(\gamma)$ we have

$$\mathrm{FV}(\mathtt{coh}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma]) = \mathrm{FV}(\gamma) = \mathrm{FV}(t[\gamma] : A[\gamma])$$

and the statement holds by taking for each term of appropriate dimension the term itself together with the empty path. Trivially, the nodes and components of the path are all elements in $\mathrm{FV}(\gamma) = \mathrm{FV}(\mathtt{coh}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. □

We are almost ready to prove that the minimal context of a term in a ps-context is again a ps-context. It only remains to prove one last property about terms and context morphisms. The usual examples of terms (e.g. the variable, horizontal composition of 2-morphisms and associativity) show that the free variables of terms never contain any non-equal parallel variables of codimension 0. This is the first half of the next lemma. The accompanying statement for context morphisms has a more technical flavor.

Lemma 2.14. *The following statements hold in* `CaTT`*.*

(i) (♣) *Let $\Gamma \vdash t : A$ be a term in a ps-context Γ and let $x_k, x_l \in \mathrm{FV}(t : A)$ be two variables of codimension 0. Then* (♣) Benjamin [9] Lemma 99

$$x_k \parallel x_l \implies x_k \equiv x_l.$$

(ii) Let $\Delta \vdash \gamma : \Gamma$ be a context morphisms with $\Delta \vdash_{\mathrm{ps}}$ and $\Gamma \vdash_{\mathrm{ps}}$. Let $x_{k_1}, x_{k_2} \in \mathrm{FV}(\Gamma)$ be two variables such that $\dim(x_k) = \dim(x_l)$. Let $y_{j_1} \in \mathrm{FV}(\gamma_{k_1})$ and $y_{j_2} \in \mathrm{FV}(\gamma_{k_2})$ be two variables of codimension 0. Then

$$y_{j_1} \parallel y_{j_2} \implies x_{k_1} \parallel x_{k_2}.$$

Proof. We prove all statements simultaneously by mutual induction on types, terms and context morphisms.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\mathrm{VAR})$$

This follows directly from the fact that $x \in \mathrm{FV}(x : A)$ is the only variable of codimension 0 in $\mathrm{FV}(x : A)$.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma,s \xrightarrow[A]{} t}[\gamma] : s[\gamma] \xrightarrow[A[\gamma]]{} t[\gamma]}\ (\mathrm{OP})$$

where we are assuming additionally that $\Delta \vdash_{\text{ps}}$ is a ps-context. So assume we are given two variables variable $y_{j_1}, y_{j_2} \in \text{FV}(\texttt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma]) = \text{FV}(\gamma)$ of codimension 0 such that $y_{j_1} \parallel y_{j_2}$. Then $y_{j_1} \in \text{FV}(\gamma_{i_1})$ and $y_{j_2} \in \text{FV}(\gamma_{i_2})$ for some $x_{i_1}, x_{i_2} \in \text{FV}(\Gamma)$. Lemma 2.1 then forces $\dim(x_{i_1}) = \dim(x_{i_2}) = \dim(\Gamma)$. By the inductive hypothesis for context morphisms $x_{i_1} \parallel x_{i_2}$ and by Lemma 1.26*(iv)* we have in fact $x_{i_1} \equiv x_{i_2}$. By the inductive hypothesis for terms, since $y_{j_1}, y_{j_2} \in \text{FV}(\gamma_{i_1})$ are parallel and of codimension 0, we must have $y_{j_1} \equiv y_{j_2}$.

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{coh}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (COH)}$$

where $\dim(\Gamma) < \dim(s \to t)$.

Since $\text{FV}(\texttt{coh}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma]) = \text{FV}(s[\gamma] \to t[\gamma])$, by Lemma 2.1 the term $\texttt{coh}_{\Gamma,s\to t}[\gamma]$ cannot contain any variables of codimension 0 so the statement is trivially true.

(ES) Consider the rule

$$\frac{\Delta \vdash}{\Delta \vdash \langle\,\rangle : \emptyset} \text{ (ES)}$$

Since $\emptyset \nvdash_{\text{ps}}$ there is nothing to check.

(SE) Consider the rule

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x_{n+1} : A_{n+1} \vdash \qquad \Delta \vdash \gamma_{n+1} : A_{n+1}[\gamma]}{\Delta \vdash \langle \gamma, \gamma_{n+1} \rangle : (\Gamma, x_{n+1} : A_{n+1})} \text{ (SE)}$$

where $\Delta \vdash_{\text{ps}}$ and $\Gamma, x_{n+1} : A_{n+1} \vdash_{\text{ps}}$. We perform an additional induction over ps-context.

First assume $\Gamma,\, x_{n+1} : A_{n+1} \equiv x_1 : \texttt{Ob}$, i.e. $\Gamma \equiv \emptyset$. Since $\text{FV}(x_1 : \texttt{Ob}) = \{x_1\}$ given two $x_{i_1}, x_{i_2} \in \text{FV}(x_1 : \texttt{Ob})$ we must have $x_{i_1} \equiv x_1 \equiv x_{i_2}$. Also, since $\dim(x_1) = 0$, the term γ_1 is necessarily a variable so that $y_{j_1}, y_{j_2} \in \text{FV}(\gamma_1)$ implies that $y_{j_1} \equiv \gamma_1 \equiv y_{j_2}$. In particular, we have $y_{j_1} || y_{j_2}$. On the other hand, $x_{i_1} \equiv x_{i_2}$ also trivially implies $x_{i_1} || x_{i_2}$.

Assume now that $(\Gamma,\, x_{n+1} : A_{n+1}) \equiv (\Gamma^{n-1},\, x_n : A_n,\, x_{n+1} : x_l \to x_n)$ where $\Gamma^{n+1} \vdash_{\text{ps}}$ and the statement is true for $\Delta \vdash \gamma^{n-1} : \Gamma^{n-1}$. There are four cases for the two variables x_{i_1}, x_{i_2} to consider

(1) $x_{i_1}, x_{i_2} \in \text{FV}(\Gamma^{n-1})$

Since $\Gamma^{n-1} \vdash_{\text{ps}}$, this statement holds by the inductive hypothesis for context morphisms.

(2) $x_{i_1} \in \text{FV}(\Gamma^{n-1})$ and $x_{i_2} \equiv x_n$

Assume $y_{j_1} \in \text{FV}(\gamma_{i_1})$ and $y_{j_2} \in \text{FV}(\gamma_n)$ are two variables both of codimension 0 respectively such that $y_{i_1} \parallel y_{i_2}$. Then, since we have $\gamma_{n+1} : \gamma_l \to \gamma_n$, by Lemma 2.13 there exists a $y_{j^-} \in \text{FV}(\gamma_l)$ and a path $y_{j^-} \rightsquigarrow y_{j_2}$. In particular, $y_{j^-} \parallel y_{j_2}$ so that $y_{j_1} \parallel y_{j^-}$. Since $x_l \in \text{FV}(\Gamma^{n-1})$ and $y_{j_1} \parallel y_{j^-}$ the case for the pairs (x_{i_1}, y_{j_1}) and (x_l, y_{j^-}) reduces to the situation of part (1) from which we deduce that $x_{i_1} \parallel x_l$. But by definition $x_l \parallel x_n$ so that $x_{i_1} \parallel x_n$.

(3) $x_{i_1} \in \text{FV}(\Gamma^{n-1})$ and $x_{j_2} \equiv x_{n+1}$

Let $y_{j_1} \in \text{FV}(\gamma_{i_1})$ and $y_{j_2} \in \text{FV}(\gamma_{n+1})$ be two variables of codimension 0 respectively and such that $y_{j_1} \parallel y_{j_2}$. The goal is to derive a contradiction, from which we conclude that there can be no such variables y_{j_1}, y_{j_2} which are parallel.

Since $\dim(\gamma_{n+1}) > 0$, it follows that $\dim(y_{j_2}) > 0$. Let $y_s \to y_t$ be the type of y_{j_1} and y_{j_2}. Then, by Lemma 2.1 we must have $y_s, y_t \in \text{FV}(\gamma_{n+1} : \gamma_l \to \gamma_n)$. By Lemma 2.13 there is a $y_{s^-} \in \text{FV}(\gamma_l)$ and a path $y_{s^-} \rightsquigarrow y_s$, which, when concatenated with $(y_{j_2}) : y_s \rightsquigarrow y_t$ gives a nonempty path $y_{s^-} \rightsquigarrow y_t$ in $\Delta \vdash_{\text{ps}}$. In particular $y_{s^-} \not\equiv y_t$ by Lemma 1.26*(v)*. In what follows we will also construct a path $y_t \rightsquigarrow y_{s^-}$ in $\Delta \vdash_{\text{ps}}$ which, by Lemma 1.26*(i)* delivers the contradiction.

Now $y_s \to y_t$ is also the type of $y_{j_1} \in \text{FV}(\gamma_{i_1})$ where $x_{i_1} \in \text{FV}(\Gamma^{n-1})$. So, by Lemma 2.9 $y_t \in \text{FV}(\gamma_{i_1} : \gamma_{\partial^- i_1} \to \gamma_{\partial^+ i_1})$ where $x_{\partial^- i_1}, x_{\partial^+ i_1} \in \text{FV}(\Gamma^{n-1})$. Since $\dim(y_t) = \dim(y_{j_1}) - 1 = \dim(x_{i_1}) - 1$, it follows from Lemma 2.13 that there exists a $y_{t^+} \in \text{FV}(\gamma_{\partial^- i_1})$ and a path $y_t \rightsquigarrow y_{t^+}$. In particular $y_{t^+} \in \text{FV}(\gamma_{\partial^+ i_1})$ is of codimension 0.

But we also have $y_{s^-} \in \text{FV}(\gamma_l)$ of codimension 0 where $x_l \in \text{FV}(\Gamma^{n-1})$ and where $y_{s^-} \parallel y_s \parallel y_t \parallel y_{t^+}$. By the inductive hypothesis for context morphisms on $\Delta \vdash \gamma^{n-1} : \Gamma^{n-1}$ we must have $x_l \parallel x_{\partial^+ i_1}$. Since by Lemma 1.40*(iv)* $\Gamma^{n-1} \blacktriangleright x_l$, we have a path $x_{\partial^+ i_1} \rightsquigarrow x_l$ in Γ^{n-1}. Now, starting with $y_{t^+} \in \text{FV}(\gamma_{\partial^+ i_1})$, using Lemma 2.13 we can go through all the nodes of the path $x_{\partial^+ i_1} \rightsquigarrow x_l$ step by step and find a parallel variable $y_{t^*} \in \text{FV}(\gamma_l)$ as visualized in the following diagram

$$\gamma_{t^+} \to \cdots \to \gamma_l$$

$$\text{FV}(\gamma_{\partial^+ i_1}) \ni y_{t^+} \rightsquigarrow y_{t*} \in \text{FV}(\gamma_l)$$

But then $y_{t*} \parallel y_{s^-}$ in $\mathrm{FV}(\gamma_l)$ and since they are of codimension 0, the inductive hypothesis for terms implies that $y_{t*} \equiv y_{s^-}$. Thus we have found a path $y_t \rightsquigarrow y_{s^-}$ in $\Delta \vdash_{\mathrm{ps}}$, with which we have derived the contradiction.

(4) $x_{i_1} \equiv x_n$ and $x_{i_2} \equiv x_{n+1}$

In this case $\dim(x_{i_1}) \neq \dim(x_{i_2})$ so there is nothing to check. □

We are finally ready to prove that the minimal context of a term in a ps-context is again a ps-context. But before we proceed with the proof, we offer some justification as to why one might expect this to be true.

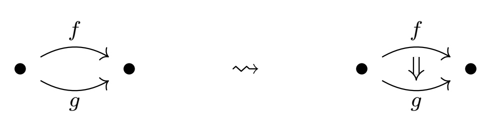
The idea is that all term constructions essentially take place inside a ps-context. For the variable rule this follows from the fact that the minimal context of a variable in a globular context is simply a globe. For the other rules, the intuition comes from first ignoring substitution, upon which the rules take the following form

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A}{\Gamma \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s} : s \underset{A}{\to} t} \qquad\qquad \frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t}{\Gamma \vdash \mathtt{coh}_{\Gamma, s \to t} : s \underset{A}{\to} t}$$

By definition $\mathrm{FV}(\mathtt{op}_{\Gamma, s \to t}) = \mathrm{FV}(\Gamma)$ and $\mathrm{FV}(\mathtt{coh}_{\Gamma, s \to t}) = \mathrm{FV}(\Gamma)$ so that the minimal context is precisely $\Gamma \vdash_{\mathrm{ps}}$. If we include substitution, the statement would continue to hold if, given a context morphism $\Delta \vdash \gamma : \Gamma$ such that $\Delta \vdash_{\mathrm{ps}}$ and $\Gamma \vdash_{\mathrm{ps}}$ we also have $\Delta_\gamma \vdash_{\mathrm{ps}}$. This statement about context morphisms is proven together with the statement about terms by mutual induction.

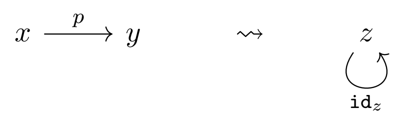
Since Δ_γ is obtained by gluing together all the Δ_{γ_i} for all $x_i \in \mathrm{FV}(\Gamma)$ assuming that $\Delta \vdash_{\mathrm{ps}}$ and $\Gamma \vdash_{\mathrm{ps}}$ it is reasonable to expect $\Delta_\gamma \vdash_{\mathrm{ps}}$ to hold. This is because, by the inductive hypothesis, since $\Delta \vdash_{\mathrm{ps}}$ each Δ_{γ_i} will be a ps-context and since $\Gamma \vdash_{\mathrm{ps}}$, all of these subcontexts are glued together in a well-formed way, according to the prescription given by Γ.

In what follows we give two counterexamples of context morphisms $\Delta \vdash \gamma : \Gamma$ exhibiting what may go wrong in the absence of the assumption of both contexts being ps-contexts. We visualize all examples diagrammatically as morphisms $\Gamma \rightsquigarrow \Delta$. In the first example



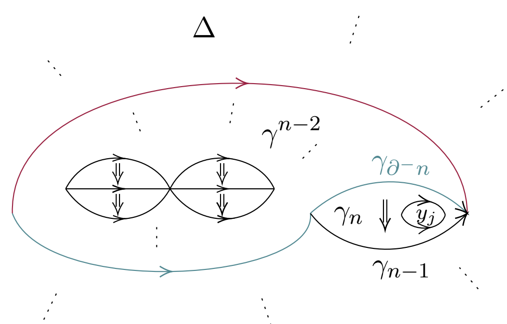
$\Gamma \not\vdash_{\mathrm{ps}}$ and since Δ_γ is equal to Γ, it follows that $\Delta_\gamma \not\vdash_{\mathrm{ps}}$. In the second example

we have that $\Delta_\gamma \nvdash_{\text{ps}}$ since $\Delta \equiv \Delta_\gamma$ and since Δ is not a ps-context. Notice however, that not all identifications are problematic. In this example



both $\Delta \vdash_{\text{ps}}$ and $\Gamma \vdash_{\text{ps}}$ and $\Delta_\gamma \equiv \Delta \equiv (z : \texttt{Ob})$ so that $\Delta_\gamma \vdash_{\text{ps}}$. Here $\texttt{id}_z$ is generated by the coherence rule.

As part of the proof for Δ_γ we will show that $\Delta_{\gamma^{n-2}} \cap \Delta_{\gamma_n} \equiv \Delta_{\gamma_{\partial^- n}}$. Here we are making use of the notation $\Gamma \equiv \Gamma^{n-2}$, $x_{n-1} : A_{n-1}$, $x_n : x_{\partial^- n} \to x_{n-1}$ which gives a corresponding factorization of the context morphism $\gamma \equiv \langle \gamma^{n-2}, \gamma_{n-1}, \gamma_n \rangle$. With this notation we have $\mathrm{FV}(\Delta_{\gamma^{n-2}}) = \mathrm{FV}(\gamma^{n-2})$, $\mathrm{FV}(\Delta_{\gamma_n}) = \mathrm{FV}(\gamma_n : \gamma_{\partial^- n} \to \gamma_{n-1})$ and $\mathrm{FV}(\Delta_{\gamma_{\partial^- n}}) = \mathrm{FV}(\gamma_{\partial^- n} : T)$. The intuition for this is captured by the following diagram



(1.11)

Here, the bulk of the diagram, outlined by the red and blue lines, forms Γ^{n-1} with γ^{n-2} substituted into it. To this diagram we glue the new cell γ_n, as in the bottom right. We see that when attaching the new morphism γ_n, if we use variables in Δ of the same dimension as γ_n, then γ_n must grow "outwards", in order to respect the inherent direction of Δ in which the whole diagram lives. The diagram shows how the only common variables that γ_n and γ_{n-1} may share with the bulk γ^{n-2} are contained in $\gamma_{\partial^- n} : T$.

Lemma 2.15. *The following statements hold in* `CaTT`

(i) Given a term $\Gamma \vdash t : A$

$$\Gamma \vdash_{\text{ps}} \implies \Gamma_t \vdash_{\text{ps}}$$

(ii) Given a substitution $\Delta \vdash \gamma : \Gamma$*, then*

$$\Big(\Delta \vdash_{\text{ps}} \quad \textit{and} \quad \Gamma \vdash_{\text{ps}}\Big) \quad \Longrightarrow \quad \Delta_\gamma \vdash_{\text{ps}}$$

Proof. We prove the statement simultaneously by mutual induction on terms and context morphisms.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \text{ (VAR)}$$

Then Γ_x is a globe and therefore a ps-context.

(OP) Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (OP)}$$

where by assumption $\Delta \vdash_{\text{ps}}$. Thus, by the inductive hypothesis we get $\Delta_\gamma \vdash_{\text{ps}}$. But $\Delta_{\texttt{op}_{\Gamma, s \to t}[\gamma]} \equiv \Delta_\gamma$ (see the proof of Lemma 2.8).

(COH) Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{coh}_{\Gamma, (t \to s)}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (COH)}$$

This is similar to the case of (OP).

(ES) Consider the rule

$$\frac{\Delta \vdash}{\Delta \vdash \langle\, \rangle : \emptyset} \text{ (ES)}$$

By assumption $\Delta \vdash_{\text{ps}}$. But $\emptyset \nvdash_{\text{ps}}$ so there is nothing to check.

(SE) Consider the rule

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma,\ x_{n+1} : A_{n+1} \vdash \qquad \Delta \vdash \gamma_{n+1} : A_{n+1}[\gamma]}{\Delta \vdash \langle \gamma, \gamma_{n+1} \rangle : (\Gamma,\ x_{n+1} : A_{n+1})} \text{ (SE)}$$

where $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n}$. By assumption $\Delta \vdash_{\text{ps}}$ and $\Gamma,\ x_{n+1} : A_{n+1} \vdash_{\text{ps}}$.

We now perform an induction on ps-contexts. So assume first that $(\Gamma, x_{n+1} : A_{n+1}) \equiv (x_1 : A_1)$, i.e. $\Gamma \equiv \emptyset$, $n = 0$, $A_1 \equiv \texttt{Ob}$ and $\gamma \equiv \gamma_1$. Since the only variables of dimension 0 are variables, it follows that $\gamma_1 \equiv y_j$ for some $y_j \in \text{FV}(\Delta)$. Since $\gamma \equiv \gamma_1 \equiv y_j$ we have $\Delta_\gamma \equiv \Delta_{y_j}$, which, being a globe is a ps-context.

Assume now that

$$\Gamma,\ x_{n+1} : A_{n+1} \quad \equiv \quad \Gamma^{n-1},\ x_n : A_n,\ x_{n+1} : x_l \to x_n \tag{1.12}$$

where we've used the notation $l = \partial^-(n+1)$ in order to reduce the clutter and where by the definition of ps-contexts $\Gamma^{n-1} \vdash_{\text{ps}}$. By the inductive hypothesis the statement holds for $\Delta \vdash \gamma^{n-1} : \Gamma^{n-1}$ so that $\Delta_{\gamma^{n-1}} \vdash_{\text{ps}}$.

Our task is to show that $\Delta_{\langle\gamma,\gamma_{n+1}\rangle} \vdash_{\text{ps}}$, where by definition

$$\Delta_{\langle\gamma,\gamma_{n+1}\rangle} \equiv \Delta_\gamma \cup \Delta_{\gamma_{n+1}}.$$

(see Lemma 2.8). But $\text{FV}(\gamma_n) \subset \text{FV}(\Delta_{\gamma_{n+1}})$ (see equation (1.13)). Thus we may write

$$\Delta_{\langle\gamma,\gamma_{n+1}\rangle} \equiv \Delta_{\gamma^{n-1}} \cup \Delta_{\gamma_{n+1}}.$$

If we can show that $\Delta_{\gamma^{n-1}} \cap \Delta_{\gamma_{n+1}} = \Delta_{\gamma_l}$, then since $\Delta_{\gamma_{n+1}} \vdash_{\text{ps}}$ and $\Delta_{\gamma_l} \vdash_{\text{ps}}$ by the inductive hypothesis for terms and since also $\Delta_{\gamma^{n-1}} \vdash_{\text{ps}}$ by the inductive hypothesis for contexts, Lemma 1.31 applies, so that $\Delta_{\langle\gamma,\gamma_{n+1}\rangle} \vdash_{\text{ps}}$.

So let us show that $\Delta_{\gamma^{n-1}} \cap \Delta_{\gamma_{n+1}} \equiv \Delta_{\gamma_l}$. By definition

$$\begin{aligned} \text{FV}(\Delta_{\gamma^{n-1}}) &= \text{FV}(\gamma^{n-1}) \\ \text{FV}(\Delta_{\gamma_{n+1}}) &= \text{FV}(\gamma_{n+1}) \cup \text{FV}(\gamma_l \to \gamma_n) \\ &= \text{FV}(\gamma_{n+1}) \cup \text{FV}(\gamma_n) \cup \text{FV}(\gamma_l : T) \\ \text{FV}(\Delta_{\gamma_l}) &= \text{FV}(\gamma_l : T) \end{aligned} \tag{1.13}$$

from which we can directly read off that $\text{FV}(\Delta_{\gamma_l}) \subset \text{FV}(\Delta_{\gamma_{n+1}})$. On the other hand, since $x_l \in \Gamma^{n-1}$ and since A_{n-1} depends only on variables which appear before $(x_{n-1} : A_{n-1}) \in \Gamma$, we also have $\text{FV}(\Delta_{\gamma_l}) \subset \text{FV}(\Delta_{\gamma^{n-1}})$. This shows that $\Delta_{\gamma^{n-1}} \cap \Delta_{\gamma_{n+1}} \supset \Delta_{\gamma_l}$. And for the other direction, considering equation (1.13), it suffices to show that

$$\text{FV}(\gamma^{n-1}) \cap \Big(\text{FV}(\gamma_{n+1}) \cup \text{FV}(\gamma_n) \Big\backslash \text{FV}(\gamma_l : T)\Big) \quad \subset \quad \text{FV}(\gamma_l : T) \tag{1.14}$$

Let us inspect the term $\gamma_{n+1} : \gamma_l \to \gamma_n$.

If $\text{FV}(\gamma_{n+1})\backslash\text{FV}(\gamma_l \to \gamma_n) = \emptyset$, then by Lemma 2.2 it follows that

$$\text{FV}(\gamma_l : T) = \text{FV}(\gamma_n : T)$$

(notice that $A_l \equiv A_n$, since $x_l \parallel x_n$) and

$$\text{FV}(\gamma_{n+1}) \subset \text{FV}(\gamma_l : T)$$

from which equation (1.14) follows.

If, on the other hand, $\mathrm{FV}(\gamma_{n+1})\backslash\mathrm{FV}(\gamma_l \to \gamma_n) \neq \emptyset$, then, first of all by Lemma 2.11 we only need worry about the variables in $\mathrm{FV}(\gamma_{n+1}) \cup \mathrm{FV}(\gamma_n)$ that are of dimension $\dim(\gamma_{n+1})$ or $\dim(\gamma_{n+1}) - 1$. This is because by Lemma 2.1 all other variables, which are of dimension less than or equal to $\dim(\gamma_{n+1}) - 2$ will already be included in $\mathrm{FV}(\gamma_l : T)$ by Lemma 2.11. Our goal is to show that if $y_j \in \mathrm{FV}(\gamma_{n+1}) \cup \mathrm{FV}(\gamma_n)$ with $\dim(y_j) = \dim(\gamma_{n+1})$ or $\dim(y_j) = \dim(\gamma_{n+1}) - 1$ and if $y_j \notin \mathrm{FV}(\gamma_l : T)$, then $y_j \notin \mathrm{FV}(\gamma^{n-1})$, as depicted in diagram (1.11).

Let us begin with the former, i.e. let $y_j \in \mathrm{FV}(\gamma_{n+1}) \cup \mathrm{FV}(\gamma_n)$ be a variable such that $\dim(y_j) = \dim(\gamma_{n+1})$. By Lemma 2.1 we automatically have $y_j \notin \mathrm{FV}(\gamma_l : T)$. First of all, since $\dim(\gamma_{n+1}) > 0$, it follows that $\dim(y_j) > 0$ so we may write $y_j : y_{\partial^- j} \to y_{\partial^+ j}$ where $y_{\partial^- j}, y_{\partial^+ j} \in \mathrm{FV}(\Delta)$. By Lemma 2.9, $y_{\partial^- j}, y_{\partial^+ j} \in \mathrm{FV}(\gamma_{n+1} : \gamma_l \to \gamma_n)$. Lemma 2.13 then implies that there exists a variable $y_{j^-} \in \mathrm{FV}(\gamma_l)$ and a path $\overline{p} : y_{j^-} \rightsquigarrow y_{\partial^- j}$. By concatenation we get a nonempty path $\overline{p} * (y_j) : y_{j^-} \rightsquigarrow y_{\partial^+ j}$. Since $\overline{p} * (y_j)$ is nonempty, we must have $y_{j^-} \not\equiv y_{\partial^+ j}$ (see Lemma 1.26). Since $y_{j^-} \in \mathrm{FV}(\gamma_l)$ is of codimension 0, it follows from Lemma 2.14 that $y_{\partial^+ j} \notin \mathrm{FV}(\gamma_l)$. Thus we have found a variable $y_{\partial^+ j} \in \mathrm{FV}(\gamma_{n+1}) \cup \mathrm{FV}(\gamma_l)$ which is of dimension $\dim(\gamma_{n+1}) - 1$ and such that $y_{\partial^+ j} \notin \mathrm{FV}(\gamma_l : T)$. If we can show that $y_{\partial^+ j} \notin \mathrm{FV}(\gamma^{n-1})$, then by Lemma 2.9 we must also have $y_j \notin \mathrm{FV}(\gamma^{n-1})$. So the argument reduces to the case of considering variables of dimension $\dim(\gamma_{n+1}) - 1$.

Consider now a variable $y_j \in \mathrm{FV}(\gamma_{n+1}) \cup \mathrm{FV}(\gamma_l)$ of dimension $\dim(\gamma_{n+1}) - 1$ and such that $y_j \notin \mathrm{FV}(\gamma_l : T)$. We will allow ourselves to recycle some of the names of variables used in the previous paragraph, since the discussions are independent. By Lemma 2.13 there exists a variable $y_{j^-} \in \mathrm{FV}(\gamma_l)$ and a path $\overline{q} : y_{j^-} \rightsquigarrow y_j$. Since $y_j \notin \mathrm{FV}(\gamma_l : T)$, we must have $y_j \not\equiv y_{j^-}$ so that the path $\overline{q}$ necessarily is nonempty (see Lemma 1.26). In order to show that equation (1.14) holds, we must show that $y_j \notin \mathrm{FV}(\gamma^{n-1})$. We will assume the contrary, namely, that $y_j \in \mathrm{FV}(\gamma^{n-1})$ and derive a contradiction. We will do this by constructing a path $y_j \rightsquigarrow y_{j^-}$ in Δ, which together with the existence of the (nonempty) path $y_{j^-} \rightsquigarrow y_j$ contradicts Lemma 1.26*(i)*.

The given relevant data are the variables $y_j \in \mathrm{FV}(\gamma^{n-1})$ and $y_{j^-} \in \mathrm{FV}(\gamma_l) \subset \mathrm{FV}(\gamma^{n-1})$ where $y_{j^-} \parallel y_j$ and $\dim(y_{j^-}) = \dim(x_l)$. So $y_j \in \mathrm{FV}(\gamma_k)$ for some $x_k \in \mathrm{FV}(\Gamma^{n-1})$. By Lemma 2.1 $\dim(y_j) \leq \dim(x_k)$. We now show that there exists a $x_{k^+} \in \mathrm{FV}(\Gamma^{n-1})$ and a $y_{j^+} \in \mathrm{FV}(\gamma_{k^+})$ of codimension 0 together with a path $y_j \rightsquigarrow y_{j^+}$. If $\dim(y_j) = \dim(x_k)$ then we may simply choose $x_{k^+} \equiv x_k$ and $y_{j^+} \equiv y_j$. If $\dim(y_j) = \dim(x_k) - 1$, then, by Lemma 2.13 there exists a $y_{j^+} \in \mathrm{FV}(\gamma_{\partial^+ k})$ of codimension 0 and a path $y_j \rightsquigarrow y_{j^+}$, where

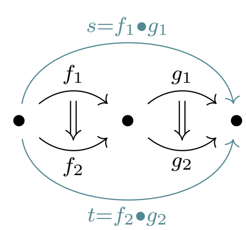
$x_k : x_{\partial^- k} \to x_{\partial^+ k}$. So in this case we may choose $x_{k^+} \equiv x_{\partial^+ k}$ and y_{j^+} itself. Finally, if $\dim(y_j) \leq \dim(x_k) - 2$, then by Lemma 2.11 $y_j \in \mathrm{FV}(\gamma_{\partial^+ k} : T')$. By recursively applying the same arguments we can reduce the dimension all the way down until we get a term of the desired form.

So at this point we have $y_{j^+} \in \mathrm{FV}(\gamma_{k^+})$ of codimension 0, a path $y_j \rightsquigarrow y_{j^+}$ and $y_{j^-} \in \mathrm{FV}(\gamma_l)$ also of codimension 0. So $y_{j^+} \parallel y_{j^-}$ and by Lemma 2.14 it follows that $x_{k^+} \parallel x_l$. Now, since by Lemma 1.40 we have $\Gamma^{n-1} \blacktriangleright x_l$ and since $\Gamma \vdash_{\mathrm{ps}}$, we must have a path $x_{k^+} \rightsquigarrow x_l$ in Γ. Using Lemma 2.13, starting with $y_{j^+} \in \mathrm{FV}(\gamma_{k^+})$ and going step by step through all nodes of this path we can find a $y_{j*} \in \mathrm{FV}(\gamma_l)$ and a path $y_{j^+} \rightsquigarrow y_{j*}$ as in the following diagram

$$\begin{array}{ccccc} & \gamma_{k^+} & \to \cdots \to & \gamma_l & \\ \mathrm{FV}(\gamma_{k^+}) \ni & y_{j^+} & \rightsquigarrow & y_{j*} & \in \mathrm{FV}(\gamma_l) \end{array}$$

But since $y_{j*}, y_{j^-} \in \mathrm{FV}(\gamma_l)$ are parallel and of codimension 0, it follows from Lemma 2.14 that $y_{j*} \equiv y_{j^-}$. So we actually have a path $y_{j^+} \rightsquigarrow y_{j^-}$. Concatenating this with the path $y_j \rightsquigarrow y_{j^+}$ we finally get the desired path $y_j \rightsquigarrow y_{j^-}$ as promised. So we have derived a contradiction and thus $y_j \notin \mathrm{FV}(\gamma^{n-1})$. □

Before ending this section we prove one final lemma which will be required in the proof of the admissibility of (OP') in `CaTT`. To build some intuition, consider the diagram from which we build horizontal composition of 2-morphisms:



Both terms s, t have all free variables of dimension 0 in common. As for the variables of dimension 1, these are in bijective correspondence, where to each variable of dimension 1 in s there is a unique parallel variable in t (e.g. f_1 corresponds to f_2). This is a low dimensional example of what the following lemma captures.

Lemma 2.16. *Let $\Gamma \vdash s, t : A$ be two terms in a ps-context. Let $x_i \in \mathrm{FV}(s : A)$ be a variable:*

(i) if $\dim(x_i) = \dim(s) - 1$, then $x_i \in \mathrm{FV}(t : A)$;

(ii) if $\dim(x_i) = \dim(s)$, then $x_i \in \mathrm{FV}(s)$ and there exists a parallel variable $x_j \in \mathrm{FV}(t)$, i.e. $x_i \parallel x_j$;

In particular, there is a bijective correspondence between variables of $\mathrm{FV}(s)$ *and* $\mathrm{FV}(t)$*.*

Proof. (i) Let $x_i \in \mathrm{FV}(s : A)$ be such that $\dim(x_i) \leq \dim(A) - 1$. If $\dim(x_i) < \dim(A) - 1$, then, by Lemma 2.11 we have $x_i \in \mathrm{FV}(A) \subset \mathrm{FV}(t : A)$. If, on the other hand, $\dim(x_i) = \dim(A) - 1$, then, since the only terms of dimension 0 are variables (the free variables of which are of codimension 0), we must have $\dim(A) > 0$. So we may write $A \equiv a_1 \to a_2$. By Lemma 2.13 there exists a variable $x_{i^-} \in \mathrm{FV}(a_1)$ and a path $\overline{p}_- : x_{i^-} \rightsquigarrow x_i$ in $\mathrm{FV}(s : A)$, as well as a variable $x_{i^+} \in \mathrm{FV}(a_2)$ and a path $\overline{p}_+ : x_i \rightsquigarrow x_{i^+}$ in $\mathrm{FV}(s : A)$. Together this gives a path $\overline{p}_- * \overline{p}_+ : x_{i^-} \rightsquigarrow x_{i^+}$ in $\mathrm{FV}(s : A)$. Since $x_{i^-}, x_{i^+} \in \mathrm{FV}(A) \subset \mathrm{FV}(t : A)$ and $\Gamma_t \vdash_{\mathrm{ps}}$ by Lemma 2.15, there is also a path $x_{i^-} \rightsquigarrow x_{i^+}$ in $\mathrm{FV}(t : A) = \mathrm{FV}(\Gamma_t)$ by Lemma 1.25. Note that the direction is fixed by Lemma 1.26*(i)* because we already have a path in this direction in Γ namely $\overline{p}_- * \overline{p}_+$. Considering these two paths in Γ, Lemma 1.25 then implies that they must have the same nodes. Thus $x_i \in \mathtt{nodes}(\overline{p}_- * \overline{p}_+) \subset \mathrm{FV}(t : A)$.

(ii) Let $x_{i_s} \in \mathrm{FV}(s)$ be a variable of codimension 0. If $\dim(A) = 0$, then both $s, t : A$ are variables and the statement holds by choosing $x_{i_t} \equiv t$. If $\dim(A) > 0$, then we may write $x_{i_s} : x_l \to x_k$ for some variables $x_l, x_k \in \mathrm{FV}(\Gamma)$ and by Lemma 2.9 we actually have $x_l, x_k \in \mathrm{FV}(s : A)$. Since $\dim(x_l) = \dim(x_k) = \dim(A) - 1$, part (i) it follows that $x_l, x_k \in \mathrm{FV}(t : A)$. Now, by Lemma 2.15, $\Gamma_t \vdash_{\mathrm{ps}}$ so that by Lemma 1.25 there exists a path between x_l and x_k in Γ_t. Since we already have such a path in Γ, namely $(x_{i_s}) : x_l \to x_k$, it follows from Lemma 1.26*(i)* and *(ii)* that the path in Γ_t must have length 1 and have the direction $x_l \rightsquigarrow x_k$. The component of this path is then our candidate for x_{i_t}. Its uniqueness follows from Lemma 1.26*(iv)* applied to Γ_t. ☐

2.2.3 (COH') is admissible in CaTT

We now finally have all the necessary tools available to us, and we are ready to prove the lemma which essentially implies the admissibility of (COH') in `CaTT`.

Lemma 2.17. *Given a derivable type* $\Gamma \vdash s \underset{A}{\to} t$*,*

$$\begin{pmatrix} \Gamma \vdash_{\mathrm{ps}} \quad \textit{and} \\ \\ \dim(\Gamma) < \dim(s \underset{A}{\to} t) \end{pmatrix} \implies \begin{pmatrix} \Gamma_{s \underset{A}{\to} t} \vdash_{\mathrm{ps}} \quad \textit{and} \\ \\ \mathrm{FV}(s : A) = \mathrm{FV}(t : A) \end{pmatrix}$$

In particular, $\Gamma \vdash s \underset{A}{\to} t$ *satisfying* (COH') *implies that* $\Gamma_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$ *satisfies* (COH)*.*

Proof. First of of all, by Lemma 2.15, $\Gamma_s \vdash_{\text{ps}} s$ and $\Gamma_t \vdash_{\text{ps}}$. By definition $\Gamma_{s\to t} :\equiv \Gamma_s \cup \Gamma_t$ and our task is to show that $\Gamma_{s\to t} \vdash_{\text{ps}}$ and that $\text{FV}(s : A) = \text{FV}(t : A)$. In fact it is enough to prove the latter. This is because by definition $\text{FV}(\Gamma_s) = \text{FV}(s : A)$ and $\text{FV}(\Gamma_t) = \text{FV}(t : A)$, so that $\text{FV}(s : A) = \text{FV}(t : A)$ would prove that $\text{FV}(\Gamma_s) \equiv \text{FV}(\Gamma_t)$. This in turns would show that $\Gamma_{s\to t} \equiv \Gamma_t$ so that by using $\Gamma_t \vdash_{\text{ps}}$, we get $\Gamma_{s\to t} \vdash_{\text{ps}}$.

As usual, let us use the notation $\Gamma \equiv (x_i : A_i)_{1\leq i\leq n}$ and let $x_i \in \text{FV}(t : A)$. If $x_i \in \text{FV}(A)$, then we also have $x_i \in \text{FV}(s : A)$. So assume there is a variable $x_i \in \text{FV}(t)\backslash\text{FV}(A)$. Then, by Lemma 2.11, $\dim(x_i) = \dim(t)$ or $\dim(x_i) = \dim(t) - 1$.

Let us begin with the former, i.e. $\dim(x_i) = \dim(t)$. Since, by assumption $\dim(\Gamma) \leq \dim(A) = \dim(t)$, and since $\dim(x_i) \leq \dim(\Gamma)$ we must have $\dim(x_i) = \dim(\Gamma)$. We now induct on the length of Γ. If $\Gamma \equiv x : \texttt{Ob}$, it follows that $t \equiv x \equiv s$ so that $\text{FV}(t : A) = \text{FV}(s : A)$. Otherwise we must have $\dim(x_i) = \dim(\Gamma) > 0$. Thus, we may write $x_i : x_{\partial^- i} \to x_{\partial^+ i}$ for some variables $x_{\partial^- i}, x_{\partial^+ i} \in \text{FV}(t : A)$. Since $\dim(t) = \dim(x_i) > 0$, we may also write $t : a_1 \underset{\partial A}{\to} a_2$ where $a_1 \to a_2 \equiv A$. Then, by Lemma 2.13 there exist variables $x_{i-} \in \text{FV}(a_1 : \partial A)$ and $x_{i+} \in \text{FV}(a_2 : \partial A)$ and paths $\overline{p}^- : x_{i-} \rightsquigarrow x_{\partial^- i}$ and $\overline{p}^+ : x_{\partial^+ i} \rightsquigarrow x_{i+}$. In particular, $x_{i-}, x_{i+} \in \text{FV}(A) \subset \text{FV}(s : A)$. Said differently, $x_{i-}, x_{i+} \in \text{FV}(\Gamma_s)$. Since $x_{i-} \parallel x_{i+}$ and $\Gamma_s \vdash_{\text{ps}}$, Lemma 1.26 then implies that there must be a path $x_{i-} \rightsquigarrow x_{i+}$ in Γ_s. But $\Gamma_s \subset \Gamma$ and we already have such a path in Γ obtained by the concatenation $\overline{p}^- * (x_i) * \overline{p}^+ : x_{i-} \rightsquigarrow x_{i+}$. Since, $\dim(x_{i-}) = \dim(x_{i+}) = \dim(\Gamma) - 1$, it follows from Lemma 1.26*(iii)* that there is at most one path $x_{i-} \rightsquigarrow x_{i+}$. Thus, $\overline{p}^- * (x_i) * \overline{p}^+$ must be also the path in Γ_s and so Γ_s must contain x_i, i.e. $x_i \in \text{FV}(s : A)$. We have shown that $\text{FV}(t : A) \subset \text{FV}(s : A)$. By symmetry $\text{FV}(s : A) = \text{FV}(t : A)$ and this takes care of the case $\dim(x_i) = \dim(t)$.

Consider now the case $\dim(x_i) = \dim(t) - 1$. Then first of all $\dim(t) > 0$, since the only terms of dimension 0 are variables which therefore as terms cannot have free variables of lower dimension. Thus, we may write $t : a_1 \underset{\partial A}{\to} a_2$. By Lemma 2.13 there are variables $x_{i^-} \in \text{FV}(a_1)$ and $x_{i^+} \in \text{FV}(a_2)$ together with two paths $\overline{q}^- : x_{i^-} \rightsquigarrow x_i$ and $\overline{q}^+ : x_i \rightsquigarrow x_{i^+}$ in $\text{FV}(t : A)$. In particular $x_{i^-}, x_{i^+} \in \text{FV}(A) \subset \text{FV}(s : A)$. Now, since $\dim(x_i) \leq \dim(\Gamma) \leq \dim(t) = \dim(x_i) + 1$, there are two cases to consider. The first case is that in which $\dim(x_i) = \dim(\Gamma)$. Then, by Lemma 1.26*(iv)* it follows that $x_{i^-} \equiv x_i \equiv x_{i^+}$. In particular $x_i \in \text{FV}(s : A)$ so that $\text{FV}(t : A) \subset \text{FV}(s : A)$. As for the second case, i.e. $\dim(x_i) = \dim(\Gamma) - 1$, we may argue in a similar fashion as in the previous paragraph. Since $x_{i^-}, x_{i^+} \in \text{FV}(s : A) = \text{FV}(\Gamma_s)$, by Lemma 1.25, there must be a path $\overline{r} : x_{i^-} \rightsquigarrow x_{i^+}$ in $\text{FV}(s : A) \subset \text{FV}(\Gamma)$. But we already have such a path in Γ, namely $\overline{q}^- * \overline{q}^+$. By Lemma 1.25 again, $\overline{r}$ and $\overline{q}^- * \overline{q}^+$ must have the same nodes. Thus, $x_i \in \texttt{nodes}(\overline{r}) \subset \text{FV}(s : A)$ so that again $\text{FV}(t : A) \subset \text{FV}(s : A)$. By a symmetrical argument $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$. □

Theorem 2.18. (COH') *is admissible in* `CaTT`*.*

Proof. Assume we are given a type $\Gamma \vdash s \underset{A}{\to} t$ satisfying all the premises of the rule (COH'), i.e. $\Gamma \vdash_{\mathrm{ps}}$ and $\dim(\Gamma) < \dim(s \underset{A}{\to} t)$. Given a context morphism $\Delta \vdash \gamma : \Gamma$, our task is to construct a term $\Delta \vdash \mathtt{coh}'_{\Gamma, s \to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$. Here we use the notation $\mathtt{coh}'$ to distinguish this from those produced by the (COH) rule.

By Lemma 2.15 we know that $\Gamma_{s\to t} \vdash_{\mathrm{ps}}$ and that $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$. Thus $\mathrm{FV}(\Gamma_{t\to s}) = \mathrm{FV}(s \underset{A}{\to} t) = \mathrm{FV}(s : A) = \mathrm{FV}(t : A)$. The context morphism $\Delta \vdash \gamma : \Gamma$ from (COH') restricts to $\Delta \vdash \gamma_{s\to t} : \Gamma_{s\to t}$ so that we may apply (COH) to $\Gamma_{s\to t} \vdash s \underset{A}{\to} t$ and $\Delta \vdash \gamma_{s\to t} : \Gamma_{s\to t}$ and produce a coherence $\Delta \vdash \mathtt{coh}_{\Gamma_{s\to t}, s\to t}[\gamma_{s\to t}] : s[\gamma_{s\to t}] \underset{A[\gamma_{s\to t}]}{\to} t[\gamma_{s\to t}]$. But, since $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma_{s\to t}) = \mathrm{FV}(t : A)$, we actually have $A[\gamma_{s\to t}] \equiv A[\gamma]$ and $s[\gamma_{s\to t}] \equiv s[\gamma] : A[\gamma]$ as well as $t[\gamma_{s\to t}] \equiv t[\gamma] : A$. Thus, we have actually produced a term $\Delta \vdash \mathtt{coh}_{\Gamma_{s\to t}, s\to t}[\gamma_{s\to t}] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$ and we define $\mathtt{coh}'_{\Gamma, s\to t}[\gamma] \equiv \mathtt{coh}_{\Gamma_{s\to t}, s\to t}[\gamma_{s\to t}]$. □

2.2.4 (OP') is admissible in `CaTT`

The following lemma is the key result required for proving the admissibility of (OP) in `CaTT`.

Lemma 2.19. *Let* $\Gamma \vdash s \underset{A}{\to} t$ *be a derivable type satisfying the premises of* (OP')*, i.e.*

$$\Gamma \vdash_{\mathrm{ps}}, \qquad \dim(\Gamma) = \dim(s \underset{A}{\to} t), \qquad \partial^-\Gamma \vdash s : A \qquad \textit{and} \qquad \partial^+\Gamma \vdash t : A.$$

(i) If $\mathrm{FV}(s : A) \neq \mathrm{FV}(t : A)$*, then there is a ps-context* $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash_{\mathrm{ps}}$ *such that*

- $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$ *is derivable;*
- $\mathrm{FV}(s : A) = \mathrm{FV}(\partial^-\overline{\Gamma}_{s \underset{A}{\to} t}) \neq \mathrm{FV}(\partial^+\overline{\Gamma}_{s \underset{A}{\to} t}) = \mathrm{FV}(t : A)$

In particular, if $\mathrm{FV}(s : A) \neq \mathrm{FV}(t : A)$*, then* $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$ *satisfies* (OP)*.*

(ii) If $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$*, then*

- $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma_{s \underset{A}{\to} t}) = \mathrm{FV}(t : A)$.

In particular, if $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$*, then* $\Gamma_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$ *satisfies* (COH)*.*

Proof. (i) By Lemma 2.15 we know that $\Gamma_s \vdash_{\text{ps}}$ and $\Gamma_t \vdash_{\text{ps}}$. Thinking of the rules for ps-contexts, by which we build larger ps-context by adjoining pairs of variables $x : T, f : x \to y$, when introducing a variable of maximal dimension, this must always be the second variable in the given pair, i.e. f, in the example. Let us begin by exposing the variables of maximal dimension in these ps-contexts. Thus, we shall write, Γ_s in the form

$$\begin{aligned} \Gamma_s \equiv\ & \Gamma_1,\ x_{i_1} : A_{i_1} \\ & \Gamma_2,\ x_{i_2} : A_{i_2}, \\ & \dots \\ & \Gamma_{\nu+1}. \end{aligned} \tag{1.15}$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_\nu}$ are all the variables of dimension $\dim(A)$. Thus, the lists Γ_k, for $1 \leq k \leq \nu$ contain all the variables of dimension up to $\dim(A) - 1$. As explained, because of the specific structure of ps-contexts, each top-dimensional variable x_{i_1} must be the second variable in its corresponding pair. (This is equivalent to saying that the length of the Γ_k is even for $1 \leq k \leq \nu$.)

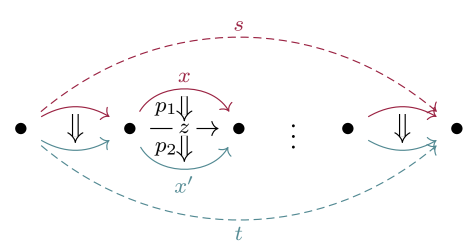
By Lemma 2.16, we know that $\mathrm{FV}(\Gamma_k) \subset \mathrm{FV}(\Gamma_t)$ for all k. In fact, by the same lemma, Γ_s and Γ_t differ only in the variables of dimension $\dim(A)$. But again, by Lemma 2.16, there is a bijective correspondence between the variables of codimension 0 in $\mathrm{FV}(s)$ and $\mathrm{FV}(t)$. Moreover, the pairs of corresponding variables are parallel. So, for each codimension 0 variable $x_{i_k} \in \mathrm{FV}(\Gamma_s)$ there is a parallel codimension 0 variable $x'_{i_k} \in \mathrm{FV}(\Gamma_t)$. So, in total, $\mathrm{FV}(\Gamma_t)$ is the same as $\mathrm{FV}(\Gamma_s)$ with each x_{i_k} replaced by a corresponding parallel x'_{i_k}.

Assume we are given a variable $x_{i_k} \in \mathrm{FV}(\Gamma_s)$ of codimension 0 and let $x'_{i_k} \in \mathrm{FV}(\Gamma_t)$ be the corresponding codimension 0 variable in Γ_t (Lemma 2.16). In particular $x_{i_k} \parallel x'_{i_k}$. Starting with x_{i_k} which has dimension $\dim(x_{i_k}) = \dim(s) = \dim(\Gamma) - 1$, by Lemma 2.13 there is a variable $x^+_{i_k} \in \mathrm{FV}(\partial^+\Gamma)$ and a path $\overline{p}_k : x_{i_k} \rightsquigarrow x^+_{i_k}$. In particular $x^+_{i_k}$ has dimension $\dim(\Gamma) - 1$. But, recalling that $\partial^+\Gamma \vdash t : A$, because of which we have $\mathrm{FV}(\Gamma_t) = \mathrm{FV}(t : A) \subset \mathrm{FV}(\partial^+\Gamma)$, we already have a variable of dimension $\dim(\Gamma) - 1$ in $\partial^+\Gamma$ which is parallel to x_{i_k}, namely x'_{i_k}. Since $x'_{i_k}, x^+ i_k \in \mathrm{FV}(\partial^+\Gamma)$ both have dimension $\dim(\Gamma) - 1 = \dim(\partial^+\Gamma)$, it follows from Lemma 1.26*(iv)* that $x'_{i_k} \equiv x^+_{i_k}$. So we actually have a path $\overline{p}_k : x_{i_k} \rightsquigarrow x'_{i_k}$, one for every k. Moreover, since the dimension of the nodes is $\dim(\Gamma) - 1$, by Lemma 1.26*(iii)* this path is unique. It is also important that the only variable shared by the path $\overline{p}$ and Γ_s is x_{i_k}. For the components this follows from Lemma 2.1, by noticing that the components have dimension $\dim(\Gamma) = \dim(\Gamma_s) - 1$. For the remaining nodes, this follows from Lemma 1.26*(iv)*, since we already know that $x_{i_k} \in \mathrm{FV}(\Gamma_s)$.

We are now ready to construct $\overline{\Gamma}_A$. It is essentially given by Γ_s where, at the end of each line, we insert the variables of the unique path $\overline{p}_k : x_{i_k} \rightsquigarrow x'_{i_k}$. To make this explicit, let us denote the nodes of the path $\overline{p}_k$ by $\texttt{nodes}(\overline{p}_k) = \{x_{i_k}, z_{k,1}, z_{k,2}, \dots, z_{k,l_k-1}, x'_{i_k}\}$ and the components by $\texttt{comp}(\overline{p}_k) = \{p_{k,1}, p_{k,2}, \cdots, p_{k,l_k}\}$. We define

$$\begin{aligned}\overline{\Gamma}_{s\to t} :\equiv \Gamma_1, \ & \overbrace{x_{i_1} : A_{i_1}, \ z_{1,1} : A_{i_1}, \ p_{1,1} : x_{i_1} \to z_{1,1}, \cdots, x'_{i_1} : A_{i_1}, \ p_{1,l_1} : z_{1,l_1-1} \to x'_{i_1}}^{\overline{p}_1}, \\ \Gamma_2, \ & \overbrace{x_{i_2} : A_{i_2}, \ z_{2,1} : A_{i_2}, \ p_{2,1} : x_{i_2} \to z_{2,2}, \cdots, x'_{i_2} : A_{i_2}, \ p_{2,l_2} : z_{2,l_2-1} \to x'_{i_2}}^{\overline{p}_2}, \\ \dots & \\ \Gamma_{\nu+1}. & \end{aligned} \tag{1.16}$$

By construction, this list of variables has the structure of a ps-context. To see this, let us explain why this is true for the first line in $\overline{\Gamma}_A$, since the argument for the others are similar. First of all, since $\Gamma_s \vdash_{\text{ps}}$, we must have $\Gamma_1, x_{i_1} : A_{i_1} \vdash_{\text{ps}}$. And indeed, by the rule (PSE) we may directly attach the next two cells $z_{1,1} : A_{i_1}, \ p_{1,1} : x_{i_1} \to z_{1,1}$. For every subsequent cell, we first apply (PSD) once, and then (PSE). As for the connection to Γ_2, since $\Gamma_s \equiv \Gamma_1, \ x_{i_1} : A_{i_1}, \ \Gamma_2, \ \dots$ is a valid ps-context, by Lemma 1.16 the first variable of Γ_2 has the same type as $(\partial^+)^q(x_{i_k})$ for some $q \in \mathbb{N}$. But then it also has the same type as $(\partial^+)^{q+1}(p_{1,l_1})$, the last element of the first line of $\overline{\Gamma}_{s \underset{A}{\to} t}$. And so we may continue the first line of equation (1.16) as we would have done in Γ_s, but by including one additional application of the rule (PSD) and then attaching Γ_2. Pictorially, what have been doing may be visualized as in the following diagram



where we have suppressed all subscripts. Here s is construct using only the top row of 1-morphisms (red, solid arrows) and t is constructed using the bottom row of 1-morphisms (blue, solid arrows). In a certain sense, what we are doing is filling out the space between s and t.

Since $\Gamma_s \vdash s : A$ and $\Gamma_t \vdash t : A$ and by construction $\Gamma_s \subset \overline{\Gamma}_{s \underset{A}{\to} t}$ and $\Gamma_t \subset \overline{\Gamma}_{s \underset{A}{\to} t}$, by weakening we get $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash s : A$ and $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash t : A$. We therefore have $\overline{\Gamma}_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$, as desired.

Now, crucially, Γ_s contains at least one variable of dimension $\dim(A)$. To see this, notice first that the hypothesis implies that either $\mathrm{FV}(s)\backslash\mathrm{FV}(A) \neq \emptyset$ or that $\mathrm{FV}(t)\backslash\mathrm{FV}(A) \neq \emptyset$. Otherwise we would have $\mathrm{FV}(s : A) = \mathrm{FV}(A) = \mathrm{FV}(t : A)$. Let us begin with the assumption that $\mathrm{FV}(s)\backslash\mathrm{FV}(A) = \emptyset$. Then, by Lemma 2.2 $\mathrm{FV}(s)$ contains a variable of codimension 0. So indeed, Γ_s contains at least one variable of dimension $\dim(A)$. As a consequence $\dim(\overline{\Gamma}_{s \underset{A}{\to} t}) = \dim(A)$.

It follows that $\partial^- \overline{\Gamma}_{s \underset{A}{\to} t} \equiv \partial^+_{\dim(A)} \overline{\Gamma}_{s \underset{A}{\to} t}$. Applying now the rules for ps-context directly to $\overline{\Gamma}_{s \underset{A}{\to} t}$, one verifies that $\partial^- \overline{\Gamma}_{s \underset{A}{\to} t} \equiv \Gamma_s$ and $\partial^+ \overline{\Gamma}_{s \underset{A}{\to} t} \equiv \Gamma_t$. Since by assumption $\mathrm{FV}(s : A) \neq \mathrm{FV}(t : A)$ we have $\mathrm{FV}(s : A) = \mathrm{FV}(\partial^- \overline{\Gamma}_{s \underset{A}{\to} t}) \neq \mathrm{FV}(\partial^+ \overline{\Gamma}_{s \underset{A}{\to} t}) = \mathrm{FV}(t : A)$, as desired.

(ii) By definition (see Lemma 2.8) $\Gamma_{s \underset{A}{\to} t} \equiv \Gamma_s \cup \Gamma_t$ and $\Gamma_{s \underset{A}{\to} t} \vdash s \underset{A}{\to} t$ is derivable. Since $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$, it follows that $\Gamma_{s \underset{A}{\to} t} \equiv \Gamma_s \equiv \Gamma_t$. In particular, since $\Gamma_s \vdash_{\mathrm{ps}}$, by Lemma 2.15 we also have $\Gamma_{s \underset{A}{\to} t} \vdash_{\mathrm{ps}}$. Moreover $\mathrm{FV}(s : A) = \mathrm{FV}(\Gamma_{s \underset{A}{\to} t}) = \mathrm{FV}(t : A)$, as desired. □

Theorem 2.20. (OP') *is admissible in* `CaTT`.

Proof. Assume we are given a type $\Gamma \vdash s \underset{A}{\to} t$ along with a substitution $\Delta \vdash \gamma : \Gamma$, such that $\Gamma \vdash_{\mathrm{ps}}$ and $\dim(\Gamma) = \dim(s \underset{A}{\to} t)$. Our task is to construct a term $\Delta \vdash \mathtt{op}'_{\Gamma, s \to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$. Here we use the notation $\mathtt{op}'_{\Gamma, s \underset{A}{\to} t}$ for the term constructed by the admissible rule (OP') in order to distinguish it from the term produced by (OP).

By Lemma 2.19, if we have $\mathrm{FV}(s : A) \neq \mathrm{FV}(t : A)$, then we may apply (OP) to $\overline{\Gamma}_{s \to t} \vdash s \to t$ and the restricted context morphism $\Delta \vdash \overline{\gamma}_{s \to t} : \overline{\Gamma}_{s \to t}$, to produce a term $\Delta \vdash \mathtt{op}_{\overline{\Gamma}_{s \to t}, s \to t}[\overline{\gamma}_{s \to t}] : s[\overline{\gamma}_{s \to t}] \to t[\overline{\gamma}_{s \to t}]$. Now, since $\overline{\Gamma}_{s \to t} \subset \Gamma$, it follows that $A[\overline{\gamma}_{s \to t}] \equiv A[\gamma]$ and $s[\overline{\gamma}_{s \to t}] \equiv s[\gamma] : A[\gamma]$ and $t[\overline{\gamma}_{s \to t}] \equiv t[\gamma] : A[\gamma]$. So we actually produced a term $\Delta \vdash \mathtt{op}_{\overline{\Gamma}_{s \to t}, s \to t}[\overline{\gamma}_{s \to t}] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$ and we define $\mathtt{op}'_{\Gamma, s \to t}[\gamma] :\equiv \mathtt{op}_{\overline{\Gamma}_{s \to t}, s \to t}[\overline{\gamma}_{s \to t}]$.

If, on the other hand, we have $\mathrm{FV}(s : A) = \mathrm{FV}(t : A)$, then Lemma 2.19 and a reasoning similar to that in the above paragraph allow us to construct a term $\Delta \vdash \mathtt{coh}_{\overline{\Gamma}_{s \to t}, s \to t}[\overline{\gamma}_{s \to t}] : s[\gamma] \to t[\gamma]$ and we define $\mathtt{op}'_{\Gamma, s \to t}[\gamma] :\equiv \mathtt{coh}_{\overline{\Gamma}_{s \to t}, s \to t}[\overline{\gamma}_{s \to t}]$. □

2.3 (OP) and (COH) are Admissible in `CaTT'`

In this section we show that (OP) and (COH) are admissible in `CaTT'`. We begin with some lemmas, most of which are necessary for the derivability of (OP'). Most of the statements are also lemmas which we have already proven in CaTT.

2.3.1 Lemmas on Dimension and the Free Variables

The following lemma has already been proven in `CaTT` (see Lemma 2.1). We now show that it also holds in `CaTT'`. From this lemma we can directly deduce the admissibility of (COH) in `CaTT'`. For (OP') on the other hand we will need to work a bit harder. As all the work in this section is performed exclusively in `CaTT'` and no confusion can arise, we drop the prime and write `op` and `coh` instead of $\mathtt{op}'$ and $\mathtt{coh}'$.

Lemma 2.21. *The following two statements hold in* `CaTT`'

(i) Given a judgment $\Gamma \vdash A$ *we have*

$$x_i : \overline{\mathrm{FV}}(A) \implies \dim(A_i) < \dim(A)$$

(ii) Given a judgment $\Gamma \vdash t : A$ *we have*

$$x_i : \overline{\mathrm{FV}}(t) \implies \dim(x_i) \leq \dim(t)$$

Proof. As already done in `CaTT` (Lemma 2.1), we prove all three statements simultaneously. In fact, most of the proof of Lemma 2.1 carries over. We only need to replace the clauses for (OP) and (COH) with clauses for (OP') and (COH').

(OP') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \partial^-\Gamma \vdash s : A \qquad \partial^+\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(OP')}$$

where $\dim(\Gamma) = \dim(s \underset{A}{\to} t)$. Now, let $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma])$ where by definition

$$\overline{\mathrm{FV}}(\mathtt{op}_{\gamma, s \underset{A}{\to} t}[\gamma]) := \overline{\mathrm{FV}}\Big(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]\Big) \cup \bigcup_{\substack{\overline{p} : y_{j_s} \rightsquigarrow y_{j_t} \\ \mathrm{codim}_s(x_{i_s}) = 0 \\ \mathrm{codim}_t(x_{i_t}) = 0}} \overline{\mathrm{FV}}(\overline{p}[\gamma]).$$

If $y_j \in \overline{\mathrm{FV}}(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]) \subset \overline{\mathrm{FV}}(\gamma)$, then $y_j \in \overline{\mathrm{FV}}(\gamma_i)$ for some $x_i \in \overline{\mathrm{FV}}(\Gamma)$ and by the inductive hypothesis for terms $\dim(y_j) \leq \dim(\gamma_i) = \dim(x_i)$.

Thus,

$$\begin{aligned}
\dim(y_j) &\leq \dim(x_i) \\
&\leq \dim(\Gamma) \\
&= \dim(s \underset{A}{\to} t), \qquad \text{by hypothesis} \\
&= \dim(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]) \\
&= \dim(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma]).
\end{aligned}$$

Otherwise, we must have $y_j \in \overline{\mathrm{FV}}(\overline{p}[\gamma])$ for some path $\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}$, where $\mathrm{codim}_s(x_{i_s}) = 0$ and $\mathrm{codim}_t(x_{i_t}) = 0$. By Definition 2.5 this means $y_j \in \overline{\mathrm{FV}}(p_a[\gamma] : T_{p_a}[\gamma])$ for some $p_a \in \mathtt{comp}(\overline{p})$. But

$$\begin{aligned}
\dim(y_j) &\leq \dim(p_a[\gamma]), \qquad \text{by the inductive hypothesis} \\
&= \dim(p_a) \\
&= \dim(s \to t) \\
&= \dim(s[\gamma] \to t[\gamma]) \\
&= \dim(\mathtt{op}_{\Gamma, s \to t}[\gamma]).
\end{aligned}$$

(COH') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, A}[\gamma] : A[\gamma]} \text{(COH')} \qquad \dim(\Gamma) < \dim(A)$$

Let $y_j \in \overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,A}[\gamma]) = \overline{\mathrm{FV}}(A[\gamma]) \subset \overline{\mathrm{FV}}(\gamma)$. Thus, there must exist a $1 \leq i \leq n$ such that $y_j \in \overline{\mathrm{FV}}(\gamma_i)$ and by the inductive hypothesis for terms we have $\dim(y_j) \leq \dim(\gamma_i) = \dim(x_i)$. Thus,

$$\begin{aligned}
\dim(y_j) &\leq \dim(x_i) \\
&\leq \dim(\Gamma) \\
&< \dim(A), \qquad \text{by hypothesis} \\
&= \dim(A[\gamma]).
\end{aligned}$$

□

Using Lemma 2.21 we can deduce the derivability of the rule (COH) in `CaTT'` (see Theorem 2.27).

For the admissibility of (OP) we again need to show that types satisfying the premises of (OP) must have the same dimension as the context (see Lemma 2.2), that is, we need to reprove Lemma 2.21 in `CaTT`'. To do this we need to work out some additional lemmas about free variables which have already been proven in `CaTT`.

Lemma 2.22. *The following statements hold in* `CaTT'`

(i) Given a type $\Gamma \vdash A$

$$x_i \in \overline{\mathrm{FV}}(A) \implies \overline{\mathrm{FV}}(A_i) \subset \overline{\mathrm{FV}}(A)$$

(ii) Given a term $\Gamma \vdash t : A$

$$x_i \in \overline{\mathrm{FV}}(t) \implies \overline{\mathrm{FV}}(A_i) \subset \overline{\mathrm{FV}}(t : A)$$

Proof. The proof is given by a mutual induction on the two statements.

(Ob) Consider the rule

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathtt{Ob}} \text{ (Ob)}$$

By definition $\overline{\mathrm{FV}}(\mathtt{Ob}) = \emptyset$ so the statement is vacuously true.

($\to$) Consider the rule

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : A}{\Gamma \vdash t \underset{A}{\to} s} \; (\to)$$

By definition $\overline{\mathrm{FV}}(t \underset{A}{\to} s) = \overline{\mathrm{FV}}(t) \cup \overline{\mathrm{FV}}(s) \cup \overline{\mathrm{FV}}(A)$. Let $x_i \in \overline{\mathrm{FV}}(t \underset{A}{\to} s)$. If $x_i \in \overline{\mathrm{FV}}(t)$, then by induction

$$\overline{\mathrm{FV}}(A_i) \overset{\text{i.h}}{\subset} \overline{\mathrm{FV}}(t : A) = \overline{\mathrm{FV}}(t) \cup \overline{\mathrm{FV}}(A) \subset \overline{\mathrm{FV}}(t \underset{A}{\to} s).$$

If, on the other hand, $x_i \in \overline{\mathrm{FV}}(A)$, then again by induction

$$\overline{\mathrm{FV}}(A_i) \overset{\text{i.h.}}{\subset} \overline{\mathrm{FV}}(A) \subset \overline{\mathrm{FV}}(t \underset{A}{\to} s).$$

(VAR) Consider the rule

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash x : A} \text{ (VAR)}$$

By definition $\overline{\mathrm{FV}}(x) = \{x\}$. So $x_i \in \overline{\mathrm{FV}}(x)$ implies $A_i \equiv A$ and $x_i \equiv A$. So our task becomes to show that $\overline{\mathrm{FV}}(A) \subset \overline{\mathrm{FV}}(t : A)$, which holds by definition of the latter.

(OP') Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \qquad \partial^- \Gamma \vdash t : A \qquad \partial^+ \Gamma \vdash s : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : t[\gamma] \underset{A[\gamma]}{\to} s[\gamma]} \text{ (OP')}$$

where $\dim(\Gamma) = \dim(s \to t)$. We need to show that for $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma])$ we have $\overline{\mathrm{FV}}(B_j) \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \to t[\gamma])$.

By definition

$$\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]) \cup \bigcup_{\substack{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t} \\ \cdots}} \overline{\mathrm{FV}}(\overline{p}[\gamma]).$$

Assume first that $y_j \in \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$. In that case $y_j \in \overline{\mathrm{FV}}(\gamma_i)$ for some $x_i \in \overline{\mathrm{FV}}(s \to t)$. Now, by the inductive hypothesis for types, $\overline{\mathrm{FV}}(A_i) \subset \overline{\mathrm{FV}}(s \to t)$. Thus,

$$\begin{aligned} \overline{\mathrm{FV}}(B_j) &\subset \overline{\mathrm{FV}}(\gamma_i : A_i[\gamma]), && \text{by the inductive hypothesis for terms} \\ &\subset \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) \\ &\subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma]), \end{aligned}$$

where in the second line we have used Lemma 1.11 and Remark **??**.

This means, given a $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \to s}[\gamma])$ we have $y_j \in \overline{\mathrm{FV}}(p_a[\gamma] : T_{p_a}[\gamma])$ for some component $p_a \in \mathtt{comp}(\overline{p})$ of some path $\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}$ where $\mathrm{codim}_s(x_{i_s}) = 0$ and $\mathrm{codim}_t(x_{i_t}) = 0$. So $y_j \in \overline{\mathrm{FV}}(\gamma_i)$ for some $x_i \in \overline{\mathrm{FV}}(p_a : T_a)$ where p_a is as before. By the inductive hypothesis for terms $\overline{\mathrm{FV}}(A_i) \subset \overline{\mathrm{FV}}(p_a : T_a)$. But by definition $\overline{\mathrm{FV}}(p_a[\gamma] : T_{p_a}[\gamma]) \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \to s}[\gamma])$ and so

$$\begin{aligned} \overline{\mathrm{FV}}(B_j) &\subset \overline{\mathrm{FV}}(\gamma_i : A_i[\gamma]), && \text{by the inductive hypothesis for terms} \\ &\subset \overline{\mathrm{FV}}(p_a[\gamma] : T_{p_a}[\gamma]) \\ &\subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma]). \end{aligned}$$

(COH') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash t \underset{A}{\to} s \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, t \underset{A}{\to} s}[\gamma] : t[\gamma] \underset{A[\gamma]}{\to} s[\gamma]} \; (\text{COH'})$$

By definition $\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma, A}[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$ and a proof for this situation is already contained in that for the rule (OP'). □

The next lemma collects together a number of statements about the free variables of terms and types, most of which have been proven in CaTT (see Lemma 2.11, Lemma 2.13, Lemma 2.14 and Lemma 2.16). Contrary to CaTT, here all the statements (at least with our current proof) are interrelated and so must be proven simultaneously by a mutual induction. The new statement about terms which was not already proven in CaTT involves parallel variables.

Lemma 2.23. *The following statements hold in* `CaTT'`*.*

(i) Let $\Gamma \vdash A$ *be a type in a ps-context* $\Gamma \vdash_{\text{ps}}$ *and let* $x_k \in \overline{\text{FV}}(A)$*. Then* $\dim(A) > 0$ *and*

(1) If $\dim(x_k) \leq \dim(A) - 2$ *then* $x_k \in \overline{\text{FV}}(a_1 : \partial A) \cap \overline{\text{FV}}(a_2 : \partial A)$*;*

(2) If $\dim(x_k) = \dim(A) - 1$ *then:*

if $x_k \in \overline{\text{FV}}(a_1)$ *then there exists a* $x_l \in \overline{\text{FV}}(a_2)$ *such that* $x_k \parallel x_l$*;*
if $x_k \in \overline{\text{FV}}(a_2)$ *then there exists a* $x_l \in \overline{\text{FV}}(a_1)$ *such that* $x_k \parallel x_l$*;*

where $A \equiv (a_1 \underset{\partial A}{\to} a_2)$*.*

(ii) Let $\Gamma \vdash t : A$ *be a term in a globular context* Γ *and let* $x_k \in \overline{\text{FV}}(t : A)$*.*

(1) If $\dim(x_k) \leq \dim(t) - 2$ *then* $x_k \in \overline{\text{FV}}(\partial^- t : \partial A) \cap \overline{\text{FV}}(\partial^+ t : \partial A)$

(2) If $\dim(x_k) = \dim(t) - 1$ *then:*

there exists a $x_{k^-} \in \overline{\text{FV}}(\partial^- t)$ *and a path* $x_{k^-} \rightsquigarrow x_k$ *in* $\overline{\text{FV}}(t : A)$*;*
there exists a $x_{k^+} \in \overline{\text{FV}}(\partial^+ t)$ *and a path* $x_k \rightsquigarrow x_{k^+}$ *in* $\overline{\text{FV}}(t : A)$*;*

where $A \equiv (\partial^- t \underset{\partial A}{\to} \partial^+ t)$*.*

Given a further variable $x_l \in \overline{\text{FV}}(t : A)$ *such that* $x_l \parallel x_k$*, assuming* $\Gamma \vdash_{\text{ps}}$ *then:*

(3) if $\dim(x_k) = \dim(t) - 1$*, then there exists a path between* x_k *and* x_l *in* $\overline{\text{FV}}(t : A)$*;*

(4) if $\dim(x_k) = \dim(t)$*, then* $x_k \equiv x_l$*.*

(iii) Let $\Delta \vdash \gamma : \Gamma$ *be a context morphisms with* $\Delta \vdash_{\text{ps}}$ *and* $\Gamma \vdash_{\text{ps}}$*. Let* $x_{k_1}, x_{k_2} \in \overline{\text{FV}}(\Gamma)$ *be two variables such that* $\dim(x_k) = \dim(x_l)$*. Let* $y_{j_1} \in \overline{\text{FV}}(\gamma_{k_1})$ *and* $y_{j_2} \in \overline{\text{FV}}(\gamma_{k_2})$ *be two variables of codimension 0. Then*

$$y_{j_1} \parallel y_{j_2} \implies x_{k_1} \parallel x_{k_2}.$$

Proof. We prove all statements simultaneously by a mutual induction on types, terms and context morphisms.

(Ob) Consider the rule

$$\frac{\Gamma \vdash}{\Gamma \vdash \texttt{Ob}} \text{ (Ob)}$$

By definition $\overline{\text{FV}}(\texttt{Ob}) = \emptyset$ so there is nothing to check.

$(\to)$ Consider the rule

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash s \underset{A}{\to} t} \ (\to)$$

where $\Gamma \vdash_{\mathrm{ps}}$. Let $x_k \in \overline{\mathrm{FV}}(s \to t)$. Since $\overline{\mathrm{FV}}(s \to t) = \overline{\mathrm{FV}}(s) \cup \overline{\mathrm{FV}}(t) \cup \overline{\mathrm{FV}}(A)$ we either have $x_k \in \overline{\mathrm{FV}}(s : A)$ or $x_k \in \overline{\mathrm{FV}}(t : A)$. By symmetry it suffices to consider the case $x_k \in \overline{\mathrm{FV}}(s : A)$.

We first deal with part (1). So assume first that $\dim(x_k) \leq \dim(s \to t) - 2 = \dim(A) - 1$. Then, by the inductive hypothesis for terms we may write $A \equiv a_1 \to a_2$ and there exist variables $x_{k^-} \in \overline{\mathrm{FV}}(a_1 : \partial A)$ and $x_{k^+} \in \overline{\mathrm{FV}}(a_2 : \partial A)$ as well as paths

$$x_{k^-} \rightsquigarrow x_k \rightsquigarrow x_{k^+}$$

in $\overline{\mathrm{FV}}(s : A)$. But $x_{k^-}, x_{k^+} \in \overline{\mathrm{FV}}(t : A)$ so by the inductive hypothesis for terms there exists a path $x_{k^-} \rightsquigarrow x_{k^+}$ in $\overline{\mathrm{FV}}(t : A)$ (here the direction of the path is determined by the fact that we already have such a path and Lemma 1.26). By Lemma 1.25 the two paths have the same nodes. Thus $x_k \in \overline{\mathrm{FV}}(t : A)$.

As for part (2), assume now that $\dim(x_k) = \dim(s \to t) - 1 = \dim(A)$. In this case we must necessarily have $x_k \in \overline{\mathrm{FV}}(s)$ by Lemma 2.21. So $\mathrm{codim}_s(x_k) = 0$.

If $\dim(A) = 0$, then, since the only variables of dimension 0 are variables, it follows that $x_k \equiv s$. We may also choose $x_l \equiv t$ which satisfies the desired properties.

If on the other hand, $\dim(A) > 0$, then we may write $A \equiv (a_1 \to a_2)$ and $x_k : x_{\partial^- k} \to x_{\partial^+ k}$. By the inductive hypothesis for terms there exists two terms $x_{k^-} \in \overline{\mathrm{FV}}(a_1)$ and $x_{k^+} \in \overline{\mathrm{FV}}(a_2)$ and two paths

$$x_{k^-} \rightsquigarrow x_{\partial^- k} \xrightarrow{x_k} x_{\partial^+ k} \rightsquigarrow x_{k^+}$$

in $\overline{\mathrm{FV}}(s : A)$. But $x_{k^-}, x_{k^+} \in \overline{\mathrm{FV}}(A) \subset \overline{\mathrm{FV}}(t : A)$, so again, by an argument completely analogous to that of part (1) there exists a path $x_{k^-} \rightsquigarrow x_{k^+}$ in $\overline{\mathrm{FV}}(t : A)$ with the same nodes. In particular $\overline{\mathrm{FV}}(t : A)$ contains a component $x_l : x_{\partial^- k} \to x_{\partial^+ k}$. By Lemma 2.21 $x_l \in \overline{\mathrm{FV}}(t)$ and this variable satisfies the desired properties.

(VAR) Consider the rule

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A} \ (\mathrm{VAR})$$

The (VAR) clauses of the proofs of Lemma 2.11 and Lemma 2.13 in `CaTT` carry over and take care of parts (1) and (2). As for part (3) if we write

$x : \partial^- x \to \partial^+ x$ where $\partial^- x, \partial^+ x$ are again variables since Γ a ps-context and thus globular, then by Lemma 2.21, $\partial^- x, \partial^+ x$ are the only free variables of $x : A$ of dimension $\dim(x) - 1$. For the pairs $(\partial^- x, \partial^- x)$ and $(\partial^+ x, \partial^+ x)$ we may choose the empty path, while for $(\partial^- x, \partial^+ x)$ we can take the path $(x) : \partial^- x \rightsquigarrow \partial^+ x$. Part (4) follows from the fact that $x \in \overline{\mathrm{FV}}(x : A)$ is the only variable of codimension 0 in $\overline{\mathrm{FV}}(x : A)$.

(OP') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \quad \partial^- \Gamma \vdash s : A \quad \partial^+ \Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{ (OP')}$$

where $\dim(\Gamma) = \dim(s \to t)$. So assume $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma])$ where by definition

$$\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \underset{A}{\to} t}[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) \cup \left(\bigcup_{\substack{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t} \\ \mathrm{codim}_s(x_{i_s}) = 0 \\ \mathrm{codim}_t(x_{i_t}) = 0}} \overline{\mathrm{FV}}(\overline{p}[\gamma]) \right).$$

We will work on parts (1) and (2) simultaneously. Consider first the situation in which $y_j \in \bigcup_{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t} \dots} \overline{\mathrm{FV}}(\overline{p}[\gamma])$. This means that there exists a path $\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}$ with $x_{i_s} \in \overline{\mathrm{FV}}(s)$ and $x_{i_t} \in \overline{\mathrm{FV}}(t)$ two variables of codimension 0 and $y_j \in \overline{\mathrm{FV}}(p_a[\gamma] : \gamma_{a-1} \to \gamma_a)$ for some $p_a \in \mathtt{comp}(\overline{p})$. If $\dim(y_j) \leq \dim(\mathtt{op}_{\Gamma, s \to t}[\gamma]) - 2$, then by the inductive hypothesis $y_j \in \overline{\mathrm{FV}}(\gamma_{a-1} : T) \cap \overline{\mathrm{FV}}(\gamma_a)$. Applying the same argument inductively to the components of the path

$$\gamma_{i_s} \to \cdots \to \gamma_{a-1} \overset{p_a[\gamma]}{\to} \gamma_a \to \cdots \to \gamma_{i_t}$$

we conclude that $y_j \in \overline{\mathrm{FV}}(\gamma_{i_s} : T) \cap \overline{\mathrm{FV}}(\gamma_{i_t} : T)$. But $\overline{\mathrm{FV}}(\gamma_{i_s} : T) \subset \overline{\mathrm{FV}}(s[\gamma] : A[\gamma])$ and $\overline{\mathrm{FV}}(\gamma_{i_t} : T) \subset \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$ so that $y_j \in \overline{\mathrm{FV}}(s[\gamma] : A[\gamma]) \cap \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$. If, on the other hand, $\dim(y_j) = \dim(\mathtt{op}_{\Gamma, s \to t}[\gamma]) - 1$, by the inductive hypothesis there exist variables $y_{a-1} \in \overline{\mathrm{FV}}(\gamma_{a-1} : T)$ and $y_a \in \overline{\mathrm{FV}}(\gamma_a : T)$ and paths $y_{a-1} \rightsquigarrow y_j$ and $y_a \rightsquigarrow y_j$ in $\overline{\mathrm{FV}}(p_a[\gamma] : \gamma_{a-1} \to \gamma_a)$. Applying this argument inductively to the above chain of terms we can at each step find a variable and a path one step further away from p_a until we reach the endpoints. Connecting all these paths gives us in total a $y_{j^-} \in \overline{\mathrm{FV}}(\gamma_{i_s})$ and a path $y_{j^-} \rightsquigarrow y_j$ in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma] : s[\gamma] \to t[\gamma])$ as well as a $y_{j^+} \in \overline{\mathrm{FV}}(\gamma_{i_t})$ and a path $y_j \rightsquigarrow y_{j^+}$ in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma] : s[\gamma] \to t[\gamma])$ as desired.

Consider now the case $y_j \in \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$. So $y_j \in \overline{\mathrm{FV}}(\gamma_k)$ for some $x_k \in \overline{\mathrm{FV}}(s \to t)$. By symmetry it suffices to consider the case $x_k \in \overline{\mathrm{FV}}(s : A)$. Lemma 2.21 then implies that $\dim(x_k) \leq \dim(s)$. If $\dim(x_k) = \dim(s)$, then, since $\dim(s) = \dim(\Gamma) - 1$ (as part of the (OP') rule), it follows from Lemma 1.27 that there exists a $x_{k*} \in \overline{\mathrm{FV}}(\partial^+\Gamma)$ and a path $x_k \rightsquigarrow x_{k*}$ by Lemma 1.27. But, by the inductive hypothesis there also exists a variable $x_l \in \overline{\mathrm{FV}}(t)$ parallel to x_k. Since $x_l \parallel x_k \parallel x_{k*}$, $\dim(x_k) = \dim(s) = \dim(\Gamma) - 1$ and $x_l, x_{k^*} \in \overline{\mathrm{FV}}(\partial^+\Gamma)$, it follows from Lemma 1.26 that $x_l \equiv x_{k*}$ so that we actually have a path $x_k \rightsquigarrow x_l$ with $y_j \in \overline{\mathrm{FV}}(\gamma_k)$. Thus, this case reduces to the previous where $y_j \in \bigcup_{\overline{p}\,:\,x_{i_s} \rightsquigarrow x_{i_t}} \overline{\mathrm{FV}}(\overline{p}[\gamma])$. If $\dim(x_k) < \dim(s)$, then, by Lemma 2.21 we have $\dim(y_j) \leq \dim(x_k) \leq \dim(s) - 1 = \dim(\mathsf{op}) - 2$. So we need worry only about part (1). But, by the inductive hypothesis for types we must also have $x_k \in \overline{\mathrm{FV}}(t)$ so that $y_j \in \overline{\mathrm{FV}}(\gamma_k) \subset \overline{\mathrm{FV}}(s[\gamma] : A[\gamma]) \cap \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$. This takes care of parts (1) and (2).

Next we work on parts (3) and (4). This means we are assuming additionally that $\Delta \vdash_{\mathrm{ps}}$ is a ps-context. We will allow ourselves to recycle some variable names from the previous paragraphs since they are independent. So assume we are given two variables variable $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\mathsf{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$ such that $y_{j_1} \parallel y_{j_2}$. Then $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ and $y_{j_2} \in \overline{\mathrm{FV}}(\gamma_{i_2})$ for some $x_{i_1}, x_{i_2} \in \overline{\mathrm{FV}}(\Gamma)$.

For part (3) assume that $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\mathsf{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$ are of codimension 1. We need to show that there is a path between these two variables in $\overline{\mathrm{FV}}(\mathsf{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. By Lemma 2.21 we must have $\dim(x_{i_1}) = \dim(\Gamma)$ or $\dim(x_{i_1}) = \dim(\Gamma) - 1$ and similarly for x_{i_2}. Consider first the case where $\dim(x_{i_1}) = \dim(x_{i_2}) = \dim(\Gamma) - 1$. First of all, by the inductive hypothesis for context morphisms we have $x_{i_1} \parallel x_{i_2}$. Second, since $\dim(x_{i_1}) = \dim(\Gamma) - 1$, the above paragraphs on parts (1) and (2) contain a proof that there exists a path $\overline{p}_1 : x_{s_1} \rightsquigarrow x_{t_1}$ where $x_{s_1} \in \overline{\mathrm{FV}}(s)$ and $x_{t_1} \in \overline{\mathrm{FV}}(t)$ are variables of codimension 0 and $x_{i_1} \in \overline{\mathrm{FV}}(\overline{p}_1)$. Similarly, there is a path $\overline{p}_2 : x_{s_2} \rightsquigarrow x_{t_2}$ with $x_{s_2} \in \overline{\mathrm{FV}}(s)$ and $x_{t_2} \in \overline{\mathrm{FV}}(t)$ variables of codimension 0 and $x_{i_2} \in \overline{\mathrm{FV}}(\overline{p}_2)$. But $x_{s_1}, x_{s_2} \in \overline{\mathrm{FV}}(s) \subset \overline{\mathrm{FV}}(\partial^-\Gamma)$, $x_{s_1} \parallel x_{s_2}$ and $\dim(x_{s_1}) = \dim(x_{s_2}) = \dim(s) = \dim(\partial^-\Gamma)$ so, by Lemma 1.26 we must have $x_{s_1} \equiv x_{s_2}$. Similarly $x_{t_1} \equiv x_{t_2}$. By Lemma 1.26*(iii)* the two paths must therefore be equal and we denote it by $\overline{p} : x_s \rightsquigarrow x_t$. Thus so far we have $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\overline{p}[\gamma]) \subset \overline{\mathrm{FV}}(\mathsf{op}_{\Gamma,s\to t}[\gamma])$. Now, by applying the inductive hypothesis for terms part (2) to all components of $\overline{p}$ step by step, we can (as in the proof for part (2) find paths

$$\gamma_{i_s} \longrightarrow \cdots \longrightarrow \gamma_{i_s}$$

$$\overline{\mathrm{FV}}(\gamma_{i_s}) \ \ni \ y_{j_1^-} \rightsquigarrow y_{j_1} \rightsquigarrow y_{j_1^+} \ \in \ \overline{\mathrm{FV}}(\gamma_{a_2})$$

If we do the same thing for y_{j_2} we will end up again with two terms $y_{j_2^-} \in \overline{\mathrm{FV}}(\gamma_{i_s})$ and $y_{j_2^+} \in \overline{\mathrm{FV}}(\gamma_{i_t})$. But $y_{j_2^-} \parallel y_{j_2} \parallel y_{j_1} \parallel y_{j_1^-}$ and since $y_{j_1^-}, y_{j_2^-} \in \overline{\mathrm{FV}}(\gamma_{i_s})$ are of codimension 0 it follows from the inductive hypothesis that $y_{j^-} :\equiv y_{j_1^-} \equiv y_{j_2^-}$. Similarly $y_{j^+} :\equiv y_{j'^+} \equiv y_{j_2^+}$. Thus, by Lemma 1.25 y_{j_1} and y_{j_2} live in the same path $y_{j^-} \rightsquigarrow y_{j^+}$ from which we can extract a path between y_{j_1} and y_{j_2}.

Assume now that $\dim(x_{i_1}) = \dim(\Gamma)$. Then since $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ is of codimension 1, by the inductive hypothesis we can write $x_{i_1} : x_{\partial^- i_1} \to x_{\partial^+ i_1}$ and there exists a $y_{j^+} \in \overline{\mathrm{FV}}(\gamma_{\partial^+ i_1})$ as well as a path $y_{j_1} \rightsquigarrow y_{j^+}$ in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. So we have a new variable $y_{j^+} \in \overline{\mathrm{FV}}(\gamma_{\partial^+ i_1})$ which is of codimension 0 and parallel to y_{j_1} and we work with y_{j^+} instead of y_{j_1}. Similarly, if $\dim(x_{i_2}) = \dim(\Gamma)$ we can find a variable $y_{j^-} \in \overline{\mathrm{FV}}(\gamma_{\partial^- i_2})$ of codimension 0 and a path $y_{j^-} \rightsquigarrow y_{j_2}$ in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. By the previous paragraph there is a path between y_{j^+} and y_{j^-} in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$. Together with the other two paths, by concatenation and extraction (potentially using Lemma 1.26*(vi)*) we get a path between y_{j_1} and y_{j_2}. If $\dim(x_{i_1}) = \dim(\Gamma) - 1$ we work with y_{j_1} directly without constructing first a y_{j^+} and similarly if $\dim(x_{i_2}) = \dim(\Gamma) - 1$ then we work with y_{j_2}. This completes part (3).

For part (4), let $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma])$ be two variables of codimension 0. Lemma 2.21 then forces $\dim(x_{i_1}) = \dim(x_{i_2}) = \dim(\Gamma)$. By the inductive hypothesis for context morphisms we have that $x_{i_1} \parallel x_{i_2}$ and by Lemma 1.26*(iv)* in fact $x_{i_1} \equiv x_{i_2}$. By the inductive hypothesis for terms, since $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ are parallel and of codimension 0 we must have $y_{j_1} \equiv y_{j_2}$. This completes part (4).

(COH') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \ (\text{COH'})$$

where $\dim(\Gamma) < \dim(s \to t)$.

Again we consider parts (1) and (2) together. So let $y_j \in \overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,s\to t}[\gamma] : s[\gamma] \to t[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$. This means $y_j \in \overline{\mathrm{FV}}(\gamma_k)$ for some $x_k \in \overline{\mathrm{FV}}(s \to t)$. By Lemma 2.21 it follows that $\dim(x_k) \leq \dim(A)$. If $\dim(x_k) = \dim(A)$, then again by Lemma 2.21 $x_k \in \overline{\mathrm{FV}}(s) \cup \overline{\mathrm{FV}}(t)$. By symmetry it suffices to consider the case $x_k \in \overline{\mathrm{FV}}(s)$. Then, by the inductive hypothesis for types there exists a parallel $x_l \in \overline{\mathrm{FV}}(t)$. But since $\dim(\Gamma) \leq \dim(A) = \dim(x_k)$ it follows from Lemma 1.26*(iv)* that $x_k \equiv x_l$ so that $x_k \in \overline{\mathrm{FV}}(s) \cap \overline{\mathrm{FV}}(t)$. If on the other hand $\dim(x_k) \leq \dim(A) - 1$, then by the inductive hypothesis for types we directly get $x_k \in \overline{\mathrm{FV}}(s : A) \cap \overline{\mathrm{FV}}(t : A)$. So in all cases $y_j \in \overline{\mathrm{FV}}(\gamma_k) \subset \overline{\mathrm{FV}}(s[\gamma] : A[\gamma]) \cap \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$. Part (1) follows directly from this while for part (2) this shows that we may simply choose y_j and the empty paths.

Regarding part (3), assume we are given two parallel variables $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma, s \to t}[\gamma] : s[\gamma] \to t[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$ of codimension 1. Then $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ and $y_{j_2} \in \overline{\mathrm{FV}}(\gamma_{i_2})$ for some $x_{i_1}, x_{i_2} \in \overline{\mathrm{FV}}(s \to t)$. By Lemma 2.21 we must have $\dim(x_{i_1}) = \dim(x_{i_2}) = \dim(A)$. Since $y_{j_1} \parallel y_{j_2}$ and additionally $\Delta \vdash_{\mathrm{ps}}, \Gamma \vdash_{\mathrm{ps}}$ by assumption, it follows from the inductive hypothesis for context morphisms that $x_{i_1} \parallel x_{i_2}$. In fact, since $\dim(\Gamma) \leq \dim(A) = \dim(x_{i_1}) = \dim(x_{i_2})$, it follows from Lemma 1.26*(iv)* that $x_{i_1} \equiv x_{i_2}$. So $y_{j_1}, y_{j_2} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ and $\dim(y_{j_1}) = \dim(y_{j_2}) = \dim(\gamma_{i_1})$. So by the inductive hypothesis for terms $y_{j_1} \equiv y_{j_2}$ and we may simply choose the trivial path.

As for part (4), since $\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma, s \to t}[\gamma] : s[\gamma] \to t[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$, by Lemma 2.21 the term $\mathtt{coh}_{\Gamma, s \to t}[\gamma]$ cannot contain any variables of codimension 0 so the statement is trivially true.

(ES) Consider the rule

$$\frac{\Delta \vdash}{\Delta \vdash \langle \rangle : \emptyset} \text{ (ES)}$$

Since $\emptyset \nvdash_{\mathrm{ps}}$ there is nothing to check.

(SE) Consider the rule

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x_{n+1} : A_{n+1} \vdash \qquad \Delta \vdash \gamma_{n+1} : A_{n+1}[\gamma]}{\Delta \vdash \langle \gamma, \gamma_{n+1} \rangle : (\Gamma, x_{n+1} : A_{n+1})} \text{ (SE)}$$

where $\Delta \vdash_{\mathrm{ps}}$ and $\Gamma, x_{n+1} : A_{n+1} \vdash_{\mathrm{ps}}$. We perform an additional induction over ps-context.

First assume $\Gamma,\, x_{n+1} : A_{n+1} \equiv x_1 : \mathtt{Ob}$, i.e. $\Gamma \equiv \emptyset$. Since $\overline{\mathrm{FV}}(x_1 : \mathtt{Ob}) = \{x_1\}$ given two $x_{i_1}, x_{i_2} \in \overline{\mathrm{FV}}(x_1 : \mathtt{Ob})$ we must have $x_{i_1} \equiv x_1 \equiv x_{i_2}$. Also, since $\dim(x_1) = 0$, the terms γ_{i_1} and γ_{i_2} are necessarily variables so that $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ implies $y_{j_1} \equiv \gamma_{i_1}$. Similarly $y_{j_2} \equiv \gamma_{i_2}$. Since $x_{i_1} \equiv x_{i_2}$ we have $y_{j_1} \equiv y_{j_2}$ so that $y_{j_1} || y_{j_2}$. But $x_{i_1} \equiv x_{i_2}$ also trivially implies $x_{i_1} || x_{i_2}$.

Assume now that $(\Gamma,\, x_{n+1} : A_{n+1}) \equiv (\Gamma^{n-1},\, x_n : A_n,\, x_{n+1} : x_l \to x_n)$ where

$\Gamma^{n+1} \vdash_{\text{ps}}$ and the statement is true for $\Delta \vdash \gamma^{n-1} : \Gamma^{n-1}$. There are four cases for the two variables x_{i_1}, x_{i_2} to consider

(1) $x_{i_1}, x_{i_2} \in \overline{\mathrm{FV}}(\Gamma^{n-1})$

Since $\Gamma^{n-1} \vdash_{\text{ps}}$, this statement holds by the inductive hypothesis for context morphisms.

(2) $x_{i_1} \in \overline{\mathrm{FV}}(\Gamma^{n-1})$ and $x_{i_2} \equiv x_n$

Assume $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ and $y_{j_2} \in \overline{\mathrm{FV}}(\gamma_n)$ are two variables both of codimension 0 respectively such that $y_{i_1} \parallel y_{i_2}$. Then, since we have $\gamma_{n+1} : \gamma_l \to \gamma_n$, by the inductive hypothesis there exists a $y_{j^-} \in \overline{\mathrm{FV}}(\gamma_l)$ and a path $y_{j^-} \rightsquigarrow y_{j_2}$. In particular, $y_{j^-} \parallel y_{j_2}$ so that $y_{j_1} \parallel y_{j^-}$. Since $x_l \in \overline{\mathrm{FV}}(\Gamma^{n-1})$ and $y_{j_1} \parallel y_{j^-}$ the case for the pairs (x_{i_1}, y_{j_1}) and (x_l, y_{j^-}) reduces to the situation of part (1) from which we deduce that $x_{i_1} \parallel x_l$. But by definition $x_l \parallel x_n$ so that $x_{i_1} \parallel x_n$.

(3) $x_{i_1} \in \overline{\mathrm{FV}}(\Gamma^{n-1})$ and $x_{j_2} \equiv x_{n+1}$

Let $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ and $y_{j_2} \in \overline{\mathrm{FV}}(\gamma_{n+1})$ be two variables of codimension 0 respectively and such that $y_{j_1} \parallel y_{j_2}$. The goal is to derive a contradiction from which we conclude that there can be no such variables y_{j_1}, y_{j_2} which are parallel.

Since $\dim(\gamma_{n+1}) > 0$, it follows that $\dim(y_{j_2}) > 0$. Let $y_s \to y_t$ be the type of y_{j_1} and y_{j_2}. Then, by Lemma 2.22 we must have $y_s, y_t \in \overline{\mathrm{FV}}(\gamma_{n+1} : \gamma_l \to \gamma_n)$. By the inductive hypothesis for terms there is a $y_{s^-} \in \overline{\mathrm{FV}}(\gamma_l)$ and a path $y_{s^-} \rightsquigarrow y_s$, which, when concatenated with $(y_{j_2}) : y_s \rightsquigarrow y_t$ gives a nonempty path $y_{s^-} \rightsquigarrow y_t$ in $\Delta \vdash_{\text{ps}}$. In particular $y_{s^-} \not\equiv y_t$ by Lemma 1.26*(v)*. In what follows we will also construct a path $y_t \rightsquigarrow y_{s^-}$ in $\Delta \vdash_{\text{ps}}$ which, by Lemma 1.26*(i)* delivers the contradiction.

Now $y_s \to y_t$ is also the type of $y_{j_1} \in \overline{\mathrm{FV}}(\gamma_{i_1})$ where $x_{i_1} \in \overline{\mathrm{FV}}(\Gamma^{n-1})$. So, by Lemma 2.22 $y_t \in \overline{\mathrm{FV}}(\gamma_{i_1} : \gamma_{\partial^- i_1} \to \gamma_{\partial^+ i_1})$ where $x_{\partial^- i_1}, x_{\partial^+ i_1} \in \overline{\mathrm{FV}}(\Gamma^{n-1})$. Since $\dim(y_t) = \dim(y_{j_1}) - 1 = \dim(x_{i_1}) - 1$, it follows from the inductive hypothesis for terms that there exists a $y_{t^+} \in \overline{\mathrm{FV}}(\gamma_{\partial^- i_1})$ and a path $y_t \rightsquigarrow y_{t^+}$. In particular $y_{t^+} \in \overline{\mathrm{FV}}(\gamma_{\partial^+ i_1})$ is of codimension 0.

But we also have $y_{s^-} \in \overline{\mathrm{FV}}(\gamma_l)$ of codimension 0 where $x_l \in \overline{\mathrm{FV}}(\Gamma^{n-1})$ and where $y_{s^-} \parallel y_s \parallel y_t \parallel y_{t^+}$. By the inductive hypothesis for context morphisms on $\Delta \vdash \gamma^{n-1} : \Gamma^{n-1}$ we must have $x_l \parallel x_{\partial^+ i_1}$. Since by Lemma 1.40*(iv)* $\Gamma^{n-1} \blacktriangleright x_l$, we have a path $x_{\partial^+ i_1} \rightsquigarrow x_l$ in Γ^{n-1}. Now, starting with $y_{t^+} \in \overline{\mathrm{FV}}(\gamma_{\partial^+ i_1})$, using the inductive hypothesis for terms we can go through all the nodes of the path $x_{\partial^+ i_1} \rightsquigarrow x_l$ step by step and finally find a parallel variable $y_{t^*} \in \overline{\mathrm{FV}}(\gamma_l)$ as visualized in the following diagram

$$\begin{array}{ccccccc} & & \gamma_{t^+} & \to \cdots \to & \gamma_l & & \\ \overline{\mathrm{FV}}(\gamma_{\partial^+ i_1}) & \ni & y_{t^+} & \rightsquigarrow & y_{t*} & \in & \overline{\mathrm{FV}}(\gamma_l) \end{array}$$

But then $y_{t*} \parallel y_{s^-}$ in $\overline{\mathrm{FV}}(\gamma_l)$ and since they are of codimension 0, the inductive hypothesis for terms implies that $y_{t*} \equiv y_{s^-}$. Thus we have found a path $y_t \rightsquigarrow y_{s^-}$ in $\Delta \vdash_{\mathrm{ps}}$, with which we have derived the contradiction.

(4) $x_{i_1} \equiv x_n$ and $x_{i_2} \equiv x_{n+1}$

In this case $\dim(x_{i_1}) \neq \dim(x_{i_2})$ so there is nothing to check. □

Corollary 2.24. *Let $\Gamma \vdash s, t : A$ be two terms in a ps-context and let $x_i \in \overline{\mathrm{FV}}(s : A)$ be a variable. In* `CaTT'`

(i) if $\dim(x_i) = \dim(s) - 1$, then $x_i \in \overline{\mathrm{FV}}(t : A)$;

(ii) if $\dim(x_i) = \dim(s)$, then $x_i \in \overline{\mathrm{FV}}(s)$ and there exists a parallel variable $x_j \in \overline{\mathrm{FV}}(t)$, i.e. $x_i \parallel x_j$;

(iii) moreover, if part (ii) holds and $\partial^-\Gamma \vdash s : A$ and $\partial^+\Gamma \vdash t : A$ with $\dim(\Gamma) = \dim(s \to t)$, then there is a (unique) path $\overline{p} : x_i \rightsquigarrow x_j$ in Γ.

Proof. Parts (i) and (ii) follow directly from Lemma 2.23. The proof for part (iii) is contained in clause (OP') in the proof of Lemma 2.23. □

Lemma 2.25. *The following statements hold in* `CaTT'`

(i) Given a type $\Gamma \vdash A$ satisfying (OP), then $\dim(\Gamma) = \dim(A)$.

(ii) Given a term $\Gamma \vdash t : A$

$$\overline{\mathrm{FV}}(t) \backslash \overline{\mathrm{FV}}(A) \neq \emptyset \quad \Longrightarrow \quad \Big(\exists\, x_i \in \overline{\mathrm{FV}}(t) \quad \textit{such that} \quad \mathrm{codim}_t(x_i) = 0\Big)$$

and

$$\overline{\mathrm{FV}}(t) \backslash \overline{\mathrm{FV}}(A) = \emptyset \quad \Longrightarrow \quad \overline{\mathrm{FV}}(\partial^- t : \partial A) = \overline{\mathrm{FV}}(\partial^+ t : \partial A).$$

Proof. We prove all statements simultaneously by inducting over all the rules for types and terms in `CaTT'`. Most of the cases have already been taken care of in the proof of Lemma 2.2 however and the only remaining cases are (OP') and (COH').

(OP') Consider the rule

$$\frac{\Gamma \vdash_{\text{ps}} \quad \partial^-\Gamma \vdash s : A \quad \partial^+\Gamma \vdash t : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{op}_{\Gamma, t \underset{A}{\to} s}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(OP')}$$

where $\dim(\Gamma) = \dim(s \to t)$ and where by definition

$$\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) \cup \bigcup_{\substack{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t} \\ \mathrm{codim}_s(x_{i_s}) = 0 \\ \mathrm{codim}_t(x_{i_t}) = 0}} \overline{\mathrm{FV}}(\overline{p}[\gamma]).$$

Assume first that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma]) \backslash \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) \neq \emptyset$. We must show that there exists a $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma])$ which is of codimension 0. Instead of this we shall however prove the contrapositive statement. So assume all variables $y_j \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma])$ have dimension $\dim(y_j) < \dim(\mathtt{op}_{\Gamma, s \to t}[\gamma])$. By Lemma 2.21, the set $\overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$ contains only variables of dimension strictly less than $\dim(\mathtt{op}_{\Gamma, s \to t}[\gamma])$ so we need only work on $\bigcup_{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}} \overline{\mathrm{FV}}(\gamma_i)$. If $\bigcup_{\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}} \overline{\mathrm{FV}}(\gamma_i)$ is empty, then we have that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t})[\gamma]) \backslash \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) = \emptyset$. Otherwise there exists a path $\overline{p} : x_{i_s} \rightsquigarrow x_{i_t}$ with $\mathrm{codim}_s(x_{i_s}) = 0$ and $\mathrm{codim}(x_{i_t}) = 0$. Let $p_a : x_{a-1} \to x_a$ be a component of this path. Applying the substitution yields the term $p_a[\gamma] : \gamma_{a-1} \to \gamma_a$, the free variables of which are contained in $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma])$. So $p_a[\gamma]$ contains no variables of codimension 0. By the inductive hypothesis $\overline{\mathrm{FV}}(p_a[\gamma]) \backslash \overline{\mathrm{FV}}(\gamma_{a-1} \to \gamma_a) = \emptyset$ and by the inductive hypothesis again $\overline{\mathrm{FV}}(\gamma_{a-1} : T) = \overline{\mathrm{FV}}(\gamma_a : T)$. Applying this argument to the whole path shows that $\overline{\mathrm{FV}}(\overline{p}[\gamma]) \subset \overline{\mathrm{FV}}(\gamma_{i_s} \to \gamma_{i_t})$. But $x_{i_s} \in \overline{\mathrm{FV}}(s)$ and $x_{i_t} \in \overline{\mathrm{FV}}(t)$ so that $\overline{\mathrm{FV}}(\gamma_{i_s}) \subset \overline{\mathrm{FV}}(s[\gamma])$ and $\overline{\mathrm{FV}}(\gamma_{i_t}) \subset \overline{\mathrm{FV}}(t[\gamma])$. By Lemma 2.22 $\overline{\mathrm{FV}}(\gamma_{i_s} : T) \subset \overline{\mathrm{FV}}(s[\gamma] : A[\gamma])$ and similarly $\overline{\mathrm{FV}}(\gamma_{i_t} : T) \subset \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$. Thus $\overline{\mathrm{FV}}(\overline{p}[\gamma]) \subset \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$. Since this is true for all paths of the above described form, it follows that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t)[\gamma]}) \backslash \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) = \emptyset$.

Assume now that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma, s \to t}[\gamma]) \backslash \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) = \emptyset$. We must show that $\overline{\mathrm{FV}}(s[\gamma] : A[\gamma]) = \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$. Equivalently, by Lemma 1.11 and Remark **??** we must show that

$$\bigcup_{x_i \in \overline{\mathrm{FV}}(s:A)} \overline{\mathrm{FV}}(\gamma_i) = \bigcup_{x_i \in \overline{\mathrm{FV}}(t:A)} \overline{\mathrm{FV}}(\gamma_i).$$

Let $x_i \in \overline{\mathrm{FV}}(s : A)$. If $\dim(x_i) \leq \dim(s) - 2$, then $x_i \in \overline{\mathrm{FV}}(A) \subset \overline{\mathrm{FV}}(t : A)$ by Lemma 2.23. If $\dim(x_i) = \dim(s) - 1$, then $x_i \in \overline{\mathrm{FV}}(t : A)$ by corollary 2.24(i). If $\dim(x_i) = \dim(s)$, then the same corollary guarantees the existence of a parallel variable $x_j \in \overline{\mathrm{FV}}(t)$ and a path $\overline{p} : x_i \rightsquigarrow x_j$. Thus we get $\overline{\mathrm{FV}}(\overline{p}[\gamma]) \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma]) \subset \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$, where the first subset follows from the definition of $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t)}[\gamma])$ and the second from our assumption. Let $p_a : x_{a-1} \to x_a$ be a component in $\overline{p}$. By definition $\dim(p_a) = \dim(s \to t)$. But, by Lemma 2.21 $\overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$ contains variables of dimension at most $\dim(s)$. Thus $\overline{\mathrm{FV}}(p_a[\gamma] : \gamma_{a-1} \to \gamma_a)$ contains no variables of codimension 0 for all components p_a in $\overline{p}$. By the inductive hypothesis $\overline{\mathrm{FV}}(\gamma_{a-1} : T) = \overline{\mathrm{FV}}(\gamma_a : T)$ for all adjacent nodes in the path $\overline{p} : x_i \rightsquigarrow x_j$. In particular $\overline{\mathrm{FV}}(\gamma_i : T) = \overline{\mathrm{FV}}(\gamma_j : T)$. So in total we have shown that for all $x_i \in \overline{\mathrm{FV}}(s : A)$ we have $\overline{\mathrm{FV}}(\gamma_i) \subset \bigcup_{x_j \in \overline{\mathrm{FV}}(t:A)} \overline{\mathrm{FV}}(x_j)$. By a symmetrical argument we get the other inclusion which completes this part of the proof.

(COH') Consider the rule

$$\frac{\Gamma \vdash_{\mathrm{ps}} \qquad \Gamma \vdash s \underset{A}{\to} t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathtt{coh}_{\Gamma,s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(COH')}$$

where $\dim(\Gamma) < \dim(s \to t)$ and where by definition $\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,s\to t}[\gamma]) = \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma])$. So $\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,s\to t}[\gamma]) \backslash \overline{\mathrm{FV}}(s[\gamma] \to t[\gamma]) = \emptyset$ and we must show that $\overline{\mathrm{FV}}(s[\gamma] : A[\gamma]) = \overline{\mathrm{FV}}(t[\gamma] : A[\gamma])$, or equivalently, that

$$\bigcup_{x_i \in \overline{\mathrm{FV}}(s:A)} \overline{\mathrm{FV}}(\gamma_i) = \bigcup_{x_i \in \overline{\mathrm{FV}}(t:A)} \overline{\mathrm{FV}}(\gamma_i).$$

Let $x_i \in \overline{\mathrm{FV}}(s)$. By corollary 2.24(i) if $\dim(x_i) \leq \dim(s) - 1$, then $x_i \in \overline{\mathrm{FV}}(t : A)$. And if $\dim(x_i) = \dim(s)$, then by Lemma 2.24(ii) there exists a parallel variable $x_j \in \overline{\mathrm{FV}}(t)$ and a path $x_i \rightsquigarrow x_j$. But the components of such a path must have dimension $\dim(s \to t)$ while $\dim(\Gamma) < \dim(s \to t)$. So this path must be empty and $x_i \equiv x_j$ showing that also in this case $x_i \in \overline{\mathrm{FV}}(t : A)$. So $\overline{\mathrm{FV}}(s : A) \subset \overline{\mathrm{FV}}(t : A)$ and by symmetry $\overline{\mathrm{FV}}(t : A) \subset \overline{\mathrm{FV}}(s : A)$ which completes the proof. $\square$

We end this subsection with a lemma about the free variables of $\mathtt{op}_{\Gamma,s\to t}[\gamma]$ and $\mathtt{coh}_{\Gamma,s\to t}[\gamma]$ which shows that in the appropriate situation, the free variable definitions of `CaTT'` agree with those of `CaTT`.

Regarding the rule (OP'), in the special case where $\overline{\mathrm{FV}}(\partial^{-}\Gamma) = \overline{\mathrm{FV}}(s : A)$ and $\overline{\mathrm{FV}}(\partial^{+}\Gamma) = \overline{\mathrm{FV}}(t : A)$, then the formula for the free variables of $\mathtt{op}_{\Gamma,s\to t}[\gamma]$ can be simplified.

Lemma 2.26. *In* `CaTT'`*,*

(i) *if* $\overline{\mathrm{FV}}(\partial^-\Gamma) = \overline{\mathrm{FV}}(s : A)$ *and* $\overline{\mathrm{FV}}(\partial^+\Gamma) = \overline{\mathrm{FV}}(t : A)$*, then*

$$\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma]) = \overline{\mathrm{FV}}(\gamma).$$

(ii) *if* $\overline{\mathrm{FV}}(s : A) = \overline{\mathrm{FV}}(\Gamma) = \overline{\mathrm{FV}}(t : A)$*, then*

$$\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,s\to t}[\gamma]) = \overline{\mathrm{FV}}(\gamma);$$

Proof. (i) Let us start with the rule (OP') applied with the identity context morphism $\Gamma \vdash \mathtt{id}_\Gamma : \Gamma$. This means we are interested in the term $\Gamma \vdash \mathtt{op}_{\Gamma,s\to t}[\mathtt{id}] : s[\mathtt{id}] \underset{A[\mathtt{id}]}{\to} t[\mathtt{id}]$ and we will show that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}]) = \overline{\mathrm{FV}}(\Gamma)$.

Now, by Proposition 1.9 and Remark **??** we know that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}]) \subset \overline{\mathrm{FV}}(\Gamma)$. For the other direction, by Lemma 1.13 and Remark **??** we know that $A[\mathtt{id}] \equiv A$, $s[\mathtt{id}] \equiv s$ and that $t[\mathtt{id}] \equiv t$. By assumption $\overline{\mathrm{FV}}(s \underset{A}{\to} t) = \overline{\mathrm{FV}}(\partial\Gamma)$, so by our definition for the free variables we have

$$\begin{aligned}\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}]) &= \overline{\mathrm{FV}}(s \underset{A}{\to} t) \cup \bigcup_{\substack{\overline{p}:x_{i_s}\rightsquigarrow x_{i_t} \\ \mathrm{codim}_s(x_{i_s})=0 \\ \mathrm{codim}_t(x_{i_t})=0}} \overline{\mathrm{FV}}(\overline{p}) \\ &= \overline{\mathrm{FV}}(\partial\Gamma) \cup \bigcup_{\substack{\overline{p}:x_{i_s}\rightsquigarrow x_{i_t} \\ \cdots}} \overline{\mathrm{FV}}(\overline{p}). \end{aligned} \tag{1.17}$$

By Lemma 1.28 and Remark **??**, we know that $\Gamma|_{\dim(\Gamma)-2} \subset \overline{\mathrm{FV}}(\partial\Gamma)$ so we already know that $\Gamma|_{\dim(\Gamma)-2} \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}])$. Thus we need only worry about the variables of dimension $(\dim(\Gamma) - 1)$ and $\dim(\Gamma)$.

Given a variable $x_k \in \overline{\mathrm{FV}}(\Gamma)$ with dimension $\dim(\Gamma)-1$, then by Lemma 1.27 there is a variable $x_{k^-} \in \overline{\mathrm{FV}}(\partial^-\Gamma) = \overline{\mathrm{FV}}(s : A)$ and a path $\overline{f}_- : x_{k^-} \rightsquigarrow x_k$, as well as a variable $x_{k^+} \in \overline{\mathrm{FV}}(\partial^+\Gamma) = \overline{\mathrm{FV}}(t : A)$ and a path $\overline{f}_+ : x_k \rightsquigarrow x_{k^+}$. Concatenation gives us a path $\overline{p} : x_{k^-} \rightsquigarrow x_{k^+}$ which passes through x_k. Now by Lemma 2.21 we necessarily have $x_{k^-} \in \overline{\mathrm{FV}}(s)$ and $x_{k^+} \in \overline{\mathrm{FV}}(t)$. By equation (1.17) we thus have $x_k \in \overline{\mathrm{FV}}(\overline{p}) \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}])$.

Given a variable $x_i \in \overline{\mathrm{FV}}(\Gamma)$ of dimension $\dim(\Gamma)$, since the source and target ps-context are defined only for ps-contexts with dimension greater than 0, we have $\dim(x_i) > 0$. This allows us to write the type of this variable as $x_i : x_{\partial^- i} \to x_{\partial^+ i}$. Using Lemma 1.27 we again get variables $x_{i^-} \in \overline{\mathrm{FV}}(\partial^-\Gamma) = \overline{\mathrm{FV}}(s : A)$ and a path $\overline{g}_- : x_{i^-} \rightsquigarrow x_{\partial^- i}$ as well as a variable $x_{i^+} \in \overline{\mathrm{FV}}(\partial^+\Gamma) = \overline{\mathrm{FV}}(t : A)$ and a path $\overline{g}_+ : x_{\partial^+ i} \rightsquigarrow x_{i^+}$. Concatenation then gives us a path $\overline{g}_- * (x_i) * \overline{g}_+ : x_{i^-} \rightsquigarrow x_{i^+}$ which we denote by $\overline{q}$. Now, by

Lemma 2.21 we necessarily have $x_{i^-} \in \overline{\mathrm{FV}}(s)$ and $x_{i^+} \in \overline{\mathrm{FV}}(t)$. By equation (1.17) we thus have $x_i \in \overline{\mathrm{FV}}(\overline{q}) \subset \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}])$.

We have shown that $\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}]) = \overline{\mathrm{FV}}(\Gamma)$. To complete the proof we use Lemma 1.11 and Remark ?? which gives us

$$\begin{aligned}\overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\gamma]) &= \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}][\gamma]) \\ &= \bigcup_{x_i \in \overline{\mathrm{FV}}(\mathtt{op}_{\Gamma,s\to t}[\mathtt{id}])} \overline{\mathrm{FV}}(\gamma_i) \\ &= \bigcup_{x_i \in \Gamma} \overline{\mathrm{FV}}(\gamma_i) \\ &= \overline{\mathrm{FV}}(\gamma).\end{aligned}$$

(ii) Assume $\overline{\mathrm{FV}}(s : A) = \overline{\mathrm{FV}}(\Gamma) = \overline{\mathrm{FV}}(t : A)$. Then $\overline{\mathrm{FV}}(s \underset{A}{\to} t) = \overline{\mathrm{FV}}(\Gamma)$. We compute

$$\begin{aligned}\overline{\mathrm{FV}}(\mathtt{coh}_{\Gamma,s\to t}[\gamma]) &= \overline{\mathrm{FV}}\Big((s \underset{A}{\to} t)[\gamma]\Big), && \text{by definition} \\ &= \bigcup_{x_i \in \overline{\mathrm{FV}}(s \underset{A}{\to} t)} \overline{\mathrm{FV}}(\gamma_i), && \text{by Lemma 1.11 and Remark ??} \\ &= \bigcup_{x_i \in \overline{\mathrm{FV}}(\Gamma)} \overline{\mathrm{FV}}(\gamma_i) \\ &= \overline{\mathrm{FV}}(\gamma).\end{aligned}$$

□

2.3.2 (COH) is admissible in CaTT'

Theorem 2.27. (COH) *is admissible in* CaTT'.

Proof. Assume that we have a type $\Gamma \vdash s \underset{A}{\to} t$ such that $\overline{\mathrm{FV}}(s : A) = \overline{\mathrm{FV}}(\Gamma) = \overline{\mathrm{FV}}(t : A)$ and a context morphism $\Delta \vdash \gamma : \Gamma$. Then, in particular $\overline{\mathrm{FV}}(\Gamma) = \overline{\mathrm{FV}}(s \underset{A}{\to} t)$. By Lemma 2.21 it follows that $\dim(\Gamma) < \dim(s \underset{A}{\to} t)$ so that we may apply the rule (COH') to produce the term $\Delta \vdash \mathtt{coh}'_{\Gamma,s\to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$ and we define $\mathtt{coh}_{\Gamma,s\to t}[\gamma] :\equiv \mathtt{coh}'_{\Gamma,s\to t}[\gamma]$. □

2.3.3 (OP) is admissible in CaTT'

Theorem 2.28. (OP) *is admissible in* CaTT'.

Proof. Assume we are given a type $\Gamma \vdash s \underset{A}{\to} t$ such that $\overline{\mathrm{FV}}(s : A) = \overline{\mathrm{FV}}(\partial^-\Gamma)$ and $\overline{\mathrm{FV}}(\partial^+\Gamma) = \overline{\mathrm{FV}}(t : A)$ and a context morphism $\Delta \vdash \gamma : \Gamma$. Then, by Lemma 2.25

it follows that $\dim(\Gamma) = \dim(s \underset{A}{\to} t)$ which allows us to apply the (OP') rule and produce the term $\Delta \vdash \mathtt{op}'_{\Gamma, s\to t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]$. We define $\mathtt{op}_{\Gamma, s\to t}[\gamma] :\equiv \mathtt{op}'_{\Gamma, s\to t}[\gamma]$. □

2.4 On the Models of `CaTT` and `CaTT'`

A natural question to ask wether `CaTT` and `CaTT'` share the same models. Admissibility of rules is a rather weak condition. For completeness we recall here the definition of admissibility, and define also the notion of derivable rules.

Definition 2.29 (Admissibility)**.** *Let $\mathcal{T}$ be some type theory. A rule*

$$\frac{J_1 \quad \cdots \quad J_n}{J}$$

where $J_1, \ldots, J_n, J$ are some judgments in $\mathcal{T}$ is said to be admissible in $\mathcal{T}$, if given a derivation of $J_1, \ldots, J_n$ in $\mathcal{T}$ we can derive J in $\mathcal{T}$. The rule is said to be derivable, if it can be represented by a derivation tree of rules in $\mathcal{T}$.

By definition, a derivable rule is also admissible. An admissible rule which is not derivable is a rule in which the derivation of the conclusion depends on the judgments in the hypothesis. Typically such an admissible rule is given by inducting over the rules by which the hypothesis are obtained.

A crucial difference is that derivable rules are automatically valid in the respective models, whereas admissible rules are not. In our case, the rules (OP) and (COH) are actually derivable in `CaTT'`. As a result, every model of `CaTT'` is also a model of `CaTT`.

In the other direction, the rule (COH') in `CaTT` can be unfolded into

$$\frac{\dfrac{\Gamma \vdash_{\mathrm{ps}}}{\Gamma_{s\to t} \vdash_{\mathrm{ps}}} \qquad \dfrac{\Gamma \vdash s \underset{A}{\to} t}{\Gamma_{s\to t} \vdash s \underset{A}{\to} t} \qquad \dfrac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \gamma|_{s\to t} : \Gamma_{s\to t}}}{\Delta \vdash \mathtt{coh}_{\Gamma_{s\to t}, s \underset{A}{\to} t}[\gamma] : s[\gamma] \underset{A[\gamma]}{\to} t[\gamma]} \text{(COH')}$$

the derivability of which depends on whether the three rules on the top are derivable. The case for (OP') is more subtle, since we need to make a distinction between the cases where the free variables of $\mathrm{FV}(s : A)$ and $\mathrm{FV}(t : A)$ agree and when they do not. When they agree the desired term in the conclusion of (OP') is derived using (COH), while when they do not agree we use (OP). A comparison of the models thus requires an understanding of the precise details of how this analysis translates under the semantics. An alternative approach, bypassing the semantics, would be to study and relate the syntactic categories. To conclude, establishing the equivalence of models requires more work, left for the future.

3 (∞, ∞)-Limits

With the advent of higher category theory, the 21st century is casting new light on abstract homotopy theory. By now the theory of $(\infty, 1)$-categories has matured into a full fledged theory thanks to the pioneering work of André Joyal and Jacob Lurie among others, providing us with a language native to ∞-groupoids.

Working with $(\infty, 1)$-categories one quickly runs into $(\infty, 2)$-categories, with the prototypical example of $(\infty, 2)$-category being that of all $(\infty, 1)$-categories. On the other hand, (∞, n)-categories appear naturally in the study of cobordisms motivating the necessity for (∞, n)-categories for arbitrary n. With these considerations in mind it thus becomes natural to consider the most general such structures namely (∞, ∞)-categories in which the existence of non-invertible morphisms is permitted in every dimension.

A number of different proposals for models for higher categories have been made. Approaches include Rezk's complete Segal Θ_n-spaces [66] and Barwick's n-fold complete Segal spaces [5], as well as Verity's n-complicial sets (see Barwick–Schommer-Pries [6] for a more extensive list and again Barwick–Schommer-Pries [6] and Loubaton [56] for the equivalence of the aforementioned examples). While non-algebraic models such as those mentioned have been the most successful so far, it is an important challenge to provide also a purely algebraic notion, as envisioned by Grothendieck. As an algebraic system, one may ask for a type-theoretic approach, which is the path taken in this work.

In his manuscript Pursuing Stacks, Grothendieck himself already spelled out an algebraic definition of ∞-groupoids in terms of globular sets (see Maltsiniotis [60]). This was then taken up by Maltsiniotis, who generalized this definition to (∞, ∞)-categories [60]. Brunerie [20] developed a type-theoretic version of the definition of ∞-groupoids with the aim of showing that the types in Homotopy Type Theory possess this structure. Building upon this and filling in the remaining slot, Finster and Mimram constructed a type-theoretic definition of (∞, ∞)-categories called `CaTT` [30], and together with Benjamin proved that its models are precisely the Grothendieck–Maltsiniotis (∞, ∞)-categories [10].

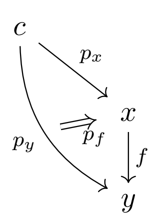
With the definition in place, it is now our task to develop theory for it. One of the most basic categorical construction is that of limits. In higher dimensional categories there is an increased level of complexity, as the higher-dimensional cells comprising cones may be noninvertible. In 2-dimensional categories, for example, we may consider lax or oplax limits, depending on the orientation of the 2-cells involved, or pseudo limits if the 2-cells are invertible. Many fundamental concepts in 2-category theory arise as lax (co)limits. The Grothendieck construction corresponding to a pseudofunctor, for example, can be described as the oplax colimit of the given pseudofunctor. Furthermore, given a monad as a lax functor $1 \to \text{Cat}$, the Eilenberg–Moore category and the Kleisli category are computed by the lax limit and lax colimit of the lax functor respectively.

The same ideas are expected to continue to hold as we increase the dimension of categories all the way up to (∞, ∞)-categories. The Grothendieck construction, for example, is again computed by the lax colimit in the setting of (∞, ∞)-categories, which motivates the study of lax limits in the setting of (∞, ∞)-categories (see Loubaton [56] for a definition of lax (co)limits and the Grothendieck construction for (∞, ∞)-categories, developed in a model independent framework). In this paper we propose a definition for lax (∞, ∞)-limits in the type theory `CaTT` of Finster and Mimram [30]. In particular, we define a new theory $\mathtt{CaTT}_{\mathtt{lim}}$ extending `CaTT`, which describes (∞, ∞)-categories with lax limits for finite computads.

`CaTT` is a dependent type theory with two type constructors. The first, functioning as the base case, introduces the type `Ob` which one may think of as the type of objects. The second one takes two terms $s, t : A$ as input, and produces the type $s \to_A t$, which is understood as the type of morphisms from $s \to_A t$. Starting with the terms of `Ob` and applying the second type constructor iteratively, one can access the type of all higher dimensional morphisms. In addition to the type constructors, `CaTT` also contains two term constructors. The first term constructor is responsible for the existence of all (∞, ∞)-categorical operations. These include all binary operations. The second term constructor is responsible for all coherences, interpolating between the different ways of composing cells. These include the unit, the associator, the unit laws and so on.

In `CaTT`, diagrams can conveniently be encoded by contexts and we will use these words interchangeably. A context is a list of variables $x_1 : A_1, \ldots, x_n : A_n$, such that the type A_i can be constructed using the variables of $x_1 : A_1, \ldots, x_{i-1} : A_{i-1}$. Now, contexts in `CaTT` are finite computads, thanks to a result by Benjamin, Markakis and Sarti [13]. This finiteness constraint is brought upon us by the finiteness of type theory's contexts. It may be possible to translate and extend these ideas to other frameworks, such as that of Dean, Finster, Markakis, Reutter and Vicary [27], in which one is liberated from this restriction.

We define the cone over a context Γ to be another context K. As a low dimensional example, consider the cone over $\Gamma \equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y$, given by



$$K \equiv \Gamma,\ c : \mathtt{Ob},\ p_x : c \to x,\ p_y : c \to y,\ p_f : p_y \to p_x \ast^1_0 f$$

The definition of cones is based on the observation that the types of the projections exhibit a certain pattern. Take, for example, the variable $f : x \to y$. First of all, the source of p_f is built out of the projections associated to the target of f, namely p_y. Second, the target of p_f is built out of the projections associated to the source of f, namely p_x, as well as f itself. Finally, the variable f appears in a certain linear way. These are the properties which appear as a set of side conditions in the recognition algorithm for cones below.

For the recognition algorithm for cones we introduce a new auxiliary judgment $K \;\; \texttt{cone} \;\; (\Gamma; c)$, subject to the following rules

$$\frac{}{c : \texttt{Ob} \;\; \texttt{cone} \;\; (\emptyset, c)}$$

$$\frac{\Gamma, c : \texttt{Ob}, \Pi \;\; \texttt{cone} \;\; (\Gamma; c) \qquad \Gamma, x : X, c : \texttt{Ob}, \Pi \vdash s \to_A t}{\Gamma, x : X, c : \texttt{Ob}, \Pi, p_x : s \to_A t \;\; \texttt{cone} \;\; ((\Gamma, x : X), c)} \qquad (+\text{ side conditions})$$

If $K\,\texttt{cone}\,(\Gamma; c)$ is derivable we say that K is a cone over Γ with apex c.

The rules exploit the inductive definition of contexts. For the empty context the cone is simply an apex. This is the first rule. In the second rule we begin with a cone K over a diagram Γ as well as a context extension $\Gamma, x : X$. The side conditions ensure that the type $s \to_A t$ is of the appropriate form so as to be the type of a projection corresponding to the appended variable $x : X$. Given this, the rule extends the original cone over Γ to a cone over $\Gamma, x : X$.

The definition of the cone involves a choice of orientation for the higher cells. The orientation chosen has the benefit of exhibiting a certain uniformity. All other choices can be obtained by making suitable adjustments to the rules. Reversing the orientation of all 1-dimensional cells turns the definition into one for colimits.

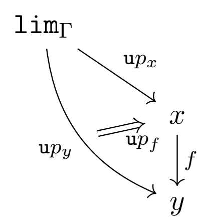
If Γ is a globular diagram, meaning that all terms involved are variables, we show that there exists a context K such that K is the cone over Γ. In particular we construct a cone over a globular diagram Γ with an explicit description of the type of the projection of a variable $(x : A) \in \Gamma$:

$$\begin{cases} c \to x, & \text{if } \dim(x) = 0 \\ p_{\tau(x)} \to p_{\sigma(x)} \overset{d}{\underset{d-1}{*}} \left(1^d_{p_{\sigma^2(x)}} \overset{d}{\underset{d-2}{*}} \cdots \left(1^d_{p_{\sigma^d(x)}} \overset{d}{\underset{0}{*}} x\right)\right), & \text{if } \dim(x) > 0. \end{cases} \tag{1.18}$$

where $d = \dim(x)$. Here $\overset{d}{\underset{k}{*}}$ denotes the binary composition of two d-dimensional cells along a k-dimensional gluing locus and 1^d_t denotes the d-dimensional iterative unit over the cell t, where $d > \dim(t)$. We then examine two classes of examples of two dimensional non-globular diagrams, the first one containing sequences of composable 1-dimensional morphisms and the second a sequence of composable

2-dimensional morphisms. Considering both examples in the strict case we show that there exists a cone with a similar description to that in the globular case. The argument relies on the fact that, given a term in the diagram $t : A$, the projections associated to the free variables of t may be composed in a way to obtain a certain term p_t, the type of which is described by a formula analogous to equation 1.18. Restricting ourselves to the strict case does not spoil the argument, as the coherences are absorbed by the terms p_t. Motivated by these examples we conjecture the existence of such terms for all $t : A$ in a diagram. Using these we construct a cone with an explicit description for arbitrary diagrams.

Given a cone K over a diagram Γ, we can build the universal cone with the help of term-constructor rules, which produce a term for the apex and for each projection. As a collection, these terms can be organized into a context morphism $\Gamma \vdash \texttt{ucone} : K$. Diagrammatically we may depict this as



Implementing the universal property amounts to asking the functor of (∞, ∞)-categories given by postcomposition with the universal cone, schematically depicted by

$$\texttt{cone}_* : \{\text{terms of } c \to \texttt{lim}_\Gamma\} \longrightarrow \{\text{cones over } \Gamma \text{ with apex } c\} \tag{1.19}$$

to an equivalence. Here, the domain is the (∞, ∞)-category of terms of the type $c \to \texttt{lim}_\Gamma$ and the codomain is given by the (∞, ∞)-category of cones over Γ with apex c. We refer to the n-dimensional cells of the codomain as $(n+1)$-transfors.[3] We define an equivalence of (∞, ∞)-categories to be a functor which is (essentially) surjective on all higher hom-(∞, ∞)-categories. To ensure that the functor in equation 1.19 is an equivalence we first spell out a set of rules which, in a manner similar to those generating cones, produce out of a given context a new context of the shape of a higher transfor between cones on that context. As in the case of the universal cone, we use this context as a template, to control the types of the terms we need to build with term constructor rules. In its first application, given a arbitrary cone over a given diagram, our constructions will produce a cone morphism (i.e. a modification) from the given cone to the universal cone.

[3]More generally, an n-dimensional cell in a functor category is called an n-transfor, this terminology being coined by Crans [25].

3.1 The Universal Cone

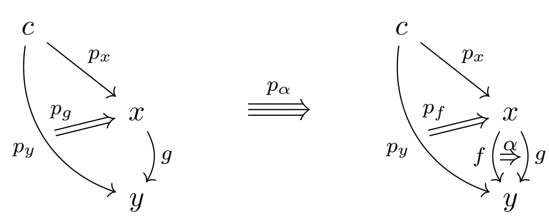
In 1-category theory, thanks to the free-forgetful adjunction between directed graphs and 1-categories, limits of functors can be equivalently described as limits over diagrams. When considering (∞, ∞)-categories, the appropriate notion encapsulating the generating data of free structures is that of computads. By a result of Benjamin, Markakis and Sarti [13], the contexts of `CaTT` are precisely finite computads. We exploit this fact to build a theory of limits over finite computads in `CaTT` in which the role of the diagram is played by the contexts. Note that the restriction to finite computads is a limitations of type theory, in which contexts are necessarily finite. In principle, the same ideas could be used to define arbitrary limits in a framework without this limitation (e.g. in that of computads as in the work of Dean, Finster, Markakis, Reutter and Vicary [27]).

The constituent cells of the cones and the corresponding limit notions we consider here are oriented. As a result, when building cones there are number of choices we can make with respect to the orientation of the terms involved, namely one for each dimension. As an example, in the 2-dimensional case we have four options. The first choice fixes the orientation of the 1-cells leading to the notion of limits and colimits respectively. The second choice fixes the orientation of the 2-cells, which are referred to as the lax and oplax versions of the corresponding (co)limit notion. In our case, for the purposes of this section we will be working with a choice of orientation with which the construction acquires a convenient uniformity. All other choices can be obtained by changing the roles of the terms s and t in the definition appropriately. Alternatively, all other choices can be obtained by taking opposites, which are studied by Benjamin and Markakis [11] for (∞, ∞)-categories in the related framework of computads of Dean et al [27].

To motivate the following proposition, let us begin with an example and build a cone over a 2-globe. The context of a 2-globe is given by

$$\Gamma \equiv x : \texttt{Ob}, y : \texttt{Ob}, f : x \to y, g : x \to y, \alpha : f \to g$$

and a cone over this diagram is given by



which as a context has the form

$$\begin{aligned} C(\Gamma) \equiv x &: \texttt{Ob},\ y : \texttt{Ob},\ f : x \to y,\ g : x \to y, \alpha : f \to g \\ c &: \texttt{Ob},\ p_x : c \to x,\ p_y : c \to y, \\ p_f &: p_y \to p_x \overset{1}{\underset{0}{*}} f,\ p_g : p_y \to p_x \overset{1}{\underset{0}{*}} g, \\ p_\alpha &: p_g \to p_f \overset{2}{\underset{1}{*}} \left(1_{p_x} \overset{2}{\underset{0}{*}} \alpha\right). \end{aligned}$$

Since we are working with a weak category, the construction of cones involves choices, as there are many ways of gluing a given diagram. The choices we made are for the sake of convenience because they fit into a pattern revealing certain features we will eventually rely on in our definition.

We can already see these features in the cone as a context in this simple example. The cone is generated by "formally adding" an apex and then freely adding a unique "projection" to each variable in Γ. Moreover, the types of each projection all have common features, which become more apparent if we increases the dimension and the complexity of the diagrams.

Take, for example, $p_\alpha : p_g \to p_f \overset{2}{\underset{1}{*}} \left(1_{p_x} \overset{2}{\underset{0}{*}} \alpha\right)$. We see, first of all, that the source of p_α involves only the projections associated to the target of α, namely g. Note that the projections associated to the dependencies of g, namely p_x and p_y are themselves dependencies of p_g and thus appear in the construction of p_g and its type. On the other hand, the target of p_α involves only projections associated to the source of α, namely f. The target of p_α also involves the variable α itself. Moreover, α appears in a certain linear way which we now make this precise.

Definition 3.1. *Let $\Gamma \vdash t : A$ be a term and let $x \in \mathrm{FV}(t)$ be a variable such that $\dim(x) = \dim(t)$. We define the relation $x \propto t$ by inducting over the term t as follows.*

(VAR) $\Gamma \vdash y : A$ *where* $(y : A) \in \Gamma$

$x \propto y$ *if* $x \equiv y$.

(OP) $\Delta \vdash \texttt{op}_{\Gamma, s \to_A t}[\gamma] : s[\gamma] \to_{A[\gamma]} t[\gamma]$

$x \propto \texttt{op}_{\Gamma, s \to_A t}[\gamma]$ *if there exists a unique* $x_i \in \mathrm{FV}(\Gamma)$ *such that* $x \propto \gamma_i$.

(COH) $\Delta \vdash \texttt{coh}_{\Gamma, s \to_A t}[\gamma] : s[\gamma] \to_{A[\gamma]} t[\gamma]$

This case is the same as that for (OP).

If $x \propto t$ we say x is linear in t.

3.1.1 Cones as Context

Motivated by the previous section we can now spell out a set of rules which build cones over a diagram. Notably we do not restrict ourselves to any particular way of gluing. All legitimate ways of gluing cells to obtain higher projections are permitted and lead to distinct but valid cones.

The idea is to induct over the underlying context, adding precisely one higher projection for each such new variable. For the sake of readability and reusability, let us introduce a shorthand for the side conditions that will appear in the rules. Given a term $K \vdash t : A$, a variable $(x : X) \in K$ and a sublist of variables Π of K, we let $\delta\text{Cond}(t : A, x : X, c : \texttt{Ob}, \Pi)$ stand for

- $t : A$ is categorical
- $\text{FV}(t : A) = \text{FV}(x : X) \cup \bigcup_{\substack{p \in \text{FV}(\Pi) \\ y \in \text{FV}(\delta(x) : \partial X) \\ y \propto \tau(p)}} \text{FV}(p : T_p)$
- $x \propto t$

where δ stands for σ or τ. We will also denote by $\delta\overline{\text{Cond}}(t : A, x : X, \Pi)$ the same set of conditions except for the linearity condition, and with $\text{FV}(x : X)$ in $\text{FV}(t : A)$ replaced by $\{c\}$. In the following rules, the set Π will be such that it contains precisely the projections. These conditions ensure that the term t is built using the appropriate projections.

Rules for cones as contexts. We introduce a new auxiliary judgment, denoted by $K \; \texttt{cone} \; (\Gamma; c)$. This judgement is subject to the rules

$$\frac{}{c : \texttt{Ob} \; \texttt{cone} \; (\emptyset; c)} \; (\text{EK})$$

$$\frac{\Gamma, c : \texttt{Ob}, \Pi \; \texttt{cone} \; (\Gamma; c) \qquad \Gamma, x : X, c : \texttt{Ob}, \Pi \vdash s \to_A t}{\Gamma, x : X, c : \texttt{Ob}, \Pi, p_x : s \to_A t \; \texttt{cone} \; ((\Gamma, x : X); c)} \; (\text{KE}) \qquad \begin{array}{l} \tau\overline{\text{Cond}}(s : A, x : X, c : \texttt{Ob}, \Pi) \\ \sigma\text{Cond}(t : A, x : X, \Pi) \end{array}$$

If the judgment $K \; \texttt{cone} \; (\Gamma; c)$ is derivable we say K is a cone over Γ with apex c. The variable p_x is called the projection corresponding to the variable x.

The first rule, to be thought of as a base case, says: a cone over the empty diagram is just the apex. The second rule, the inductive step, assumes we are given a cone K over a diagram Γ. Then, given a context extension $\Gamma, x : X$ we first of all ask whether a certain type $s \to_A t$ can be built in K extended by $x : X$. This type together with the variable $x \in \text{FV}(\Gamma)$ must satisfy certain conditions, ensuring that it is of the form as in the previous section (up to coherence). If this is the case, then we append a new variable p_x to K, which we think of as the (higher) projection corresponding to the variable $x \in \mathrm{FV}(\Gamma)$.

The following lemma assures us that the above rules are well formed, allowing us to interpret them as a recognition algorithm for context which are cones over a given context. Recall, here that the length of a context Γ is denote by $|\Gamma|$.

Lemma 3.2. *The rule*

$$\frac{K \ \texttt{cone}\ (\Gamma; c)}{K \vdash}$$

is admissible. Moreover, given a judgment K $\texttt{cone}$ $(\Gamma; c)$, the context K is of the form $\Gamma, c : \texttt{Ob}, \Pi$ for some list of variables Π such that $|\Gamma| = |\Pi|$.

Proof. By induction. □

3.1.2 Existence of Cones

In the situation where the underlying diagram Γ is globular we can give an explicit construction and provide an existence proof for the cone.

Theorem 3.3. *Let $\Gamma \vdash$ be a globular context. There exists a context*

$$C(\Gamma) \equiv \Gamma,\ c : \texttt{Ob},\ \Pi, \qquad \text{where} \qquad \Gamma \equiv (x_i : A_i)_{1 \leq i \leq n} \text{ and } \Pi \equiv (p_{x_i} : T_{x_i})_{1 \leq i \leq n}$$

such that $C(\Gamma)$ $\texttt{cone}$ (Γ, c) is derivable. Moreover, the type T_x of the projection p_x can be taken to be of the form

$$\begin{aligned} &c \to x, && \text{if } d = 0 \\ &p_{\tau(x)} \to p_{\sigma(x)} \overset{d}{\underset{d-1}{*}} \left(1^d_{p_{\sigma^2(x)}} \overset{d}{\underset{d-2}{*}} \cdots \left(1^d_{p_{\sigma^d(x)}} \overset{d}{\underset{0}{*}} x\right)\right), && \text{if } d > 0. \end{aligned}$$

where $d = \dim(x)$.

Proof. We prove the statement by induction over the length of Γ. As part of the inductive process we also show that for a given variables $(x : A) \in \Gamma$, the free variables of the corresponding projections are given by

$$\mathrm{FV}(p_x : T_x) = \bigcup_{\substack{p_{x_i} \in \mathrm{FV}(\Pi) \\ y \in \mathrm{FV}(x : A) \\ y \propto \tau(p_{x_i})}} \mathrm{FV}(p_{x_i} : T_{x_i}). \tag{1.20}$$

In addition to that we have $c \in \mathrm{FV}(p_x : T_x)$ and x is the unique variable in Γ such that $x \propto \tau(p_x)$.

If $\Gamma \equiv \emptyset$, then the rules directly give us $c : \texttt{Ob}$ as a cone over $\emptyset$ with apex c. Assume now that we have built the cone for the context Γ, that is, we have a derivation of $C(\Gamma)\ \texttt{cone}\ (\Gamma; c)$ as described in the statement. Assume moreover that we now extend the context Γ to $\Gamma, x : A$. If $\dim(x) = 0$, then $c \to x$ is derivable and satisfies the condition of the rule (KE). Thus, applying (KE) we can extend the context by $p_x : c \to x$, giving us a cone over $\Gamma, x : A$. Moreover, $p_x : c \to x$ satisfies equation 1.20.

Assume now that $d := \dim(x) > 0$ and define the terms

$$t^{d+1}(x) :\equiv x : A, \qquad t^n(x) :\equiv 1^{n-1}_{p_{\sigma^n(x)}} \mathop{*}^{d}_{d-n} t^{n+1}(x) : T_n(x) \tag{1.21}$$

for $1 \leq n \leq d$ and where T is the appropriate type. The term $t^n(x)$, which is of the form

$$t^n(x) :\equiv 1^{n-1}_{p_{\sigma^n(x)}} \mathop{*}^{d}_{d-n} \left(1^{n}_{p_{\sigma^{n+1}(x)}} \mathop{*}^{d}_{d-n-1} \cdots \left(1^{d-1}_{p_{\sigma^d(x)}} \mathop{*}^{d}_{0} x\right)\right).$$

approximates $\tau(p_x)$ as n decreases. We claim that for all $1 \leq n \leq d+1$ we have

(i) $t^n(x) : T_n(x)$ derivable

(ii) $\sigma^m(t^n(x)) \equiv t^{n-m}(\sigma^m(x))$ for all $0 \leq m < n$

(iii) $\tau(t^n(x)) \equiv t^{n-1}(\tau(x))$ if $2 \leq n$

(iv) $\mathrm{FV}(t^n(x) : T_n(x)) = \mathrm{FV}(x : A) \cup \mathrm{FV}(p_{\sigma^n(x)} : T_{\sigma^n(x)})$ if $n < d+1$

(v) $x \propto t^n(x)$

For the statements all the statements we now perform a nested induction on n starting with $n = d+1$ all the way down to $n = 1$. We start with (i) and (ii), which we prove simultaneously. For $n = d+1$ we have $t^{d+1}(x) \equiv x : A$ which is derivable so that (i) holds. Now since $\sigma^m(x)$ is of dimension $\dim(x) - m$ we also have $t^{d+1-m}(\sigma^m(x)) \equiv \sigma^m(x)$ which proves (ii). Assume now that the statements hold for some $1 < n$. For (i) we compute

$$\sigma^{n-1}(t^n(x)) \equiv t^1(\sigma^{n-1}(x)) \equiv \tau(p_{\sigma^{n-1}(x)}) \equiv \tau^{n-1}\left(1^{n-2}_{p_{\sigma^{n-1}(x)}}\right) \tag{1.22}$$

Here the first and second identities hold by the inductive hypothesis. Notice that in order to define $t^{n-1}(x)$ starting with $t^n(x)$ we also made use of $p_{\sigma^{n-1}(x)}$. But $\sigma^{n-1}(x)$ is a variable which necessarily appears before x, so that the context currently at hand already contains the corresponds projection, allowing us to form this term. From equation 1.22 we conclude that the two terms in

$$t^{n-1}(x) \equiv 1^{n-2}_{p_{\sigma^{n-1}(x)}} \mathop{*}^{d}_{d-n+1} t^n(x) : T_{n-1}(x)$$

are composable, rendering the composite derivable. Regarding (ii), for all $0 \leq m < n-1$ we have

$$\begin{aligned}
\sigma^m(t^{n-1}(x)) &\equiv \sigma^m\Big(1^{n-2}_{p_{\sigma^{n-1}(x)}} \underset{d-n+1}{\overset{d}{*}} t^n(x)\Big) \\
&\equiv \sigma^m\Big(1^{n-2}_{p_{\sigma^{n-1}(x)}}\Big) \underset{d-n+1}{\overset{d-m}{*}} \sigma^m(t^n(x)) \\
&\equiv 1^{n-m-2}_{p_{\sigma^{n-1}(x)}} \underset{d-n+1}{\overset{d-m}{*}} t^{n-m}(\sigma^m(x)) \\
&\equiv t^{n-m-1}(\sigma^m(x))
\end{aligned}$$

where in the second line we used the fact that (ii) holds for $t^n(x)$.

Now, for (iii), an argument similar to that for the base case of (ii) shows that (iii) holds which applies since $n = d+1 \geq 2$. Now, assuming $n-1 \geq 2$, we compute

$$\begin{aligned}
\tau(t^{n-1}(x)) &\equiv \tau\Big(1^{n-2}_{p_{\sigma^{n-1}(x)}} \underset{d-n+1}{\overset{d}{*}} t^n(x)\Big) \\
&\equiv \tau\Big(1^{n-2}_{p_{\sigma^{n-1}(x)}}\Big) \underset{d-n+1}{\overset{d-1}{*}} \tau(t^n(x)) \\
&\equiv 1^{n-3}_{p_{\sigma^{n-1}(x)}} \underset{d-n-1}{\overset{d-1}{*}} t^{n-1}(\tau(x)) \\
&\equiv 1^{n-3}_{p_{\sigma^{n-2}(\tau(x))}} \underset{d-n-1}{\overset{d-1}{*}} t^{n-1}(\tau(x)) \\
&\equiv t^{n-2}(\tau(x))
\end{aligned}$$

For (iv) we have $\mathrm{FV}(t^d(x)) = \mathrm{FV}(1^{d-1}_{p_{\sigma^d(x)}} \overset{d}{\underset{0}{*}} x) = \mathrm{FV}(p_{\sigma^d(x)} : T_{\sigma^d(x)}) \cup \mathrm{FV}(x : A)$. For the inductive step,

$$\begin{aligned}
\mathrm{FV}(t^n(x) : T_n(x)) &= \mathrm{FV}\Big(1^{n-1}_{p_{\sigma^n(x)}} \underset{d-n}{\overset{d}{*}} t^{n+1}(x) : T_n(x)\Big) \\
&= \mathrm{FV}(p_{\sigma^n(x)} : T_{\sigma^n(x)}) \cup \mathrm{FV}(t^{n+1}(x) : T_n(x)) \\
&= \mathrm{FV}(p_{\sigma^n(x)} : T_{\sigma^n(x)}) \cup \mathrm{FV}(p_{\sigma^{n+1}(x)} : T_{\sigma^{n+1}(x)}) \cup \mathrm{FV}(x : A) \\
&= \mathrm{FV}(p_{\sigma^n(x)} : T_{\sigma^n(x)}) \cup \mathrm{FV}(x : A)
\end{aligned}$$

where we used the fact that $\mathrm{FV}(p_{\sigma^{n+1}(x)} : T_{\sigma^{n+1}(x)}) \subset \mathrm{FV}(p_{\sigma^n(x)} : T_{\sigma^n(x)})$.

As for (v), by definition $t^{d+1}(x) \equiv x : A$, and indeed, x is linear in x by definition. Now assume that $x \propto t^n(x)$ for some $1 \leq n$ and we ask if x is linear in $t^{n+1}(x) \equiv 1^{n-1}_{p_{\sigma^n}(x)} \underset{d-n}{\overset{d}{*}} t^{n+1}(x) : T_{n-1}(x)$, which is the case, as it only appears in $t^{n+1}(x)$.

Using the properties (ii) and (iii), we find

$$\tau(t^1(x)) \equiv \tau\Big(p_{\sigma(x)} \underset{d-1}{\overset{d}{*}} t^2\Big) \equiv \tau(t^2(x)) \equiv t^1(\tau(x)) \equiv \tau(p_{\tau(x)})$$

$$\sigma(t^1(x)) \equiv \sigma\Big(p_{\sigma(x)} \underset{d-1}{\overset{d}{*}} t^2\Big) \equiv \sigma\big(p_{\sigma(x)}\big) \equiv p_{\sigma^2(x)} \equiv p_{\sigma\tau(x)} \equiv \sigma\big(p_{\tau(x)}\big)$$

from which we conclude $t^1(x) \parallel p_{\tau(x)}$. This allows us to derive the judgment $p_{\tau(x)} \to t^1(x)$.

Now, by the inductive hypothesis

$$\mathrm{FV}(p_{\tau(x)} : T_{\tau(x)}) = \mathrm{FV}(c : \mathtt{Ob}) \cup \bigcup_{\substack{p_{x_i} \in \mathrm{FV}(\Pi) \\ y \in \mathrm{FV}(\tau(x) : T_{\tau(x)}) \\ y \propto \tau(p_{x_i})}} \mathrm{FV}(p_{x_i} : T_{x_i}).$$

On the other hand, using (iv) we have

$$\begin{aligned} \mathrm{FV}(t^1(x) : T_1(x)) &= \mathrm{FV}(x : A) \cup \mathrm{FV}(p_{\sigma(x)} : T_{\sigma(x)}) \\ &= \mathrm{FV}(x : A) \cup \bigcup_{\substack{p_{x_i} \in \mathrm{FV}(\Pi) \\ y \in \mathrm{FV}(\sigma(x) : T_{\sigma(x)}) \\ y \propto \tau(p_{x_i})}} \mathrm{FV}(p_{x_i} : T_{x_i}) \end{aligned}$$

which shows that $p_{\tau(x)} \to t^1(x)$ satisfies the conditions of (KE) allowing us to produce the cone $C(\Gamma, x : A) \equiv \Gamma, x : A, c : \mathtt{Ob}, \Pi, p_x : p_{\tau(x)} \to t^1(x)$.

To complete the induction we need to show that equation 1.20 still holds for all variables in Γ. On top of that we need to show that $x : A$ satisfies equation 1.20 as well as the two properties stated right after. We start with the latter.

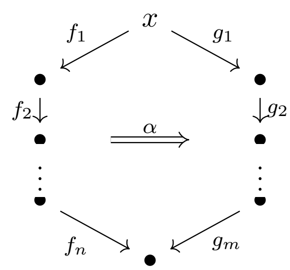
A quick inductive argument shows that $c \in \mathrm{FV}(p_x : T_x)$. Also, by (iv)

$$\begin{aligned} \mathrm{FV}(p_x : T_x) &= \mathrm{FV}(p_{\tau(x)} : T_{\tau(x)}) \cup \mathrm{FV}(t^1(x) : T_1(x)) \\ &= \mathrm{FV}(p_{\tau(x)} : T_{\tau(x)}) \cup \mathrm{FV}(p_{\sigma(x)} : T_{\sigma(x)}) \cup \mathrm{FV}(x : A). \end{aligned}$$

Since $\sigma(x) \propto \tau(p_{\sigma(x)})$ and $\tau(x) \propto \tau(p_{\tau(x)})$, we know that the variables $p_{\sigma(x)}, p_{\tau(x)}$ and x all have the same dimension. Moreover, by lemma 2.1 these are the only variables of dimension $\dim(x)$. Thus x is the unique variable of Γ such that $x \propto \tau(p_x)$. This fact guarantees that p_x satisfies equation 1.20 and that all the projections in Π continue to satisfy the same equation in the extended cone over $\Gamma, x : A$. □

It is possible to give algorithmic constructions for the cones also for more complicated diagrams in the strict world. In the following examples we will suppress the labels of the binary operations in the interest of making the equations more readable and the global form more evident.

Example 3.4. Consider for example a diagram of the form



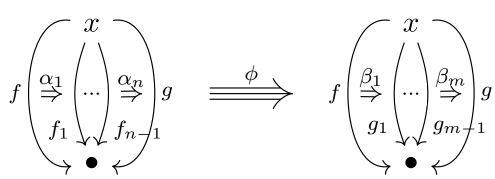
Then, in the strict case, the projection corresponding to the variable α can be taken to be of the type:

$$p_{g_m} \cdot (p_{g_{m-1}} * 1_{g_m}) \cdot (p_{g_{m-2}} * 1_{g_{m-1} \cdot g_m}) \cdots (p_{g_1} * 1_{g_2 \cdots g_m})$$
$$\downarrow$$
$$p_{f_n} \cdot (p_{f_{n-1}} * 1_{f_n}) \cdot (p_{f_{n-2}} * 1_{f_{n-1} \cdot f_n}) \cdots (p_{f_1} * 1_{f_2 \cdots f_n}) \cdot (1_{p_x} * \alpha).$$

By defining the terms $p_{f_1 \cdots f_n}$ and $p_{g_1 \cdots g_m}$ appropriately we can rewrite this as

$$p_{g_1 \cdots g_m} \to p_{f_1 \cdots f_n} \cdot (1_{p_x} * \alpha).$$

Example 3.5. Consider the diagram



For this diagram, again in the strict case, the projection corresponding to the variable ϕ can be taken to be of the type:

$$p_{\beta_m} \cdot (p_{\beta_{m-1}} * 1_{1_{p_x} * \beta_m}) \cdot (p_{\beta_{m-2}} * 1_{1_{p_x} * (\beta_{m-1} \cdot \beta_m)}) \cdots (p_{\beta_1} * 1_{1_{p_x} * (\beta_2 \cdots \beta_m)})$$
$$\downarrow$$
$$p_{\alpha_n} \cdot (p_{\alpha_{n-1}} * 1_{1_{p_x} * \alpha_n}) \cdot (p_{\alpha_{n-2}} * 1_{1_{p_x} * (\alpha_{n-1} \cdot \alpha_n)}) \cdots (p_{\alpha_1} * 1_{1_{p_x} * (\alpha_2 \cdots \alpha_n)}) \cdot (1_{p_f} * (1_{1_{p_x}} * \phi))$$

Again, by defining $p_{\alpha_1 \cdots \alpha_n}$ and $p_{\beta_1 \cdots \beta_m}$ appropriately we can rewrite this as

$$p_{\beta_1 \cdots \beta_m} \to p_{\alpha_1 \cdots \alpha_n} \cdot (1_{p_f} * (1_{1_{p_x}} * \phi)).$$

Returning to the weak higher categorical world, for low dimensional cases it is possible to construct a term p_t also for the case where t is a coherence. The examples suggest that, given a term $\Gamma \vdash t : A$ in a context, it is possible to glue the projections corresponding to the variables of $t : A$ appropriately, and build a term p_t, the type of which is of a type analogous to that in Theorem 3.3. Based on this we formulate the conjecture.

We prove the statement for globular context.

Conjecture 3.6. *Let K be a cone over a diagram Γ with apex c. Given a term $\Gamma \vdash t : A$ of dimension d, there exists a term $K \vdash p_t : T_t$ such that T_t is given by*

$$\begin{cases} c \to t & \text{if } \dim(t) = 0 \\ p_{\tau(t)} \to p_{\sigma(t)} \overset{d}{\underset{d-1}{*}} \left(1^1_{p_{\sigma^2(t)}} \overset{d}{\underset{d-2}{*}} \left(1^2_{p_{\sigma^3(t)}} \overset{d}{\underset{d-3}{*}} \cdots \left(1^{d-1}_{p_{\sigma^d(t)}} \overset{d}{\underset{0}{*}} t\right)\right)\right), & \text{if } \dim(t) > 0. \end{cases}$$

Moreover, writing $K \equiv \Gamma, c : \mathtt{Ob}, \Pi$, we have

$$\mathrm{FV}(p_t : T_t) = \bigcup_{\substack{p_{x_i} \in \mathrm{FV}(\Pi) \\ y \in \mathrm{FV}(t : A) \\ y \propto \tau(p_{x_i})}} \mathrm{FV}(p_{x_i} : T_{x_i}).$$

Assuming the conjecture we can now prove the existence of cones as contexts for all diagrams.

Theorem 3.7. *Given a context Γ there exists a context $\Gamma, c : \mathtt{Ob}, \Pi$ such that the judgment $\Gamma, c : \mathtt{Ob}, \Pi \,\mathtt{cone}\, \Gamma$ is derivable.*

Proof. This proof is similar to that of Theorem 3.3. □

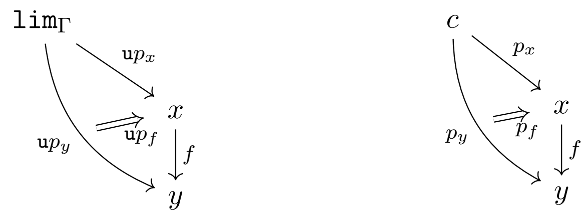
3.1.3 The Universal Cone

The rules of the judgment $K \ \mathtt{cone}\ (\Gamma; c)$ allow us to generate cones as contexts for a given diagram. We now want to use this to build the universal cone. Given a diagram Γ and a cone K over it as a context K, the universal cone is given by a collection of terms that can be assembled into a context morphism $\Gamma \vdash \mathtt{ucone} : K$.

Example 3.8. Consider for example the context $\Gamma \equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y$. All the terms of the universal cone assemble into the context morphisms

$$\Gamma \vdash \left\langle \begin{matrix} x \\ y \\ f \\ \mathtt{lim}_\Gamma \\ \mathtt{u}p_x \\ \mathtt{u}p_y \\ \mathtt{u}p_f \end{matrix} \right\rangle : \left(\begin{matrix} x : \mathtt{Ob} \\ y : \mathtt{Ob} \\ f : x \to y \\ c : \mathtt{Ob} \\ p_x : c \to x \\ p_y : c \to y \\ p_f : p_y \to p_x \cdot f \end{matrix} \right)$$

Pictorially we have:



(1.23)

It is important to note that we explicitly build in new terms only for the cone apex and the projections in the context morphism. The cells in the context morphism corresponding to the underlying diagram are given by variables. Intuitively we think of this as allowing us to build a cone over a diagram of arbitrary objects and morphisms. By substitution this can be instantiated to any other specifically chosen objects and morphisms.

Unfortunately, we cannot simply spell out a rule which builds such a context morphism in one go. Due to the nature of type theory, any context morphism must be built step by step, otherwise there would be no way of accessing the terms it consists of. We are therefore forced to build the terms of the universal cone one by one with help of term constructor rules. Crucially, however, the higher cells of the universal cone depend on those of lower dimension, as dictated by the underlying diagram Γ. As a result, when constructing a given term in the universal cone, we need to have kept track of all terms of the universal cone corresponding to the dependencies of the term at hand. We tackle this by collecting all terms at each step into a context morphism, which step by step approximates the context morphism $\Gamma \vdash \texttt{ucone} : K$.

Rules for the universal cone. The terms of the universal cone are built using the following term constructor

$$\frac{(\Theta, x : X, \Theta') \;\; \texttt{cone} \;\; (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta}{\Gamma \vdash \texttt{ucone}_{\Gamma,x} : X[\kappa]}$$

The above term constructor refers to a new auxiliary judgment $\Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta$ which we introduce and which is subject to the rules

$$\frac{\Gamma \vdash}{\Gamma \vdash^{\texttt{uni}}_{\Gamma} \texttt{id}_{\Gamma} : \Gamma}$$

$$\frac{(\Theta, x : X, \Theta') \;\; \texttt{cone} \;\; (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta}{\Gamma \vdash^{\texttt{uni}}_{\Gamma} \langle \kappa, \texttt{ucone}_{\Gamma,x} \rangle : (\Theta, x : X)}$$

The term $\texttt{ucone}_{\Gamma,x}$ also depends on the cone $\Theta, x : X, \Theta'$ which should therefore also appear as a subscript in the term constructor. We suppress this for the sake of readability.

As in Example 3.8, we will also write $\texttt{lim}_\Gamma :\equiv \texttt{ucone}_{\Gamma,c}$ where $c : \texttt{Ob}$ is the variable such that $K \equiv \Gamma, c : \texttt{Ob}, \Pi$. When no confusion can arise we will also write $\texttt{u}x :\equiv \texttt{ucone}_{\Gamma,x}$, again as in the example.

Lemma 3.9. *Given a judgment* $\Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta$ *we have*

$$\Gamma \vdash, \qquad \Theta \vdash, \qquad \Gamma \vdash \kappa : \Theta.$$

Moreover, the context $\Gamma \vdash \kappa : \Theta$ *is of the form* $\Gamma \vdash \langle \texttt{id}_\Gamma, \texttt{lim}_\Gamma, \texttt{u}p_1, \ldots, \texttt{u}p_m \rangle : (\Gamma, c : \texttt{Ob}, p_1 : T_1, \ldots, p_m : T_m)$ *for some* $1 \leq m \leq |\Gamma|$.

Proof. By induction. □

The process terminates once we have derived the judgment $\Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : K$ where K is some cone over Γ, producing a context morphism as in Example 3.8. In the same way that K factorizes as $\Gamma, c : \texttt{Ob}, \Pi$, we have a corresponding factorization $\kappa \equiv \langle \texttt{id}_\Gamma, \lim_\Gamma, \overline{\texttt{u}p} \rangle$. Here we use the notation $\overline{\texttt{u}p}$ to emphasize that we have a string of terms of the form $\texttt{u}p$.

Definition 3.10. *If* K *is a cone over* Γ*, given a derivable judgment* $\Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : K$ *we say* $\Gamma \vdash \kappa : K$*, or simply* κ*, is a universal cone over* Γ *of shape* K*.*

Remark 3.11. A cone morphism contains the same information as a cone on a cone on the underlying diagram. With minor modifications it is possible to make use of the rules for cones, to generate the cells of the universal cone morphism from an arbitrary cone to the universal cone. In fact, cones might suffice to encode all higher transforms as well, making it possible to spell out the entire universal property only in terms of cones. Instead of this we will pursue another path by giving a rules which produce contexts with the shape of modifications between cones, as well as all higher transfors in a uniform way.

3.2 The Universal Property

3.2.1 Gray Operations

In this section we spell out a set of rules which generate the cells of the Gray tensor product of a diagram with a d-globe. Following Loubaton [56] we refer to these constructions as Gray operations. We think of these as shapes for natural transformations and all higher transfors. The natural transformation is between two diagrams represented as contexts, one of which is the copy of the other. Note that at this stage we are only interested in generating the shape as a context.

In the next section we will modify the rules so as to produce higher transfors between cones (thought of as natural transformations) and use this later to define the higher coherences of the universal cone as a context morphism.

The starting point is a diagram Γ given to us as a context. We then proceed in two steps: first each variable $x \in \mathrm{FV}(\Gamma)$ is duplicated and second for each such variable x and its duplicate x' we append a new cell relating x and x' in a specified way. Let us showcase this with an example.

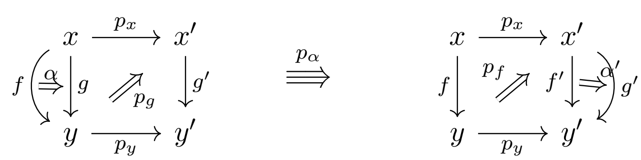
Example 3.12. Consider the diagram $\Gamma \equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y, g : x \to y, \alpha : f \to g$. Duplication yields:

$$\begin{aligned}\Gamma, \Gamma' :\equiv\ & x : \mathtt{Ob},\ \ y : \mathtt{Ob},\ \ f : x \to y, \quad g : x \to y, \quad \alpha : f \to g, \\ & x' : \mathtt{Ob},\ y' : \mathtt{Ob},\ f' : x' \to y',\ g' : x' \to y',\ \alpha' : f' \to g'.\end{aligned}$$

Next for each variable in Γ and its corresponding duplicate we add a cell as follows:

$$\begin{aligned}\Gamma, \Gamma',\ & p_x : x \to x',\ \ p_y : y \to y', \\ & p_f : f \cdot p_y \to p_x \cdot f',\ \ p_g : g \cdot p_y \to p_x \cdot g', \\ & p_\alpha : (\alpha * 1_{p_y}) \cdot p_g \to p_f \cdot (1_{p_x} * \alpha').\end{aligned}$$

For later use we may also denote this context by Γ, Γ', P. Pictorially we have



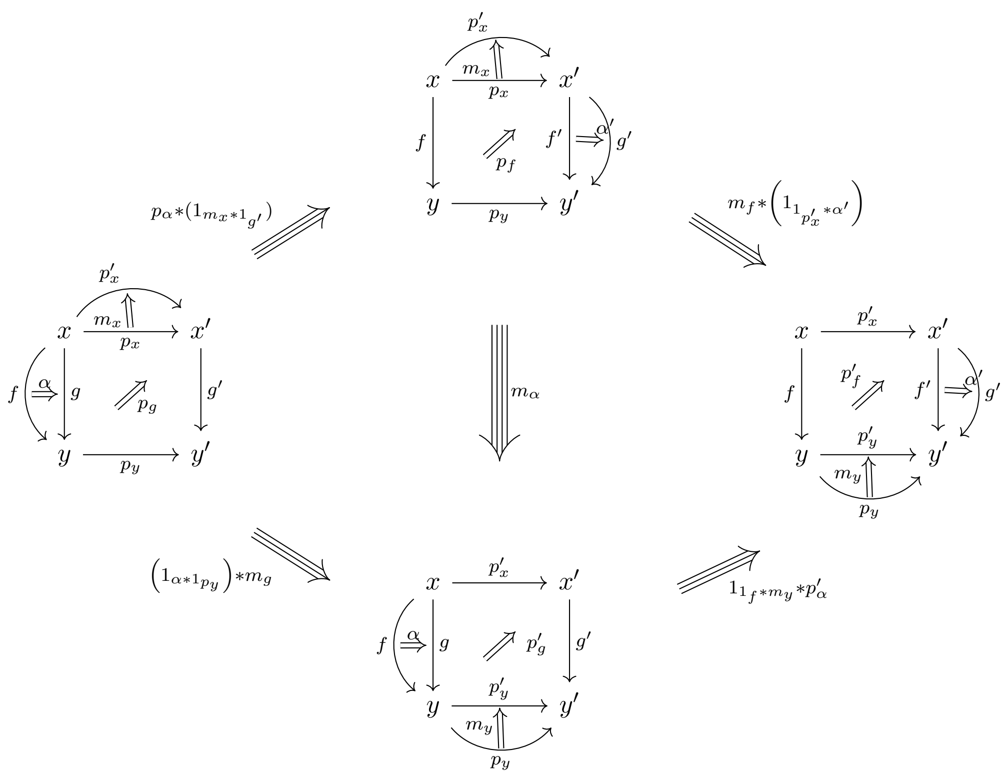
Example 3.13. Continuing Example 3.12, let us now examine modifications. For this we begin with Γ, Γ', P and extend this context by adding the appropriate new variables. As for natural transformations we begin by duplicating certain cells. The cells we duplicate turn out to be precisely the variables in P. Indeed we have

$$\begin{aligned}\Gamma, \Gamma',\ & p_x : x \to' x,\ \ p_y : y \to y', \\ & p_f : f \cdot p_y \to p_x \cdot f',\ \ p_g : g \cdot p_y \to p_x \cdot g', \\ & p_\alpha : (\alpha * 1_{p_y}) \cdot p_g \to p_f \cdot (1_{p_x} \cdot \alpha'), \\ & p'_x : x \to x',\ \ p'_y : y \to y', \\ & p'_f : f \cdot p'_y \to p'_x \cdot f',\ \ p'_g : g \cdot p'_y \to p'_x \cdot g', \\ & p'_\alpha : (\alpha * 1_{p'_y}) \cdot p'_g \to p'_g \cdot (1_{p'_x} \cdot \alpha').\end{aligned}$$

As a short hand, let us denote this context by Γ, Γ', P, P' where P' contains all the duplicates of P. Again, as for the natural transformation we now need to build in cells relating the variables in P with their duplicates, as follows:

$$\begin{aligned}
&\Gamma, \Gamma', P, P', m_x : p_x \to p'_x,\ \ m_y : p_y \to p'_y \\
&\qquad\qquad m_f : p_f \cdot (m_x * 1_{f'}) \to (1_f * m_y) \cdot p'_f \\
&\qquad\qquad m_g : p_g \cdot (m_x * 1_{g'}) \to (1_g * m_y) \cdot p'_g \\
&\qquad\qquad m_\alpha : (p_\alpha * 1_{m_x * 1_{g'}}) \cdot (m_f * 1_{1_{p'_x} * \alpha'}) \to (1_{\alpha * 1_{p_y}} * m_g) \cdot (1_{1_f * m_y} * p'_\alpha).
\end{aligned}$$

which we may also abbreviate as $\Gamma, \Gamma', P, P', M$. Diagrammatically the cell m_α can be visualized as follows:



We now formalize this procedure in terms of type-theoretic rules producing a context of the form as in Examples 3.12 and 3.13.

To keep track of the structure of these contexts we will use a semicolon instead of a comma to separate the constituent parts. The variables in between will be separated by commas as usual. With this convention, the context of Example 3.12 will be denoted by $\Gamma; \Gamma'; P; P'; M$.

Rules for transfors as contexts. We introduce two new auxiliary judgment, $M_1; \ldots; M_{2n+1}$ `gray` Γ and $M_1; \ldots; M_{2n+1}$ `pgray` M_1, where $n \in \mathbb{N}^{>}$, subject to the rules

$$\frac{}{\emptyset; \ldots; \emptyset \ \texttt{gray} \ \emptyset}$$

$$\frac{M_1; \ldots; M_{2n+1} \ \texttt{gray} \ M_1 \qquad M_1 \vdash X}{M_1, x : X; \ldots; M_{2n+1} \ \texttt{pgray} \ M_1, x : X}$$

$$\frac{\begin{array}{l} M_1; \ldots; M_{2i-1}, x : X;\ M_{2i}; \ldots; M_{2n+1} \ \texttt{pgray} \ M_1 \\ M_1, \ldots, M_{2i-1}, x : X,\ M_{2i}, x' : X,\ M_{2i+1} \vdash s \to_A t \end{array}}{M_1; \ldots; M_{2i-1}, x : X;\ M_{2i}, x' : X;\ M_{2i+1}, p_x : s \to_A t;\ M_{2i+2}; \ldots; M_{2n+1} \ \texttt{pgray} \ M_1} \quad \begin{array}{c} 1 \le i \le n \\ |M_{2i-1}| = |M_{2i}| \\ \tau\mathrm{Cond}(s : A, x : X, M_{2i+1}) \\ \sigma\mathrm{Cond}(t : A, x' : X, M_{2i+1}) \end{array}$$

$$\frac{M_1; \ldots; M_{2n+1} \ \texttt{pgray} \ M_1}{M_1; \ldots; M_{2n+1} \ \texttt{gray} \ M_1} \qquad |M_1| = |M_{2n+1}|$$

Lemma 3.14. *The rules*

$$\frac{M_1; \ldots; M_{2n+1} \ \texttt{gray} \ M_1}{M_1, \ldots, M_{2n+1} \vdash} \qquad \frac{M_1; \ldots; M_{2n+1} \ \texttt{pgray} \ M_1}{M_1, \ldots, M_{2n+1} \vdash}$$

are admissible.

Proof. By induction. □

If $M_1; \ldots; M_{2n+1}$ `gray` M_1 is derivable, then we say the context $M_1, \ldots, M_{2n+1}$ is the Gray tensor product of M_1 with an n-globe. We interpret these above rules as a recognition algorithm for those context which have the shape of a transfor over the given context.

As for cones we can formulate an existence proof for the Gray tensor product with 1-globes. The pattern that emerges simply generalizes that of cones.

Theorem 3.15. *Let Γ be a globular context. There exists a context*

$$T(\Gamma) \equiv \Gamma, \Gamma', M$$

such that $\Gamma; \Gamma'; M$ `gray` *Γ is derivable. If $\Gamma \equiv (x_i : A_i)_{1 \le i \le n}$, then Γ' is of the form $(x'_i : A_i)_{1 \le i \le n}$ and Π of the form $(p_{x_i} : T_{x_i})_{1 \le i \le n}$. Moreover, the type T_x of the projection p_x of a variable x of dimension d can be taken to be*

$$\begin{array}{ll} x \to x' & \textit{if } d = 0 \\ \sigma(p_x) \to \tau(p_x) & \textit{if } d > 0 \end{array}$$

where

$$\sigma(p_x) = \left(\left(\left(x \overset{d}{\underset{0}{*}} 1^{d-1}_{p_{\tau^d(x)}}\right) \cdots \overset{d}{\underset{d-3}{*}} 1^2_{p_{\tau^3(x)}}\right) \overset{d}{\underset{d-2}{*}} 1^1_{p_{\tau^2(x)}}\right) \overset{d}{\underset{d-1}{*}} p_{\tau(x)}$$

$$\tau(p_x) = p_{\sigma(x)} \overset{d}{\underset{d-1}{*}} \left(1^1_{p_{\sigma^2(x)}} \overset{d}{\underset{d-2}{*}} \left(1^2_{p_{\sigma^3(x)}} \overset{d}{\underset{d-3}{*}} \cdots \left(1^{d-1}_{p_{\sigma^d(x)}} \overset{d}{\underset{0}{*}} x'_i\right)\right)\right)$$

Proof. This can be proven in complete analogy to Proposition 3.3. □

Remark 3.16. A similar formula has been derived using Steiner complexes by Ara and Maltsiniotis [2] (see Appendix B) in the context of strict ∞-categories.

3.2.2 Higher Transfors between Cones

We can repeat everything done in the previous section, and apply it to cones. The rules essentially have the same form, the only difference being that the context begins with a set of variables which define a cone.

Rules for higher transfors between cones. We introduce two new auxiliary judgments $\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}\ \mathtt{ctrf}\ (\Gamma; c)$ and $\Gamma; c : \mathtt{Ob}, M_1; \ldots; M_{2n+1}\ \mathtt{pctrf}\ (\Gamma; c)$, where $n \in \mathbb{N}^{>}$ is a strictly positive number, subject to the rules

$$\frac{}{\emptyset; c : \mathtt{Ob}; \emptyset; \ldots; \emptyset\ \mathtt{ctrf}\ (\emptyset, c)}$$

$$\frac{\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}\ \mathtt{ctrf}\ (\Gamma; c) \qquad \Gamma, x : X, c : \mathtt{Ob}, M_1, p : T\ \mathtt{cone}\ ((\Gamma, x : X), c)}{\Gamma, x : X; c : \mathtt{Ob}; M_1, p : T; M_2; \ldots; M_{2n+1}\ \mathtt{pctrf}\ ((\Gamma, x : X), c)}$$

$$\frac{\begin{array}{l}\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}; \ldots; M_{2n+1}\ \mathtt{pctrf}\ (\Gamma; c) \\ \Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}, x' : X; M_{2i+1} \vdash s \to_A t\end{array}}{\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}, x' : X; M_{2i+1}, p_x : s \to_A t; \ldots; M_{2n+1}\ \mathtt{pctrf}\ (\Gamma; c)} \quad \begin{array}{c} 1 \le i \le n \\ |M_{2i-1}| = |M_{2i}| \\ \tau\mathrm{Cond}\Big(s : A, x : X, \mathrm{FV}(M_{2i+1})\Big) \\ \sigma\mathrm{Cond}\Big(t : A, x' : X, \mathrm{FV}(M_{2i+1})\Big)\end{array}$$

$$\frac{\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}\ \mathtt{pctrf}\ (\Gamma; c)}{\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}\ \mathtt{ctrf}\ (\Gamma; c)} \qquad |M_1| = |M_{2n+1}|$$

If the judgment $\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}\ \mathtt{ctrf}\ (\Gamma; c)$ is derivable we say that $\Gamma; c : \mathtt{Ob}; M_1; \ldots; M_{2n+1}$ is a conical $(n+1)$-transfor over Γ. A conical 2-transfor over Γ is a modification of cones over Γ.

The lists of variables generated by the above rules are contexts, as the following lemma affirms.

Lemma 3.17. *The rule*

$$\frac{\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1} \ \texttt{ctrf}\ (\Gamma; c)}{\Gamma, c : \texttt{Ob}, M_1, \ldots, M_{2n+1} \vdash}$$

is admissible.

Proof. By induction. □

Lemma 3.18. *Given a derivable judgment* $\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1}\ \texttt{ctrf}\ (\Gamma, c)$ *we have* $|\Gamma| = |M_i|$ *for all* $1 \leq i \leq 2n+1$. *Moreover* $K :\equiv \Gamma, c : \texttt{Ob}, M_1$ *is a cone over* Γ *with apex* c.

In some cases, instead of the full $\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1}\ \texttt{ctrf}\ (\Gamma; c)$, we will also write $M\ \texttt{ctrf}\ (\Gamma; c)$. In such situation we will write $n_M \in \mathbb{N}^{>}$ instead of n, to make clear which natural number we are referring to.

3.2.3 Composing with Cones

Let Γ be a diagram and assume we are given a universal cone $\Gamma \vdash \texttt{ucone} : K$ with apex $\lim_\gamma : \texttt{Ob}$. Given a morphism of type $c \to \texttt{lim}_\Gamma$, where $c : \texttt{Ob}$ is some variable, we can compose this with `ucone` and form a cone with apex c. More generally, given any cell in the (∞, ∞)-category of morphisms given by the type $c \to \texttt{lim}_\Gamma$, by composition we get a term in the (∞, ∞)-category of cones over Γ with apex c. Schematically there exists a functor of (∞, ∞)-categories

$$\texttt{ucone}_* : \{\text{terms of } c \to \texttt{lim}_\Gamma\} \longrightarrow \{\text{cones } K \text{ over } \Gamma \text{ with apex } c\}.$$

The cone `ucone` is a limiting cone if this functor is an equivalence.

Spelling out this universal property requires us to be able to refer to the image of this functor. We encode this again by a context morphism.

Let Γ be a context and let $\Gamma \vdash \texttt{ucone} : K$ be a context morphism encoding a universal cone obtained from the rules in Subsection 3.1.3. The cone will be of the form $K \equiv \Gamma, c : \texttt{Ob}, \Pi$ where Π contains all the projection variables. The context morphism `ucone` can then be split accordingly and we denote by $\overline{\texttt{up}}$ the list of terms in `ucone` corresponding to Π. Now, given a context $\Gamma, c : \texttt{Ob}, f : c \to \texttt{lim}_\Gamma$, applying $\texttt{ucone}_*$ to f gives a context morphism schematically denoted by

$$\Gamma, c : \texttt{Ob}, f : c \to \texttt{lim}_\Gamma \vdash \langle \texttt{id}_\Gamma, c, f * \overline{\texttt{up}} \rangle : K.$$

Here $f * \overline{\texttt{up}}$ stands for a list of terms of the same length as $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n}$, the i-th term of which is given by composing $\texttt{up}_{x_i}$ with f appropriately.

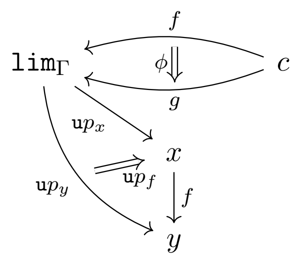
More generally, let M be a conical n-transfor over Γ. If $\Gamma, c : \mathtt{Ob}, D^n(c, \mathtt{lim}_\Gamma, \phi)$ is a context extending Γ where $D^n(c, c', \phi)$ is a

$$D^n(c, \mathtt{lim}_\Gamma, \phi) :\equiv f : c \to \mathtt{lim}_\Gamma,\ g : c \to \mathtt{lim}_\Gamma, \alpha : f \to g, \beta : f \to g, \ldots, \phi : A_\phi$$

has the shape of an n-dimensional globe between $\mathtt{lim}_\Gamma$ and some variable $c : \mathtt{Ob}$ with maximal cell ϕ, we would like to obtain a context morphism of the form

$$\Gamma, c : \mathtt{Ob}, D^n(c, \mathtt{lim}_\Gamma, \phi) \vdash \langle \mathtt{id}_\Gamma, c, f * \overline{\mathtt{up}},\ g * \overline{\mathtt{up}},\ \ldots\ ,\ \phi * \overline{\mathtt{up}} \rangle : M.$$

Up to a manageable dimension we can depict this pictorially as



In the above diagram, composing the universal cone with f and g respectively gives two cones. Composing with ϕ produces a modification between these two cones.

To obtain such a context morphism, we use the same trick as before and work with an arbitrary cone in the form of a context instead of the universal cone. This allows us to control which projections contribute in which terms by fixing the free variables. In a bit more detail, assume we are given a judgment $K\ \mathtt{cone}\ \Gamma$, that is K is a cone over Γ. It will necessarily be of the form $K \equiv \Gamma, c : \mathtt{Ob}, \Pi$, where Π contains all the projection variables. Then, what we are looking for a context morphism which schematically is of the form

$$K, c' : \mathtt{Ob}, D^n(c', c) \vdash \langle \mathtt{id}_\Gamma, c, f * \mathtt{id}_\Pi, g * \mathtt{id}_\Pi,\ \ldots\ , \phi * \mathtt{id}_\Pi \rangle : M$$

where $\mathtt{id}_\Pi$ denotes the variables $\mathrm{FV}(\Pi)$ as a sequence of terms with the induced order. In the next subsection we give a set of rules which produce such context morphisms.

3.2.4 Composing with Cones and the Universal Property

Let $\Gamma \vdash \kappa : K$ be a universal cone over a diagram Γ. As explained in subsection 3.2.3, the idea is to build into CaTT the required terms to make the functor

$$\mathtt{ucone}_* : \{\text{terms of } c \to \mathtt{lim}_\Gamma\} \longrightarrow \{\text{cones } K \text{ over } \Gamma \text{ with apex } c\}$$

into an equivalence of (∞, ∞)-categories. Switching momentarily to non type-theoretic notation, a functor $F : \mathcal{C} \to \mathcal{D}$ of (∞, ∞)-categories is said to be an equivalence if the maps

(i) $F : \mathcal{C}_0 \to \mathcal{D}_0$;

(ii) $F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ $x, y \in \mathcal{C}$ for all $x, y \in \mathcal{C}$;

(iii) $F_{f,g} : \mathrm{Hom}_{\mathrm{Hom}(x,y)}(f, g) \to \mathrm{Hom}_{\mathrm{Hom}(Fx,Fy)}(Ff, Fg)$ for all $f, g \in \mathrm{Hom}(x, y)$;

(iv) and so on.

are surjective.

By surjectivity we mean, essential surjectivity, i.e. surjectivity up to equivalence. By equivalence we mean coinductive equivalence. Coinductive equivalences have been studied by Benjamin and Markakis [12] in the related framework of computads due to Dean et al. [27]. Note that every coherence is a coinductive equivalence.

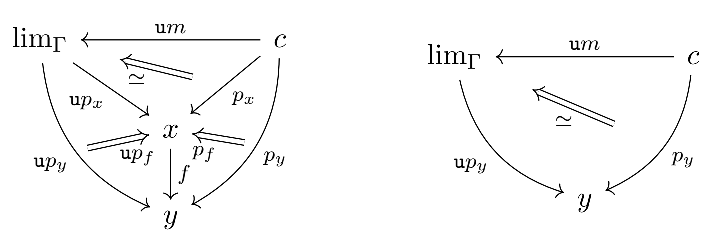
Rules for invertible morphisms. Given a term $\Gamma \vdash u : s \to t$ in `CaTT`, we can ensure this is an equivalence by introducing a new judgment $\Gamma \vDash u : s \to t$ which obeys the rules

$$\frac{\Gamma \vDash u : s \to t}{\Gamma \vdash u : s \to t \qquad\qquad \Gamma \vdash \mathtt{inv}(u) : t \to s}$$

$$\frac{\Gamma \vDash u : s \to t}{\Gamma \vDash \mathtt{eta}(u) : 1_s \to u \cdot \mathtt{inv}(u) \qquad\qquad \Gamma \vDash \mathtt{eps}(u) : \mathtt{inv}(u) \cdot u \to 1_t}$$

We say that an n-transfor is invertible, if all of its components are invertible.

Returning to $\mathtt{ucone}_*$, on objects we must require the following: given a universal cone $\Gamma \vdash \mathtt{ucone} : K$ over Γ, where $K \equiv (\Gamma, c : \mathtt{Ob}, \Pi)$, there exists a term $\mathtt{u}m : c \to \mathtt{lim}_\Gamma$ and an invertible modification $\overline{\mathtt{u}m}$ between the cone K and the universal cone composed with $\mathtt{u}m$. Working again with our example $\Gamma \equiv x : \mathtt{Ob}, y : \mathtt{Ob}, f : x \to y$, pictorially we have:



(1.24)

If $M \equiv (\Gamma; c : \texttt{Ob}; M_1; M_2; M_3)$ is a modification of cones, that is M ctrf $(\Gamma; c)$, then the whole data of diagram 1.24 can be organized into the context morphism

$$K \vdash \langle \texttt{id}_\Gamma, c, \texttt{id}_\Pi, \texttt{u}m * \overline{\texttt{u}p}, \overline{\texttt{u}m} \rangle : (\Gamma, c : \texttt{Ob}, M_1, M_2, M_3).$$

Here, if $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq n}$, $\texttt{u}m * \overline{m}$ denotes a list of $|\Gamma|$ terms, the i-th term of which is given by composing $\texttt{u}m$ with $\texttt{u}p_{x_i}$ appropriately. So in the above example we have $\texttt{u}m * \overline{\texttt{u}p} \equiv (\texttt{u}m \cdot \texttt{u}p_x, \texttt{u}m \cdot \texttt{u}p_y, 1_{\texttt{u}m} \overset{2}{\underset{0}{*}} \texttt{u}p_f)$. The notation $\overline{\texttt{u}m}$, on the other hand, contains all the components of the modification. In the above example this includes the two invertible cells, depicted in the diagram, as well as a 3-dimensional cell interpolating between the two diagrams.

At the next level we are first given two maps $f, g : c \to \texttt{lim}_\Gamma$ with which we obtain two cones $f * \overline{\texttt{u}p}$ and $g * \overline{\texttt{u}p}$ with apex c by composition. Then, given any modification between these two cones, namely a list of variables denoted by the shorthand notation $\overline{m} : f * \overline{\texttt{u}p} \to g * \overline{\texttt{u}p}$, there should exist a cell $\texttt{u}q : f \to g$, which when composed with the universal cone yields an modification $\texttt{u}q * \overline{\texttt{u}p}$ equivalent to $\overline{m}$. Using again a shorthand notation we denote this invertible map by $\overline{\texttt{u}q}$: $\overline{m} \to \texttt{u}q * \overline{\texttt{u}p}$. As a context morphism this is given by

$$\Gamma,\, c : \texttt{Ob},\, f : c \to \texttt{lim}_\Gamma,\, g : c \to \texttt{lim}_\Gamma,\, \overline{m} : f * \overline{\texttt{u}p} \to g * \overline{\texttt{u}p} \vdash \left\langle \begin{matrix} \texttt{id}_\Gamma \\ c \\ f * \overline{\texttt{u}p} \\ g * \overline{\texttt{u}p} \\ \overline{m} \\ \texttt{u}q * \overline{\texttt{u}p} \\ \overline{\texttt{u}q} \end{matrix} \right\rangle : \begin{pmatrix} \Gamma \\ c : \texttt{Ob} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} \tag{1.25}$$

where $M \equiv (\Gamma; c : \texttt{Ob}; M_1; \ldots; M_5)$ encodes a perturbation between modifications of cones. To obtain such a context morphism, we will work with an arbitrary cone in the form of a context K over Γ (instead of the universal cone) where we can control the variables, and subsequently substitute a universal cone into the construction. As an example, the context morphism in equation (1.25) is obtained as the composite of the two context morphisms

$$\begin{pmatrix} \Gamma \\ c : \mathtt{Ob} \\ f : c \to \mathtt{lim}_\Gamma \\ g : c \to \mathtt{lim}_\Gamma \\ \overline{m} : f * \overline{\mathtt{up}} \to g * \overline{\mathtt{up}} \end{pmatrix} \vdash \left\langle \begin{matrix} \kappa \\ c \\ f \\ g \\ \overline{m} \\ \mathtt{u}q \\ \overline{\mathtt{u}q} \end{matrix} \right\rangle : \begin{pmatrix} K \\ c' : \mathtt{Ob} \\ f' : c' \to c \\ g' : c' \to c \\ \Delta \\ \alpha' : f' \to g' \\ \Delta' \end{pmatrix} \tag{1.26}$$

$$\begin{pmatrix} K \\ c' : \mathtt{Ob} \\ f' : c' \to c \\ g' : c' \to c \\ \Delta \\ \alpha' : f' \to g' \\ \Delta' \end{pmatrix} \vdash \left\langle \begin{matrix} \mathtt{id}_\Gamma \\ c' \\ f' * \mathtt{id}_\Pi \\ g' * \mathtt{id}_\Pi \\ \mathtt{id}_\Delta \\ \alpha' * \mathtt{id}_\Pi \\ \mathtt{id}_{\Delta'} \end{matrix} \right\rangle : \begin{pmatrix} \Gamma \\ c : \mathtt{Ob} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} \tag{1.27}$$

The order in which the variables appear in the context $K, c' : \mathtt{Ob}, f' : c' \to c, g' : c' \to c, \Delta, \alpha' : f' \to g', \Delta'$ guarantees that we can later associate the correct free variables to the term $\mathtt{u}q$ and those in the list of terms $\overline{\mathtt{u}q}$.

The context morphisms are built inductively by moving step by step through all the variables of M. Even though it is possible to do without, let us introduce a new judgment $\Delta \vdash_\Gamma x : A$, which we will use as a convenient device to keep track of the process while keeping the notation more compact. This judgment is subject to the rules

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash_\Gamma x : A} \qquad \frac{\Delta \vdash_\Gamma x : A \qquad \Delta \vdash B}{\Delta, y : B \vdash_\Gamma x : A.}$$

If $\Delta \vdash_\Gamma x : A$ is derivable, we say Δ is an extension of Γ.

Lemma 3.19. *Given the judgement $\Delta \vdash_\Gamma x : A$, then the following judgments are derivable:*

$$\Gamma \vdash, \qquad \Delta \vdash, \qquad \Delta \vdash x : A.$$

Moreover, $\Delta \equiv \Gamma, x : A, \Delta'$ for some list of variables Δ'.

Proof. By induction. □

We now spell out a set of rules which generate the context morphism of the form of equation (1.27). The first context morphism of equation (1.26), on the other hand, is what will be generated by the rules for the universal property in the next subsection.

Rules for postcomposition with a cone. We introduce a new judgment $W \vdash^{\texttt{star}}_{K;\alpha} w : M$ subject to the rules

$$\frac{K \ \texttt{cone}\ (\Gamma, c)}{K, c' : \texttt{Ob}, D^n(c, c', \alpha) \vdash^{\texttt{pstar}}_{K;\alpha} \langle \texttt{id}_\Gamma, c' \rangle : (\Gamma, c : \texttt{Ob})}$$

$$\frac{\begin{array}{l} K \ \texttt{cone}\ (\Gamma, c) \\ M \ \texttt{ctrf}\ (\Gamma, c) \quad W \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta \quad M \vdash_\Theta x : X \quad \Theta \vdash u : X[w] \end{array}}{W \vdash^{\texttt{pstar}}_{K;\alpha} \langle w, u \rangle : (\Theta, x : X)} \quad \begin{array}{c} \dim(\alpha) = n_M \\ x \in \mathrm{FV}(M_j),\ j \in \{1, \ldots, 2n_M - 2, 2n_M\} \\ \mathrm{Star}(\Gamma, K, M, x, u : X[w], \alpha : A) \end{array}$$

$$\frac{\begin{array}{l} K \ \texttt{cone}\ (\Gamma, c) \\ M \ \texttt{ctrf}\ (\Gamma, c) \quad W \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta \quad M \vdash_\Theta x : X \end{array}}{W, x' : X[w] \vdash^{\texttt{pstar}}_{K;\alpha} \langle w, x' \rangle : (\Theta, x : X)} \quad \begin{array}{c} \dim(\alpha) = n_M \\ x \in \mathrm{FV}(M_j),\ j \in \{2n_M - 1, 2n_M + 1\} \end{array}$$

$$\frac{W, \alpha : A, x : X, W' \vdash^{\texttt{pstar}}_{K,\alpha} w : \Theta}{W, x : X, \alpha : A, W' \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta} \quad \alpha \notin \mathrm{FV}(X)$$

$$\frac{W, \alpha : A, x : X, W' \vdash^{\texttt{pstar}}_{K,\alpha} w : \Theta}{W, \alpha : A, x : X, W' \vdash^{\texttt{star}}_{K;\alpha} w : \Theta} \quad \alpha \in \mathrm{FV}(X)$$

where, with the notation $\Gamma \equiv (x_i : A_i)_{1 \leq i \leq l}$ and $K \equiv (\Gamma, c : \texttt{Ob}, (p_i : T_{x_i})_{1 \leq i \leq l})$, as well as $M_j \equiv (x_{j,i} : T_{j,i})_{1 \leq i \leq l}$, the shorthand $\mathrm{Star}(\Gamma, K, M, x_{j,i}, u : X, \alpha : T_\alpha)$ stands for

$$\mathrm{FV}(u : X) = \mathrm{FV}(\alpha : T_\alpha) \cup \mathrm{FV}(p_i : T_i)$$

$$\mathrm{FV}(u : X) = \begin{cases} \mathrm{FV}(\sigma^{n-j}(\alpha) : \partial T_\alpha) \cup \mathrm{FV}(p_i : T_{x_i}), & i \text{ odd} \\ \mathrm{FV}(\tau^{n-j}(\alpha) : \partial T_\alpha) \cup \mathrm{FV}(p_i : T_{x_i}), & i \text{ even} \end{cases}$$

Lemma 3.20. *Given a derivable judgment $W \vdash^{\texttt{star}}_{K;\alpha} w : \Theta$, the following judgments are derivable*

$$W \vdash, \qquad \Theta \vdash, \qquad W \vdash w : \Theta.$$

Proof. By induction. □

3.2.5 The Universal Property

We are now ready to formulate the universal property.

Rules for the universal property. The terms generated by the universal property are obtained by the term constructor rules

$$\frac{\begin{array}{lll} K\ \texttt{cone}\ (\Gamma; c) & \Lambda, \alpha : T_\alpha, \Delta \vdash_{K,\Theta} x : X & \\ M\ \texttt{ctrf}\ (\Gamma, c) & \Lambda, \alpha : T_\alpha, \Delta \vdash^{\texttt{star}}_{K;\alpha} w : M & \Omega \vdash^{\texttt{uni}}_{\Gamma} \theta : (K, \Theta) \end{array}}{J_1 \qquad J_2} \qquad \dim(\alpha) = n_M$$

where the J_1, J_2 together with additional side conditions are given by:

$$J_1: \qquad \Omega \vdash \texttt{uni}_{\Gamma,x} : X[\theta] \qquad x \in \{\alpha\}$$

$$J_2: \qquad \Omega \vDash \texttt{uni}_{\Gamma,x} : X[\theta] \qquad x \in \mathrm{FV}(\Delta)$$

These term constructor rules refer to a new judgment $\Omega \vdash^{\texttt{uni}}_{\Gamma} \theta : \Theta$, generalizing the judgment $\Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta$ introduced in the rules for the universal cone, and which is subject to the rules

$$\frac{\begin{array}{lll} K\ \texttt{cone}\ (\Gamma; c) & \Lambda, \alpha : T_\alpha, \Delta \vdash_{K,\Theta} x : X & \\ M\ \texttt{ctrf}\ (\Gamma, c) & \Lambda, \alpha : T_\alpha, \Delta \vdash^{\texttt{star}}_{K;\alpha} w : M & \Omega \vdash^{\texttt{uni}}_{\Gamma} \theta : (K, \Theta) \end{array}}{J_3 \qquad J_4} \qquad \dim(\alpha) = n_M$$

$$J_3: \qquad \Omega, x' : X[\theta] \vdash^{\texttt{uni}}_{\Gamma} \langle \theta, x' \rangle : (\Omega, x : X) \qquad x \in \mathrm{FV}(\Lambda)$$

$$J_4: \qquad \Omega \vdash^{\texttt{uni}}_{\Gamma} \langle \theta, \texttt{uni}_{\Gamma,x} \rangle : (\Theta, x : X) \qquad x \in \{\alpha\} \cup \mathrm{FV}(\Delta)$$

The terms $\texttt{uni}_{\Gamma,x}$ also depend on M and K as well as the context used in the `star` judgment, which should therefor also appear in the term constructor as subscripts. This, however would make the notation unwieldy, because of which we suppress these dependencies.

Example 3.21. Consider the empty diagram $\emptyset \vdash$. Applying the rules for cones we can derive the judgment $(c : \texttt{Ob})$ `cone` $(\emptyset, c)$, that is, a cone over the empty diagram is just a 0-cell, the apex. A universal cone over $\emptyset$ of shape $c : \texttt{Ob}$ is then given by a context morphism $\emptyset \vdash^{\texttt{uni}}_{\emptyset} \langle \texttt{lim}_\emptyset \rangle : (c : \texttt{Ob})$. Let us write $\top$ instead of $\texttt{lim}_\emptyset$.

Regarding the universal property, a conical 2-transfor over $\emptyset$ is given by the context $c : \texttt{Ob}$, or more precisely by the judgment $(\emptyset; c : \texttt{Ob}; \emptyset, \emptyset, \emptyset)$ `ctrf` $(\emptyset, c)$. The `star` judgement required by the rule must be of the form $c : \texttt{Ob}, c' : \texttt{Ob}, f : c' \to c \vdash \langle c' \rangle : (c : \texttt{Ob})$. The only judgment of the form $\vdash^{\texttt{uni}}_{\emptyset}$ available is $\emptyset \vdash^{\texttt{uni}} \langle \top \rangle : (c : \texttt{Ob})$. Putting everything together, the first application of the rule gives

$$\frac{(c:\mathtt{Ob})\ \mathtt{cone}\ (\emptyset, c) \quad c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c \vdash_{c:\mathtt{Ob}} c':\mathtt{Ob} \\ (c:\mathtt{Ob})\ \mathtt{ctrf}\ (\emptyset, c) \quad c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c \vdash^{\mathtt{star}}_{c:\mathtt{Ob}, f:c'\to c} \langle c' \rangle : (c:\mathtt{Ob}) \quad \emptyset \vdash^{\mathtt{uni}}_{\emptyset} \langle \top \rangle : (c:\mathtt{Ob})}{c':\mathtt{Ob} \vdash^{\mathtt{uni}}_{\emptyset} \langle \top, c' \rangle : (c:\mathtt{Ob}, c':\mathtt{Ob})}$$

and the second and final application gives

$$\frac{(c:\mathtt{Ob})\ \mathtt{cone}\ (\emptyset, c) \quad c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c \vdash_{c:\mathtt{Ob}, c':\mathtt{Ob}} f: c' \to c \\ (c:\mathtt{Ob})\ \mathtt{ctrf}\ (\emptyset, c) \quad c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c \vdash^{\mathtt{star}}_{c:\mathtt{Ob}, f:c'\to c} \langle c' \rangle : (c:\mathtt{Ob}) \quad c':\mathtt{Ob} \vdash^{\mathtt{uni}}_{\emptyset} \langle \top, c' \rangle : (c:\mathtt{Ob}, c':\mathtt{Ob})}{c':\mathtt{Ob} \vdash^{\mathtt{uni}}_{\emptyset} \langle \top, c', \mathtt{uni}_{\emptyset,f} \rangle : (c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c)}$$

Pictorially we have

$$\top \xleftarrow[\mathtt{uni}_{\emptyset,f}]{} c'$$

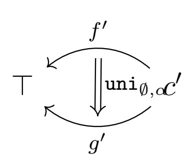
Example 3.22. Consider again the diagram $\emptyset \vdash$. This time we consider $(c:\mathtt{Ob})$ as a 3-transfor over $\emptyset$. Application of the rule then gives

$$\frac{\begin{array}{l} (c:\mathtt{Ob})\ \mathtt{cone}\ (\emptyset, c) \\ (c:\mathtt{Ob})\ \mathtt{ctrf}\ (\emptyset, c) \\ c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c, g: c' \to c, \alpha: f \to g \vdash_{c:\mathtt{Ob}} c':\mathtt{Ob} \\ c:\mathtt{Ob}, c':\mathtt{Ob}, f: c' \to c, g: c' \to c, \alpha: f \to g \vdash^{\mathtt{star}}_{c:\mathtt{Ob};\alpha} \langle c' \rangle : (c:\mathtt{Ob}) \\ c':\mathtt{Ob} \vdash^{\mathtt{uni}}_{\emptyset} \langle \top \rangle : (c:\mathtt{Ob}) \end{array}}{c':\mathtt{Ob} \vdash^{\mathtt{uni}}_{\emptyset} \langle \top, c' \rangle : (c:\mathtt{Ob}, c':\mathtt{Ob})}$$

Three more applications give

$$c':\mathtt{Ob}, f': c' \to \top, g': c' \to \top \vdash \left\langle \begin{array}{c} \top \\ c' \\ f' \\ g' \\ \mathtt{uni}_{\emptyset,\alpha} \end{array} \right\rangle : \begin{pmatrix} c:\mathtt{Ob} \\ c':\mathtt{Ob} \\ f: c' \to c \\ g: c' \to c \\ \alpha: f \to g \end{pmatrix}$$

Pictorially we have



3.2.6 Free Variables and Admissibility of Cut

For the type theory to function properly we need to make sure the cut rule is admissible, i.e., we can perform substitution. To ensure the admissibility of the cut rule we do the usual trick: introduce just enough cut into the remaining rules. In our case, it suffices to do this for all the rules introducing new terms. For the universal cone, the rules take the form

$$\frac{(\Theta, x : X, \Theta') \ \texttt{cone} \ (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{ucone}_{\Gamma,x}[\gamma] : X[\kappa \circ \gamma]}$$

$$\frac{(\Theta, x : X, \Theta') \ \texttt{cone} \ (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta}{\Gamma \vdash^{\texttt{uni}}_{\Gamma} \langle \kappa, \texttt{ucone}_{\Gamma,x}[\texttt{id}_{\Gamma}] \rangle : (\Theta, x : X)}$$

For the universal property we make similar modifications. For J_1 and J_2 rule we add in a cut and modify the rule accordingly

$$\frac{\begin{array}{llll} K \ \texttt{cone} \ (\Gamma; c) & \Lambda, \alpha : T_\alpha, \Delta \vdash_{K,\Theta} x : X & & \\ M \ \texttt{ctrf} \ (\Gamma, c) & \Lambda, \alpha : T_\alpha, \Delta \vdash^{\texttt{star}}_{K;\alpha} w : M & \Omega \vdash^{\texttt{uni}}_{\Gamma} \theta : (K, \Theta) & \Phi \vdash \omega : \Omega \end{array}}{J_1 \qquad\qquad J_2} \quad \dim(\alpha) = n_M$$

where

$$J_1 : \qquad \Phi \vdash \texttt{uni}_{\Gamma,x}[\omega] : X[\theta \circ \omega] \qquad x \in \{\alpha\}$$

$$J_2 : \qquad \Omega \vDash \texttt{uni}_{\Gamma,x}[\texttt{id}_{\Gamma}] : X[\theta] \qquad x \in \mathrm{FV}(\Delta)$$

The rule for J_3 remains unchanged, while for J_4 the premise of the rules stays the same but the conclusion becomes

$$J_4 : \qquad \Omega \vdash^{\texttt{uni}}_{\Gamma} \langle \theta, \texttt{uni}_{\Gamma,x}[\texttt{id}_{\Gamma}] \rangle : (\Theta, x : X) \qquad x \in \{\alpha\} \cup \mathrm{FV}(\Delta)$$

Finally, we add a cut to the rule producing the coinductive inverses:

$$\frac{\Gamma \vDash u : s \to_A t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vDash \texttt{inv}(u[\gamma]) : t[\gamma] \to_{A[\gamma]} s[\gamma]}$$

$$\frac{\Gamma \vDash u : s \to_A t \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vDash \texttt{eta}(u[\gamma]) : 1_{s[\gamma]} \to u[\gamma] \cdot \texttt{inv}(u[\gamma]) \qquad \Delta \vDash \texttt{eps}(u[\gamma]) : \texttt{inv}(u[\gamma]) \cdot u[\gamma] \to 1_{t[\gamma]}}$$

To make substitution admissible we then make the following definition.

Definition 3.23. *Substitution on term constructors is defined by*

$$\begin{aligned} \texttt{ucone}_{\Gamma,x}[\gamma][\delta] &\equiv \texttt{ucone}_{\Gamma,x}[\gamma \circ \delta], & \texttt{inv}(u)[\gamma] &\equiv \texttt{inv}(u[\gamma]) \\ \texttt{uni}_{\Gamma,x}[\gamma][\delta] &\equiv \texttt{uni}_{\Gamma,x}[\gamma \circ \delta], & \texttt{eta}(u)[\gamma] &\equiv \texttt{eta}(u[\gamma]) \\ & & \texttt{eps}(u)[\gamma] &\equiv \texttt{eps}(u[\gamma]). \end{aligned}$$

With this at hand one can prove:

Lemma 3.24. *The following rules are admissible in* `CaTT`

(i) For types:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]} \qquad \textit{and} \qquad A[\gamma][\delta] \equiv A[\gamma \circ \delta] \qquad \textit{for all} \qquad \Phi \vdash \delta : \Delta.$$

(ii) For terms:

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash t[\gamma] : A[\gamma]} \qquad \textit{and} \qquad t[\gamma][\delta] \equiv t[\gamma \circ \delta] \qquad \textit{for all} \qquad \Phi \vdash \delta : \Delta,$$

and the same rule holds with $\vdash$ *replaced with* $\vDash$.

(iii) For contexts:

$$\frac{\Gamma \vdash \theta : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \theta \circ \gamma : \Theta} \qquad \textit{and} \qquad (\theta \circ \gamma) \circ \delta \equiv \theta \circ (\gamma \circ \delta) \quad \textit{for all} \quad \Phi \vdash \delta : \Delta.$$

Finally we give the definition of the free variables of the new term constructors.

Definition 3.25. *The free variables of the term constructors are defined by*

$$\begin{aligned} \mathrm{FV}(\texttt{ucone}_{\Gamma,x}[\gamma]) &:= \mathrm{FV}(\gamma) & \mathrm{FV}(\texttt{inv}(u)[\gamma]) &:= \mathrm{FV}(u[\gamma]) \\ \mathrm{FV}(\texttt{uni}_{\Gamma,x}[\omega]) &:= \mathrm{FV}(\omega) & \mathrm{FV}(\texttt{eta}(u)[\gamma]) &:= \mathrm{FV}(u[\gamma]) \\ & & \mathrm{FV}(\texttt{eps}(u)[\gamma]) &:= \mathrm{FV}(u[\gamma]) \end{aligned}$$

In total, we define a type theory $\texttt{CaTT}_{\texttt{lim}}$, extending `CaTT` and describing (∞,∞)-categories with lax limits for finite computads.

Definition 3.26. *The type theory* $\texttt{CaTT}_{\texttt{lim}}$ *given by the rules of* `CaTT`*, together with the rules:*

Term constructor for the universal cone:

$$\frac{(\Theta, x : X, \Theta') \;\; \texttt{cone} \;\; (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \texttt{ucone}_{\Gamma,x}[\gamma] : X[\kappa \circ \gamma]}$$

Term constructors for the universal property:

$$\frac{\begin{array}{llll} K\ \texttt{cone}\ (\Gamma;c) & \Lambda,\alpha:T_\alpha,\Delta\vdash_{K,\Theta} x:X & & \\ M\ \texttt{ctrf}\ (\Gamma,c) & \Lambda,\alpha:T_\alpha,\Delta\vdash^{\texttt{star}}_{K;\alpha} w:M & \Omega\vdash^{\texttt{uni}}_{\Gamma}\theta:(K,\Theta) & \Phi\vdash\omega:\Omega\end{array}}{J_1 \qquad\qquad J_2}\ \dim(\alpha)=n_M$$

where

$$J_1:\qquad \Phi\vdash \texttt{uni}_{\Gamma,x}[\omega]:X[\theta\circ\omega]\qquad x\in\{\alpha\}$$

$$J_2:\qquad \Omega\vDash \texttt{uni}_{\Gamma,x}[\texttt{id}_\Gamma]:X[\theta]\qquad x\in \mathrm{FV}(\Delta)$$

Term constructors for invertible morphisms:

$$\frac{\Gamma\vDash u:s\to_A t}{\Gamma\vdash u:s\to_A t}$$

$$\frac{\Gamma\vDash u:s\to_A t\qquad \Delta\vdash\gamma:\Gamma}{\Delta\vDash \texttt{inv}(u[\gamma]):t[\gamma]\to_{A[\gamma]} s[\gamma]}$$

$$\frac{\Gamma\vDash u:s\to_A t\qquad \Delta\vdash\gamma:\Gamma}{\Delta\vDash \texttt{eta}(u[\gamma]):1_{s[\gamma]}\to u[\gamma]\cdot\texttt{inv}(u[\gamma])\qquad \Delta\vDash \texttt{eps}(u[\gamma]):\texttt{inv}(u[\gamma])\cdot u[\gamma]\to 1_{t[\gamma]}}$$

These rules are accompanied by a set of recognition rules for special contexts and context morphisms, listed below.

The rules for cone as contexts:

$$\frac{}{c:\texttt{Ob}\ \texttt{cone}\ (\emptyset,c)}\ (\mathrm{EK})$$

$$\frac{\Gamma,c:\texttt{Ob},\Pi\ \texttt{cone}\ (\Gamma;c)\qquad \Gamma,x:X,c:\texttt{Ob},\Pi\vdash s\to_A t}{\Gamma,x:X,c:\texttt{Ob},\Pi,p_x:s\to_A t\ \texttt{cone}\ ((\Gamma,x:X),c)}\ (\mathrm{KE})\qquad \begin{array}{l}\tau\overline{\mathrm{Cond}}(s:A,x:X,c:\texttt{Ob},\Pi)\\ \sigma\mathrm{Cond}(t:A,x:X,\Pi)\end{array}$$

where for $\delta\in\{\sigma,\tau\}$ *the notation* $\delta\mathrm{Cond}(s:A,x:X,\Pi)$ *stands for* $t:A$ *being categorical,* $x\propto t$ *and*

$$\mathrm{FV}(t:A)=\mathrm{FV}(x:X)\cup\bigcup_{\substack{p\in\mathrm{FV}(\Pi)\\ y\in\mathrm{FV}(\delta(x):\partial X)\\ y\propto\tau(p)}}\mathrm{FV}(p:T_p)$$

while $\tau\overline{\text{Cond}}(s : A, x : X, c : \texttt{Ob}, \Pi)$ *stands for* $t : A$ *being categorical and*

$$\text{FV}(t : A) = \text{FV}(c : \texttt{Ob}) \cup \bigcup_{\substack{p \in \text{FV}(\Pi) \\ y \in \text{FV}(\delta(x) : \partial X) \\ y \propto \tau(p)}} \text{FV}(p : T_p)$$

The rules for higher conical transfors as contexts:

$$\frac{}{\emptyset; c : \texttt{Ob}; \emptyset; \ldots; \emptyset \ \texttt{ctrf} \ (\emptyset, c)}$$

$$\frac{\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1} \ \texttt{ctrf} \ (\Gamma; c) \qquad \Gamma, x : X, c : \texttt{Ob}, M_1, p : T \ \texttt{cone} \ ((\Gamma, x : X), c)}{\Gamma, x : X; c : \texttt{Ob}; M_1, p : T; M_2; \ldots; M_{2n+1} \ \texttt{pctrf} \ ((\Gamma, x : X), c)}$$

$$\frac{\begin{array}{l}\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}; \ldots; M_{2n+1} \ \texttt{pctrf} \ (\Gamma; c) \\ \Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}, x' : X; M_{2i+1} \vdash s \to_A t\end{array}}{\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2i-1}, x : X; M_{2i}, x' : X; M_{2i+1}, p_x : s \to_A t; \ldots; M_{2n+1} \ \texttt{pctrf} \ (\Gamma; c)} \quad \begin{array}{c} 1 \le i \le n \\ |M_{2i-1}| = |M_{2i}| \\ \tau\text{Cond}\Big(s : A, x : X, \text{FV}(M_{2i+1})\Big) \\ \sigma\text{Cond}\Big(t : A, x' : X, \text{FV}(M_{2i+1})\Big)\end{array}$$

$$\frac{\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1} \ \texttt{pctrf} \ (\Gamma; c)}{\Gamma; c : \texttt{Ob}; M_1; \ldots; M_{2n+1} \ \texttt{ctrf} \ (\Gamma; c)} \qquad |M_1| = |M_{2n+1}|$$

The rules for postcomposing with a cone:

$$\frac{K \ \texttt{cone} \ (\Gamma, c)}{K, c' : \texttt{Ob}, D^n(c, c', \alpha) \vdash^{\texttt{pstar}}_{K;\alpha} \langle \texttt{id}_\Gamma, c' \rangle : (\Gamma, c : \texttt{Ob})}$$

$$\frac{\begin{array}{l}K \ \texttt{cone} \ (\Gamma, c) \\ M \ \texttt{ctrf} \ (\Gamma, c) \qquad W \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta \qquad M \vdash_\Theta x : X \qquad \Theta \vdash u : X[w]\end{array}}{W \vdash^{\texttt{pstar}}_{K;\alpha} \langle w, u \rangle : (\Theta, x : X)} \quad \begin{array}{c}\dim(\alpha) = n_M \\ x \in \text{FV}(M_j), \ j \in \{1, \ldots, 2n_M - 2, 2n_M\} \\ \text{Star}(\Gamma, K, M, x, u : X[w], \alpha : A)\end{array}$$

$$\frac{\begin{array}{l}K \ \texttt{cone} \ (\Gamma, c) \\ M \ \texttt{ctrf} \ (\Gamma, c) \qquad W \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta \qquad M \vdash_\Theta x : X\end{array}}{W, x' : X[w] \vdash^{\texttt{pstar}}_{K;\alpha} \langle w, x' \rangle : (\Theta, x : X)} \quad \begin{array}{c}\dim(\alpha) = n_M \\ x \in \text{FV}(M_j), \ j \in \{2n_M - 1, 2n_M + 1\}\end{array}$$

$$\frac{W, \alpha : A, x : X, W' \vdash^{\texttt{pstar}}_{K,\alpha} w : \Theta}{W, x : X, \alpha : A, W' \vdash^{\texttt{pstar}}_{K;\alpha} w : \Theta} \qquad \alpha \notin \text{FV}(X)$$

$$\frac{W, \alpha : A, x : X, W' \vdash^{\texttt{pstar}}_{K,\alpha} w : \Theta}{W, \alpha : A, x : X, W' \vdash^{\texttt{star}}_{K;\alpha} w : \Theta} \qquad \alpha \in \mathrm{FV}(X)$$

where, with the notation $\Gamma \equiv (x_i : A_i)_{1 \le i \le l}$ *and* $K \equiv (\Gamma, c : \texttt{Ob}, (p_i : T_{x_i})_{1 \le i \le l})$, *as well as* $M_j \equiv (x_{j,i} : T_{j,i})_{1 \le i \le l}$, *the shorthand* $\mathrm{Star}(\Gamma, K, M, x_{j,i}, u : X, \alpha : T_\alpha)$ *stands for*

$$\mathrm{FV}(u : X) = \mathrm{FV}(\alpha : T_\alpha) \cup \mathrm{FV}(p_i : T_i)$$

$$\mathrm{FV}(u : X) = \begin{cases} \mathrm{FV}(\sigma^{n-j}(\alpha) : \partial T_\alpha) \cup \mathrm{FV}(p_i : T_{x_i}), & i \text{ odd} \\ \mathrm{FV}(\tau^{n-j}(\alpha) : \partial T_\alpha) \cup \mathrm{FV}(p_i : T_{x_i}), & i \text{ even} \end{cases}$$

Rules for context morphisms the term constructors:

$$\frac{\Gamma \vdash}{\Gamma \vdash^{\texttt{uni}}_{\Gamma} \texttt{id}_\Gamma : \Gamma}$$

$$\frac{(\Theta, x : X, \Theta') \;\; \texttt{cone} \;\; (\Gamma; c) \qquad \Gamma \vdash^{\texttt{uni}}_{\Gamma} \kappa : \Theta}{\Gamma \vdash^{\texttt{uni}}_{\Gamma} \langle \kappa, \texttt{ucone}_{\Gamma,x}[\texttt{id}_\Gamma] \rangle : (\Theta, x : X)}$$

$$\frac{\begin{array}{lll} K \;\; \texttt{cone} \;\; (\Gamma; c) & \Lambda, \alpha : T_\alpha, \Delta \vdash_{K,\Theta} x : X & \\ M \;\; \texttt{ctrf} \;\; (\Gamma, c) & \Lambda, \alpha : T_\alpha, \Delta \vdash^{\texttt{star}}_{K;\alpha} w : M & \Omega \vdash^{\texttt{uni}}_{\Gamma} \theta : (K, \Theta) \end{array}}{J_3 \qquad\qquad J_4}$$

where

$$J_3 : \qquad \Omega, x' : X[\theta] \vdash^{\texttt{uni}}_{\Gamma} \langle \theta, x' \rangle : (\Omega, x : X) \qquad x \in \mathrm{FV}(\Lambda)$$

$$J_4 : \qquad \Omega \vdash^{\texttt{uni}}_{\Gamma} \langle \theta, \texttt{uni}_{\Gamma,x}[\texttt{id}_\Gamma] \rangle : (\Theta, x : X) \qquad x \in \{\alpha\} \cup \mathrm{FV}(\Delta)$$

Decomposition Spaces

Contents

About this chapter

The main body of this chapter (starting with Section 1) is comprised of a joint paper with Joachim Kock, which has been posted to the arXiv [51] and submitted to the journal Algebraic & Geometric Topology. Both authors contributed equally to the paper. For completeness sake, this chapter also includes an opening section with the basics of decomposition spaces, used throughout the remaining sections. Everything discussed here can be found in Gálvez–Kock–Tonks [33] and [35], Kock [50] and Carlier [21].

Basics in Decomposition Space Theory

A basic ingredient in algebraic combinatorics is the notion of incidence coalgebras. They were originally introduced by Rota [69] for posets, to provide a unifying framework in which an abstract Möbius inversion principle can be formulated.

This definition of incidence coalgebras requires a finiteness condition: a poset P is said be *finitely local* if every interval $[x, y] = \{ z \in P \,|\, x \leq z \leq y\}$ is finite. The *incidence coalgebra* of P is the free $\Bbbk$-vector space C_P (for some fixed ground field $\Bbbk$) spanned by all intervals, endowed with the structure of a coalgebra, whose comultiplication is given by

$$[x, y] \longmapsto \sum_{z \in [x,y]} [x, z] \otimes [z, y].$$

Möbius inversion then takes place in the *incidence algebra of* P, which is defined to be the convolution algebra of C_P. In a bit more detail, consider the zeta function

$$\zeta : C_P \longrightarrow \Bbbk; \quad [x, y] \longmapsto 1.$$

A theorem by Rota [69] then states that ζ admits a convolution inverse, which is precisely the Möbius function (see Kock [50] for more details).

The incidence coalgebras that arise from posets in this way cover a wide range of coalgebras in combinatorics, but not all, as it turns out. Now, posets can be seen as special simplicial sets. This shift of perspective gives access to more general structures, since the simplicial viewpoint includes categories and even $(\infty, 1)$-categories (if we allow our presheaves to take values in spaces) and indeed the basic constructions all generalize to these more general structures. Nevertheless, even with such generalizations, not all coalgebras in combinatorics are covered, motivating the need for a more general class of simplicial spaces. It is for this purpose that Gálvez, Kock and Tonks introduced the notion of decomposition space [33]: the axioms of decomposition space imposed on a simplicial space are precisely those guaranteeing coassociativity and counitality of the incidence coalgebra.

By the time they were introduced by Gálvez-Kock and Tonks, decomposition spaces had already appeared in the literature under the name 2-Segal spaces. Dyckerhoff and Kapranov arrived at the notion of 2-segal spaces by observing that Hall algebras can be understood as being induced from the richer homotopical structure encoded in 2-Segal spaces. Nevertheless, the two definitions are different in appearance and the theory developed by Gálvez–Kock–Tonks and that by Dyckerhoff–Kapranov were initially rather orthogonal to each other.

Remark. We use the convention of speaking about decomposition spaces in the combinatorial setting. In more general settings, such as in the work included in this thesis (starting with Section 1), we use the 2-Segal terminology.

Model-independent Higher Category Theory

Throughout the years, many models for higher categories have been proposed and much progress has been made in proving them to be equivalent. Somewhat remarkably, at the same time, a different understanding and reasoning with higher categories emerged. It is referred to as model independent and it captures the essence of higher categories by abstracting away from the implementation details specific to any given model. In this chapter we will consistently make use the model independent reasoning. In fact, working with decomposition spaces was partly an excuse to gain more experience with the model independent language to strengthen the intuition and to complement the work of the first chapter on the foundations of higher categories. Despite its success, the model independent language has not yet been grounded and pinned down by an axiomatic description, and typically, when pressed, higher category theorists will resort back to quasicategories.

According to the homotopy hypothesis we may use the terminology spaces and ∞-groupoids interchangeably. Now, in the transition from classical categories to higher categories, sets give way to spaces. One important insight in higher category theory is that much of classical category theory can be lifted to the ∞-world (see Lurie [58]), simply by attaching the ∞ prefix or sprinkling around the adjective homotopy appropriately. In particular, to a large extent spaces can be manipulated as if they were sets. There are, however, some important difference. One such difference is the lose of the ability to define, say, categories "manually" by specifying the objects and the morphisms, or similarly functors by specifying its action on objects and morphisms. Rather, the construction of higher structures typically relies on abstract theorems and universal properties. The equivalence arising in the following sections, for example, are all induced by Kan extension.

The decomposition space axioms are phrased in terms of certain (homotopy) pullback diagrams. (From here on, wherever possible, we will suppress instances of the adjective ‘homotopy’ and the ∞ prefix.) As a result pullbacks are featured very prominently in the theory of decomposition spaces. One basic tool used throughout is the pullback prism lemma, spelled out below. The prism lemma has a “forward” direction, if one of the constituent morphisms is effective epi. The general definition of effective epis in $(\infty, 1)$-categories requires the computation of the Čech nerve, but acquires a simple characterization in the $(\infty, 1)$-category of spaces:

Lemma. *A morphism $f : X \to Y$ in the $(\infty, 1)$-category of spaces is an effective epimorphism if and only if it induces an epimorphism on connected components, i.e. $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a surjection.*

Lemma (Prism Lemma)**.** *Consider the following diagram of spaces*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X & \underset{f}{\longrightarrow} & Y & \longrightarrow & Z \end{array}$$

(i) Assume the inner right square are pullbacks. Then the inner left square is a pullback if and only if the outer rectangle is a pullback.

(ii) Assuming f is an effective epimorphism, if the inner left square and outer rectangle are are a pullback, so is the inner right square.

Decomposition Spaces

Active and inert maps. The simplex category $\mathbb{\Delta}$ has an active-inert factorization system. A map $f : [n] \to [m]$ is said to be active if it is end-point preserving. The active maps are generated by the inner coface maps and the codegeneracies. A map is said to be inert $f : [n] \to [m]$ if it is distance preserving, i.e. $f(i+1) = f(i)+1$ for $0 \leq i \leq n-1$. The inert maps are generated by the top and bottom coface maps.

Lemma. *Active and inert maps admit pushouts along each other. Moreover, the pullback of an active map along an inert map is again active, and the pullback of an inert map along an active map is again inert.*

Definition. *A decomposition space is a simplicial spaces X which maps all pushouts of active maps along inert maps in $\mathbb{\Delta}$ to pullbacks.*

The property of being a decomposition space can be checked on the face maps (being sections of face maps, the pullbacks along degeneracies are automatic):

Lemma. *A simplicial space is a decomposition space if and only if, for all $n \geq 2$ and all $0 < i < n$, the squares*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array} \qquad\qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} \end{array}$$

are pullbacks.

Definition. *A Segal spaces is a simplicial space X such that all the squares*

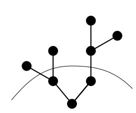
$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_\perp} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_\perp]{} & X_{n-1} \end{array} \tag{2.28}$$

are pullbacks.

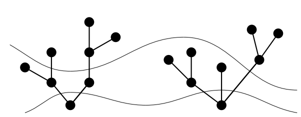
As mentioned in the introduction, decomposition spaces are indeed more general than categories:

Proposition (Gálvez, Kock and Tonks [33], Proposition 3.7)**.** *Every Segal spaces is a decomposition space.*

Example (Gálvez, Kock and Tonks [33], Example 3.3)**.** We give an example of a decomposition spaces which is not Segal, using forests and admissible cuts. A cut on a tree is said to be admissible if it splits the tree into a lower tree with the same root and an upper forest.



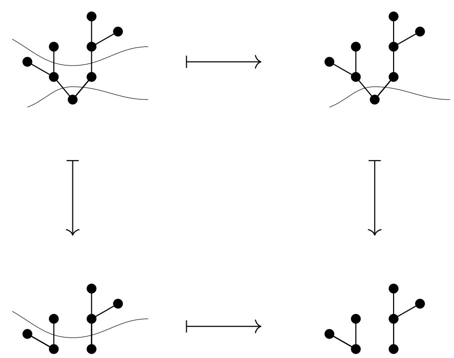
A cut on a forest is said to be admissible if it is admissible on all its trees. A set of $k-1$ cuts is said to be compatible if it splits the forest into k layers (which may be empty), as in the following diagram



Consider the simplicial groupoid $\mathbf{H}$ where, for $k > 0$, $\mathbf{H}_k$ is the groupoid of all forests with $k-1$ compatible admissible cuts and $\mathbf{H}_0$ is the trivial groupoid. As for the simplicial operators, the top face map removes the top layer and the bottom face map removes the bottom layer, whereas the i-th active face map forgets the i-th cut. The top and bottom degeneracy maps insert an empty layer, while all other degeneracy maps duplicate a cut. This simplicial groupoid $\mathbf{H}$ is not a Segal space. To see this consider for example the Segal square given by diagram (2.28) for $n = 1$ and suppose we have a tree with a cut in $\mathbf{H}_2$. The Segal condition then asks wether we can reconstruct (uniquely) this tree from its two layers. But this is not possible, since we've lost the information of the edges joining the two layers. The simplicial groupoid $\mathbf{H}$ is however a decomposition space. Pictorially, one of the low-dimensional pullbacks

$$\begin{array}{ccc} \mathbf{H}_3 & \xrightarrow{d_1} & \mathbf{H}_2 \\ {\scriptstyle d_3}\downarrow & \lrcorner & \downarrow{\scriptstyle d_2} \\ \mathbf{H}_2 & \xrightarrow[d_1]{} & \mathbf{H}_1 \end{array}$$

can be visualized as follows:



Indeed, the full diagram in the top left corner can be reconstructed uniquely from the remaining diagrams and the way the map onto each other.

The Waldhausen $S_\bullet$-construction An important source of examples of decomposition spaces is the Waldhausen $S_\bullet$-construction. In fact, when generalized appropriately, all decomposition spaces arise in this way. This is the content of the equivalence due to Bergner et al [16],[15], which will play an important role in the following sections. Here we only briefly sketch out the Waldhausen $S_\bullet$-construction for stable ∞-categories. For $\mathcal{A}$ a stable ∞-category, $S_n\mathcal{A}$ is given by the space of all staircase diagrams, which in the case $n = 3$ are diagrams of the shape

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_{01} & \longrightarrow & A_{02} & \longrightarrow & A_{03} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & A_{12} & \longrightarrow & A_{13} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & \longrightarrow & A_{23} \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

where all squares are bipullbacks. Informally, the i-th face map is given by deleting all entries with an index i. The i-th degeneracy map is given by repeating the i-th row and the i-th column.

Theorem (Dyckerhoff–Kapranov [28] Theorem 7.3.3, see also [33])**.** *The Waldhausen $S_\bullet$-construction of a stable ∞-category is a decomposition space.*

Incidence Coalgebras and Maps between Decomposition Spaces

We now make precise the statement, that decomposition spaces induce incidence coalgebras. First of all, we need the category ***LIN*** whose objects are all slices $\mathcal{S}_{/S}$ and whose morphisms are linear functors.[4] Here linear functors $\mathcal{S}_{/S} \to \mathcal{S}_{/T}$, can be described as spans: a span $S \xleftarrow{f} M \xrightarrow{g} T$ induces, by way of the "pull-push"-construction, a morphism

$$\mathcal{S}_{/S} \xrightarrow{f^*} \mathcal{S}_{/M} \xrightarrow{g_!} \mathcal{S}_{/T}$$

where f^* is the pullback and $g_!$ is the left Kan extension. Composition of spans exploits the Beck–Chevalley condition. Formally, the category ***LIN*** can be constructed as a full subcategory of $\boldsymbol{Pr}^L$, the category of presentable categories and colimit preserving functors (see Gálvez–Kock–Tonks [35]). It inherits a symmetric monoidal structure from $\boldsymbol{Pr}^L$, which admits the pleasing description

$$\mathcal{S}_{/S} \otimes \mathcal{S}_{/T} \simeq \mathcal{S}_{/(S\times T)}.$$

The main result is then

Theorem 0.27 (Gálves–Kock–Tonks [33], Theorem 7.4)**.** *Given a decomposition space X, the category $\mathcal{S}_{/X_1}$ has the structure of a comonoid in the category **LIN**, the comultiplication of which is given by*

[4] Originally the spaces S over which the slices are formed were taken to satisfy a certain finiteness condition. Such finiteness conditions are necessary when one wishes to apply the cardinality. If one stays at the "objective level", no such restrictions are required and all constructions go through.

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

The coassociativity on the level of π_0 can be derived very directly from the decomposition axioms. It requires showing the two composites of spans outlining the square

$$\begin{array}{ccccc}
X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2,d_0)} & X_1 \times X_1 \\
{\scriptstyle d_1}\uparrow & & {\scriptstyle d_1}\uparrow & \heartsuit & \uparrow{\scriptstyle d_1 \times \mathrm{id}} \\
X_2 & \xleftarrow{d_2} & X_3 & \xrightarrow[(d_3,d_0d_1)]{} & X_2 \times X_1 \\
{\scriptstyle (d_2,d_0)}\downarrow & \spadesuit & \downarrow{\scriptstyle (d_2^2,d_0)} & & \downarrow{\scriptstyle (d_2,d_0)\times \mathrm{id}} \\
X_1 \times X_1 & \xleftarrow[\mathrm{id} \times d_1]{} & X_1 \times X_2 & \xrightarrow[\mathrm{id} \times (d_2,d_0)]{} & X_1 \times X_1 \times X_1
\end{array}$$

are equivalent. (Here all inner square commute, being simplicial identities.) This is the case precisely if the squares ♠ and ♡ are pullbacks, so that we may apply Beck–Chevalley. This in turn is equivalent to asking that the two squares

$$\begin{array}{ccc}
X_3 & \xrightarrow{d_1} & X_2 \\
{\scriptstyle d_3}\downarrow & & \downarrow{\scriptstyle d_2} \\
X_2 & \xrightarrow[d_1]{} & X_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_3 & \xrightarrow{d_2} & X_2 \\
{\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\
X_2 & \xrightarrow[d_1]{} & X_1
\end{array}$$

are pullbacks. But this follows directly from the decomposition space axiom.

Definition. *Let $F : X \to Y$ be map of simplicial spaces and consider a commutative square where p is some simplicial operator. Then F is said to be*

— *cartesian, if all of its naturality squares are pullback;*

— *a left fibration, if it is cartesian on all top face maps;*

— *a right fibration, if it is cartesian on all bottom face maps;*

— *culf, if is is cartesian on all active maps.*

Example. Left fibrations and right fibrations are culf. The proof makes use of the prism lemma. It follows that a map which is both a left fibration and a right fibration is cartesian.

By design, decomposition spaces induce coalgebra structures. The structure of comodules (see Carlier [21]) is encoded in culf maps. Although we won't need this, we include here the relevant result, to emphasize the importance of culf maps in the theory of decomposition spaces.

Lemma (Carlier [21] Proposition 2.1.1)**.** *Let C be a Segal, X a decomposition space and let $f : C \to X$ be a culf map. Then the spanned*

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

induces on $\mathcal{S}_{/C_0}$ *the structure of a left* $\mathcal{S}_{/X_0}$*-comodule (on the* π_0*-level) and the span*

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(f_1,d_1)} C_0 \times X_1$$

induces on $\mathcal{S}_{/C_0}$ *the structure of a right* $\mathcal{S}_{/X_0}$*-comodule.*

The datum $f : C \to X$ satisfying the properties of the above lemma is called a *comodule configuration*. Comodules were studies by Tashi [78] and Young [81] where they appeared under the name of relative 2-Segal spaces.

Carlier went one step further and considered bicomodule structures. These are encoded in certain bisimplicial spaces and will be discussed in the following sections.

1 Introduction

1.1 Background

1.1. 2-Segal spaces and $S_\bullet$-constructions. One of the main motivations for Dyckerhoff and Kapranov to introduce the notion of 2-Segal space [28] was Waldhausen's $S_\bullet$-construction [79] and Hall algebras. They showed [28, Theorem 7.3.3] that for any any proto-exact ∞-category $\mathcal{A}$ (such as for example a stable ∞-category), the Waldhausen $S_\bullet$-construction $S_\bullet(\mathcal{A})$ is a 2-Segal space (see also [33]), and that the Hall-algebra construction on $\mathcal{A}$ factors through $S_\bullet(\mathcal{A})$.

Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [16], [15] (BOORS) made the surprising discovery that *every* 2-Segal space arises as an $S_\bullet$-construction, if just the $S_\bullet$-construction is extended to more general inputs. They identified certain augmented stable double Segal spaces as the appropriate input to produce (all) 2-Segal spaces, and showed that this generalized $S_\bullet$-construction is part of an equivalence. The BOORS theorem is quite remarkable, as it gives a completely new perspective on 2-Segal spaces, and at the same time gives new insight on the $S_\bullet$-construction (see [17]), by staging it in a setting where it has an inverse. It is striking that this inverse is another well-appreciated construction: the inverse takes a 2-Segal space to its total decalage, suitably augmented. Waldhausen's original $S_\bullet$-construction dates back to 1983, while the total decalage construction is credited to Illusie [46] (1972). For an overview of how various Waldhausen constructions relate in this perspective, see [17]; for an introduction to the BOORS equivalence, see [70].

1.2. Decomposition spaces and bicomodules. Independently of the work of Dyckerhoff and Kapranov, the notion of decomposition space was introduced by Gálvez, Kock, and Tonks [33] designed to be the most general setting for the incidence coalgebra construction, originally introduced by Rota [69] for posets. An important motivation for this generalization was Möbius inversion [34]. It was quickly realized (first by Anel) that decomposition spaces are essentially the same thing as 2-Segal spaces (although it took some years before the last bit of the equivalence fell into place, namely the statement that unitality of a 2-Segal space is automatic [29]).

A first goal of the theory of decomposition spaces was to upgrade the classical theory of incidence algebras and Möbius inversion from posets to decomposition spaces [32]. Carlier [21] took an important step in this direction with a generalization of Rota's formula for the Möbius functions of two posets related by a Galois connection [69]. He generalized Rota's formula to adjunctions of ∞-categories, and went further to the situation of certain correspondences between decomposition spaces, which were shown to induce *bicomodule configurations*, namely suitably augmented stable double Segal spaces designed to have incidence bicomodules. Carlier established a Möbius inversion principle for certain *Möbius* bicomodule configurations, which reduces to Rota's formula in the case of a Galois connection. Möbius bicomodule configurations feature vertical top splittings and horizontal bottom splittings, or equivalently, so-called *abacus* maps, which are additional operators $B_{i+1,j} \to B_{i,j+1}$ on a bisimplicial space B. A key ingredient in the theory of Carlier is a construction which to any correspondence of decomposition spaces defines a bicomodule configuration, and to any functor between ∞-categories defines a bicomodule configuration with abacus maps.

1.3. Stability and augmentations. The notion of stable double Segal space is central to both the BOORS equivalence and Carlier's theory of bicomodules, but with very different motivations. A double Segal space is called stable if certain vertical-horizontal bisimplicial identities are pullback squares. For BOORS, the purpose was to capture certain bipullback features of stable or proto-exact ∞-categories as used in the classical $S_\bullet$-construction; for Carlier the purpose was to encode the bicomodule condition of a left and a right coaction. (Carlier [21] learned about the stability condition from BOORS [16], but reformulated it in a way suitable for ∞-categories.)

However, the notions of augmentation used by BOORS and Carlier are very different. For Carlier, an augmentation of a bisimplicial space B consists of an augmentation (in the usual sense of simplicial spaces) of each row and each column, assembling into two extra new simplicial spaces: one forming an augmentation column $B_{\bullet,-1}$ and the other forming an augmentation row $B_{-1,\bullet}$; these play the role of the coalgebras that the bicomodule is over. A bicomodule configuration in the sense of Carlier thus has the shape

$$
\begin{array}{ccccccccc}
 & & B_{-1,0} & \leftrightarrows & B_{-1,1} & \leftrightarrows\!\!\leftarrow & B_{-1,2} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & \\
B_{0,-1} & \longleftarrow & B_{0,0} & \leftrightarrows & B_{0,1} & \leftrightarrows\!\!\leftarrow & B_{0,2} & \cdots \\
\uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & \\
B_{1,-1} & \longleftarrow & B_{1,0} & \leftrightarrows & B_{1,1} & \leftrightarrows\!\!\leftarrow & B_{1,2} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\tag{2.29}
$$

Technically, this is the shape of a presheaf on the category $\mathbb{\Delta}_{/[1]}$. It should still obey a number of axioms: stability, double Segalness, and some exactness conditions imposed on the augmentations.

In contrast, what for BOORS is called an augmentation of a bisimplicial space B is the addition of a single extra space B_{-1} with a morphism *to* B_{00}, subject to some pullback conditions. This space parametrizes the objects which play the role of the zero object in a proto-exact ∞-category. A BOORS-augmented bisimplicial space thus has the shape

$$
\begin{array}{ccccccc}
B_{-1} & & & & & & \\
 & \searrow & & & & & \\
 & & B_{0,0} & \leftrightarrows & B_{0,1} & \leftrightarrows\!\!\leftarrow & B_{0,2} & \cdots \\
 & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & & \uparrow\downarrow\uparrow & \\
 & & B_{1,0} & \leftrightarrows & B_{1,1} & \leftrightarrows\!\!\leftarrow & B_{1,2} & \cdots \\
 & & \vdots & & \vdots & & \vdots &
\end{array}
$$

— it is a presheaf on a certain category Σ.

In the present work (outside of this introduction), in order to avoid confusion regarding the word “augmentation”, we will instead refer to the map $B_{-1} \to B_{0,0}$ as a *pointing*, and call the pullback conditions the *pointing axioms*. For all such larger diagram shapes extending bisimplicial spaces, we call the nonnegatively-indexed part the *bulk*.

1.4. Augmentations of the total decalage. Before we embark on a deeper comparison between the two notions of augmentation, let us note how the total decalage exemplifies both. For X a simplicial space, the *total decalage* $\mathrm{Tot}(X)$ is the bisimplicial space obtained by pulling back X along the ordinal sum functor $\mathbb{\Delta} \times \mathbb{\Delta} \to \mathbb{\Delta}$ (see for example [76]). It has the upper decalage $\mathrm{Dec}_\top(X)$ as its zeroth column and the lower decalage $\mathrm{Dec}_\perp(X)$ as its zeroth row, so a picture of $\mathrm{Tot}(X)$ starts like this (suppressing degeneracy maps):

$$\begin{array}{ccccc}
X_1 & \overset{d_2}{\underset{d_1}{\leftleftarrows}} & X_2 & & \cdots \\
{\scriptstyle d_0}\upuparrows{\scriptstyle d_1} & & {\scriptstyle d_0}\upuparrows{\scriptstyle d_1} & & \\
X_2 & \overset{d_3}{\underset{d_2}{\leftleftarrows}} & X_3 & & \cdots \\
\vdots & & \vdots & &
\end{array}$$

(It also contains the edgewise subdivision $\mathrm{sd}(X)$ as its diagonal. This has recently turned out to be of relevance for 2-Segal spaces: Bergner et al. [18] show that X is 2-Segal if and only if $\mathrm{sd}(X)$ is 1-Segal, and Hackney–Kock [40] show that a simplicial map $F : X \to Y$ is culf if and only if $\mathrm{sd}(F)$ is a right fibration.)

The Tot construction can be refined to provide either a BOORS augmentation (i.e. pointing) or a row-and-column augmentation à la Carlier. In the BOORS case this is just to take the degeneracy map $s_0 : X_0 \to X_1$ as pointing. The row-and-column-augmented Tot instead simply puts the original simplicial space X both in the augmentation column and in the augmentation row, where they fit in by simplicial operators.

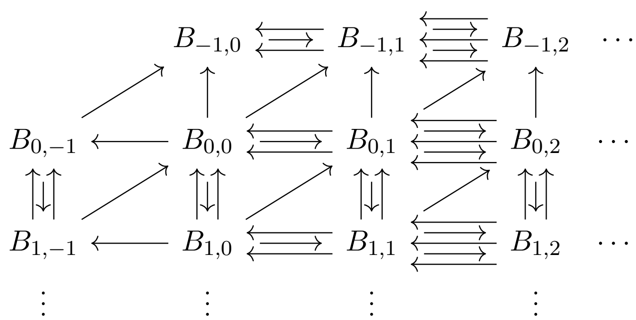
The BOORS version is $\mathrm{Tot} = \mathsf{p}^*$, where $\mathsf{p} : \Sigma \to \mathbb{\Delta}$ extends the ordinal-sum functor. This is important because the BOORS equivalence is established in [15] as a restriction of the adjunction $\mathsf{p}^* \dashv \mathsf{p}_*$, with p_* being interpreted as a generalized Waldhausen construction. The similarly defined functor $\mathbb{\Delta}_{/[1]} \to \mathbb{\Delta}$, the pullback along which is the row-and-column-augmented Tot, has not been studied from a similar viewpoint, as far as we are aware. Nevertheless, the Tot example does play a role in Carlier's theory: it is the result of applying his constructions to the identity correspondence or the identity functor of an ∞-category. This observation is the starting point of our paper.

1.2 Contributions of this paper

We have two main contributions (as well as the theory building up to these results): one is that we upgrade some of Carlier's constructions to equivalences, by identifying the conditions allowing one to "go back". Our equivalences go between certain simplicial maps and certain abacus bicomodule configurations. Secondly we explain the relationship between the two notions of augmentation, so as to allow the BOORS equivalence to be derived from more general equivalences involving simplicial maps and bicomodule configurations.

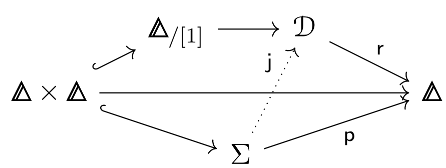
1.5. Simplicial infrastructure. The paper is primarily about bisimplicial spaces. However, we do need some groundwork which is simplicial rather than bisimplicial, culminating with a proposition (2.17) comparing the BOORS augmentation axioms with certain coalgebras for the lower-decalage comonad. This is a generalization of the observation of Garner–Kock–Weber [36] that Dec-coalgebra structure on a 1-category expresses a local-initial-objects structure (or local-terminal-objects structure, which plays a key role in the theory of operadic categories of Batanin and Markl [8].)

1.6. Abacus maps. A prominent role is played throughout by the so-called *abacus maps*: these are a family of diagonal maps $B_{i+1,j} \to B_{i,j+1}$ in a row-and-column-augmented bisimplicial space B satisfying a number of relations. Such an object thus looks like this:



It is a presheaf on a suitable index category $\mathcal{D}$, which we study in some detail. As part of the description of $\mathcal{D}$, the data of abacus maps is shown to be equivalent to having horizontal splittings (i.e. $\mathrm{Dec}_\perp$-coalgebra structures) on all bulk rows. Abacus maps appear already in Carlier's first paper [21] and were named and axiomatized in his second paper [22].

The shapes mentioned so far fit into the diagram



(The subtle functor j is explained further on.)

1.7. Equivalences in the relative setting. On the presheaf level, the abacus maps encode a simplicial map from the augmentation column to the augmentation row. Conversely, $\mathcal{D}$-presheaves arise naturally from general simplicial maps (that is, presheaves on $\mathbb{\Delta} \times \Delta^1$), by way of the functor q_* in the adjunction

$$\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) \underset{\mathsf{q}^*}{\overset{\mathsf{q}_*}{\rightleftarrows}} \boldsymbol{Pr}(\mathcal{D}),$$

given by right Kan extension along a functor $\mathsf{q} : \mathbb{\Delta} \times \Delta^1 \to \mathcal{D}$. In Theorem 4.6 we identity a certain condition $(\star)$ on $\mathcal{D}$-presheaves, required for the adjunction to restrict to an equivalence

$$\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) \simeq \boldsymbol{Pr}^{\star}(\mathcal{D}).$$

This has the flavor of the BOORS equivalence, but in a relative settings, and with more elaborate categories in place of $\mathbb{\Delta}$ and Σ. From here we gradually impose more conditions on the two sides of the equivalence to arrive at objects of special interest. We introduce the notion of relatively upper 2-Segal simplicial maps between 2-Segal spaces, identified as precisely those simplicial maps which correspond to Carlier's bicomodule configurations (with abacus maps satisfying $(\star)$). In particular, the adjunction $\mathsf{q}^* \dashv \mathsf{q}_*$ restricts to an equivalence (Theorem 4.14)

$$\boldsymbol{Pr}^{\text{up 2-Seg}}(\mathbb{\Delta} \times \Delta^1) \simeq \boldsymbol{ABC}^{\star},$$

where $\boldsymbol{ABC}^{\star}$ stands for the full subcategory of $\boldsymbol{Pr}^{\star}(\mathcal{D})$ spanned by abacus bicomodule configurations. By establishing an equivalence, this improves upon a result of Carlier [22], who showed how to construct an abacus bicomodule configuration from any functor of ∞-categories. (Note that between ∞-categories, any functor is relatively upper 2-Segal.) Finally, restricting further, to the full subcategory of $\boldsymbol{Pr}(\mathbb{\Delta})$ spanned by the 2-Segal spaces (interpreted as equivalences of 2-Segal spaces), and to the full subcategory of bicomodule configurations with invertible abacus maps, we arrive at an equivalence (Theorem 4.19)

$$\boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \simeq \boldsymbol{ABC}^{\simeq},$$

which we interpret as a BOORS-type equivalence but with Σ-presheaves replaced by $\mathcal{D}$-presheaves. In fact, the functor $\boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \to \boldsymbol{ABC}^{\simeq}$ is equivalent to the total decalage, version r^*.

1.8. Relating the two notions of augmentation. The results above prompt a closer analysis of the relationship between the two notions of augmentation, leading finally to a derivation of the BOORS equivalence from equivalences involving bicomodules.

The comparison of the two notions of augmentation is mediated by a functor $\mathsf{j} : \Sigma \to \mathcal{D}$, which maps $[-1]$ to $[0, -1]$. In outline, the translation from a Σ-presheaf satisfying the BOORS axioms to a bicomodule configuration (with invertible abacus maps) then runs as follows. First, with the help of Proposition 2.17, we reinterpret the BOORS axioms on augmentation (pointing) as providing horizontal bottom splitting (more precisely, $\mathrm{Dec}_{\perp}$-coalgebra structure) on the zeroth row, and by stability on all rows. Similarly, the pointing also induces vertical top splittings (more precisely, $\mathrm{Dec}_{\top}$-coalgebra structure) on all columns. Next, we can take geometric realization of all rows and columns to produce the Carlier augmentations. A descent argument in Proposition 3.6 involving stability ensures that the augmentations satisfy Carlier's bicomodule axioms (2-Segal augmentation column and row and culf augmentation maps). Furthermore, since both the splittings were induced from the single pointing $B_{-1} \to B_{0,0}$, both the augmentation row and the augmentation column must have B_{-1} in degree zero. As a matter of fact, the induced abacus maps are forced to be invertible, so that altogether, starting with a Σ-presheaf satisfying the BOORS axioms we have produced a bicomodule configuration with invertible abacus maps. Moreover, when restricted along j, this bicomodule configuration recovers the Σ-presheaf we started with.

A similar reasoning shows that the restriction of a bicomodule configuration with invertible abacus maps along j yields a Σ-presheaf satisfying the BOORS axioms. All told, j^* restricts to an equivalence (Theorem 5.9)

$$\boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma) \xrightarrow{\simeq} \boldsymbol{ABC}^{\simeq}$$

between Σ-presheaves satisfying the BOORS axioms and bicomodule configurations with invertible abacus maps. This equivalence, together with those described in 1.7 can be organized into the following diagram

$$\begin{array}{ccccc}
\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) & \overset{\mathsf{q}_*}{\underset{\mathsf{q}^*}{\rightleftarrows}} & \boldsymbol{Pr}(\mathcal{D}) & \xrightarrow{\mathsf{j}^*} & \boldsymbol{Pr}(\Sigma) \\
\| & & \uparrow & & \uparrow \\
\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) & \underset{4.6}{\xrightarrow{\simeq}} & \boldsymbol{Pr}^{\star}(\mathcal{D}) & & \\
\uparrow & & \uparrow & & \\
\boldsymbol{Pr}^{\mathrm{up\ 2\text{-}Seg}}(\mathbb{\Delta} \times \Delta^1) & \underset{4.14}{\xrightarrow{\simeq}} & \boldsymbol{ABC}^{\star} & & \\
\uparrow & & \uparrow & & \\
\boldsymbol{Pr}^{2\text{-}\mathrm{Seg}}(\mathbb{\Delta}) & \underset{4.19}{\xrightarrow{\simeq}} & \boldsymbol{ABC}^{\simeq} & \underset{5.9}{\xrightarrow{\simeq}} & \boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma)
\end{array} \tag{2.30}$$

where all the vertical inclusions are full. The composite of the bottom two horizontal functors turns out to be precisely the total decalage p^*, whereby we recover the original BOORS equivalence (Corollary 5.10). The final result of the paper (Theorem 5.12) states a more refined equivalence, which does not fit into the diagram above. It goes between Σ-presheaves that satisfy "half" of the BOORS axioms and certain $\mathcal{D}$-presheaves but without the augmentation row.

1.9. Organization of the paper. We begin in Section 2 with some basic facts on split simplicial spaces and the decalage comonad. This is an essential ingredient in the theory, and we need to set up precise notation. This section includes also new results on the notion of rigid $\mathrm{Dec}_{\perp}$-coalgebras, which is shown to provide an equivalent description of local-initial-objects structure (Proposition 2.17).

In Section 3 we introduce bisimplicial spaces with augmentations in the sense of Carlier. We show how, in the case where the augmentations are obtained by colimits, the augmentations inherit properties from the bulk. We then describe Carlier's index category $\mathcal{D}$, a distinctive feature of which are the abacus maps, and develop theory for $\mathcal{D}$-presheaves.

Section 4 contains our contributions to the theory of bicomodule configurations. Starting with a very general adjunction between $\mathcal{D}$-presheaves and simplicial maps, we first restrict this to an equivalence (Theorem 4.6). We then extract from this the equivalences relating bicomodule configurations and 2-Segal spaces (Theorem 4.14, Theorem 4.19). We also explain how the results of this section relate to cocartesian correspondences with 2-Segal total spaces, studied by Carlier as a source of bicomodule configurations.

Finally in Section 5 we relate the previous results to Σ-presheaves, and establish in particular a comparison between the two notions of augmentation (Theorem 5.9, Theorem 5.12). As a corollary we derive the original BOORS equivalence (Corollary 5.10).

After the main body of the paper, we include a short little cheat sheet with standard facts about 2-Segal spaces and various classes of simplicial maps.

Acknowledgments. We thank Steve Lack for some fruitful discussions. Subsections 2.1 and 2.2 owe a lot to forthcoming work of Batanin–Kock–Weber [7].

2 Splittings, decalage, and rigid $\mathrm{Dec}_\perp$-coalgebras

2.1 Split simplicial spaces

As always, let $\mathbb{\Delta}$ be the category of non-empty finite standard ordinals and monotone maps. We work with simplicial spaces, meaning presheaves with values in the ∞-category of spaces $\mathcal{S}$. They form the ∞-category $s\mathcal{S} := \boldsymbol{Pr}(\mathbb{\Delta}) := \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$. For a space C we denote by $\bar{C}$ the simplicial space with constant value C.

We shall need also several other base categories closely related to $\mathbb{\Delta}$. Let $\mathbb{\Delta}^{\mathsf{b}}$ be the category of non-empty finite standard ordinals with a bottom element and bottom-preserving monotone maps. Let $\mathbb{\Delta}_{\mathsf{b}}$ be the full subcategory of $\mathbb{\Delta}^{\mathsf{b}}$ consisting of those objects where the bottom element is not alone. The forgetful functor $\mathsf{u} : \mathbb{\Delta}^{\mathsf{b}} \to \mathbb{\Delta}$ has a left adjoint which freely adds a bottom element. The resulting monad on $\mathbb{\Delta}$ is denoted b. The forgetful functor u is in fact monadic, so that $\mathbb{\Delta}^{\mathsf{b}}$ becomes the category of Eilenberg–Moore algebras for b, and $\mathbb{\Delta}_{\mathsf{b}}$ is the Kleisli category (the full subcategory of free algebras), hence the notation for these two categories. The left adjoint free functor thus factors as

$$\mathbb{\Delta} \xrightarrow{\mathsf{k}} \mathbb{\Delta}_{\mathsf{b}} \xrightarrow{\mathsf{i}} \mathbb{\Delta}^{\mathsf{b}},$$

where k is identity-on-objects and i is full. We let these functors dictate the naming conventions for the objects in the three categories:

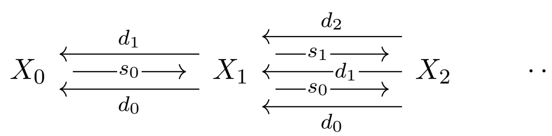
$$[n] \mapsto [n] \mapsto [n].$$

The category Δ^{b} thus has an extra object denoted $[-1]$ corresponding to the linear-order-with-bottom-element consisting of only the bottom element.

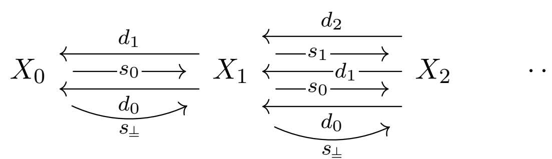
We now take presheaves (of spaces), to get functors

$$\boldsymbol{Pr}(\Delta) \xleftarrow{\mathsf{k}^*} \boldsymbol{Pr}(\Delta_{\mathsf{b}}) \xleftarrow{\mathsf{i}^*} \boldsymbol{Pr}(\Delta^{\mathsf{b}}).$$

Objects in $\boldsymbol{Pr}(\Delta)$ are of course simplicial spaces; we picture them by drawing the first few face and degeneracy maps

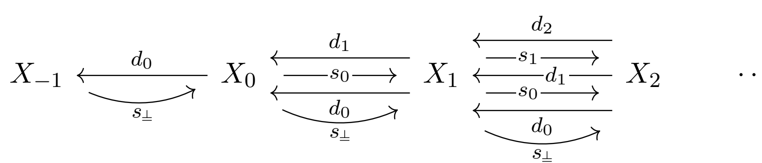


Objects in $\boldsymbol{Pr}(\Delta_{\mathsf{b}})$ are called *bottom-split simplicial spaces*; the diagram of their generating maps starts like this:

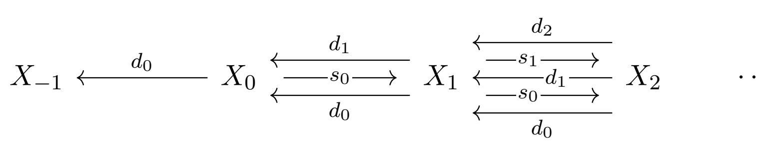


The extra bottom sections satisfy the simplicial identities corresponding to a section to $d_\perp$ "below", and we will always use the symbol $s_\perp$ for these "sub-bottom" degeneracy maps.

Objects in $\boldsymbol{Pr}(\Delta^{\mathsf{b}})$ are called *bottom-split augmented simplicial spaces*; they are drawn with



An *augmented simplicial space* is a presheaf on Δ_+, where the index convention is this:



It is often practical to interpret this data as a simplicial map

$$\bar{X}_{-1} \leftarrow X$$

from the underlying simplicial space to the constant simplicial space on X_{-1}.

For any simplicial space X, we can take the geometric realization (colimit) X_{-1} forming altogether an augmented simplicial space with augmentation map $X_{-1} \leftarrow X_0$. In the situation where X is split by extra bottom degeneracy maps (that is, X is the underlying simplicial space of a $\mathbb{\Delta}_{\mathsf{b}}$-presheaf), the augmentation map $X_{-1} \overset{d_0}{\leftarrow} X_0$ acquires a section $s_\perp : X_{-1} \to X_0$, so as to form altogether a $\mathbb{\Delta}^{\mathsf{b}}$-presheaf. In fact, this shape of diagram is an absolute colimit: every $\mathbb{\Delta}^{\mathsf{b}}$-presheaf A exhibits A_{-1} as the geometric realization (colimit) of $\mathsf{i}^*(A)$, or equivalently, as the geometric realization (colimit) of the underlying simplicial space $\mathsf{k}^*\mathsf{i}^*(A)$ (see [58, p. 6.1.3.16]). This is one interpretation of the fact that i^* is an equivalence.

2.2 Decalage

The adjunction

$$\begin{array}{ccc} & & \boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}}) \\ & \swarrow^{\mathsf{i}^*} & \uparrow \\ \boldsymbol{Pr}(\mathbb{\Delta}_{\mathsf{b}}) & \dashv & \mathsf{u}^* \\ & \searrow_{\mathsf{k}^*} & | \\ & & \boldsymbol{Pr}(\mathbb{\Delta}) \end{array}$$

defines a comonad on $\boldsymbol{Pr}(\mathbb{\Delta})$ which is the lower decalage comonad

$$\mathrm{Dec}_\perp := \mathsf{b}^* = (\mathsf{ik})^*\mathsf{u}^* : \boldsymbol{Pr}(\mathbb{\Delta}) \to \boldsymbol{Pr}(\mathbb{\Delta}).$$

We write

$$\varepsilon : \mathrm{Dec}_\perp(X) \to X$$

for its counit; it is given in each degree by bottom face maps. We write $\delta : \mathrm{Dec}_\perp(X) \to \mathrm{Dec}_\perp \mathrm{Dec}_\perp(X)$ for the comultiplication; it is given in each degree by bottom degeneracy maps. We shall also use the canonical augmentation of $\mathrm{Dec}_\perp(X)$, namely the simplicial map

$$\bar{X}_0 \overset{\zeta}{\longleftarrow} \mathrm{Dec}_\perp(X)$$

given in each degree by composites of top face maps; in particular, in degree zero it is the map $X_0 \overset{d_1}{\leftarrow} X_1$.

The functor $\mathsf{k}^* \circ \mathsf{i}^*$ is comonadic, so as to identify $\boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}})$ with the category of $\mathrm{Dec}_\perp$-coalgebras. In explicit terms, a $\mathbb{\Delta}^{\mathsf{b}}$-presheaf A has an underlying simplicial space $X := \mathsf{k}^*\mathsf{i}^*(A)$ possessing extra bottom degeneracies; these assemble into a simplicial map $\gamma : X \to \mathrm{Dec}_\perp(X)$ which is the structure map of a $\mathrm{Dec}_\perp$-coalgebra: the split-simplicial identities satisfied by the extra bottom degeneracies correspond precisely to the coalgebra axioms.

It is also the case that u^* is monadic, so that the induced monad $\widetilde{\mathrm{Dec}}_\perp := \mathsf{u}^*(\mathsf{ik})^*$ on $\boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}}) \simeq \mathrm{Dec}_\perp$-$\boldsymbol{Coalg}$ has the category $\boldsymbol{Pr}(\mathbb{\Delta})$ as category of algebras [36]. We shall not need that fact, but we shall reference the monad $\widetilde{\mathrm{Dec}}_\perp$ a few times. We write $\eta_A : A \to \widetilde{\mathrm{Dec}}_\perp(A)$ for its unit.

2.1. Observation. If $X \in \boldsymbol{Pr}(\mathbb{\Delta})$ has a $\mathrm{Dec}_\perp$-coalgebra structure, so as to constitute a $\mathbb{\Delta}^{\mathsf{b}}$-presheaf A (that is, $X = (\mathsf{ik})^*(A)$), then the structure map $\gamma : X \to \mathrm{Dec}_\perp(X)$ underlies a $\mathrm{Dec}_\perp$-coalgebra map, namely $\gamma = (\mathsf{ik})^*(\eta_A)$. (This is a general fact about comonads.) We say loosely that γ is itself a $\mathrm{Dec}_\perp$-coalgebra map (with the standard blurring of what kind of objects we are talking about).

For 1-categories (and in fact for ∞-categories), $\mathrm{Dec}_\perp$-coalgebras can be interpreted as specifying local initial objects, meaning an initial object in each connected component (cf. Garner–Kock–Weber [36] for the dual case of local terminal objects). The degeneracy map $s_\perp : A_{-1} \to A_0$ picks out an object in each connected component, and the remaining coalgebra axioms imply that these objects are in fact initial in each component.

Since our main interest is in simplicial objects, we use the underlying simplicial objects of $\mathbb{\Delta}^{\mathsf{b}}$-presheaves to dictate how we extend the various notions of simplicial map to bottom-split simplicial objects: a morphism of bottom-split (possibly augmented) simplicial spaces $F : A \to B$ is called *left fibration*, *right fibration*, *or culf* if the map of underlying simplicial spaces

$$(\mathsf{k}^* \circ \mathsf{i}^*)(A) \to (\mathsf{k}^* \circ \mathsf{i}^*)(B)$$

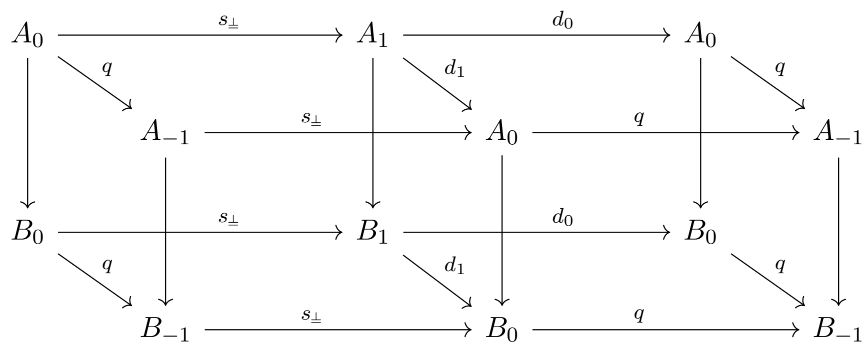
is a left fibration, a right fibration, or culf, respectively. For the property "cartesian" there is a potential conflict of meanings: it could mean cartesian on the simplicial part in line with the convention, or it could mean truly cartesian (on all arrows). The following lemma resolves this conflict, as it implies that the two notions agree:

Lemma 2.2. *A left fibration $F : A \to B$ of bottom-split simplicial spaces is automatically cartesian. If the bottom-split simplicial spaces are augmented, then F is cartesian also on the augmentation map, here temporarily denoted $A_{-1} \stackrel{q}{\leftarrow} A_0$. In particular*

$$\begin{array}{ccc} A_{-1} & \xleftarrow{q} & A_0 \\ \downarrow & & \downarrow \\ B_{-1} & \xleftarrow[q]{} & B_0 \end{array} \tag{2.31}$$

is a pullback.

Proof. We can assume that $F : A \to B$ is augmented (since it can be augmented uniquely by taking geometric realization). We first prove that then also the square (2.31) is a pullback. This is a retract argument: the diagram



is easily seen to commute, and it exhibits the square (2.31) as a retract of the middle vertical square in the diagram. This middle square is a pullback, since the simplicial map is assumed to be a left fibration. Since limits are stable under retracts, it follows that also the square (2.31) is a pullback, as asserted. Now since F is cartesian on both q and on d_1, it is cartesian on $q \circ d_1 = q \circ d_0$ and therefore it must also be cartesian on d_0; this argument can be repeated for all face maps. So now F is cartesian on all face maps. But then it is also cartesian on all degeneracy maps, since these are sections. □

Remark 2.3. The above lemma is an abstraction of the fact for 1-categories that if a left fibration preserves local initial objects, then it is also a right fibration.

Lemma 2.4. *Let $F : A \to B$ be a right fibration of bottom-split simplicial spaces. Then F is also cartesian on all splitting maps, including the augmentation splitting.*

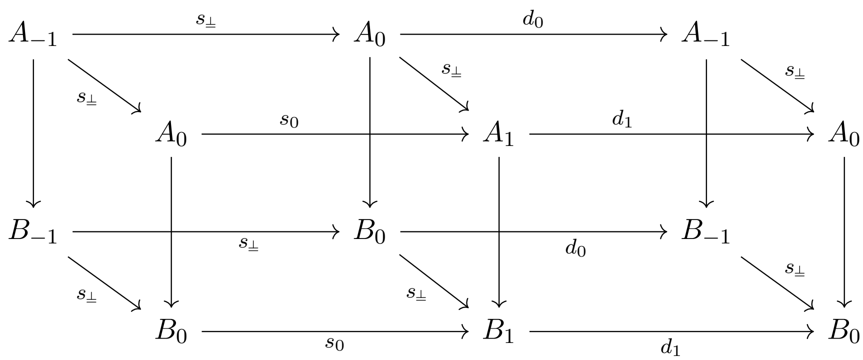
Proof. (Although the proof is similar to the previous proof, there are subtle differences, and in particular the square (2.31) is not always a pullback.) That F is a right fibration means that it is cartesian on bottom face maps. Since the splittings

$$A_i \xrightarrow{s_\perp} A_{i+1}$$

are sections to bottom face maps for all $i \geq 0$, it follows that F is also cartesian on all these splittings. It remains to show that F is cartesian on the augmentation splitting, meaning that the square

$$\begin{array}{ccc} A_{-1} & \xrightarrow{s_\perp} & A_0 \\ \downarrow & & \downarrow \\ B_{-1} & \xrightarrow[s_\perp]{} & B_0 \end{array} \qquad (2.32)$$

is a pullback. This is a retract argument: the diagram

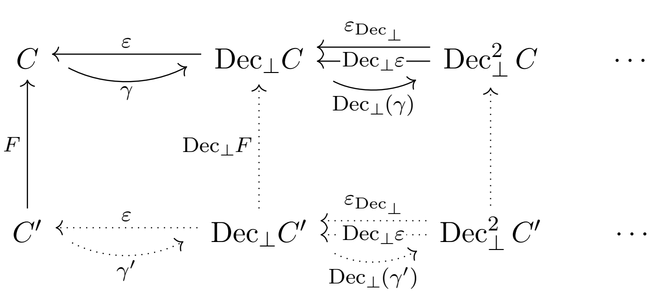


is easily seen to commute, and it exhibits the square (2.32) as a retract of the middle vertical square in the diagram. But the middle square is a pullback since p is a right fibration and by the first argument of the proof. Since limits are stable under retracts, it follows that also the square (2.32) is a pullback, as asserted. □

Remark 2.5. It is standard fact for 1-categories that if $\mathcal{C}' \to \mathcal{C}$ is a right fibration of categories, and if $\mathcal{C}$ has local initial objects, then also $\mathcal{C}'$ has local initial objects, and they are just the preimages of the local initial objects of $\mathcal{C}$. The next lemma generalizes this to state that $\mathrm{Dec}_\perp$-coalgebra structure can be pulled back along right fibrations. We will need this result again in Section 5, where we need it in a slightly more general version, so we state it in that generality: Let $\mathcal{E}$ be an ∞-category with pullbacks. Then we may speak of right fibrations in the ∞-category $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{E})$, and we can also speak of the lower decalage comonad.

Lemma 2.6. *Let $\mathcal{E}$ be a category with pullbacks and let $F : C' \to C$ be a right fibration in* $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{E})$*. Then a* $\mathrm{Dec}_\perp$*-coalgebra structure on C induces a* $\mathrm{Dec}_\perp$*-coalgebra structure on C' (which is in fact unique with the property that F becomes a morphism of* $\mathrm{Dec}_\perp$*-coalgebras).*

Proof. The $\mathrm{Dec}_\perp$-coalgebra structure on C is given by the bottom splitting of the bar resolution of the comonad $\mathrm{Dec}_\perp$, as pictured as the (horizontal) solid part of the diagram



In the solid diagram, all the $\mathrm{Dec}_\perp^k(\varepsilon)$ are instances of d_0, and all the $\varepsilon_{\mathrm{Dec}_\perp^k}$ are instances of d_k (which are active). Since F is a right fibration, it forms pullbacks with all of them, which means that when pulling back the whole diagram, a new $\mathrm{Dec}_\perp$-coalgebra structure results on C'. □

2.3 Rigid $\mathrm{Dec}_\perp$-coalgebras

Among all $\mathrm{Dec}_\perp$-coalgebras, of particular importance for this article are the rigid $\mathrm{Dec}_\perp$-coalgebras:

Definition 2.7. *A* $\mathrm{Dec}_\perp$*-coalgebra* $\gamma : X \to \mathrm{Dec}_\perp(X)$ *on a simplicial space* X *is said to be* rigid *if* γ *is cartesian.*

Remark 2.8. Equivalently, *a* $\mathrm{Dec}_\perp$*-coalgebra* $A \in \boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}})$ *(with underlying structure map* γ*) is rigid if and only if the* A*-component* $\eta_A : A \to \widetilde{\mathrm{Dec}}_\perp(A)$ *of the unit of the monad* $\widetilde{\mathrm{Dec}}_\perp$ *is cartesian.* Indeed, we have $\gamma \simeq (\mathsf{ik})^*(\eta_A)$ by Observation 2.1, saying that γ is the restriction of η_A to the underlying simplicial space. So if η_A is cartesian, then also γ is cartesian. Conversely, if γ is cartesian and hence a left fibration, then η_A is a left fibration (by our convention for left fibrations in $\boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}})$), but now Lemma 2.2 guarantees that η_A is actually cartesian.

The notion of rigidity is motivated by the notion of local initial objects, cf. the next subsection. For 1-categories (or ∞-categories), $\mathrm{Dec}_\perp$-coalgebra structure encodes a choice of local initial objects [36], but for more general simplicial spaces, the further property of rigidity must be imposed separately in order to get a reasonable notion of local initial objects, as we shall see. The reason the rigidity condition does not turn up in category theory [36] is that it is automatic for Segal spaces:

Lemma 2.9. *For a Segal space* X*, any* $\mathrm{Dec}_\perp$*-coalgebra structure is rigid.*

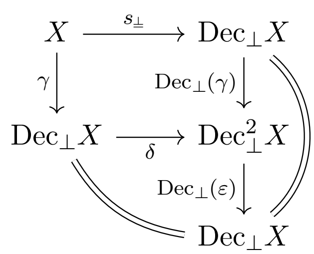
Proof. Let $\gamma : X \to \mathrm{Dec}_\perp(X)$ be a $\mathrm{Dec}_\perp$-coalgebra. The counit axiom for coalgebras gives the equivalence $\varepsilon \circ \gamma \simeq \mathrm{id}$. Now X is Segal if and only if ε is a left fibration (A.1), so in this case also γ is a left fibration. On the other hand, by Observation 2.1, γ is actually a morphism of bottom-split simplicial spaces, and we know from Lemma 2.2 that a left fibration of bottom-split simplicial spaces is in fact cartesian. □

Example 2.10. Let Y be lower 2-Segal, and consider the 1-Segal space $A := \mathrm{Dec}_\perp Y$. Then A has a canonical extra bottom degeneracy map $s_\perp : A \to \mathrm{Dec}_\perp A$, and this is cartesian.

For the next lemma, recall that $\mathrm{Dec}_\perp$-coalgebra structure can be pulled back along cartesian maps (see Lemma 2.6). It turns out the rigidity accounts for all the higher coherences in the notion of $\mathrm{Dec}_\perp$-coalgebra:

Lemma 2.11. *Let $\gamma : X \to \mathrm{Dec}_\perp X$ be a simplicial map satisfying the equation $\varepsilon \circ \gamma \simeq 1$. If γ is cartesian, then γ is obtained by pulling the canonical* $\mathrm{Dec}_\perp$*-coalgebra on* $\mathrm{Dec}_\perp(X)$ *along* γ*. In particular,* γ *is a rigid* $\mathrm{Dec}_\perp$*-coalgebra structure.*

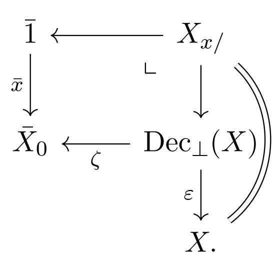
Proof. First of all, since γ is cartesian, it follows from Lemma 2.6 that X inherits a $\mathrm{Dec}_\perp$-coalgebra structure $s_\perp : X \to \mathrm{Dec}_\perp X$ from the canonical one on $\mathrm{Dec}_\perp X$, given by the comultiplication of the comonad $\delta : \mathrm{Dec}_\perp X \to \mathrm{Dec}^2_\perp X$. According to the same lemma, the top square in the diagram



commutes (and is in fact a pullback). By the commutativity of the whole diagram we see that γ is in fact equivalent to $s_\perp$, so since $s_\perp$ is a coalgebra structure, also γ is a coalgebra structure. ☐

2.4 Local initial objects

The property characterizing an initial object x in a Segal space X is that the projection $X_{x/} \to X$ is an equivalence. The coslice under x is formally defined as the pullback in the diagram below, and the projection required to be an equivalence is the vertical composite in the diagram



If it exists, an initial object is essentially unique. (The resulting map $\gamma : X \to \mathrm{Dec}_\perp(X)$ is the corresponding $\mathrm{Dec}_\perp$-coalgebra structure.) For local initial objects, there is instead an initial object in each connected component, and the situation is rather

$$
\begin{array}{ccc}
\bar{C} & \longleftarrow & X_{a/} \\
{\scriptstyle \bar{a}}\downarrow & \lrcorner & \downarrow \\
\bar{X}_0 & \xleftarrow[\zeta]{} & \mathrm{Dec}_\perp(X) \\
 & & {\scriptstyle \varepsilon}\downarrow \\
 & & X.
\end{array}
\qquad X_{a/} = X
\tag{2.33}
$$

Again, the choice of the pointing $a : C \to X_0$ with this property is essentially unique, and by absoluteness, C must be the geometric realization of X. This definition is appropriate also for general simplicial spaces:

Definition 2.12. *A* local-initial-objects structure *on a simplicial space X consists of a pointing $a : C \to X_0$ for which the map*

$$\bar{C} \times_{\bar{X}_0} \mathrm{Dec}_\perp(X) \xrightarrow{pr} \mathrm{Dec}_\perp(X) \xrightarrow{\varepsilon} X$$

is an equivalence (as in diagram (2.33)).

Dually, it is said to be a local-terminal-objects structure on X *if* $\bar{C} \times_{\bar{X}_0} \mathrm{Dec}_\top(X) \xrightarrow{pr} \mathrm{Dec}_\top(X) \xrightarrow{\varepsilon} X$ *is an equivalence.*

Once we pass to the bisimplicial setting, this condition will appear in the pointing axiom 5.1 of BOORS [16], [15].

We now head towards proving that local-initial-objects structures and rigid $\mathrm{Dec}_\perp$-coalgebras are in fact equivalent notions. The underlying combinatorial shape of local-initial-objects structures is captured by the category $\mathbb{\Delta}^{\mathrm{pt}}$, given by adjoining a terminal object to $\mathbb{\Delta}$. In terms of generators and relations, $\mathbb{\Delta}^{\mathrm{pt}}$ has, in addition to the generators and relations of $\mathbb{\Delta}$, an object $[-1]$ and a new morphism $[-1] \leftarrow [0]$. We call the objects in $\boldsymbol{Pr}(\mathbb{\Delta}^{\mathrm{pt}})$ *pointed simplicial spaces.* A pointed simplicial space X thus consists of a simplicial space, also denoted by X, and a map $a : C \to X_0$ called the *pointing*. A convenient way of organizing the data of such a pointed simplicial space is by a simplicial map $\bar{C} \to X$.

The comparison of rigid $\mathrm{Dec}_\perp$-coalgebras and local-initial-objects structures spaces is based on the inclusion of categories

$$
\begin{array}{ccccccccc}
\mathbb{\Delta}^{\mathrm{pt}} & & [-1] & \leftarrow & [0] & \rightrightarrows & [1] & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & [2] \quad \cdots \\
{\scriptstyle \mathsf{h}}\downarrow & & & & & & & & \\
\mathbb{\Delta}^{\mathrm{b}} & & [-1] & \rightleftarrows & [0] & \substack{\rightrightarrows\\\leftarrow} & [1] & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\leftarrow} & [2] \quad \cdots
\end{array}
$$

and the induced adjunction

$$\mathrm{Pr}(\mathbb{\Delta}^{\mathsf{b}}) \underset{\mathsf{h}_*}{\overset{\mathsf{h}^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Pr}(\mathbb{\Delta}^{\mathrm{pt}}).$$

We first describe h_*. Given a simplicial space X, we can consider the bottom-split augmented simplicial space $\mathsf{u}^*(X)$ (which amounts to removing the bottom face maps, but not the bottom degeneracy maps, and shifting down everything so that the original degree 0 becomes the new degree -1). There is a canonical morphism of bottom-split augmented simplicial space $\bar{X}_0 \overset{\iota}{\longleftarrow} \mathsf{u}^*(X)$, which in degree -1 is given by the identity. The right Kan extension of a pointed simplicial space $\bar{C} \to X$, denoted by $\mathsf{h}_*(X)$ for short, can now be described as the pullback of the entire $\mathbb{\Delta}^{\mathsf{b}}$-presheaf $\mathsf{u}^*(X)$ along the pointing $a : C \to X_0$:

$$\begin{array}{ccc} \bar{C} & \longleftarrow & \mathsf{h}_*(X) \\ {\scriptstyle \bar{a}}\downarrow & \lrcorner & \downarrow \\ \bar{X}_0 & \underset{\iota}{\longleftarrow} & \mathsf{u}^*(X), \end{array}$$

where $\bar{a}$ is a map of constant bottom-split simplicial spaces. Expanding these bottom-split augmented simplicial spaces, the construction can be depicted in more detail as in the diagram

$$\begin{array}{ccccccc} C & \leftrightarrows & \mathsf{h}_*(X)_0 & \leftleftarrows & \mathsf{h}_*(X)_1 & & \cdots \\ {\scriptstyle a}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\ X_0 & \underset{s_0}{\overset{d_1}{\leftrightarrows}} & X_1 & \underset{s_0}{\overset{d_2, d_1}{\leftleftarrows}} & X_2 & & \cdots \end{array}$$

where $\mathsf{h}_*(X)_{-1} \simeq C$.

The counit $\mathsf{h}^*\mathsf{h}_* \to 1$ of the adjunction evaluated at a pointed simplicial space X is given by id_C on the augmentation and by

$$\bar{C} \times_{\bar{X}_0} \mathrm{Dec}_\bot(X) \xrightarrow{\mathrm{pr}} \mathrm{Dec}_\bot(X) \xrightarrow{\varepsilon} X$$

on the underlying simplicial space.

With the adjunction $\mathsf{h}_* \vdash \mathsf{h}^*$ at hand we can now reformulate the definition of local-initial-objects structure as in the following lemma, which is a tautology.

Lemma 2.13. *A pointed simplicial space $\bar{C} \to X$ is a local-initial-objects structure on X if and only if the counit of the adjunction $\mathsf{h}_* \vdash \mathsf{h}^*$ evaluated at X is invertible.*

Lemma 2.14. *Let $A \in \boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}})$ be a rigid $\mathrm{Dec}_\bot$-coalgebra. Then $\mathsf{h}^*(A)$ is a local-initial-objects structure on the underlying simplicial space of A.*

Proof. Let $X := (\mathsf{ik})^*(A)$ denote the underlying simplicial space of A and let $\gamma : X \to \mathrm{Dec}_\bot X$ denote the corresponding $\mathrm{Dec}_\bot$-coalgebra structure map. We must show that the composite

$$\bar{A}_{-1} \times_{\bar{A}_0} \mathrm{Dec}_\perp(X) \xrightarrow{\mathrm{pr}} \mathrm{Dec}_\perp(X) \xrightarrow{\varepsilon} X$$

is invertible. The fact that A is rigid means that the pullback projection $\bar{A}_{-1} \times_{\bar{A}_0} \mathrm{Dec}_\perp(A') \xrightarrow{\mathrm{pr}} \mathrm{Dec}_\perp(A')$ is computed by $\gamma : X \to \mathrm{Dec}_\perp X$, as can be seen in the diagram

$$\begin{array}{ccccccc} A_{-1} & \longleftarrow & A_0 & \overset{d_1}{\underset{d_0}{\leftleftarrows}} & A_1 & & \cdots \\ {\scriptstyle\gamma_{-1}}\downarrow & \llcorner & \downarrow{\scriptstyle\gamma_0} & \llcorner & \downarrow{\scriptstyle\gamma_1} & & \\ A_0 & \longleftarrow & A_1 & \overset{d_2}{\underset{d_1}{\leftleftarrows}} & A_2 & & \cdots \end{array}$$

As a result, the above composite reduces to

$$X \xrightarrow{\gamma} \mathrm{Dec}_\perp X \xrightarrow{\varepsilon} X$$

which is equivalent to the identity by the $\mathrm{Dec}_\perp$-coalgebra counit law and thereby invertible. □

Lemma 2.15. *Let $a : C \to X_0$ be a local-initial-objects structure on a simplicial space X. Then $\mathsf{h}_*(X)$ is a rigid $\mathrm{Dec}_\perp$-coalgebra.*

Proof. The assumption that $\bar{C} \to X$ is a local-initial-objects structure says that we have a pullback square

$$\begin{array}{ccc} \bar{C} & \longleftarrow & X \\ {\scriptstyle\bar{a}}\downarrow & \llcorner & \downarrow{\scriptstyle\gamma} \\ \bar{X}_0 & \underset{\zeta}{\longleftarrow} & \mathrm{Dec}_\perp X \end{array} \tag{2.34}$$

where γ satisfies the unit law $\varepsilon \circ \gamma \simeq 1$. Since $\bar{a}$ is cartesian so is γ. But, by definition, the pullback is precisely the underlying simplicial space of $\mathsf{h}_*(X)$, and the $\mathrm{Dec}_\perp$-coalgebra structure on $\mathsf{h}_*(X)$ is precisely that induced by pulling back the canonical $\mathrm{Dec}_\perp$-coalgebra on $\mathrm{Dec}_\perp(X)$ along γ. By Lemma 2.11, the $\mathrm{Dec}_\perp$-coalgebra structure map of $\mathsf{h}_*(X)$ is identified with γ, rendering $\mathsf{h}_*(X)$ into a rigid $\mathrm{Dec}_\perp$-coalgebra. □

Thanks to these two lemmas we may restrict the functors $\mathsf{h}^* \dashv \mathsf{h}_*$ to an adjunction between the full subcategory of rigid $\mathrm{Dec}_\perp$-coalgebras and the full subcategory of local-initial-objects structures. Lemma 2.13 characterizes the fixpoint locus of the counit. We now analyze the unit.

The unit $1 \to \mathsf{h}_* \mathsf{h}^*$ evaluated on $A \in \boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}})$ is given by

$$A \longrightarrow \bar{A}_{-1} \times_{\bar{A}_0} \widetilde{\mathrm{Dec}}_{\perp} A,$$

where $\bar{A}_{-1}$ and $\bar{A}_0$ are interpreted as $\mathbb{\Delta}^{\mathsf{b}}$-presheaves. This map is induced by the universal property of the pullback, as detailed in the diagram

$$\begin{tikzcd} & & A \arrow[dll, bend right, "\iota"'] \arrow[dl, dashed] \arrow[ddl, bend left] \\ \bar{A}_{-1} \arrow[d, "\bar{s}_{\perp}"'] & \mathsf{h}_* \mathsf{h}^*(A) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] & \\ \bar{A}_0 & \widetilde{\mathrm{Dec}}_{\perp}(A). \arrow[l, "\iota"] & \end{tikzcd} \tag{2.35}$$

Here the arrows labeled ι are the canonical maps of augmented (split) simplicial spaces given by the identity in degree -1, and where the map $A \to \widetilde{\mathrm{Dec}}_{\perp}(A)$ is the unit for the $\widetilde{\mathrm{Dec}}_{\perp}$ monad; it is given by $s_{\perp}$ in each degree, so the outer square commutes.

Lemma 2.16. *The unit of the adjunction* $\mathsf{h}^* \dashv \mathsf{h}_*$ *evaluated on a* $\mathrm{Dec}_{\perp}$*-coalgebra* A *is an equivalence if and only if* A *is rigid.*

Proof. The unit of $\mathsf{h}^* \dashv \mathsf{h}_*$ is the dashed arrow in diagram (2.35), so it is an equivalence if and only if the outer square is a pullback, which is to say that the map $A \to \widetilde{\mathrm{Dec}}_{\perp}(A)$ is cartesian. But according to Remark 2.8 this is precisely to say that A is rigid. □

The last four lemmas (2.13–2.16) together establish the following result.

Proposition 2.17. *The adjunction* $\mathsf{h}^* \vdash \mathsf{h}_*$ *restricts to an equivalence of full subcategories*

$$\{\textit{rigid } \mathrm{Dec}_{\perp}\textit{-coalgebras}\} \underset{\mathsf{h}_*}{\overset{\mathsf{h}^*}{\rightleftarrows}} \{\textit{local-initial-objects structures}\}.$$

Example 2.18. Let X be a lower 2-Segal simplicial space. Then by definition $\mathrm{Dec}_{\perp}(X)$ is Segal. Now, $\mathrm{Dec}_{\perp}(X)$ supports a canonical $\mathrm{Dec}_{\perp}$-coalgebra structure given by the comonad comultiplication. Since $\mathrm{Dec}_{\perp}(X)$ is 1-Segal, it follows from Lemma 2.9 that $\mathrm{Dec}_{\perp}(X)$ is a rigid $\mathrm{Dec}_{\perp}$-coalgebra. Its restriction to $\mathbb{\Delta}^{\mathrm{pt}}$ is a local-initial-objects structure on X by Proposition 2.17.

Remark 2.19. According to Proposition 2.17, the data of local initial objects can be understood from two different viewpoints. While the notion in terms of a pointing is more economical, the viewpoint of rigid $\mathrm{Dec}_{\perp}$-coalgebras has other advantages by relating to various functorialities of interest, as exploited in this article.

3 Augmented bisimplicial spaces with abacus maps

From now on we will be considering bisimplicial spaces, that is presheaves on $\mathbb{\Delta}\times\mathbb{\Delta}$. For a bisimplicial space $B : (\mathbb{\Delta}\times\mathbb{\Delta})^{\mathrm{op}} \to \mathcal{S}$, we use matrix convention for rows and columns, but starting of course the indexations from 0. So $B_{\bullet,0}$ is referred to as the zeroth column and $B_{0,\bullet}$ as the zeroth row. Following Carlier [21] we use the convention of using the letters

> d and s for face and degeneracy maps in the horizontal direction (second index)
>
> e and t for face and degeneracy maps in the vertical direction (first index).

3.1 Stability

A bisimplicial space is called *upper stable* if all $e_\perp$ and $d_\perp$ form pullbacks with each other; it is called *lower stable* if all $e_\top$ and $d_\top$ form pullbacks with each other; it is called *stable* if it is both upper stable and lower stable. Upper stable can also be formulated as saying that all $e_\perp$ (or equivalently all $d_\perp$) are right fibrations; lower stable can be formulated as saying that all $e_\top$ (or equivalently all $d_\top$) are left fibrations.

Remark 3.1. The stability condition is due to BOORS [16], see also Carlier [21]. The separate notions of lower and upper stable are required in a few places below. It may be a bit confusing that lower stable relates to top face maps while upper stable relates to bottom face maps, but the terminology matches the usage for upper and lower 2-Segal conditions under the chosen convention for total decalage (cf. 1.4), as expressed by this straightforward lemma:

Lemma 3.2. *If a simplicial space X is 2-Segal (respectively upper 2-Segal, respectively lower 2-Segal), then* $\mathrm{Tot}(X)$ *is stable (respectively upper stable, respectively lower stable).*

Lemma 3.3. *Let B be a bisimplicial space.*

1. *If B is upper stable, then*

   — all horizontal active maps are right fibrations (considered as simplicial maps between columns),

   — all vertical active maps are right fibrations (considered as simplicial maps between rows).

2. *If B is lower stable, then*

   — all horizontal active maps are left fibrations (considered as simplicial maps between columns),

   — all vertical active maps are left fibrations (considered as simplicial maps between rows).

3. *If B is stable then*

   — every vertical active map is cartesian (considered as a simplicial map between rows),

   — every horizontal active map is cartesian (considered as a simplicial map between columns).

Proof. This follows from standard pullback-prism-lemma arguments (in analogy with the so-called bonus pullbacks holding for decomposition spaces [33, Lemma 3.10]). □

Definition 3.4. *A bisimplicial space is called* double Segal *if every row and every column is a* 1*-Segal space.*

Lemma 3.5 (Carlier [21, Lemma 2.3.3])**.** *If a bisimplicial space is double Segal, then for it to be stable it is enough for the two squares*

$$\begin{array}{ccc} B_{0,0} & \xleftarrow{d_0} & B_{0,1} \\ {\scriptstyle e_0}\uparrow & \lrcorner & \uparrow{\scriptstyle e_0} \\ B_{1,0} & \xleftarrow[d_0]{} & B_{1,1} \end{array} \qquad\qquad \begin{array}{ccc} B_{0,0} & \xleftarrow{d_1} & B_{0,1} \\ {\scriptstyle e_1}\uparrow & \lrcorner & \uparrow{\scriptstyle e_1} \\ B_{1,0} & \xleftarrow[d_1]{} & B_{1,1} \end{array}$$

to be pullbacks. (To check upper stability, it is enough to check the square on the left, and to check lower stability, it is enough to check the square on the right.)

3.2 Augmentations

Let B be a bisimplicial space. Then by taking the geometric realization of each row we obtain an augmentation column $X = B_{\bullet,-1}$. The simplicial operators inside X are induced by the vertical simplicial operators of B and the universal property of the geometric realization. Similarly, by taking geometric realization of each column, we obtain an augmentation row $Y = B_{-1,\bullet}$, where the simplicial operators inside Y are induced by the horizonal simplicial operators of B.

The total shape of such a diagram (see picture (2.29) in the introduction) is that of a presheaf on

$$(\mathbb{\Delta}_+ \times \mathbb{\Delta}_+) \setminus ([-1],[-1]) \simeq \mathbb{\Delta}_{/[1]}$$

(see Carlier [21]). Under this isomorphism, an object $[i, j]$ on the left corresponds to the map $(s^{\perp})^{\circ i}(s^{\top})^{\circ j} : [i{+}1{+}j] \to [1]$ in $\mathbb{\Delta}$ on the right. Here one of i and j, but not both, could be equal to -1, corresponding to an empty fibre. The left-hand category motivates our notation (which agrees with that of $\mathbb{\Delta}{\times}\mathbb{\Delta}$), whereas the $\mathbb{\Delta}_{/[1]}$ viewpoint is more elegant, and becomes highly relevant from Subsection 4.3.

Proposition 3.6. *If a stable double* 2*-Segal space B is endowed with colimit augmentation column X and colimit augmentation row Y, then X and Y are again* 2*-Segal, and the augmentation maps are culf.*

Proof. Let $a : Y_k \to Y_n$ be an active map of Y. It is induced by the corresponding simplicial map $a_\bullet : B_{\bullet,k} \to B_{\bullet,n}$ between columns. Since B is stable, the simplicial map $a_\bullet$ is cartesian by Lemma 3.3. By descent (see Lurie [58, Theorem 6.1.3.9, Proposition 6.1.3.10]), this implies that also the augmentation square

$$\begin{array}{ccc} Y_k & \xrightarrow{\;a\;} & Y_n \\ \uparrow & & \uparrow \\ B_{0k} & \xrightarrow[a_0]{} & B_{0n} \end{array}$$

is a pullback. Since this argument works for every active map, this means that the simplicial map $B_{0\bullet} \to Y$ is culf.

It is a general fact that the augmentation map from a simplicial space to its geometric realization is an effective epimorphism. To see this, note first that the induced map from the space of all simplices $\coprod_{[n]\in\mathbb{\Delta}} B_{\bullet,n} \to Y_n$ is effective epi (by [58, p. 6.2.3.13]), but this map factors through the augmentation map $B_{0,n} \to Y_n$ which must therefore be an effective epi too (see [58, p. 6.2.3.12]). Since the augmentation maps are effective epimorphisms, and since the zeroth row is 2-Segal, it follows from A.7 that also Y is 2-Segal. □

Remark 3.7. Note that both lower stable and upper stable are required in the proposition, because this is what makes the active maps cartesian (3.3) so that we can apply descent. (Our attempts to prove separate versions for lower stable and 2-Segal rows did not work out.) In the applications of Proposition 3.6, the bisimplicial space B will actually be 1-Segal in each column and each row, but the augmentation row and column will still only be 2-Segal. The key example of this situation is when $B = \mathrm{Tot}(X)$ for X a 2-Segal space.

Definition 3.8 (Carlier [21])**.** *A* bicomodule configuration *is a presheaf $B \in$* ***Pr***$(\mathbb{\Delta}_{/[1]})$ *whose underlying bisimplicial space is stable, double Segal, where the augmentation row and column are* 2*-Segal and where the augmentation maps are culf.*

Remark 3.9. The role of the augmentations, which we often denote $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$, is to encode the coalgebras that B is a bicomodule over: being a left X-comodule and a right Y-comodule requires the 1-Segal and culf conditions imposed (cf. Walde [78] and Young [81]), whereas the bicomodule condition is expressed by stability, cf. Carlier [21].

3.3 The abacus category $\mathcal{D}$

In this subsection we describe the category $\mathcal{D}$ that captures the combinatorics of bicomodule configurations with abacus maps. The category $\mathcal{D}$ was first studied by Carlier [21], as far as we know (he denotes it $\overline{\Delta}_{/[1]}$). Carlier [21, §3.2] defined it in terms of certain mapping cylinders and cocartesian fibrations. We give a slightly more elementary definition, more closely reflecting Carlier's suggested combinatorial interpretation in terms of black and white beads ([21, p. 3.2.1]). The elementary combinatorial/graphical interpretation is useful in practice to quickly verify some identities.

Definition 3.10. *We define $\mathcal{D}$ formally as a full subcategory of the arrow category of $\mathbb{\Delta}_+$: the objects are the maps $(d^\top)^{\circ(1+j)} : [i] \to [i+1+j]$ for all $i, j \in \mathbb{\Delta}_+$ except for the case $i = j = -1$. The object $(d^\top)^{\circ(1+j)} : [i] \to [i+1+j]$ will be denoted by $[i, j]$.*

The morphisms $[i, j] \to [i', j']$ in $\mathcal{D}$ are thus commutative diagrams

$$\begin{array}{ccc} [i] & \longrightarrow & [i'] \\ {\scriptstyle (d^\top)^{\circ(1+j)}}\downarrow & & \downarrow{\scriptstyle (d^\top)^{\circ(1+j')}} \\ [i{+}1{+}j] & \longrightarrow & [i'{+}1{+}j'], \end{array}$$

where the horizontal arrows are monotone maps.

3.11. Coface maps. The category $\mathcal{D}$ contains the category $\mathbb{\Delta}_{/[1]} \simeq (\mathbb{\Delta}_+ \times \mathbb{\Delta}_+) \setminus \{(-1, -1)\}$, and they have the same objects. The *vertical coface maps* in $\mathcal{D}$ are given by

$$e^k : [i, j] \to [i{+}1, j] \qquad \begin{array}{ccc} [i] & \xrightarrow{d^k} & [i+1] \\ {\scriptstyle (d^\top)^{\circ(1+j)}}\downarrow & & \downarrow{\scriptstyle (d^\top)^{\circ(1+j)}} \\ [i{+}1{+}j] & \xrightarrow[d^k]{} & [(i{+}1){+}1{+}j]. \end{array}$$

$$0 \leq k \leq i + 1$$

The *horizontal coface maps* in $\mathcal{D}$ are given by

$$d^k : [i,j] \to [i,j{+}1] \qquad \begin{array}{ccc} [i] & = & [i] \\ \downarrow{\scriptstyle (d^\top)^{\circ(1+j)}} & & \downarrow{\scriptstyle (d^\top)^{\circ(1+j+1)}} \\ [i{+}1{+}j] & \xrightarrow[d^{k+i+1}]{} & [i{+}1{+}(j{+}1)]. \end{array}$$

$$0 \leq k \leq j+1$$

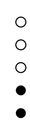
The vertical and horizontal codegeneracy maps can be described similarly. Following the global convention we use the letter s for the horizontal codegeneracy maps and the letter t for the vertical codegeneracy maps.

A distinctive feature of $\mathcal{D}$ are the abacus maps, which we now define.

Definition 3.12. *The* abacus maps *are the maps in* $\mathcal{D}$ *given by*

$$f : [i,j+1] \to [i+1,j] \qquad \begin{array}{ccc} [i] & \xrightarrow{d^\top} & [i+1] \\ \downarrow{\scriptstyle (d^\top)^{\circ(1+j+1)}} & & \downarrow{\scriptstyle (d^\top)^{\circ(j+1)}} \\ [i{+}1{+}(j{+}1)] & = & [(i{+}1){+}1{+}j]. \end{array}$$

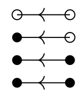
There is a useful graphical interpretation of $\mathcal{D}$ (mentioned in passing by Carlier [21, Remark 3.2.1]) in terms of black and white beads: the objects in $\mathcal{D}$ are columns of beads, one for each element in the codomain $[i+1+j]$, first black then white: the object $[i,j]$ has $i+1$ black beads followed by $j+1$ white beads, such as



representing the object $[1,2]$. (We read from the bottom to the top.) The black beads are the elements in the image of the map (that is, the elements in $[i]$), whereas the white beads are the elements in $[i+1+j]$ that are not in the image. The graphical interpretation of a map in $\mathcal{D}$ is that it is monotone and maps black beads to black beads, but may map white beads to black beads. The maps that preserve colors are the maps that belong to the subcategory $\mathbb{\Delta}_{/[1]}$. The maps that are the identity on white beads are the vertical maps; the maps that are the identity on black beads are the horizontal maps, such as for example

$$[2,1] \xleftarrow{e^2} [1,1] \qquad\qquad [1,2] \xleftarrow{d^1} [1,1]$$

The abacus maps f are those for which the underlying map is the identity and whose only effect is to turn the bottom white bead into a black bead (loosely speaking, swiping it from the white part to the black part, hence Carlier's terminology "abacus map"), such as

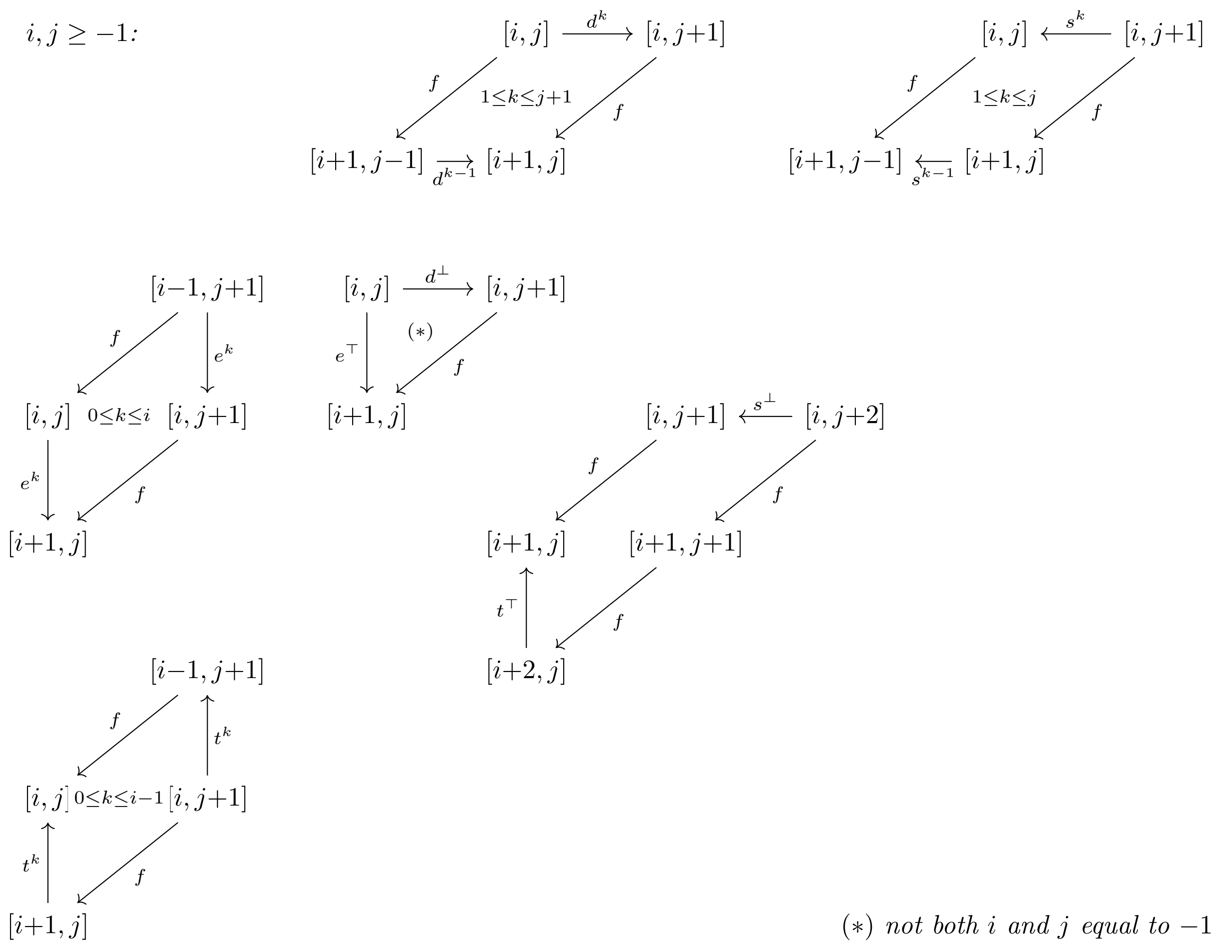


$$[2,0] \xleftarrow{f} [1,1]$$

The abacus maps together with the two commuting sets of (augmented) cosimplicial operators on the black and white beads form a generating set of morphisms for $\mathcal{D}$. To see this, let g be a morphism in $\mathcal{D}$. If g maps a white bead to a black bead, then g factors uniquely through an abacus map as $g = g' \circ f$, where if we forget the colors, g and g' are equal as monotone maps. We can repeat this process and extract from this a factorization $g = g_{\mathrm{simp}} \circ g_{\mathrm{ab}}$ where g_{ab} is a composite of abacus maps and g_{simp} is a composite of (augmented) bisimplicial operators. This shows that the (augmented) bisimplicial operators, together with the abacus maps, generate $\mathcal{D}$. To put any morphism in $\mathcal{D}$ in this canonical form it suffices to know how this factorization applies to the case where an (augmented) cosimplicial operator is followed by an abacus map. These identities, together with the usual identities for bisimplicial operators of $\mathbb{\Delta}_{/[1]}$ therefore give a full description of the category $\mathcal{D}$:

Lemma 3.13. *The category $\mathcal{D}$ is presented in terms of generators and relations as having the generators and relations of $\mathbb{\Delta}_{/[1]}$ and in addition to that, abacus maps* *f, subject to the relations*

$i, j \geq -1$:

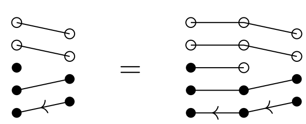


(∗) *not both i and j equal to* -1

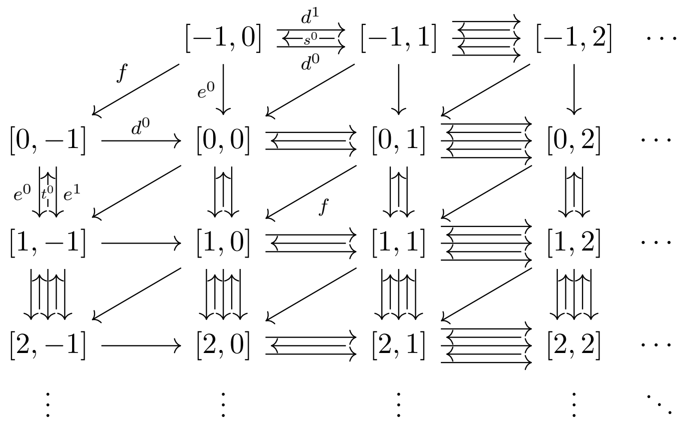
The commutation relations of the abacus map and the cosimplicial operators, displayed above can be read off the columns of beads. As an example, the equation

$$e^{\top} = f \circ d^{\perp} \tag{2.36}$$

is pictorially represented by



One can visualize $\mathcal{D}$ as

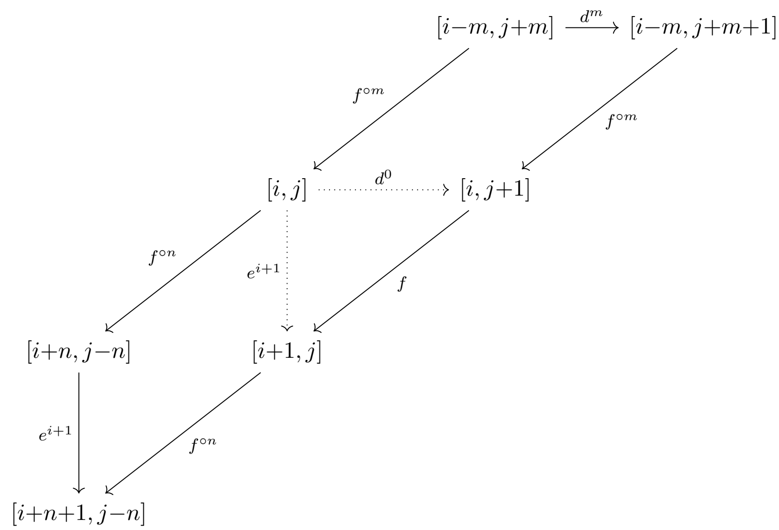


Remark 3.14. The terminology *abacus maps* is from Carlier's second paper [22], except that his abacus maps were slightly more general, and do not admit any easy combinatorial description. What we call abacus map corresponds to what Carlier called *perfect abacus map*. One of the constructions in his second paper was concerned with modifying a diagram with the more general abacus maps to one with perfect abacus maps, that is, a presheaf on $\mathcal{D}$. Since the perfect abacus maps correspond to the combinatorial axioms holding in $\mathcal{D}$, we prefer to use the name for these.

3.15. Trapezium equations. Several other useful equations hold in $\mathcal{D}$, including the *trapezium equations*

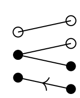
$$f^{\circ(m+n+1)} \circ d^m = e^{i+1} \circ f^{\circ(m+n)}, \qquad \text{where} \qquad \begin{array}{rrcl} d^m : & [i-m, j+m] & \to & [i-m, j+m+1] \\ e^{i+1} : & [i+n, j-n] & \to & [i+n+1, j-n], \end{array} \tag{2.37}$$

valid for all $i, j \geq -1$ (but not both equal to -1), and all $m \leq i+1$ and $n \leq j+1$. They owe their name to the shape they trace out in $\mathcal{D}$



The diagram is factored so as to contain already the proof: the triangle and the two parallelograms are instances of the equations of Lemma 3.13.

There is an alternative presentation of $\mathcal{D}$ in terms of generators and relations, where instead of abacus maps, we describe extra bottom codegeneracy maps $s^{\perp}$ in each row (except the augmentation row). Pictorially, these are given by joining the last black and the first white bead to a single black bead, as in this example, picturing $[1,0] \xleftarrow{s^{\perp}} [1,1]$:



It follows readily that these $s^{\perp}$ are actually extra bottom codegeneracy maps in each row in the sense that they satisfy the identities for split cosimplicial objects.

One should be aware that the extra bottom codegeneracy maps are not compatible with vertical top coface maps. They *are* compatible with all the other vertical cosimplicial operators (including the vertical top codegeneracy maps). We omit the proof of the following lemma which summarizes this.

Lemma 3.16. *The category $\mathcal{D}$ is presented in terms of generators and relations as having the generators and relations of $\mathbb{\Delta}_{/[1]}$ and in addition to that, extra bottom codegeneracy maps in all rows (except the augmentation row). The equations satisfied by these extra bottom codegeneracy are the split cosimplicial identities*

$$\begin{array}{ccc} [i,j] & \xleftarrow{s^{\perp}} & [i,j{+}1] \\ {\scriptstyle d^k}\big\downarrow & {\scriptstyle 0\le k\le j+1} & \big\downarrow{\scriptstyle d^{(k+1)}} \\ [i,j{+}1] & \xleftarrow[s^{\perp}]{} & [i,j] \end{array} \qquad \begin{array}{ccc} [i,j] & \xleftarrow{s^{\perp}} & [i,j{+}1] \\ {\scriptstyle s^k}\big\uparrow & {\scriptstyle 0\le k\le j} & \big\uparrow{\scriptstyle s^{(k+1)}} \\ [i,j{+}1] & \xleftarrow[s^{\perp}]{} & [i,j{+}2] \end{array}$$

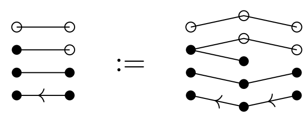
as well as those stating that they are compatible with all vertical cosimplicial operators except the top vertical coface maps

$$\begin{array}{ccc} [i,j] & \xleftarrow{s^{\perp}} & [i,j{+}1] \\ {\scriptstyle e^k}\big\downarrow & {\scriptstyle 0\le k\le i+1} & \big\downarrow{\scriptstyle e^{k}} \\ [i{+}1,j] & \xleftarrow[s^{\perp}]{} & [i{+}1,j{+}1] \end{array} \qquad \begin{array}{ccc} [i,j] & \xleftarrow{s^{\perp}} & [i,j{+}1] \\ {\scriptstyle t^k}\big\uparrow & {\scriptstyle 0\le k\le i} & \big\uparrow{\scriptstyle t^{k}} \\ [i{+}1,j] & \xleftarrow[s^{\perp}]{} & [i{+}1,j{+}1] \end{array}$$

where in all diagrams $i \geq 0, j \geq -1$.

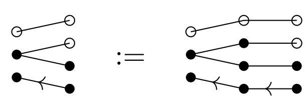
One can pass back and forth between the abacus viewpoint and the extra-bottom-codegeneracy viewpoint. Given $s^{\perp}$, the abacus map f is defined as

$$f := s^{\perp} \circ e^{\top}$$



Conversely, given f, the extra bottom codegeneracy map $s^{\perp}$ is defined as

$$s^{\perp} := t^{\top} \circ f$$



3.4 $\mathcal{D}$-presheaves

3.17. Abacus maps. Let B be a $\mathcal{D}$-presheaf. The relations holding in $\mathcal{D}$ (Lemma 3.13) translate into equations for the abacus maps f at the presheaf level, which can be interpreted either in terms of rows or columns of B. Precisely, the abacus maps form simplicial maps between rows

$$f : B_{i+1,\bullet} \to \mathrm{Dec}_{\perp} B_{i,\bullet} \tag{2.38}$$

and simplicial maps between columns

$$f : \mathrm{Dec}_{\top} B_{\bullet,j} \to B_{\bullet,j+1}.$$

In these formulae, we allow $i = -1$, so that f goes from the zeroth row to the decalage of the augmentation row, and we also allow $j = -1$, so that f goes from the decalage of the augmentation column to the zeroth column.

Remark 3.18. Purely combinatorially, from the relations holding in $\mathcal{D}$ (again Lemma 3.13), we have the important equation holding inside a $\mathcal{D}$-presheaf:

$$f = e_\top \circ s_\perp.$$

This means that each abacus map, viewed as a simplicial map between rows, $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$, can be described as the composite

$$B_{i+1,\bullet} \xrightarrow{s_\perp} \mathrm{Dec}_\perp B_{i+1,\bullet} \xrightarrow{\mathrm{Dec}_\perp(e_\top)} \mathrm{Dec}_\perp B_{i,\bullet}. \tag{2.39}$$

Conversely, from the equation $s_\perp = f \circ t_\top$ holding in any $\mathcal{D}$-presheaf (which again follows from Lemma 3.13), we see that we can write the simplicial map $s_\perp : B_{i,\bullet} \to \mathrm{Dec}_\perp(B_{i,\bullet})$ (between rows) as the composite

$$B_{i,\bullet} \xrightarrow{t_\top} B_{i+1,\bullet} \xrightarrow{f} \mathrm{Dec}_\perp(B_{i,\bullet}). \tag{2.40}$$

From the viewpoint of columns we can write instead the simplicial map $f : \mathrm{Dec}_\top B_{\bullet,j} \to B_{\bullet,j+1}$ (between columns) as the composite

$$\mathrm{Dec}_\top B_{\bullet,j} \xrightarrow{s_\perp} \mathrm{Dec}_\top B_{\bullet,j+1} \xrightarrow{\varepsilon} B_{\bullet,j+1}. \tag{2.41}$$

Finally, still from the equations holding in $\mathcal{D}$ (again Lemma 3.13), we have the following useful equation holding in any $\mathcal{D}$-presheaf:

$$e_\top = d_\perp \circ f.$$

In particular, we can describe the vertical augmentation map $e_0 : B_{0,\bullet} \to Y$ as the composite

$$B_{0,\bullet} \xrightarrow{f} \mathrm{Dec}_\perp Y \xrightarrow{\varepsilon} Y.$$

3.5 Stability, and how it affects abacus structure

3.19. Lower and upper stable $\mathcal{D}$-presheaves. A $\mathcal{D}$-presheaf is called upper stable or lower stable if the underlying bisimplicial space is so (cf. 3.1). Thus upper stable means that all simplicial maps $e_\perp$ between bulk rows are right fibrations, and all simplicial maps $d_\perp$ between bulk columns are right fibrations. Similarly lower stable means that all simplicial maps $e_\top$ between bulk rows are left fibrations, and all simplicial maps $d_\top$ between bulk columns are left fibrations. Note that nothing is said about the augmentation maps $e_\perp : B_{0,\bullet} \to B_{-1,\bullet}$ or $d_\top : B_{\bullet,0} \to B_{\bullet,-1}$.

Lemma 3.20. *In a $\mathcal{D}$-presheaf B, if all bulk rows are 1-Segal, then for all $i \geq 0$, the simplicial map between rows $s_\perp : B_{i,\bullet} \to \mathrm{Dec}_\perp(B_{i,\bullet})$ is cartesian.*

Proof. This is the statement that for 1-Segal spaces, $\mathrm{Dec}_\bot$-coalgebras are always rigid (Lemma 2.9). □

Lemma 3.21. *If a $\mathcal{D}$-presheaf B is lower stable and all its bulk rows are 1-Segal, then for all $i \geq 0$, the abacus map $f : B_{i+1,\bullet} \to \mathrm{Dec}_\bot(B_{i,\bullet})$ is cartesian.*

Proof. By Equation (2.39) we can write $f : B_{i+1,\bullet} \to \mathrm{Dec}_\bot(B_{i,\bullet})$ as the composite

$$B_{i+1,\bullet} \xrightarrow{s_\perp} \mathrm{Dec}_\bot(B_{i+1,\bullet}) \xrightarrow{\mathrm{Dec}_\bot(e_\top)} \mathrm{Dec}_\bot(B_{i,\bullet}).$$

But $s_\perp$ is cartesian as a consequence of Lemma 3.20 (since the bulk rows are 1-Segal), and $\mathrm{Dec}_\bot(e_\top)$ is cartesian since $e_\top$ is a left fibration by the lower stability assumption (here we use $i \geq 0$) and since the lower decalage of a left fibration is cartesian (A.3). □

Remark 3.22. The conclusion of the lemma is almost Condition ($\star$) from 4.3 below, but without saying anything about the augmentation row. Note that we cannot use the same argument in the case $i = -1$. The argument would involve the augmentation map $e_0 : B_{0,\bullet} \to Y$ which cannot be assumed to be a left fibration. In fact the augmentation map is not even a left fibration for $B = \mathrm{Tot}(Y)$ except when Y is 1-Segal. (Nevertheless, for $B = \mathrm{Tot}(Y)$, we do have that f is cartesian (it is even invertible).)

The previous results concern the bottom splittings $s_\perp$ as simplicial maps between rows, as well as the interpretation of the abacus maps as simplicial maps between rows. We now turn to the interpretation of $s_\perp$ and f as going between columns. This is trickier, since $s_\perp$ is *not* a simplicial map between columns (cf. 3.16): it fails to be compatible with the top face maps $e_\top$. But after taking upper decalage, we do get a simplicial map $s_\perp : \mathrm{Dec}_\top(B_\bullet, j) \to \mathrm{Dec}_\top(B_{\bullet,j+1})$ for each $j \geq -1$.

Lemma 3.23. *In an upper-stable $\mathcal{D}$-presheaf B, the $s_\perp$ constitute cartesian simplicial maps between upper-decs of columns*

$$\mathrm{Dec}_\top(B_{\bullet,j}) \xrightarrow{s_\perp} \mathrm{Dec}_\top(B_{\bullet,j+1}).$$

This is for $j \geq -1$, so as to include the case of the augmentation column $\mathrm{Dec}_\top X \xrightarrow{s_\perp} \mathrm{Dec}_\top B_{\bullet,0}$.

Proof. The simplicial maps $e_\bot$ are right fibrations between (bulk) rows by virtue of upper stability. Lemma 2.4 tells us that each $e_\bot$ is also cartesian on all bottom splittings (including $X_i \xrightarrow{s_\perp} B_{i,0}$). Concretely this means that all the squares

$$\begin{array}{ccc} B_{i,j} & \xrightarrow{s_\perp} & B_{i,j+1} \\ {\scriptstyle e_k}\uparrow & & \uparrow{\scriptstyle e_k} \\ B_{i+1,j} & \xrightarrow[s_\perp]{} & B_{i+1,j+1} \end{array}$$

are pullbacks for all $i \geq 0$, $j \geq -1$, and for $k = 0$ (that's $e_\perp$). A standard pullback argument shows that we then get pullback squares also for e_k for all $0 \leq k \leq i$, but for $k = i+1$ the square does not even commute. But after taking upper decalage, the $s_\perp$ do form simplicial maps $s_\perp : \mathrm{Dec}_\top(B_{\bullet,j}) \xrightarrow{s_\perp} \mathrm{Dec}_\top(B_{\bullet,j+1})$, whose component on face maps are the pullback squares above. Therefore these simplicial maps are cartesian. □

Proposition 3.24. *Let B be a $\mathcal{D}_{i\geq 0}$-presheaf. If B is upper stable and if the bulk columns are* 1*-Segal, then the abacus maps regarded as simplicial maps between columns*

$$f : \mathrm{Dec}_\top(B_{\bullet,j}) \to B_{\bullet,j+1}$$

are right fibrations. This is for $j \geq -1$, so as to include the case of $\mathrm{Dec}_\top(X) \xrightarrow{f} B_{\bullet,0}$*, but it does* not *involve the abacus maps to the augmentation row.*

Proof. By (2.41) in Remark 3.18, f factors as

$$\mathrm{Dec}_\top(B_{\bullet,j}) \xrightarrow{s_\perp} \mathrm{Dec}_\top(B_{\bullet,j+1}) \xrightarrow{\varepsilon} B_{\bullet,j+1}.$$

The first map is cartesian by Lemma 3.23 (since B is upper stable), and the second map is a right fibration since we assume that the bulk columns $B_{\bullet,j+1}$ are 1-Segal (A.1). □

Definition 3.25. *A $\mathcal{D}$-presheaf B is said to be an* abacus bimodule configuration *if its underlying $\Delta_{/[1]}$-presheaf (augmented bisimplicial space) is stable and double Segal, the augmentation row and column are 2-Segal, and the augmentation maps are culf. We write **ABC** for the full subcategory of $\mathbf{Pr}(\mathcal{D})$ spanned by the abacus bicomodule configurations in the sense of Carlier.*

Remark 3.26. We should warn that the general bisimplicial maps are not the "correct" notion of morphism of bicomodules: the correct maps are certain spans where the left bisimplicial map is a left fibration on columns and a right fibration on rows, and where the right bisimplicial map is a right fibration on columns and a left fibration on rows.

4 Simplicial maps vs. $\mathcal{D}$-presheaves

4.1 Right Kan extension along q

Consider the functor $\mathsf{q} : \mathbb{\Delta} \times \Delta^1 \to \mathcal{D}$ that includes $\mathbb{\Delta} \times \{0\}$ as the augmentation row of $\mathcal{D}$ and includes $\mathbb{\Delta} \times \{1\}$ as the augmentation column of $\mathcal{D}$, and sends all maps $[i] \times a$ to the long composite abacus maps.

The right Kan extension has an explicit formula: given a simplicial map $F \in \boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1)$, the right Kan extension $B := \mathsf{q}_*(F)$ is given by

$$B_{i,j} \simeq \mathrm{Map}_{\boldsymbol{Pr}(\mathbb{\Delta}\times\Delta^1)}(\mathsf{q}^* y_{\mathcal{D}}[i,j], F).$$

In fact, a direct computation gives $\mathsf{q}^* y_{\mathcal{D}}[i,j] \simeq (\Delta^i \xrightarrow{(d^\top)^{\circ(1+j)}} \Delta^{i+1+j})$ where by convention we take $\Delta^{-1} \simeq \emptyset$. (Note that since we dealing with simplicial spaces, Δ^{-1} is not a representable functor, but it is convenient notation.) With this the right Kan extension becomes

$$B_{i,j} \simeq \mathrm{Map}_{\boldsymbol{Pr}(\mathbb{\Delta}\times\Delta^1)}(\Delta^i \to \Delta^{i+1+j}, F). \tag{2.42}$$

This formula is essentially Carlier's cocartesian nerve [21].

An object of $B_{i,j}$ is thus a commutative square

$$\begin{array}{ccc} \Delta^i & \longrightarrow & X \\ {\scriptstyle (d^\top)^{\circ(1+j)}}\downarrow & & \downarrow{\scriptstyle F} \\ \Delta^{i+1+j} & \longrightarrow & Y. \end{array}$$

For $i, j \geq 0$ we equivalently have $B_{ij} = X_i \times_{Y_i} Y_{i+1+j}$ by Yoneda. More precisely, and exhibiting more of the structure, we have the following proposition.

Proposition 4.1. *For any simplicial map $F : X \to Y$ considered as an object in $\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1)$, put $B := \mathsf{q}_*(F)$ Then we have:*

1. *The ith row of B is $B_{i,\bullet} = X_i \times_{Y_i} \mathrm{Dec}_\perp^{i+1}(Y)$, which is more precisely given by*

$$\begin{array}{ccc} \bar{Y}_i & \xleftarrow{\ \zeta\ } & \mathrm{Dec}_\perp^{i+1} Y \\ {\scriptstyle \bar{F}_i}\uparrow & \lrcorner & \uparrow{\scriptstyle f^{\circ(i+1)}} \\ \bar{X}_i & \xleftarrow[\ \zeta\ ]{} & B_{i,\bullet}. \end{array}$$

   (Recall that overline means constant simplicial space and that the zeroth component of ζ is $d_\top$.)

2. *The jth column of B is $B_{\bullet,j} = X \times_Y \mathrm{Dec}_\top{}^{1+j}(Y)$, which is more precisely*

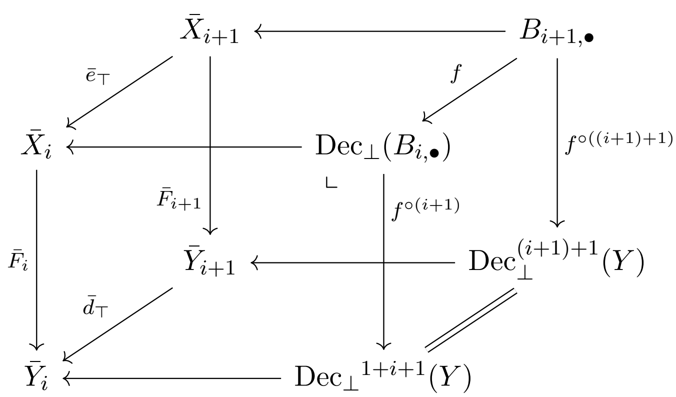
$$\begin{array}{ccc} Y & \xleftarrow{d_\top^{\circ(1+j)}} & \mathrm{Dec}_\top{}^{1+j}Y \\ {\scriptstyle F}\uparrow & \ulcorner & \uparrow{\scriptstyle f^{\circ(\bullet+1)}} \\ X & \xleftarrow[d_\top^{\circ(1+j)}]{} & B_{\bullet,j}. \end{array}$$

This includes the case $j = -1$ which shows that X itself appears as the augmentation column of $\mathsf{q}_*(F)$; the case $j = 0$ exhibits the horizontal augmentation map $X \xleftarrow{d_0} B_{\bullet,0}$ as the projection seen in the pullback diagram.

3. *The abacus map (Equation (2.38)) $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp(B_{i,\bullet})$ is*

$$\bar{X}_{i+1} \underset{\bar{Y}_{i+1}}{\times} \mathrm{Dec}_\perp^{(i+1)+1}(Y) \xrightarrow{\bar{e}_\top \underset{\bar{d}_\top}{\times} \mathrm{id}} \bar{X}_i \underset{\bar{Y}_i}{\times} \mathrm{Dec}_\perp^{(i+1)+1}(Y).$$

Proof. The first two claims follow directly from Equation (2.42). As for the third claim, consider the commutative diagram



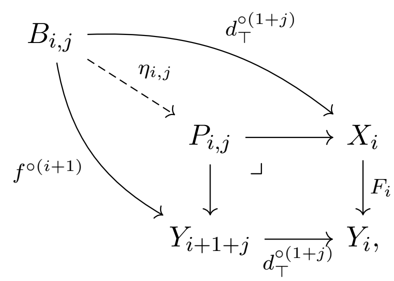
Here the front and back faces are the pullback squares of the first claim, where in the case of the front face we first applied $\mathrm{Dec}_\perp$ to the whole square. The result follows from the uniqueness of the induced map into the front pullback. ☐

Remark 4.2. The simplicial map $B_{\bullet,k} \to \mathrm{Dec}_\top{}^{1+k}Y$ is given in degree p by $f^{\circ(p+1)}$. The fact that these components form a simplicial map is an expression of the trapezium equations of 3.15. Specifically, evaluated at the mth coface map $[p] \to [p+1]$ we get the face maps $e_m : B_{p+1,k} \to B_{p,k}$ and $d_m : Y_{p+k+2} \to Y_{p+k+1}$; the simplicial-map equation $d_m \circ f^{\circ(p+2)} = f^{\circ(p+1)} \circ e_m$ now appears as an instance of the trapezium equations (by taking $m = m$, $n = p - m + 1$, $i = m - 1$, $j = k + p - m + 1$, in the notation of 3.15).

Since q is fully faithful, so is q_*. We now characterize the image by analyzing the unit for the $\mathsf{q}^* \dashv \mathsf{q}_*$ adjunction. For $B \in \boldsymbol{Pr}(\mathcal{D})$, let $\eta : B \to \mathsf{q}_*\mathsf{q}^*(B)$ be the unit. Evaluated on $[i, -1]$ (for $i \geq 0$) the unit reduces to the identity $\mathrm{id}_{X_i} : X_i \to X_i$. Similarly, when evaluated on $[-1, j]$ it reduces to the identity $\mathrm{id}_{Y_j} : Y_j \to Y_j$. For the remaining values $i, j \geq 0$, the unit is given by the pullback induced map,

$$\eta_{i,j} : B_{i,j} \longrightarrow X_i \times_{Y_i} Y_{i+1+j},$$

which, in more detail, is constructed as in the diagram



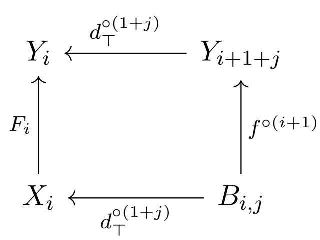
where $P_{i,j}$ is the pullback characterizing $(\mathsf{q}_*\mathsf{q}^*(B))_{ij}$.

4.3. Condition "star". We say a $\mathcal{D}$-presheaf B satisfies Condition $(\star)$ if the abacus maps are cartesian, regarded as a simplicial map between rows:

$$(\star) \qquad f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet} \quad \text{is cartesian for all } i \geq -1.$$

Lemma 4.4. *The unit $B \to \mathsf{q}_*\mathsf{q}^* B$ is invertible if and only if B satisfies Condition $(\star)$ of 4.3*

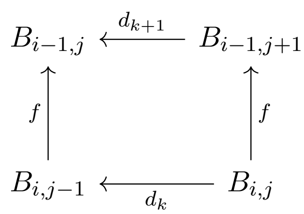
Proof. Assume that $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$ is cartesian. The unit is invertible if the square



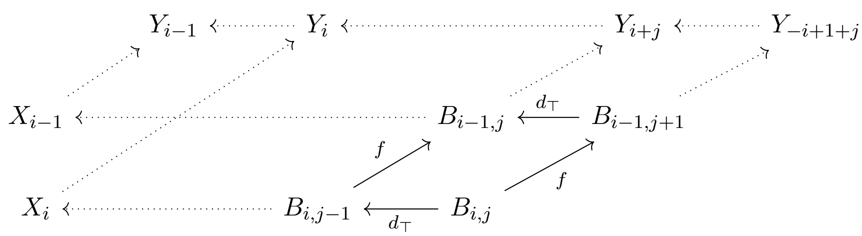
(2.43)

is a pullback for all $i, j \geq 0$. But this square can be broken down into squares with abacus maps against top face maps, which are pullbacks because $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$ is cartesian. Thus the square (2.43) is a pullback by the pullback prism lemma.

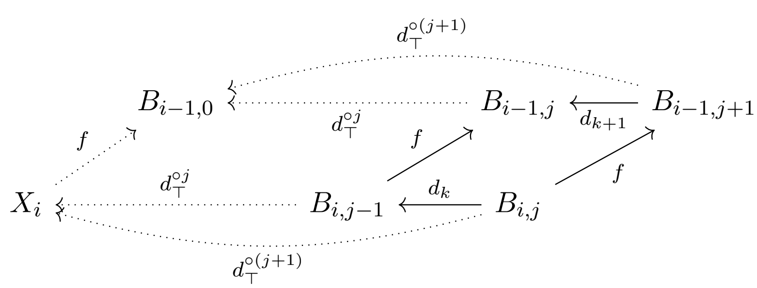
For the other direction, assume that the unit is invertible, i.e. the square (2.43) is a pullback for all $i, j \geq 0$. To show that $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$ is cartesian, it suffices to show that the abacus maps form pullbacks against all face maps, so we must show that the square



is a pullback for all $i, j \geq 0$ and all $0 \leq k \leq j$. For the case of top face maps, consider the diagram



Here all the horizontal dotted arrows are composites of $d_\top$ maps, and the diagonal dotted maps are composites of abacus maps. All the big parallelograms that end at the augmentation row and column are pullbacks by assumption. Chopping up all the parallelograms appropriately and using the pullback prism lemma repeatedly, we see that the bottom right solid parallelogram is a pullback. For the squares with the remaining (non-top) face maps consider the commutative diagram



The outer and inner left parallelogram are pullbacks by arguments similar to that above. By the pullback prism lemma the solid parallelogram is a pullback. Thus, in total, $f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$ is cartesian for all $i \geq -1$. □

Corollary 4.5. *The image of* q_* *is characterized by Condition* ($\star$) *(of 4.3), so as to consist of those presheaves* $B \in \boldsymbol{Pr}(\mathcal{D})$ *for which*

$$f : B_{i+1,\bullet} \to \mathrm{Dec}_\perp B_{i,\bullet}$$

is cartesian (for all $i \geq -1$*).*

Let $\boldsymbol{Pr}^\star(\mathcal{D})$ denote the full subcategory $\boldsymbol{Pr}(\mathcal{D})$ consisting of the $\mathcal{D}$-presheaves that satisfy Condition ($\star$). We summarize the results in the following theorem.

Theorem 4.6. *The* $\mathsf{q}^* \dashv \mathsf{q}_*$ *adjunction restricts to an equivalence*

$$\boldsymbol{Pr}(\mathbb{\Delta} \times \Delta^1) \simeq \boldsymbol{Pr}^\star(\mathcal{D}).$$

4.2 Dictionary between conditions on F and on $B := \mathsf{q}_*(F)$

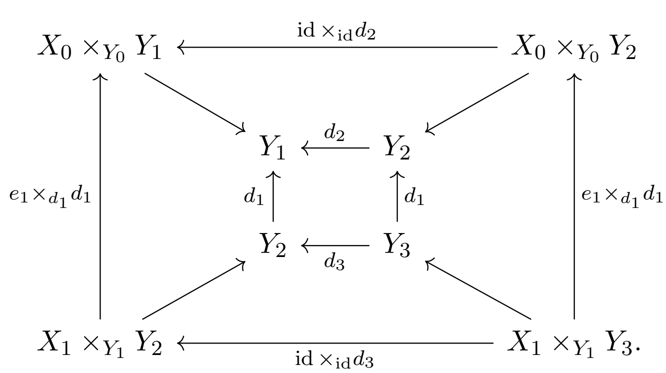
In this subsection, we consider a simplicial map $F : X \to Y$ and the corresponding $\mathcal{D}$-presheaf $B := \mathsf{q}_*(F)$, and explain how various conditions match up.

Lemma 4.7. *Given a simplicial map* $F : X \to Y$*, if* Y *is lower* 2*-Segal, then* $B := \mathsf{q}_*(F)$ *is lower stable. If* Y *is upper* 2*-Segal, then* $\mathsf{q}_*(F)$ *is upper stable.*

Proof. For the first claim, it is a question of checking each individual square in the condition of being lower stable. Consider the square

$$\begin{array}{ccc} B_{00} & \xleftarrow{d_1} & B_{01} \\ {\scriptstyle e_1}\uparrow & & \uparrow{\scriptstyle e_1} \\ B_{10} & \xleftarrow[d_1]{} & B_{11} \end{array}$$

which, in terms of the data of F is given by the outer square of the diagram



Here all diagonals are projections and the top and bottom inner trapezoids are pullbacks. Now the central square is a pullback since Y is lower 2-Segal. Thus, by the pullback prism lemma, the outer square is again a pullback. The same argument works for all other $e_\top$-against-$d_\top$ squares.

The proof of the second statement is similar. $\square$

Lemma 4.8. *If Y is lower 2-Segal, then in $B := \mathsf{q}_*(F)$ all bulk rows are 1-Segal, and as a result B has cartesian $s_\perp$ in all bulk rows.*

Proof. By Lemma 4.1(1) we have a map $f^{\circ(i+1)} : B_{i,\bullet} \to \mathrm{Dec}_\perp^{i+1} Y$, which by Corollary 4.5 is cartesian. Since Y is 2-Segal, $\mathrm{Dec}_\perp Y$ is 1-Segal, and since $B_{i,\bullet}$ is cartesian over $\mathrm{Dec}_\perp Y$, it is 1-Segal too (by A.6). The second statement follows from the fact that every 1-Segal space with bottom splittings is automatically rigid by Lemma 2.9. $\square$

Discussion 4.9. *One way to interpret the upper-2-Segal condition (see A.8) on a simplicial space X is that one cannot quite compose arrows in X, so as to get a 2-simplex from a pair of composable arrows $\cdot \to \cdot \to \cdot$, but if everything has a further arrow down to a common point $z \in X_0$, then it* is *possible to compose. In other words, one can compose in the slice over z. Indeed, one formulation of the upper-2-Segal condition is that every slice is a 1-Segal space. More uniformly, X is upper 2-Segal iff $\mathrm{Dec}_\top X$ is 1-Segal.*

Now consider $F : X \to Y$. Rather than asking that $X_{/x}$ be 1-Segal for every point $x \in X$, we may ask instead that the slice $X_{/y}$ — or more precisely comma category $F{\downarrow}y$ — be 1-Segal for every $y \in Y$. That is, we demand that the X-arrows can be composed in Y if just they are over some point $y \in Y$. This discussion motivates the following relative notion.

Definition 4.10. *A simplicial map $F : X \to Y$ is called* relatively upper 2-Segal *when the pullback $X \times_Y \mathrm{Dec}_\top Y$ (of ε along F) is 1-Segal.*

Example 4.11. If Y is upper 2-Segal and $F : X \to Y$ is a left fibration or a right fibration, then F is relatively upper 2-Segal. Indeed, $\mathrm{Dec}_\top(Y)$ is then 1-Segal, and $X \times_Y \mathrm{Dec}_\top Y$ will then be a left or right fibration over $\mathrm{Dec}_\top(Y)$ and hence 1-Segal (by A.6). (The same arguments work in the situation where F is ikeo, or just semi-ikeo; see [31] for these notions.)

Lemma 4.12. *If Y is upper 2-Segal and if $F : X \to Y$ is relatively upper 2-Segal (meaning that $X \times_Y \mathrm{Dec}_\top Y$ is 1-Segal), then also $X \times_Y \mathrm{Dec}_\top{}^{1+j} Y$ is 1-Segal (for all $j \geq 0$).*

Proof. There are two maps $\mathrm{Dec}_\top Y \leftarrow \mathrm{Dec}_\top \mathrm{Dec}_\top Y$. One is $\varepsilon_{\mathrm{Dec}_\top Y}$ which is a right fibration since $\mathrm{Dec}_\top Y$ is 1-Segal (A.1). The other is $\mathrm{Dec}_\top(\varepsilon_Y)$ which is a left fibration since ε_Y is culf since Y is upper 2-Segal (A.2 and A.10). So we can work either with one or the other. Now take $X \times_Y \_$ on that fibration to obtain that $X \times_Y \mathrm{Dec}_\top{}^{1+1} Y$ is a fibration over $X \times_Y \mathrm{Dec}_\top{}^1 Y$, and so on. Therefore, all the higher versions are 1-Segal too. □

Corollary 4.13. *If Y is upper 2-Segal and if $F : X \to Y$ is relatively upper 2-Segal, then $B = \mathsf{q}_*(F)$ has 1-Segal bulk columns.*

Proof. By Proposition 4.1(2) we have $B_{\bullet,j} \simeq X \times_Y \mathrm{Dec}_\perp^{j+1} Y$ which is 1-Segal for $j \geq 0$ by Lemma 4.12. □

We can now state the main result of this subsection:

Theorem 4.14. *Let $B = \mathsf{q}_*(F)$ be the $\mathcal{D}$-presheaf corresponding to a simplicial map $F : X \to Y$ as in 4.6. Then $F : X \to Y$ is 2-Segal and relatively upper 2-Segal if and only if B is a bicomodule configuration. In particular, the functor q_* restricts to an equivalence*

$$\boldsymbol{Pr}^{\mathrm{up\,2\text{-}Seg}}(\mathbb{\Delta} \times \Delta^1) \xrightarrow{\simeq} \boldsymbol{ABC}^{\star}.$$

Proof. Suppose X and Y are 2-Segal and that F is relatively upper 2-Segal. Now we start checking the axioms for B being a comodule configuration.

The augmentation row is 2-Segal with culf augmentation. Indeed, the augmentation row is Y itself, and the augmentation map $B_{0,\bullet} \to Y$ is given by $B_{0,\bullet} \xrightarrow{f} \mathrm{Dec}_\perp Y \xrightarrow{\varepsilon} Y$, where f is cartesian by Condition ($\star$) (which holds by 4.5 since we are in the image of q_*) and ε is culf because Y is 2-Segal (see A.10).

The augmentation column is 2-Segal and the augmentation map is culf. Indeed, the augmentation column is X itself, and the augmentation map $X \leftarrow B_{\bullet,0}$ is the pullback of $Y \xleftarrow{\varepsilon} \mathrm{Dec}_\top Y$ (see Lemma 4.1(2)), which is culf since Y is 2-Segal.

All bulk rows are 1-Segal. This follows from Lemma 4.8.

All bulk columns are 1-Segal. This follows from Corollary 4.13.

Stability. This follows from Lemma 4.7.

For the converse implication, assume $B := \mathsf{q}_*(F)$ is an abacus bicomodule configuration. Then X and Y are already 2-Segal as part of the assumptions, and the relative upper-2-Segal-ness of F is precisely the assumption that the zeroth column is 1-Segal (see Proposition 4.1(2)). □

4.3 2-Segal cocartesian correspondences

The relatively upper 2-Segal condition featured in the previous subsection may appear a bit mysterious, but it can be explained in connection with Carlier's viewpoint of cocartesian fibrations over Δ^1, which is the purpose of this subsection. We'll see (in Proposition 4.16) that $F : X \to Y$ is relatively upper 2-Segal (with both X and Y 2-Segal) if and only if the associated cocartesian fibration $M \to \Delta^1$ has M 2-Segal. We will be slightly sketchy in this subsection, so as not to deviate too much from the main thread.

Given any simplicial map $F : X \to Y$, there is associated a cocartesian fibration $M \to \Delta^1$, in the sense of simplicial spaces (see Carlier [21, §3.2]). This should be a kind of Grothendieck construction of a functor $\Delta^1 \to s\mathcal{S}$, but we are not aware of the general theory of such a thing for $s\mathcal{S}$-valued functors, so we just spell out the construction by hand in this very simple case where the base is just Δ^1. The 0-simplices of M are given by

$$M_0 := X_0 + Y_0,$$

with X_0 mapping to $0 \in \Delta^1$ and Y_0 mapping to $1 \in \Delta^1$. The 1-simplices of M are

$$M_1 := X_1 + [X_0, Y_1] + Y_1,$$

where $[X_0, Y_1]$ is temporary notation for the space of 1-simplices in Y that start in a 0-simplex of X. Formally it is the pullback

$$\begin{array}{ccc} [X_0, Y_1] & \longrightarrow & Y_1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_1} \\ X_0 & \xrightarrow[F]{} & Y_0. \end{array}$$

The map to Δ^1 is given as follows: the 1-simplices in X_1 map to $\mathrm{id}_0 \in \Delta^1$, the 1-simplices in Y_1 map to $\mathrm{id}_1 \in \Delta^1$, and the 1-simplices in $[X_0, Y_1]$ map to the nontrivial edge $a \in \Delta^1$.

More generally, an n-simplex of M is either an n-simplex of X, an n-simplex of Y, or an n-simplex of Y together with a specification of how an initial-segment i-simplex comes from X. Formally this component of M_n is the pullback

$$\begin{array}{ccc} [X_i, Y_n] & \longrightarrow & Y_n \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (d_\top)^{\circ(n-i)}} \\ X_i & \xrightarrow[F]{} & Y_i. \end{array}$$

We recognize that this is precisely

$$M_n = \sum_{i+1+j=n} B_{i,j}, \tag{2.44}$$

with reference to $B := \mathsf{q}_*(F)$ as in Subsection 4.1.

The simplicial map $M \to \Delta^1$ is given as follows. An n-simplex in $B_{i,j}$ (that is, $i+1+j=n$) maps to $s_0{}^{\circ i} s_1{}^{\circ j}(a) \in (\Delta^1)_n$, meaning that out of the $i+1+j$ edges the first i edges contract to 0 and the last j edges contract to 1, whereas the remaining middle edge maps to the nontrivial edge a. In the special case where $i=-1$, this means that all edges map to $1 \in \Delta^1$ and in the special case where $j=-1$, it means that all edges map to $0 \in \Delta^1$.

The simplicial structure of M is also readily described in terms of B: the $n+1$ face maps

$$M_{n-1} \xleftarrow{\bar{d}_k} M_n, \qquad 0 \le k \le n,$$

are given on the $B_{i,j}$-component as the $n+1$ face maps going out of $B_{i,j}$:

$$\begin{array}{ccc} & & B_{i-1,j} \\ & & {\scriptstyle 0}\Big\uparrow \cdots \Big\uparrow{\scriptstyle i} \\ B_{i,j-1} & \begin{array}{c}\xleftarrow{\;i+1+j\;} \\ \vdots \\ \xleftarrow[\;i+1\;]{} \end{array} & B_{i,j}. \end{array}$$

More precisely,

$$\bar{d}_k := \begin{cases} e_k & \text{for } k = 0, \dots, i \\ d_{k-i-1} & \text{for } k = i+1, \dots, n. \end{cases} \tag{2.45}$$

In particular, in the case $i=-1$, this reduces to $Y_{n-1} \xleftarrow{d_k} Y_n$, and in the case $j=-1$ this reduces to $X_{n-1} \xleftarrow{e_k} X_n$. Similarly, the degeneracy maps $\bar{s}_k : M_n \to M_{n+1}$ are given by

$$\bar{s}_k := \begin{cases} t_k & \text{for } k = 0, \dots, i \\ s_{k-i-1} & \text{for } k = i+1, \dots, n. \end{cases}$$

This construction is in fact nothing but the canonical equivalence between presheaves on a slice and slice of the presheaf category

$$\boldsymbol{Pr}(\mathbb{\Delta}_{/[1]}) \xrightarrow{\sim} \boldsymbol{Pr}(\mathbb{\Delta})_{/\Delta^1}.$$

In the direction indicated, this is precisely the assignment $B \mapsto M$ explained above. In the other direction, given $M \to \Delta^1$, one can extract each individual B_{ij} as the pullback

$$\begin{array}{ccc} B_{ij} & \longrightarrow & \mathrm{Map}(\Delta^{i+1+j}, M) \\ \big\downarrow & \lrcorner & \big\downarrow \\ 1 & \xrightarrow[\ulcorner \mathrm{id} \urcorner]{} & \mathrm{Map}(\Delta^1, \Delta^1). \end{array}$$

The vertical map is postcomposition with $M \to \Delta^1$ and precomposition with $(d^\perp)^{\circ i}(d^\top)^{\circ j}$, so it amounts to picking out the edge of the $(i+1+j)$-simplex that lies over the nontrivial arrow $a \in \Delta^1$.

Lemma 4.15. *Starting with $F : X \to Y$, the associated simplicial map $M \to \Delta^1$ is a cocartesian fibration.*

Proof. The cocartesian edges of M (see [21, §3.2]) are the elements in $[X_0, Y_1]$ corresponding to invertible maps $Fx \xrightarrow{\simeq} y$. The canonical cocartesian lift of $a : 0 \to 1$ in Δ^1 to an object $x \in X$ is simply the identity map $Fx \xrightarrow{=} Fx$. □

Proposition 4.16. *A simplicial map $F : X \to Y$ is relatively upper 2-Segal between 2-Segal spaces if and only if the associated cocartesian fibration $M \to \Delta^1$ has 2-Segal total space.*

Remark 4.17. It is very likely that this result can be improved to an equivalence of ∞-categories between a category of 2-Segal cocartesian fibrations over Δ^1 and a category of abacus bicomodule configurations satisfying Condition $(\star)$, but the arguments employed here are not quite enough, since these categories are not full within the categories of the equivalence $\boldsymbol{Pr}(\mathbb{\Delta}_{/[1]}) \xrightarrow{\simeq} \boldsymbol{Pr}(\mathbb{\Delta})_{/\Delta^1}$.

Proof of Proposition 4.16. Theorem 4.14 says that the condition "relatively upper 2-Segal between 2-Segal spaces" means that B is a bicomodule configuration. We claim that this condition in turn matches up with the 2-Segal condition on M. As an illustration of the arguments, let us check the square

$$\begin{array}{ccc} M_3 & \xrightarrow{d_2} & M_2 \\ {\scriptstyle d_0}\big\downarrow & & \big\downarrow{\scriptstyle d_0} \\ M_2 & \xrightarrow[d_1]{} & M_1. \end{array} \tag{2.46}$$

In terms of B_{ij}, this expands to

$$\begin{array}{ccc} X_3{+}B_{20}{+}B_{11}{+}B_{02}{+}Y_3 & \longrightarrow & X_2{+}B_{10}{+}B_{01}{+}Y_2 \\ \big\downarrow & & \big\downarrow \\ X_2{+}B_{10}{+}B_{01}{+}Y_2 & \longrightarrow & X_1{+}B_{00}{+}Y_1, \end{array}$$

and by following through the definition of the face maps in M (Equation (2.45)), this square is the sum of three squares:

$$\begin{array}{ccc} X_3 & \xrightarrow{e_2} & X_2 \\ {\scriptstyle e_0}\downarrow & & \downarrow{\scriptstyle e_0} \\ X_2 & \xrightarrow[e_1]{} & X_1 \end{array} \quad + \quad \begin{array}{ccc} B_{20}+B_{11} & \xrightarrow{e_2|d_0} & B_{10} \\ {\scriptstyle e_0+e_0}\downarrow & & \downarrow{\scriptstyle e_0} \\ B_{10}+B_{01} & \xrightarrow[e_1|d_0]{} & B_{00} \end{array} \quad + \quad \begin{array}{ccc} B_{02}+Y_3 & \xrightarrow{d_1+d_2} & B_{01}+Y_2 \\ {\scriptstyle e_0|d_0}\downarrow & & \downarrow{\scriptstyle e_0|d_0} \\ Y_2 & \xrightarrow[d_1]{} & Y_1. \end{array}$$

The first square is a pullback since X is 2-Segal. The second square is a combination of two pullbacks: one because $B_{\bullet,0}$ is 1-Segal, the other an upper stability square. The third is a combination of two pullbacks: one is a pullback since the augmentation map to Y is culf, the other because Y is 2-Segal. So if B is a bicomodule configuration, then (2.46) is a pullback. Similarly, the mirror image of (2.46) is a pullback because of other bicomodule axioms — and all the bicomodule axioms are actually used. The fact that conversely 2-Segalness of M implies the bicomodule axioms was proved already by Carlier [21, Proposition 3.1.1]. □

4.4 Invertible abacus maps and 2-Segal spaces

So far we restricted the adjunction $\mathsf{q}^* \dashv \mathsf{q}_*$ to an equivalence between 2-Segal relatively upper-2-Segal simplicial maps and abacus bicomodule configurations satisfying Condition ($\star$), as on the top row of the diagram

$$\begin{array}{ccc} \boldsymbol{Pr}^{\text{up 2-Seg}}(\blacktriangle \times \Delta^1) & \xrightarrow[4.14]{\simeq} & \boldsymbol{ABC}^{\star} \\ \uparrow & & \uparrow \\ \boldsymbol{Pr}^{\text{2-Seg}}(\blacktriangle) & \xrightarrow{\simeq} & \boldsymbol{ABC}^{\simeq} \end{array}$$

We now show how this restricts further to an equivalence between 2-Segal spaces and bicomodule configurations with invertible abacus maps, as indicated above in the second row. (In reality, the category $\boldsymbol{Pr}^{\text{2-Seg}}(\blacktriangle)$ appears as the full subcategory of $\boldsymbol{Pr}(\blacktriangle \times \Delta^1)$ consisting of invertible simplicial maps between 2-Segal spaces.)

Lemma 4.18. *Let $F : X \to Y$ be a map of simplicial spaces. Then $\mathsf{q}_*(F)$ has invertible abacus maps if and only if F is an equivalence.*

Proof. First note that the each component $F_i : X_i \to Y_i$ of F is identified with the composite of abacus maps $f^{\circ(i+1)} : X_i \to Y_i$ in $B := \mathsf{q}_*(F)$. If all abacus maps are invertible, then it follows immediately that F is invertible.

For the other direction, assume that F is invertible and consider some abacus map f. If f ends in the augmentation row, i.e. it is of the form $f : B_{0,j} \to Y_{j+1}$, then by Proposition 4.1(1), f is a pullback of F_0 and is therefore invertible. If f is in the bulk, i.e. it is of the form $f : B_{i+1,j} \to B_{i,j+1}$, then we first postcompose with abacus maps until we reach the augmentation row, giving the map $B_{i+1,j} \xrightarrow{f} B_{i,j+1} \xrightarrow{f^{\circ(i+1)}} Y_{i+j+2}$. Again by the same proposition and the same argument, we find that $f^{\circ(i+1)} \circ f : B_{i+1,j} \to Y_{i+j+2}$ as well as $f^{\circ(i+1)} : B_{i,j+1} \to Y_{i+j+2}$ are invertible. By the 2-out-of-3 property, it follows that $f : B_{i+1,j} \to B_{i,j+1}$ is invertible. A similar argument works in the last case in which f starts in the augmentation column. □

Theorem 4.19. *The adjunction* $\mathsf{q}^* \dashv \mathsf{q}_*$ *restricts to an equivalence between* 2*-Segal spaces and bicomodule configurations with invertible abacus maps:*

$$\boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \leftrightarrows \boldsymbol{ABC}^{\simeq}.$$

Proof. This follows by putting together Theorem 4.14 and Lemma 4.18, noting that invertible simplicial maps of 2-Segal spaces are trivially relatively upper 2-Segal. □

Remark 4.20. From the formula for q_* in Equation (2.42) we see that the equivalence is actually the total decalage functor $\text{Tot} = \mathsf{r}^*$ induced by $\mathsf{r} : \mathcal{D} \to \mathbb{\Delta}$, as discussed in the introduction (cf. 1.4).

5 Σ-presheaves vs. $\mathcal{D}$-presheaves, and the BOORS equivalence

The goal of this section is to relate the disparate notions of augmentation of BOORS and Carlier, and as a result derive the BOORS equivalence, using the results already established in the previous sections.

We first analyze the consequences of having both the horizontal and the vertical pointing axiom, and in particular we show how, as a result, the abacus maps acquire inverses. We then use this to derive an equivalence between Σ-presheaves satisfying the BOORS axioms and bicomodule configurations with invertible abacus maps. Composing this equivalence with that between bicomodule configurations with invertible abacus and 2-Segal spaces (Theorem 4.19) establishes the original BOORS equivalence.

We end this section with a finer analysis of the relation between Σ-presheaves and $\mathcal{D}$-presheaves, establishing an equivalence between Σ-presheaves that satisfy only half of the BOORS axioms (horizontal pointing, upper stable, Segal rows) and $\mathcal{D}_{i\geq 0}$-presheaves that are upper stable and have Segal rows (the $i \geq 0$ decoration means that the augmentation row is missing).

5.1 From the pointing axioms to invertible abacus maps

We first recall the BOORS equivalence: Let Σ be the category obtained by taking the cocone on $\mathbb{\Delta}\times\mathbb{\Delta}$. In terms of generators and relations this category is described by the generators and relations of $\mathbb{\Delta} \times \mathbb{\Delta}$ together with an object $[-1]$ and a morphism $[0,0] \to [-1]$.

Definition 5.1. *A presheaf* $B \in \boldsymbol{Pr}(\Sigma)$ *is said to satisfy the* horizontal pointing axiom *if the pointing* $B_{-1} \to B_{0,0}$ *constitutes a local-initial-objects structure on the zeroth row. Dually,* $B \in \boldsymbol{Pr}(\Sigma)$ *is said to satisfy the* vertical pointing axiom *if the pointing* $B_{-1} \to B_{0,0}$ *constitutes a local-terminal-objects structure on the zeroth column.*

Remark 5.2. Bergner et al. [16, 15] use the terminology *preaugmented bisimplicial spaces* for general presheaves on Σ, and say *augmented bisimplicial spaces* for presheaves that satisfy both the horizontal and vertical pointing axioms of Definition 5.1.

Let B be an upper stable Σ-presheaf satisfying the horizontal pointing axiom. By Proposition 2.17, which identifies local-initial-objects structure with rigid $\mathrm{Dec}_\perp$-coalgebra structure, the zeroth row of B is endowed with a bottom-split structure. This is the first step towards relating Σ-presheaves and $\mathcal{D}$-presheaves. Upper stability, which says that each simplicial map $e_\perp : B_{i+1,\bullet} \to B_{i,\bullet}$ is a right fibration between rows, induces the remaining extra bottom sections in the bulk: applying Lemma 2.6 produces first a $\mathrm{Dec}_\perp$-coalgebra structure on $B_{1,\bullet}$ which is compatible with $e_\perp$, and with the help of an inductive argument we can propagate down the bottom-split structure to all rows.

A priori, the aforementioned construction of the bottom sections $s_\perp$ only guarantees that they commute with the bottom face maps $e_\perp$. It remains to show that the bottom sections are compatible also with the active part of columns, i.e. that the bottom sections $s_\perp$ form simplicial maps between columns after applying $\mathrm{Dec}_\top$ to the columns. To see that the bottom sections are compatible with the active part of each column we provide a construction that produces all the bottom sections uniformly. In this construction we invoke Lemma 2.6 again, but this time take $\mathcal{E}$ to be $s\mathcal{S}$ (simplicial spaces). The simplicial object C of the lemma is taken to be the composite

$$\mathbb{\Delta}^{\mathrm{op}} \xrightarrow{B_{0,\bullet}} \mathcal{S} \xrightarrow{\mathrm{const}} s\mathcal{S}$$

which is a $\mathrm{Dec}_\perp$-coalgebra in $\mathcal{E}$. For C' we take the transpose of

$$\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Dec}_\top \times \mathrm{id}} \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \xrightarrow{B_{\bullet,\bullet}} \mathcal{S}$$

in the first coordinate, which gives $C' : \mathbb{\Delta}^{\mathrm{op}} \to s\mathcal{S};\ j \mapsto {\mathrm{Dec}_\top}^{\mathrm{vert}} B_{\bullet,j}$. The simplicial map $C' \to C$ is given by the vertical augmentation map for $\mathrm{Dec}_\top$ (which in degree 0 is the map $e_\bot$). So far, the argument leaves out the compatibility of the horizontal bottom sections $s_{\underline{\bot}}$ with the vertical top degeneracies $t_\top$ (as these are discarded by the vertical $\mathrm{Dec}_\top$); however, this compatibility will be automatic thanks to the top degeneracy $t_\top$ being a section of a face map $e_{\top-1}$, which is contained in $\mathrm{Dec}_\top B_{\bullet,0}$.

The above discussion can be distilled into a proof of the following lemma.

Lemma 5.3. *Let B be a Σ-presheaf which is upper stable and satisfies the horizontal pointing axiom. Then B has induced extra bottom sections in every row.*

Remark 5.4. As a matter of fact, the argument above Lemma 5.3 applies to any bisimplicial space B with extra bottom sections in the first row. In other words, any upper stable bisimplicial space with a $\mathrm{Dec}_\bot$-coalgebra structure on its zeroth row has a canonical extension to a $\mathcal{D}_{\geq 0}$-presheaf, where $\mathcal{D}_{\geq 0}$ is the category $\mathcal{D}$ with the augmentations removed.

The equivalence of Bergner et al. [16, 15] says that if a Σ-presheaf is stable, double Segal and satisfies both horizontal and vertical pointing axioms, then it is the total decalage of a 2-Segal spaces. In the total decalage of a simplicial space the abacus maps are the identities, and in particular invertible. We show directly how for a Σ-presheaf with all the above properties the induced abacus maps are invertible.

Let B be a Σ-presheaf which is stable, double Segal and satisfies both horizontal and vertical pointing axioms. By Lemma 5.3 and its dual, B is endowed both with extra bottom sections in all bulk rows and with extra top sections $t_{\overline{\top}}$ in all bulk columns. Taking colimits row-wise gives an augmentation column, where the augmentation maps inherit a bottom section from the bottom-split structure of the rows. Similarly, taking colimits column-wise gives an augmentation row, where the augmentation map is equipped with extra top sections. The following lemma expresses a key compatibility between the horizontal bottom splittings and the vertical top splittings.

Lemma 5.5. *Let B be a Σ-presheaf which is stable, double Segal and satisfies both horizontal and vertical pointing axioms. Then for the induced splittings we have*

$$t_\top s_{\underline{\bot}} = t_{\overline{\top}}\, s_{\underline{\bot}}\,. \tag{2.47}$$

This holds also in the augmentation column which is obtained by taking colimits.

Proof. The proof relies on an inductive argument. First of all, consider the diagram

$$\begin{array}{ccc} & & B_{-1} \\ & /\!\!/ & \Big\downarrow{\scriptstyle t_{\overline{\top}}} \\ B_{-1} & \xrightarrow[s_{\underline{\perp}}]{} & B_{0,0} \\ & & {\scriptstyle t_{\top}}\Big\downarrow\!\Big\downarrow{\scriptstyle t_{\overline{\top}}} \\ & & B_{1,0} \end{array}$$

The top triangle commutes by definition, where both $s_{\underline{\perp}}$ and $t_{\overline{\top}}$ are given by the pointing in Σ. Since $t_{\overline{\top}} : B_{-1} \to B_{0,0}$ equalizes the two maps $t_{\top}, t_{\overline{\top}} : B_{0,0} \to B_{1,0}$, (this is a simplicial identity), it follows that also $s_{\underline{\perp}}$ equalizes $t_{\top}$ and $t_{\overline{\top}}$. This gives us the first instance of Equation (2.47) and forms the base case for the inductive proof.

To allow for a uniform argument, let us denote the objects of the augmentation column by $B_{i,-1}$. In particular, $B_{0,-1} = B_{-1}$. For the inductive step, assume that

$$B_{i,j} \xrightarrow{s_{\underline{\perp}}} B_{i,j+1} \underset{t_{\top}}{\overset{t_{\overline{\top}}}{\rightrightarrows}} B_{i+1,j+1}$$

commutes for some $i \geq 0$ and some $j \geq -1$. We show that this continues to commute in the next row, i.e. with i replaced by $i+1$. For this consider the diagram

$$\begin{array}{ccccccc} B_{i,j} & \xrightarrow{s_{\underline{\perp}}} & B_{i,j+1} & \underset{t_{\top}}{\overset{t_{\overline{\top}}}{\rightrightarrows}} & B_{i+1,j+1} & \xrightarrow{e_{\top}} & B_{i,j+1} \\ {\scriptstyle e_{\perp}}\Big\uparrow & & {\scriptstyle e_{\perp}}\Big\uparrow & & {\scriptstyle e_{\perp}}\Big\uparrow & \lrcorner & \Big\uparrow{\scriptstyle e_{\perp}} \\ B_{i+1,j} & \xrightarrow[s_{\underline{\perp}}]{} & B_{i+1,j+1} & \underset{t_{\top}}{\overset{t_{\overline{\top}}}{\rightrightarrows}} & B_{i+2,j+1} & \xrightarrow[e_{\top}]{} & B_{i+1,j+1}. \end{array}$$

The two outer squares commute, and so does the middle square if we consider either both $t_{\top}$ or both $t_{\overline{\top}}$ degeneracies. The right-most square is a pullback by the Segal condition in column $j+1$. Since both $t_{\top}$ and $t_{\overline{\top}}$ are sections of $e_{\top}$ and we are assuming that the equation holds on the first row, by the uniqueness of the pullback-induced maps it follows that Equation (2.47) holds also in the next row.

Finally we show that it also holds in the next column, i.e. with j replaced by $j+1$. For this we use the same argument as for the induction along rows, now applied to the the diagram

$$
\begin{array}{ccccccc}
B_{i,j} & \xrightarrow{s_{\underline{\bot}}} & B_{i,j+1} & \overset{t_{\overline{\top}}}{\underset{t_{\top}}{\rightrightarrows}} & B_{i+1,j+1} & \xrightarrow{e_{\top}} & B_{i,j+1} \\
{\scriptstyle d_{\top}}\uparrow & & {\scriptstyle d_{\top}}\uparrow & & {\scriptstyle d_{\top}}\uparrow \ \lrcorner & & \uparrow{\scriptstyle d_{\top}} \\
B_{i,j+1} & \xrightarrow[s_{\underline{\bot}}]{} & B_{i,j+2} & \overset{t_{\overline{\top}}}{\underset{t_{\top}}{\rightrightarrows}} & B_{i+1,j+2} & \xrightarrow[e_{\top}]{} & B_{i,j+2}
\end{array}
$$

□

Corollary 5.6. *Let B be a Σ-presheaf which is stable, double Segal and satisfies the horizontal and vertical pointing axioms. Then the induced abacus maps are invertible. In particular, the pair of maps*

$$
B_{i+1,j} \ \underset{g:=d_{\bot}t_{\overline{\top}}}{\overset{f:=e_{\top}s_{\underline{\bot}}}{\rightleftarrows}} \ B_{i,j+1}
$$

constitute a pair of inverse maps for all $i, j \geq -1$, where the $B_{\bullet,-1}$ is obtained by taking colimits row-wise, and $B_{-1,\bullet}$ is obtained by taking colimits column-wise.

Remark 5.7. The maps $g := d_{\bot}t_{\overline{\top}}$ are the "dual abacus maps" obtained from the vertical extra top degeneracy maps induced by the local-terminal-objects structure corresponding to the vertical pointing axiom.

Proof of Corollary 5.6. We compute

$$
\begin{aligned}
gf &= d_{\bot}t_{\overline{\top}}\, e_{\top}s_{\underline{\bot}} \\
&= d_{\bot}e_{\top-1}\, t_{\overline{\top}}\, s_{\underline{\bot}} \\
&= d_{\bot}e_{\top-1}t_{\top}\, s_{\underline{\bot}}, && \text{by Lemma 5.5} \\
&= d_{\bot}s_{\underline{\bot}} \\
&= 1.
\end{aligned}
$$

where every line except the first and third uses a simplicial identity. A dual argument shows that $fg = 1$. □

Remark 5.8. The fact that in a BOORS-augmented stable double Segal space B there is an equivalence $B_{10} \simeq B_{01}$ was observed in [15, Remark 2.31]: they establish a zig-zag of weak equivalences between the two spaces, using the pointing axiom and the stability axiom. From the viewpoint of that zig-zag, it is perhaps surprising that half of the axioms are enough to get the abacus map directly, not as a zigzag (although of course both sides of the axioms are needed to establish that it is invertible, as we have just seen). On the other hand, a posteriori, the BOORS equivalence tells us of course that a canonical equivalence must exist, since in the case (which is every case) of $B = \mathrm{Tot}(X)$ both B_{10} and B_{01} are identified with X_2.

5.2 The equivalence $\boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma) \simeq \boldsymbol{ABC}^{\simeq}$

Consider the inclusion $\mathsf{j} : \Sigma \to \mathcal{D}$, which is defined as the identity on the underlying bisimplicial space and which maps the pointing of Σ onto the map $s^{\perp\!\!\!\perp} : [0,0] \to [0,-1]$ in $\mathcal{D}$. Recall that $\boldsymbol{ABC}^{\simeq} \subset \boldsymbol{Pr}(\mathcal{D})$ is the full subcategory spanned by those presheaves which are stable, double Segal and have invertible abacus maps. For $B \in \boldsymbol{ABC}^{\simeq}$, the restriction $\mathsf{j}^*(B)$ will be stable and double Segal, as these conditions do not refer to the augmentations. The zeroth row in B is equipped with a $\mathrm{Dec}_\perp$-coalgebra structure. Since B is double Segal, this coalgebra structure will automatically be rigid by Lemma 2.9. It follows from Proposition 2.17 that the zeroth row together with the map $s_{\perp\!\!\!\perp} : X_0 \to B_{0,0}$ satisfies the horizontal pointing axiom. On the other hand, the invertibility of the abacus maps provides extra top degeneracy maps in every column, and in particular a $\mathrm{Dec}_\top$-coalgebra structure

$$B_{\bullet,j} \xrightarrow{s_0} B_{\bullet,j+1} \xrightarrow{f^{-1}} \mathrm{Dec}_\top(B_{\bullet,j}).$$

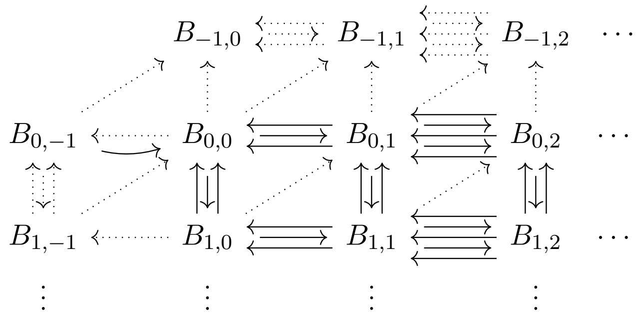
This is the description of $t_{\top\!\!\!\top}$, dual to Equation (2.40) in Remark 3.18. In particular we get a $\mathrm{Dec}_\top$-coalgebra on the zeroth column, which is rigid (in the sense of top splittings) by the double Segal condition again and by the dual of 2.9. From Proposition 2.17, it follows that the zeroth column satisfies the vertical pointing axiom. Altogether we have shown that the functor j^* restricts to

$$\mathsf{j}^* : \boldsymbol{ABC}^{\simeq} \to \boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma).$$

Theorem 5.9. *The functor* $\mathsf{j}^* : \boldsymbol{ABC}^{\simeq} \to \boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma)$ *is an equivalence.*

Proof. Essential surjectivity: Starting with a Σ-presheaf $A' \in \boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma)$ we shall extend this to a $\mathcal{D}$-presheaf in $\boldsymbol{ABC}^{\simeq}$ which restricts to A' under j^*. We obtain all the bottom splittings in the bulk by the horizontal pointing axiom and upper stability, via Lemma 5.3. Taking colimits row-wise allows us to build the complete augmentation column along with the augmentation maps. Since the rows are split by bottom sections, so will the augmentation maps be. Taking colimits column-wise constructs the augmentation row together with the augmentation maps. So far we have built a $\mathcal{D}$-diagram A which restricts to A' when pulled back along j. Now, A inherits the properties of being stable and double Segal from A'. The augmentation row and column are 2-Segal and the augmentation maps are culf by Proposition 3.6. By Theorem 5.6 all abacus maps are invertible.

Faithfulness: Consider a functor $G \in \mathrm{Map}_{\boldsymbol{ABC}^{\simeq}}(A, B)$. Every object in $\mathcal{D}$ which is not in the image of $\mathsf{j} : \Sigma \hookrightarrow \mathcal{D}$ is connected to an object in the image of j by moving along an abacus map. We visualize this on the level of presheaves

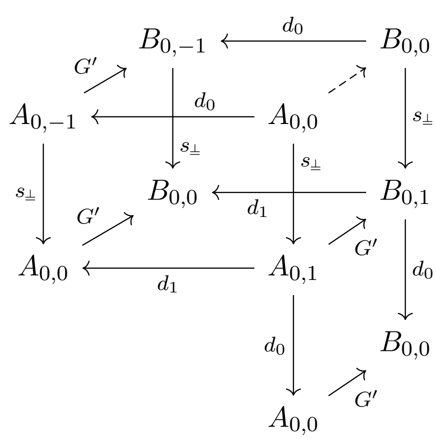


where the restriction to $\boldsymbol{Pr}(\Sigma)$ under j^* is depicted using solid arrows. Since the abacus maps are assumed to be invertible, the value of any natural transformation on the image of j^* fixes its values outside of it, as can be seen for example in the square

$$\begin{array}{ccc} A_{-1,0} & \xrightarrow{G} & B_{-1,0} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ A_{0,-1} & \xrightarrow[G]{} & B_{0,-1} \end{array}$$

As a result, if two functors $G, H \in \mathrm{Map}_{\boldsymbol{ABC}^{\simeq}}(A, B)$ agree on Σ, that is $\mathsf{j}^*(G) \simeq \mathsf{j}^*(H)$, then they also agree on $\mathcal{D}$, that is $G \simeq H$, proving faithfulness.

Fullness: Let A, B be two $\mathcal{D}$-presheaves in $\boldsymbol{ABC}^{\simeq}$ and let $G' \in \mathrm{Map}_{\boldsymbol{Pr}^{\mathrm{BOORS}}(\Sigma)}(\mathsf{j}^*(A), \mathsf{j}^*(B))$ be a functor. Our goal is to construct a functor $G : A \to B$ which when pulled back along j recovers G'. Let us begin with the diagram



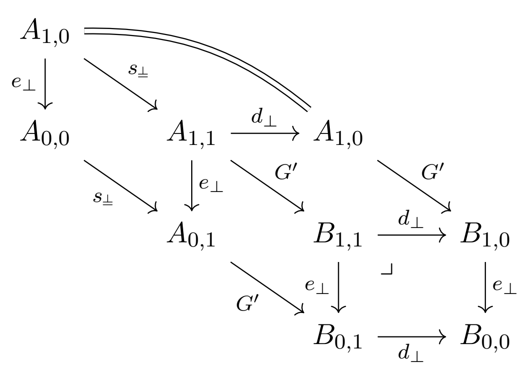
Here all the solid squares involving the diagonal maps G' commute since G' is a map of Σ-presheaves. Since all rows are 1-Segal and bottom-split, the $\mathrm{Dec}_{\perp}$-

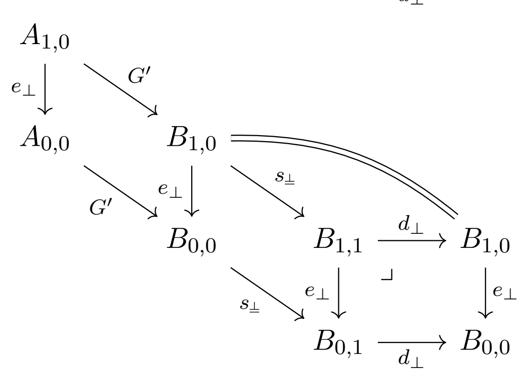
coalgebra structure is rigid by Lemma 2.9. It follows that the front and back face of the cube are pullbacks, with the back face pullback inducing the dashed morphism. By construction this induced morphism is compatible with the augmentation map and with the extra bottom degeneracy $[0,0] \leftarrow [0,1]$. Composing the right face of the cube with the hanging square $G'd_\perp \simeq d_\perp G'$, as in the diagram, shows that the dashed morphism is necessarily G', since the two vertical composites are the identities. Working instead with the pullback $\bar{A}_{0,-1} \times_{\bar{A}_{0,0}} \mathrm{Dec}_\perp A_{0,\bullet}$ of simplicial spaces and the corresponding one for B, a similar argument allows us to conclude that G' is in fact compatible with all extra bottom sections of the complete zeroth row as well as with the augmentation map in the zeroth row.

Next we show that G' is compatible with all the extra bottom sections in the bulk (which are obtained by pullback using upper stability). By assumption, the square

$$\begin{array}{ccc} A_{0,0} & \xrightarrow{s_\perp} & A_{0,1} \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G'} \\ B_{0,0} & \xrightarrow[s_\perp]{} & B_{0,1} \end{array} \tag{2.48}$$

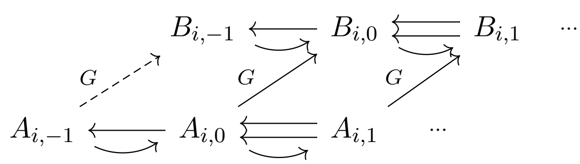
commutes. To start with, we claim that the same square in the row below commutes too. To see this consider the two commutative diagrams



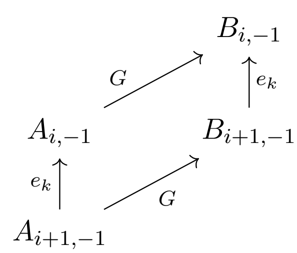


where the pullback is an upper stability square. By the commutativity of diagram 2.48 the two diagonal maps $A_{1,0} \to B_{1,1}$ agree, when composed with either of the two pullback projections. Thus, by the uniqueness in universal property of the pullback, they must agree, proving the claim. The same argument can be used in any row and any column. Thus, starting with the compatibility of G' with the extra bottom sections in the zeroth row, we can deduce the compatibility of G' with all the extra bottom sections in all other rows by applying an inductive argument. We define G on the bulk and on $A_{0,-1}$ to be equal to G'.

Next we turn our attention to the augmentation column. Since the rows of A and B are (absolute) colimits, we can define the value of G on the augmentation by the colimit-induced map as in the diagram



where $i > 0$. This automatically makes G compatible with the augmentation map and its extra bottom section. Thus defined, G will be compatible also with the simplicial operators between rows, that is, all the squares



commute for all $0 \leq k \leq \top$ and similarly for all degeneracies. This is because the same equation holds in the bulk and as a result, by the uniqueness of induced maps on colimits, also on the augmentation.

Finally, the augmentation row can be addressed in a similar fashion as we did for the augmentation column. All in all we have extended G' to a map $G : A \to B$, which by construction restricts to $G' : \mathsf{j}^*(A) \to \mathsf{j}^*(B)$. □

The following corollary is the BOORS equivalence.

Corollary 5.10 (BOORS [15])**.** *The functor* $\mathsf{p}^* : \boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \longrightarrow \boldsymbol{Pr}^{\text{BOORS}}(\Sigma)$ *is an equivalence.*

Proof. Combining Theorem 4.19 and Theorem 5.9 gives an equivalence

$$\boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \xrightarrow{\mathsf{q}_*} \boldsymbol{ABC}^{\simeq} \xrightarrow{\mathsf{j}^*} \boldsymbol{Pr}^{\text{BOORS}}(\Sigma).$$

According to Remark 4.20, the functor $\mathsf{q}_* : \boldsymbol{Pr}^{\text{2-Seg}}(\mathbb{\Delta}) \to \boldsymbol{ABC}^{\simeq}$ is equivalent to the total decalage r^*, induced by $\mathsf{r} : \mathcal{D} \to \mathbb{\Delta}$, which when composed with j^* gives precisely p^* (see 1.6). □

5.3 More detailed comparison between Σ-presheaves and $\mathcal{D}$-presheaves

The category Σ as well as the BOORS axioms are symmetric with respect to the diagonal. The category $\mathcal{D}$ on the other hand, is asymmetric forcing the inclusion $\mathsf{j} : \Sigma \hookrightarrow \mathcal{D}$ to be asymmetric as well. We now embrace the asymmetry and study the inclusion $\mathsf{j} : \Sigma \hookrightarrow \mathcal{D}$ more carefully by factorizing it into smaller steps:

$$\Sigma \xrightarrow{\simeq} \mathbb{\Delta}^{\text{pt}} \sqcup_{\mathbb{\Delta}} (\mathbb{\Delta}\times\mathbb{\Delta}) \xrightarrow{\mathsf{h}\sqcup_{\text{id}}\text{id}} \mathbb{\Delta}^{\mathsf{b}} \sqcup_{\mathbb{\Delta}} (\mathbb{\Delta}\times\mathbb{\Delta}) \xrightarrow{\mathsf{w}} \mathcal{D}_{i\geq 0} \longrightarrow \mathcal{D}$$

It follows from Proposition 2.17 that the map

$$\boldsymbol{Pr}(\mathbb{\Delta}^{\mathsf{b}}) \times_{\boldsymbol{Pr}(\mathbb{\Delta})} \boldsymbol{Pr}(\mathbb{\Delta} \times \mathbb{\Delta}) \xrightarrow{\mathsf{h}^* \times_{\text{id}} \text{id}} \boldsymbol{Pr}(\mathbb{\Delta}^{\text{pt}}) \times_{\boldsymbol{Pr}(\mathbb{\Delta})} \boldsymbol{Pr}(\mathbb{\Delta} \times \mathbb{\Delta})$$

restricts to an equivalence after imposing the horizontal pointing axiom on the domain and imposing rigidity on the first row on the codomain. By Lemma 2.9 the rigidity becomes automatic if we ask for the rows to be 1-Segal (or at least the zeroth row). We now turn our attention to w^*. Let $\boldsymbol{Pr}^{\text{up st}}(\mathcal{D}_{i\geq 0})$ and $\boldsymbol{Pr}^{\text{up st}}(\mathbb{\Delta}^{\mathsf{b}} \sqcup_{\mathbb{\Delta}} (\mathbb{\Delta} \times \mathbb{\Delta}))$ be the full subcategories of the domain and the codomain of w^* respectively consisting of the presheaves that are upper stable.

Lemma 5.11. *The functor*

$$\boldsymbol{Pr}^{\text{up st}}(\mathcal{D}_{i\geq 0}) \xrightarrow{\mathsf{w}^*} \boldsymbol{Pr}^{\text{up st}}(\mathbb{\Delta}_{\mathsf{b}} \sqcup_{\mathbb{\Delta}} (\mathbb{\Delta}\times\mathbb{\Delta}))$$

is an equivalence.

Proof. This proof relies on similar arguments as those appearing in the proof of Theorem 5.9.

Essential surjectivity: Starting with a presheaf B in the codomain of w^*, Lemma 5.3 together with Remark 5.4 imply extra bottom sections can be propagated down starting from the zeroth row, thus producing a $\mathcal{D}_{\geq 0}$-presheaf with a pointing, where $\mathcal{D}_{\geq 0}$ is $\mathcal{D}$ but with both augmentations removed. Taking colimits row-wise (which are absolute) we fill in the augmentation column producing the desired $\mathcal{D}_{i\geq 0}$-presheaf which restricts B under w^*.

Fullness and Faithfulness: The argument is the same as that in the proof of Theorem 5.9. □

Putting everything together gives the following theorem.

Theorem 5.12. *The functor* $\boldsymbol{Pr}(\mathcal{D}_{i\geq 0}) \to \boldsymbol{Pr}(\Sigma)$ *induced by the inclusion* $\Sigma \hookrightarrow \mathcal{D}_{i\geq 0}$ *restricts to an equivalence on the full subcategories*

$$\{B \in \boldsymbol{Pr}(\mathcal{D}_{i\geq 0}) \,|\, \textit{upper stable, Segal rows}\} \xrightarrow{\simeq} \left\{ B' \in \boldsymbol{Pr}(\Sigma) \,\middle|\, \begin{array}{c} \textit{upper stable, Segal rows,} \\ \textit{horizontal pointing axiom} \end{array} \right\}.$$

Remark 5.13. Note that Proposition 3.6, where we deduced the 2-Segalness of the augmentations and the culfness of the augmentation maps from properties of the bulk, does not apply in Theorem 5.12, since we do not have the full stability (see also Remark 3.7). As a result, the $\mathcal{D}_{i\geq 0}$-presheaves which are upper stable with Segal rows, appearing in Theorem 5.12, cannot be ensured to have 2-Segal augmentation column or culf augmentation map.

A 2-Segal cheat sheet

All the following are standard facts.

Fact A.1. *The following are equivalent conditions on a simplicial space X:*

— *X is 1-Segal*

— *the counit $\varepsilon : \mathrm{Dec}_\top X \to X$ is a right fibration (meaning cartesian on $d_\bot$).*

— *the counit $\varepsilon : \mathrm{Dec}_\bot X \to X$ is a left fibration (meaning cartesian on $d_\top$).*

Fact A.2 (Cf. [33, Prop.4.13])**.** *If $F : Y \to X$ is culf, then*

— *$\mathrm{Dec}_\top(F)$ is a left fibration.*

— *$\mathrm{Dec}_\bot(F)$ is a right fibration.*

Fact A.3. *If $F : Y \to X$ is a left fibration, then $\mathrm{Dec}_\bot(F)$ is cartesian*

Fact A.4. *If $F : Y \to X$ is a right fibration, then $\mathrm{Dec}_\top(F)$ is cartesian.*

Fact A.5. *If $F : Y \to X$ is a left fibration, then*

$$\begin{array}{ccc} Y & \xleftarrow[\varepsilon]{} & \mathrm{Dec}_\top Y \\ \downarrow & & \downarrow \\ X & \xleftarrow[\varepsilon]{} & \mathrm{Dec}_\top X \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \bar{Y}_0 & \xleftarrow[\zeta]{} & \mathrm{Dec}_\bot Y \\ \downarrow & & \downarrow \\ \bar{X}_0 & \xleftarrow[\zeta]{} & \mathrm{Dec}_\bot X \end{array}$$

are a pullbacks.

Dually, if $F : Y \to X$ is a right fibration, then

$$\begin{array}{ccc} Y & \xleftarrow[\varepsilon]{} & \mathrm{Dec}_\bot Y \\ \downarrow & & \downarrow \\ X & \xleftarrow[\varepsilon]{} & \mathrm{Dec}_\bot X \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \bar{Y}_0 & \xleftarrow[\zeta]{} & \mathrm{Dec}_\top Y \\ \downarrow & & \downarrow \\ \bar{X}_0 & \xleftarrow[\zeta]{} & \mathrm{Dec}_\top X \end{array}$$

are pullbacks.

Fact A.6. *If $Y \to X$ is a left or a right fibration and X is 1-Segal, then also Y is 1-Segal.*

Fact A.7. *If $Y \to X$ is culf and an effective epi, then X is 2-Segal if and only if Y is 2-Segal.*

A.8. Upper and lower 2-Segal spaces. Recall that a simplicial space Y is called *upper 2-Segal* if $\mathrm{Dec}_\top(Y)$ is 1-Segal. In particular, the following square is then a pullback:

$$\begin{array}{ccc} Y_2 & \xleftarrow{d_2} & Y_3 \\ {\scriptstyle d_0}\downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ Y_1 & \xleftarrow[d_1]{} & Y_2. \end{array}$$

Similarly, a simplicial space Y is called *lower* 2-*Segal* if $\mathrm{Dec}_\bot(Y)$ is 1-Segal. In particular, the following square is then a pullback:

$$\begin{array}{ccc} Y_2 & \xleftarrow{d_1} & Y_3 \\ {\scriptstyle d_2}\downarrow & \lrcorner & \downarrow{\scriptstyle d_3} \\ Y_1 & \xleftarrow[d_1]{} & Y_2. \end{array}$$

Fact A.9. *If $Y \to X$ is culf and X is* 2*-Segal (resp. lower* 2*-Segal, resp. upper* 2*-Segal), then also Y is* 2*-Segal (resp. lower* 2*-Segal, resp. upper* 2*-Segal).*

Fact A.10. *If Y is upper* 2*-Segal, then the counit $\varepsilon : \mathrm{Dec}_\bot(Y) \to Y$ is culf. If Y is lower* 2*-Segal, then the counit $\varepsilon : \mathrm{Dec}_\top(Y) \to Y$ is culf. (In particular, if Y is* 2*-Segal, then both the counits $\mathrm{Dec}_\top Y \to Y$ and $\mathrm{Dec}_\bot Y \to Y$ are culf [33].)*

(This fact is slightly trickier than the others, as the proof depends on a retract argument, as in [29].)

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