Characterizing symmetric powers bialgebraically

A first step into the logic of usual linear mathematics

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Introduction

W. Lawvere (p.213 in **Foundations and applications: axiomatization and education**, Bulletin of Symbolic Logic 213-224, 2003):

> *In my own education I was fortunate to have two teachers who used the term "foundations" in a common-sense way (rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition). This way is exemplified by their work in Foundations of Algebraic Topology, published in 1952 by Eilenberg (with Steenrod), and The Mechanical Foundations of Elasticity and Fluid Mechanics, published in the same year by Truesdell. The orientation of these works seemed to be "concentrate the essence of practice and in turn use the result to guide practice".*

Internal vs External

J. B. Watson (**Behaviorism**, 1924):

> *Behaviorism claims that 'consciousness' is neither a definable nor a usable concept; that it is merely another word for the 'soul' of more ancient times. The old psychology is thus dominated by a subtle kind of religious philosophy.*

L. Wittgenstein (**Philisophical Investigations** § 43, 1953):

> *For a large class of cases of the employment of the word 'meaning'—though not for all—this word can be explained in this way: the meaning of a word is its use in the language.*

Either Watson or Wittgenstein claim for an external point of view of psychology/language. People, like words must be understood by what an external observer can tell about them when they are in action.

That's the opposite of an "internal" approach where we think about what's going on inside the person, or "inside" the word ie. thinking of the word as an abstract independent concept, which can be different from its concrete use.

In mathematics: External $\approx$ Algebraic $\approx$ Logical

Both approaches also exist in mathematics. Take the real numbers.

- Internal/Rigorist/Mystical approach: we create the real numbers from more elementary pieces: packs of empty sets, rational numbers, then Dedekind cuts... to have strong foundations.
- External/Relaxed/Practical approach: What matter are the usable properties. Real number are: a complete space, a field, ordered and archimedian. With this definition, we now a lot about the real numbers, from the start. We can then use it without thinking about foundations.

These two definitions are equivalent. But to do proof-theoretic logic, we need the second type of definitions. We'll see this with homogenous polynomials/symmetric powers. We're going to give an external characterization which is equivalent to the classical definition but is better to build a logic of them.

Finite exponentials vs Infinite exponentials

J.-Y. Girard (p.6 in **Bounded linear logic: a modular approach to polynomial-time computability** J.-Y. Girard, P. J. Scott and A. Scedrov, 1992):

> *In these times of great utopias falling, "forever" is no longer a viable expression, and in bounded linear logic (BLL) it is replaced by more realistic goals: reuse will be possible, but only a certain number of times limited in advance.*

We will use graded/bounded exponentials. They are more concrete, but most important, we can characterize symmetric powers as a particular graded exponential but I don't know how to characterize symmetric algebras as a non-graded exponential.

Linear mathematics and Graded Bialgebraic Linear Logics

Linear Logic is a logic about these symbols:

$$\otimes \quad \& \quad \oplus \quad ! \quad ? \quad \_^{\perp}$$

It allows to prove eye-catching isomorphisms:

$$!(A \mathbin{\&} B) \cong\, !A \otimes\, !B$$

$$?A \cong (!A^{\perp})^{\perp}$$

Graded Linear Logic is a logic about these symbols:

$$\otimes \quad \& \quad \oplus \quad !_n \quad ?_n \quad \_^{\perp}$$

It allows to prove eye-catching isomorphisms:

$$!_n(A \& B) \cong \bigoplus_{0 \leq k \leq n} !_k A \otimes !_{n-k} B$$

[1]

$$?_n A \cong (!_n A^{\perp})^{\perp}$$

[1] In fact, we need a graded differential/bialgebraic linear logic for this one

Linear mathematics is about these functors:

$$\otimes \quad \& \quad \oplus \quad \_^{\perp}$$

and these ones:

$$\_^{\otimes n} : Vect \to Vect$$
$$S_n : Vect \to Vect$$
$$\Lambda_n : Vect \to Vect$$
$$S_\lambda : Vect \to Vect$$
$$\Gamma_n : Vect \to Vect$$
$$H_n : Top \to Vect$$
$$\ldots$$

It allows to prove eye-catching isomorphisms:

$$S_n(A \oplus B) \cong \bigoplus_{0 \leq k \leq n} S_k A \otimes S_{n-k} B$$

$$H_n(A \,\&\, B) \cong \bigoplus_{0 \leq k \leq n} H_k A \otimes H_{n-k} B$$

$$\Gamma_n A \cong (S_n A^{\perp})^{\perp}$$

$$A \otimes A \cong S_2 A \oplus \Lambda_2 A$$

$$A^{\otimes n} \cong \bigoplus_{\lambda \vdash n} (S_\lambda V)^{\oplus m_\lambda}$$

...

Note that in Vec, $\oplus$ is a biproduct
but in Linear Logic, $\oplus$ is the coproduct and & is the product.

However in Differential Linear Logic, $\oplus$ is a biproduct.

Morover, in Linear Logic $!A$ is a coalgebra
but in Differential Linear Logic, $!A$ is a bialgebra.

It seems that the ideas of Differential Linear Logic and Graded Linear Logic are useful to make some logic of linear mathematics. Because there are biproducts and graded bialgebras in all this stuff.

I'd like to name the vanilla such logic Graded Bialgebraic Linear Logic.

Symmetric powers are a model of Graded Bialgebraic Linear Logic

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The specialty property of symmetric powers

We have a connected graded bialgebra ie. a family $(A_n)_{n\geq 0}$ of objects such that $A_0 \cong I$ and two families

$$(\nabla_{n,p\geq 0} : A_n \otimes A_p \to A_{n+p})_{n,p\geq 0}$$

$$(\Delta_{n,p\geq 0} : A_{n+p} \to A_n \otimes A_p)_{n,p\geq 0}$$

which verifies some equations akin to the one of a bialgebra.

We'll see an exact definition later. But when A_n is the n^{th} symmetric power of A_1, there is a more surprising "specialty" equation which simplifies

$$(\Delta_{n,p}; \nabla_{n,p}) : S_{n+p}A \to S_{n+p}A$$

Let's see how it works in vector spaces.

First, I recall what are the symmetric powers of a vector space.

If A is a vector space, the n^{th} symmetric power S_nA is the quotient of $A^{\otimes n}$ by the action of the symmetric group $\mathfrak{S}_n$ by permutation:

$$\sigma.x_1 \otimes \ldots \otimes x_n = x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$$

ie. S_nA is spanned by the vectors

$$x_1 \otimes_s \ldots \otimes_s x_n$$

such that

$$x_{\sigma(1)} \otimes_s \ldots \otimes_s x_{\sigma(n)} = x_1 \otimes_s \ldots \otimes_s x_n$$

for every $\sigma \in \mathfrak{S}_n$.

Alternatively, in characteristic 0, it can be seen as the subspace of $A^{\otimes n}$ constituted by the vectors which are invariant under this action.

If we have indeterminates $(X_i)_{i\in I}$, then an homogeneous polynomial $P \in \mathbb{K}_n[X_i, i \in I]$ is a polynomial which is a sum of monomials of degree n.

If we have a basis $(X_i)_{i\in I}$ of A, then it gives an isomorphism

$$S_n A \cong \mathbb{K}_n[X_i, i \in I]$$

If we write the coordinates

$$x_k = \sum_{i\in I} a_k^i . X_i$$

it is given by

$$x_1 \otimes_s \dots \otimes_s x_n \mapsto \prod_{1\le k\le n} \left(\sum_{i\in I} a_k^i . X_i \right)$$

$$\sum_{\substack{(k_i)\in\mathbb{N}^I \\ k_1+\dots+k_i=n}} a_{(k_i)} . \prod_{i\in I} X_i^{k_i} \mapsto \sum_{\substack{(k_i)\in\mathbb{N}^I \\ k_1+\dots+k_i=n}} a_{(k_i)} . \bigotimes_{s,i\in I} X_i^{\otimes_s k_i}$$

The comultiplication is like this:

$$\begin{aligned}\mathbb{K}_5[X, Y, Z] &\to \mathbb{K}_2[X, Y, Z] \otimes \mathbb{K}_3[X, Y, Z] \\ X^2Y^2Z &\mapsto X^2 \otimes Y^2Z + 4XY \otimes XYZ \\ &\quad + 2XZ \otimes XY^2 + Y^2 \otimes X^2Z + 2YZ \otimes X^2Y\end{aligned}$$

How does it work exactly?

A multiset $M \in \mathcal{M}_n(X, Y, Z)$ is the same thing as a monic monomial of degree n with variable in X, Y, Z.

For every $M \in \mathcal{M}(X, Y, Z)$ and $P \in \mathbb{K}[X, Y, Z]$, $D_M(P)$ is equal to "the number of ways to extract M from P" $* \frac{P}{M}$.

Example: $D_{X^2Y}(X^2Y^2Z) = \binom{2}{1}YZ = 2YZ$

(You have to consider that you chose one of the two Y in P)

Example: $D_{X^2Y^3}(X^4Y^5Z^2) = \binom{4}{2}\binom{5}{3}X^2Y^2Z^2 = 60X^2Y^2Z^2$

This is the Hasse-Schmidt derivative of P with respect to M.

Compare w/ $\frac{\partial^5 X^4Y^5Z^2}{\partial X^2 \partial Y^3} = (4*3)(5*4*3)X^2Y^2Z^2 = 720X^2Y^2Z^2$.

Finally we have

$$D_{X_1^{m_1}...X_q^{m_q}}(X_1^{n_1}...X_q^{n_q}) = \binom{n_1}{m_1}...\binom{n_q}{m_q} X_1^{n_1-m_1}...X_q^{n_q-m_q}$$

and $\Delta_{n,p} : \mathbb{K}_{n+p}[X_1, ..., X_q] \to \mathbb{K}_n[X_1, ..., X_q] \otimes \mathbb{K}_p[X_1, ..., X_q]$ is given on monomials by

$$\begin{aligned}
\Delta_{n,p}(P = X_1^{n_1}...X_q^{n_q}) &= \sum_{\substack{M \in \mathcal{M}_p(X_1,...,X_q) \\ M|P}} D_M(P) \otimes M \\
&= \sum_{\substack{0 \leq m_1 \leq n_1 \\ ... \\ 0 \leq m_q \leq n_q}} D_{X_1^{m_1}...X_q^{m_q}}(P) \otimes X_1^{(n_1-m_1)}...X_q^{(n_q-m_q)} \\
&= \sum_{\substack{0 \leq m_1 \leq n_1 \\ ... \\ 0 \leq m_q \leq n_q}} \binom{n_1}{m_1}...\binom{n_q}{m_q} X_1^{n_1-m_1}...X_q^{n_q-m_q} \otimes X_1^{(n_1-m_1)}...X_q^{(n_q-m_q)}
\end{aligned}$$

It is much simpler to write it with symmetric tensors:

$$\Delta_{n,p}(y_1 \otimes_s \dots \otimes_s y_{n+p}) = \sum_{X \in \mathcal{P}_p([1,n+p])} y_{[1,n+p]\setminus X} \otimes y_X$$

It shows directly that it is a natural transformation:

$$\begin{CD} S_{n+p}E @>{\Delta_{n,p}E}>> S_nE \otimes S_pE \\ @V{S_{n+p}\phi}VV @VV{S_n\phi\otimes S_p\phi}V \\ S_{n+p}F @>>{\Delta_{n,p}E}> S_nF \otimes S_pF \end{CD}$$

It would be more difficult to show directly that for any linear map

$$u : \mathbb{K}_1[X_1, ..., X_q] \to \mathbb{K}_1[Y_1, ..., Y_r]$$

this diagram commute:

$$\begin{array}{ccc}
\mathbb{K}_{n+p}[X_1, ..., X_q] & \xrightarrow{\Delta_{n,p}} & \mathbb{K}_n[X_1, ..., X_q] \otimes \mathbb{K}_p[X_1, ..., X_q] \\
\Big\downarrow {\scriptstyle \mathbb{K}_{n+p}(u)} & & \Big\downarrow {\scriptstyle \mathbb{K}_n(u) \otimes \mathbb{K}_p(u)} \\
\mathbb{K}_{n+p}[Y_1, ..., Y_r] & \xrightarrow[\Delta_{n,p}]{} & \mathbb{K}_n[Y_1, ..., Y_r] \otimes S_p[Y_1, ..., Y_r]
\end{array}$$

by using the first definition of $\Delta_{n,p}$ and the matrix of u...

With polynomials: $\mathbb{K}[X_i, i \in I]$

$$\Delta_{n,1} : \mathbb{K}_{n+1}[X_i, i \in I] \to \mathbb{K}_n[X_i, i \in I] \otimes \mathbb{K}_1(X_i, i \in I)$$

is given by

$$\Delta_{n,1}(P) = \sum_{i \in I} \frac{\partial P}{\partial X_i} \otimes X_i$$

We then have $\nabla_{n,1}(\Delta_{n,1}(P)) = \sum_{i \in I} \frac{\partial P}{\partial X_i} X_i = (n+1).P$ by a theorem of Euler which says that this identity is a characterization of homogeneous polynomials of degree n among smooth functions!

It is much easier to view the identity without coordinates:

$$\Delta_{n,1} : S_{n+1}E \to S_nE \otimes E$$

is given by

$$\Delta_{n,1}(x_1 \otimes_s \ldots \otimes_s x_n) = \sum_{1 \leq i \leq n} (x_1 \otimes_s \ldots \otimes_s \widehat{x_i} \otimes_s \ldots \otimes_s x_n) \otimes x_i$$

and thus

$$\begin{aligned} \nabla_{n,1}(\Delta_{n,1}(x_1 \otimes_s \ldots \otimes_s x_n)) &= \sum_{1 \leq i \leq n} x_1 \otimes_s \ldots \otimes_s x_i \otimes_s \ldots \otimes_s x_n \\ &= n.x_1 \otimes_s \ldots \otimes_s x_n \end{aligned}$$

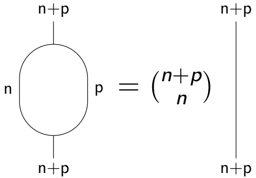
More generally, we have:

$$\begin{aligned}\nabla_{n,p}(\Delta_{n,p}(x_1 \otimes_s \dots \otimes_s x_{n+p})) &= \sum_{X \in \mathcal{P}_p([1,n+p])} y_1 \otimes_s \dots \otimes_s y_{n+p} \\ &= |\mathcal{P}_p([1, n+p])|.y_1 \otimes_s \dots \otimes_s y_{n+p} \\ &= \binom{n+p}{p}.y_1 \otimes_s \dots \otimes_s y_{n+p}\end{aligned}$$

We thus have:

$$S_{n+p}A \xrightarrow{\Delta_{n,p}} S_nA \otimes S_pA \xrightarrow{\nabla_{n,p}} S_{n+p}A, \qquad \nabla_{n,p} \circ \Delta_{n,p} = \binom{n+p}{n} Id$$

In string diagrams, it looks:



We'll see in a minute how to characterize symmetric powers combining this with the graded bialgebraic structure.

But before, let's see something else...

Binomial theorem, ideas of polynomial linear logic and vectorial categories

Binomial theorem. In every commutative ring:

$$(x+y)^n = \sum_{0\leq k\leq n} \binom{n}{k} x^k y^{n-k}$$

How to code this into linear logic? First, write it:

$$(x+y)^{\otimes_s n} = \sum_{0\leq k\leq n} \binom{n}{k} x^{\otimes_s k} \otimes_s y^{\otimes_s (n-k)}$$

Now we are in vector spaces. But, is

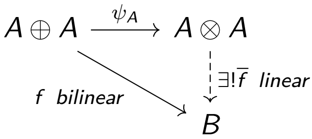
$$f : (x,y) \mapsto \sum_{0\leq k\leq n} \binom{n}{k} x^{\otimes_s k} y^{\otimes_s (n-k)}$$

a linear map? No!?! But it is a polynomial map. More precisely, an homogeneous polynomial map of degree n. Because

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Can I see this map as a proof in an appropriate logic?

Remember how the tensor product is defined in **Vec**.



where $\psi_A : (x, y) \mapsto x \otimes y$ is bilinear.

I suggest to make ψ_A a rule in a polynomial linear logic.

We'll see in a minute which categories are the models.

In this logic, we would have a "binomial proof":

$$A \oplus A \xrightarrow{sum} A \xrightarrow{copy^n} A^{\oplus n} \xrightarrow{\psi^n} A^{\otimes n} \xrightarrow{r_n} S_n A$$

which does that:

$$(x, y) \mapsto x + y \mapsto (x + y, ..., x + y) \mapsto (x + y)^{\otimes n} \mapsto (x + y)^{\otimes_s n}$$

It should be equivalent (by cut elimination/rewriting) to the other binomial proof:

$$(x, y) \mapsto \sum_{0 \leq k \leq n} \binom{n}{k} x^{\otimes_s k} \otimes_s y^{\otimes_s (n-k)}$$

Cartesian left additive categories and biadditve maps

A left additive category is a **CMon**-category $\mathcal{C}$ such that morphisms are left additive ie. $f; (g+h) = (f; g) + (f; h)$ and $f; 0 = 0$.

Additive morphisms are then morphisms f such that $(g+h); f = (g; f) + (h; f)$ and $0; f = 0$.

Proposition: Additive morphisms form a wide subcategory $\mathcal{C}_+$ which is also a **CMon**-category.

A cartesian left additive category is a left additive category with binary products, such that π_1, π_2, Δ are additive and $f \times g$ is additive whenever f, g are additive.

In a cartesian left additive category, we define a biadditive map $A \times B \to C$ as a map $f : A \times B \to C$ such that:

- $((u_1 + u_2) \times v); f = (u_1 \times v); f + (u_2 \times v); f$
- $(u \times (v_1 + v_2)); f = (u \times v_1); f + (u \times v_2); f$

Proposition:

- If $u : A \to C$ and $v : B \to D$ are additive and $f : C \times D \to E$ is biadditive, then $(u \times v); f$ is biadditive.
- If $f : A \times B \to C$ is biadditive and $u : C \to D$ is additive, then $f; u$ is biadditive.

A **vectorial** category is a cartesian left additive category $\mathcal{C}$, such that $\mathcal{C}_+$ is symmetric monoidal category with bilinear tensor product , together with for every $A, B \in \mathcal{C}$, a biadditive map $\psi_{A,B} : A \times B \to A \otimes B$, such that:

- for every biadditive map $f : A \oplus B \to C$, there exists a unique additive map $\overline{f} : A \otimes B \to C$ such that:

$$\begin{array}{ccc} A \oplus B & \xrightarrow{\psi_{A,B}} & A \otimes B \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & C \end{array}$$

- for every additive maps $u : A \to C$, $v : B \to D$:

$$\begin{array}{ccc} A \times B & \xrightarrow{\psi_{A,B}} & A \otimes B \\ {\scriptstyle u \times v}\downarrow & & \downarrow{\scriptstyle u \otimes v} \\ C \times D & \xrightarrow[\psi_{C,D}]{} & C \otimes D \end{array}$$

  Warning: $\psi_{A.B}$ is neither a natural transformation in $\mathcal{C}$, nor in $\mathcal{C}_+$. We can say it is a natural transformation in $\mathcal{C}$ with respect to $\mathcal{C}_+$.

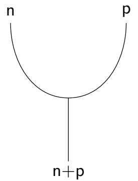
- probably some additional boring conditions.

In a vectorial category $\mathcal{C}$ such that $\mathcal{C}_+$ has the symmetric powers, we should have the binomial theorem verified...

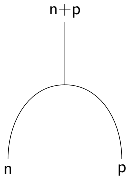
Characterization of symmetric powers: statement ~~and overview of the proof~~

Definition: Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal $\mathbb{Q}^+$-linear category. A **symmetric bialgebra** is a family $(A_n)_{n \geq 0}$ of objects with:

$(\nabla_{n,p} \colon A_n \otimes A_p \to A_{n+p})_{n,p \geq 0}$:



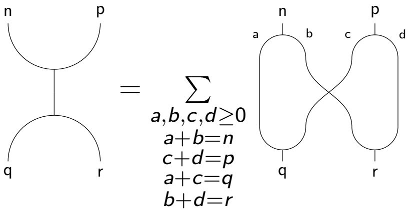
$(\Delta_{n,p} \colon A_{n+p} \to A_n \otimes A_p)_{n,p \geq 0}$:



$\eta \colon I \to A_0$:

$\epsilon \colon A_0 \to I$:

such that:



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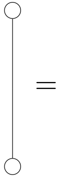

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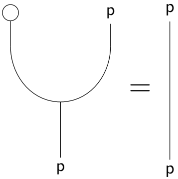

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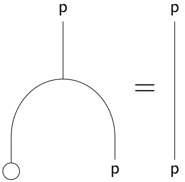

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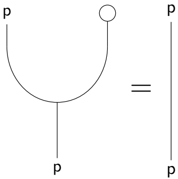
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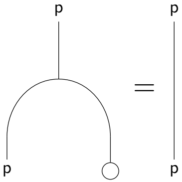
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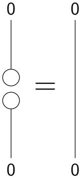
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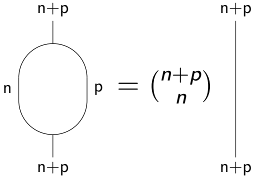
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$\binom{n+p}{n}$
n+p
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Definition: Let $\mathcal{C}$ be a symmetric monoidal $\mathbb{Q}^+$-linear category. Define a **family of symmetric powers** as a family $(A_n)_{n\geq 0} \in \mathcal{C}$ together with morphisms

$$\left(A_1^{\otimes n} \underset{s_n}{\overset{r_n}{\rightleftarrows}} A_n \right)_{n\in\mathbb{N}\backslash\{1\}}$$

such that:

$$\forall n \in \mathbb{N}\backslash\{1\} \quad \begin{cases} r_n; s_n & = \frac{1}{n!} \sum\limits_{\sigma\in\mathfrak{S}_n} \sigma \\ s_n; r_n & = Id \end{cases}$$

Theorem:

1) If we have a symmetric bialgebra with the preceding notations. If we define for $n \geq 2$, by induction ,

$$\nabla^n : A_1^{\otimes n} \to A_n$$

$$\Delta^n : A_n \to A_1^{\otimes n}$$

like this:

$$\nabla^2 := \nabla \qquad \nabla^{n+1} := (\nabla^n \otimes Id); \nabla_{n,1}$$

$$\Delta^2 := \Delta \qquad \nabla^{n+1} := \Delta_{n,1}; (\Delta^n \otimes Id)$$

and

$$\nabla^0 := \eta : I \to A_0$$

$$\Delta^0 := \epsilon : A_0 \to I$$

we obtain that $(A_n)_{n \geq 0}$ together with

$$\left(\begin{array}{ccc} A_1^{\otimes n} & \underset{\Delta^n}{\overset{\nabla^n}{\rightleftarrows}} & A_n \end{array} \right)_{n \in \mathbb{N} \setminus \{1\}}$$

is a family of symmetric powers.

2) If we have a family of symmetric powers with the preceding notations. If we define

$$\nabla^*_{n,p} = s_n \otimes s_p ; r_{n+p}$$

$$\Delta^*_{n,p} = \binom{n+p}{n} . s_{n+p} ; r_n \otimes r_p$$

$$\eta^* = r_0$$

$$\epsilon^* = s_0$$

Then $(A_n)_{n \geq 0}$ together with $\eta^*, \epsilon^*, (\nabla^*_{n,p})_{n,p \geq 0}$, $(\Delta^*_{n,p})_{n,p \geq 0}$ is a symmetric bialgebra.

3) Given a family $(A_n)_{n\geq 0}$ of objects, the two preceding transformations give a bijection between the sets of morphisms which define a structure of symmetric bialgebra and the sets of morphisms which define a structure of family of symmetric powers.

Proof:

The proof is about showing that the combinatorics of "paths with fixed flow" is equivalent to the combinatorics of symmetrization.

It is really interesting but quite long. And I'm still trying to really finish it and to polish it.

Maybe I could talk of the proof another day :) because it seems to be a technique applicable to a lot of situations (I talk of that in a minute), so it's useful to make it crystal clear.

Towards further characterizations: Schur functors, $(A^{\otimes n})^{G_n} = (A^{\otimes n})_{G_n}$, Cyclic homology, Positive characteristic...

Theorem: Given a subgroup $G_n \leq \mathfrak{S}_n$ and an object $A \in \mathcal{C}$ a symmetric monoidal $\mathbb{Q}^+$-linear category, these morphisms are in bijection:

- a limit $(A^{\otimes n})^{G_n} \xrightarrow{s_n} A^{\otimes n}$ of this diagram:

$$A^{\otimes n} \underset{\dots}{\overset{g}{\rightrightarrows}} A^{\otimes n}$$

- a colimit $A^{\otimes n} \xrightarrow{r_n} (A^{\otimes n})_{G_n}$ of this diagram:

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- a splitting $A^{\otimes n} \xrightarrow{r_n} B \xrightarrow{s_n} A^{\otimes n}$ of this idempotent:

$$\frac{1}{|G_n|} \sum_{g \in G_n} g : A^{\otimes n} \to A^{\otimes n}$$

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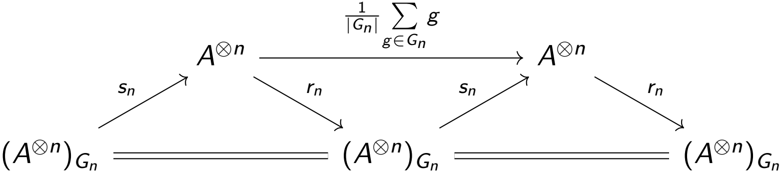
We can then resume the useful equations by this diagram:



And define a "unit", a "counit", "multiplications" and "comultiplications":

$$((A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \xrightarrow{\nabla_{n,p}} (A^{\otimes n+p})_{G_{n+p}})_{n,p\geq 0}$$

$$((A^{\otimes n+p})_{G_n} \xrightarrow{\Delta_{n,p}} (A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p})_{n,p\geq 0}$$

$$I \xrightarrow{\eta} (A^{\otimes 0})_{G_0}$$

$$(A^{\otimes 0})_{G_o} \xrightarrow{\epsilon} I$$

by:

$$\begin{array}{ccc} (A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} & \xrightarrow{s_n \otimes s_p} & A^{\otimes(n+p)} \\ \nabla_{n,p} := \downarrow & \swarrow_{r_{n+p}} & \\ (A^{\otimes(n+p)})_{G_{n+p}} & & \end{array}$$

$$\begin{array}{ccc} (A^{\otimes(n+p)})_{G_{n+p}} & \xrightarrow{\frac{|G_{n+p}|}{|G_n|.|G_p|} s_{n+p}} & A^{\otimes(n+p)} \\ \Delta_{n,p} := \downarrow & \swarrow_{r_n \otimes r_p} & \\ (A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} & & \end{array}$$

$$I \xrightarrow{\eta := r_0} (A^{\otimes 0})_{G_0}$$

$$(A^{\otimes 0})_{G_0} \xrightarrow{\epsilon := s_0} I$$

We then get the left/right unitality, left/right counitality, $\eta;\epsilon = Id_I$ and $\epsilon;\eta = Id_{(A^{\otimes 0})_{G_0}}$.

But we only get:

$$
\begin{array}{c}
(A^{\otimes(n+p)})_{G_{n+p}} \\
\Delta_{n,p}\big\downarrow \\
(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \\
\nabla_{n,p}\big\downarrow \\
(A^{\otimes(n+p)})_{G_{n+p}}
\end{array}
= \frac{|G_{n+p}|}{|G_n|^2.|G_p|^2} \sum_{\substack{g\in G_n \\ h\in G_p}}
\begin{array}{c}
(A^{\otimes(n+p)})_{G_{n+p}} \\
s_{n+p}\big\downarrow \\
A^{\otimes(n+p)} \\
g\otimes h\big\downarrow \\
A^{\otimes(n+p)} \\
r_{n+p}\big\downarrow \\
(A^{\otimes(n+p)})_{G_{n+p}}
\end{array}
$$

and:

$$\begin{array}{c}(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \\ \Big\downarrow \nabla_{n,p} \\ (A^{\otimes (n+p)})_{G_{n+p}} \\ \Big\downarrow \Delta_{q,r} \\ (A^{\otimes q})_{G_q} \otimes (A^{\otimes r})_{G_r}\end{array} = \frac{1}{|G_q|.|G_r|} \sum_{g \in G_{n+p}} \begin{array}{c}(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \\ \Big\downarrow s_n \otimes s_p \\ A^{\otimes (n+p)} \\ \Big\downarrow g \\ A^{\otimes (n+p)} \\ \Big\downarrow r_q \otimes r_r \\ (A^{\otimes q})_{G_q} \otimes (A^{\otimes r})_{G_r}\end{array}$$

- By putting $G = \mathbb{Z}/n\mathbb{Z}$, we obtain $(A^{\otimes n})_{\mathbb{Z}/n\mathbb{Z}} =$ "cyclic n^{th} tensor power of A" ie. the set spanned by vectors of the form:

$$\begin{aligned} x_1 \otimes_{\mathbb{Z}_n} \dots \otimes_{\mathbb{Z}_n} x_n &= x_n \otimes_{\mathbb{Z}_n} x_1 \otimes_{\mathbb{Z}_n} \dots \otimes_{\mathbb{Z}_n} x_{n-1} \\ &= x_{n-1} \otimes_{\mathbb{Z}_n} x_n \otimes_{\mathbb{Z}_n} x_1 \otimes_{\mathbb{Z}_n} \dots \otimes_{\mathbb{Z}_n} x_{n-2} \\ &\dots \\ &= x_2 \otimes_{\mathbb{Z}_n} \dots \otimes_{\mathbb{Z}_n} x_n \otimes_{\mathbb{Z}_n} x_1 \end{aligned}$$

- Exterior powers are different because we need a symmetric monoidal $\mathbb{Q}$-linear category and we must put signs. We look at splitting of the idempotents:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma)\sigma : A^{\otimes n} \to A^{\otimes n}$$

- Symmetric powers and exterior powers are example of Schur functors which can be defined in any symmetric monoidal $\mathbb{Q}$-linear category as a functor $S_\lambda : \mathcal{C} \to \mathcal{C}$ such that $S_\lambda A$ is the intermediate object in the splitting of some idempotent

$$e_\lambda : A^{\otimes n} \to A^{\otimes n}$$

for every partition $\lambda \vdash n$

- In every symmetric monoidal $\mathbb{Q}$-linear category, we can look at the n^{th} object of the cyclic homology complex of an object $A^{\otimes n}$. It is the set spanned by vectors of the form:

$$\begin{aligned} x_1 \otimes^a_{\mathbb{Z}_n} \dots \otimes^a_{\mathbb{Z}_n} x_n &= (-1)^{n-1} x_n \otimes^a_{\mathbb{Z}_n} x_1 \otimes^a_{\mathbb{Z}_n} \dots \otimes^a_{\mathbb{Z}_n} x_{n-1} \\ &= (-1)^{n-1} x_{n-1} \otimes^a_{\mathbb{Z}_n} x_n \otimes^a_{\mathbb{Z}_n} x_1 \otimes^a_{\mathbb{Z}_n} \dots \otimes^a_{\mathbb{Z}_n} x_{n-2} \\ &\dots \\ &= (-1)^{n-1} x_2 \otimes^a_{\mathbb{Z}_n} \dots \otimes^a_{\mathbb{Z}_n} x_n \otimes^a_{\mathbb{Z}_n} x_1 \end{aligned}$$

- In a symmetric monoidal **CMon**-category (ie. possibly in positive characteristic), we no longer have the previous equivalence between equalizer, coequalizer and split idempotents. Hence, symmetric powers divide into symmetric powers:

$$A^{\otimes n} \underset{\dots}{\overset{\sigma}{\rightrightarrows}} A^{\otimes n} \xrightarrow{s_n} S_n A$$

and divided powers:

$$\Gamma_n A \xrightarrow{r^n} A^{\otimes n} \underset{\dots}{\overset{\sigma}{\rightrightarrows}} A^{\otimes n}$$

therefore, the multiplications and comultiplications would be of the form:

$$\Gamma_{n+p} A \xrightarrow{\nabla_{n,p}} S_n A \otimes S_p A$$

$$\Gamma_n A \otimes \Gamma_p A \xrightarrow{\Delta_{n,p}} S_{n+p} A$$

at first sight...

Because we also get (non-idempotent) splittings:

$$\overset{\sum_{\sigma\in\mathfrak{S}_n}\sigma}{\overbrace{A^{\otimes n} \xrightarrow[r_n]{} S_nA \xrightarrow[s_n]{} A^{\otimes n}}} \qquad \overset{\sum_{\sigma\in\mathfrak{S}_n}\sigma}{\overbrace{A^{\otimes n} \xrightarrow[r^n]{} \Gamma_nA \xrightarrow[s^n]{} A^{\otimes n}}}$$

And thus, we have in fact all this stuff:

$$\begin{array}{ccccc}
\Gamma_nA \otimes \Gamma_pA & & & & \\
\uparrow\downarrow & & & & \\
\Gamma_{n+p}A & \rightleftarrows & A^{\otimes(n+p)} & \rightleftarrows & S_{n+p}A \\
 & & & & \downarrow\uparrow \\
 & & & & S_nA \otimes S_pA
\end{array}$$

We should have two combined graded bialgebras $(S_nA) \leftrightarrow (\Gamma_nA)$ and also

$$!A = SA = \bigoplus_{n\geq 0} S_nA \leftrightarrow TA = \bigoplus_{n\geq 0} A^{\otimes n} \leftrightarrow \bigoplus_{n\geq 0} \Gamma_nA = \Gamma A = ?A$$

Compare to what we get with the language of differential linear logic:

$$!A = SA = \bigoplus_{n\geq 0} S_nA \leftrightarrow S_1A \cong A \cong \Gamma_1A \leftrightarrow \bigoplus_{n\geq 0} \Gamma_nA = \Gamma A = ?A$$

- In the same way, in a symmetric monoidal **Ab**-category $\mathcal{C}$, Schur functors would divide into Schur functors $S_\lambda : \mathcal{C} \to \mathcal{C}$ (eg. symmetric powers) and Weyl or co-Schur functors $S^\lambda : \mathcal{C} \to \mathcal{C}$ (eg. divided powers).
- I've also seen things like skew Schur functors $S_{\lambda,\mu}$...
- And we can maybe look at more complicated groups than $\mathfrak{S}_n$ or $\mathbb{Z}_n$ acting on $A^{\otimes n}$...

There is (a lot of) work to do!