Inter-Categorical Subobject Analysis

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Abstract

We introduce categorical definitions of smallness and density which generalize the instantiations of these notions in many mathematical areas, as well as derived definitions of reducedness and satedness and auxiliary notions of tablets and scales. The resulting theory is distinguished in that it is *inter-categorical* in character: its definitions can not just be applied to smallness and density of subobjects in any category and classify morphisms by how they handle them, but also to compare categories based on their measurements and functors and natural transformations by how they transform these. We provide examples of how these notions instantiate in several areas of mathematics and outline an intricate broken symmetry in the unfolding of them throughout category theory. We then apply our results to generalize topological open/closed duality and quantity based on measures. Central to this interaction is the subobject 2-functor and the 2-category of upward-bounded preorders. We close with a representation theorem for this 2-functor and 2-category and a conjecture about how to generalize this theorem to n-categories.

Contents

1 Introduction

We ask the reader to picture a simple shape, say, a cone $\boldsymbol{C}$. "The same" cone is an object in quite a lot of categories: it is a combinatorial structure, a continuous one, a topological one, an algebro-geometric one, and many more, but it also is a subobject of $\boldsymbol{R}^3$, which can have, say, a linear structure that casts its own shadows onto its subsets. Picture, say, the subobjects of $\boldsymbol{C}$ in each kind of category, or the equivalences of $\boldsymbol{C}$ in its topological, smooth, and combinatorial incarnations, or how the different embeddings of $\boldsymbol{C}$ cast shadows through outside shapes, such as the sections of $\boldsymbol{C}$ by planes in $\boldsymbol{R}^3$, or perhaps even how $\boldsymbol{C}$ degenerates into itself, like a film passing by, illuminating $\boldsymbol{C}$ in a different color and shade at each moment.

> [Reason] is often taken ... [as] something separate from positive understanding. But in truth reason is spirit... It is the negative which characterizes the quality of both dialectical reason and understanding: it negates the simple, and thus sets the difference of understanding, but resolves it just as much, thus it is dialectic. It does however not remain in the nothingness of this result but restores the first simple through it, but as something general, which is in itself concrete; under this is not something particular subsumed, but in this determination and its resolution has the particular already determined itself.

- G. W. F. Hegel, [17], personally shortened and translated.[1]

So what we are after is a kind of re-connection of what appears to be very separate, and a kind of abstraction that is different from that which is more commonplace in category theory; where we put the focus onto the object again and how it externalizes itself in different categories. We do this using a different and oft-used but somewhat underappreciated fact: the subobjects of each object in each category form a preorder, and not just any preorder but one with an upper bound, so even though all these objects live in different categories and are thus of different types, they have something in common, which is a subobject preorder. In other words, they are all *wholes* consisting of *parts*.

What can we do with this? Surprisingly much. Consider the most elementary notion that drove this investigation, that of *smallness*. What does it mean for a part to be small? The answer we provide is so simple it could already have been stated by Aristotle:

> That part is small which cannot be made whole by a proper part.

or, in more succinct modern language:

$$x \text{ small} :\Leftrightarrow \forall y : (x \vee y = \top \Rightarrow y = \top).$$

In the lattice of closeds of a topological space this is the case if and only if x is a nowhere dense subset, but our definition can be applied in any preorder with a top element $\top$ - an upward-bounded preorder. So even though $\mathscr{C}$ and $\mathscr{D}$ might be categories housing objects of a wholly different type, their monic parts are all mapped into the category of upward-bounded preorders and order-(not necessarily top-)preserving morphisms **UBP**. We will show that this category, which has so far gone unnamed to the best knowledge of the author, is an object of fundamental importance in mathematics.

This overall system of smallness, among other things, generalizes:

1. the topological notion of nowhere density (see Example 3.17),
2. lower-dimensionality of manifolds with boundary (see Example 3.18),

3. measure-theoretic negligibility (up to a small difference, see Example 4.12),
4. a variant of Frattini theory (see Section 3.1.4),
5. finitude (see in particular Section 3.4),
6. nonstandard infinitesimality (see Example 4.25),
7. and provides a sensible notion of smallness for schemes and semialgebraic varieties (see Example 3.43 and Example 3.45).

From our notion of smallness we can derive one of reducedness, which generalizes regularity in a way that can also be applied to any object, and which we will use to decompose objects of suitably closed categories into their small and reduced parts in later work. But we can also dualize smallness to obtain a direct generalization of topological *density* that is applicable in every downward-bounded preorder, so in particular in the subobject preorder of any object in a category with an initial object - a notion slightly less general but just as rich. Density itself has a notion analogous to reducedness called *satedness*, which generalizes a topological relative version of Cauchy completeness. These are the four basic notions whose incarnations we study throughout mathematics. We will also introduce a system of *tablets*, which generalizes how density is inferred for arbitrary subsets of a topological space based on its open or closed subsets, and a *scale*, which generalizes measure-theoretic notions of smallness.

This opens up another perspective on what we are doing as a reconsideration of some fundamental topological notions to get at *what they really mean*. We might summarize this in a slogan similar to that of pointless topology: *topology without opens*, at least initially. But since opens are the distinguishing feature of topology, removing them universalizes a notion so it can be applied *everywhere*.

Smallness and reducedness are abstractly dual to density and satedness on an order-theoretic level. It is perhaps the most aesthetically pleasing aspect of the current theory that, due to interaction between the conditions necessary for the preservation of smallness/density and the particularities of the subobject functor, this abstract duality is broken as soon as the category-theoretic context is considered, so that overall the preservation and reflection properties of four notions have to be considered individually in both a co- and contravariant context. The detailed description of the intricate unfolding of this broken duality under increasing categorical structure in Section 6 forms the heart of this text and the obtained results show that the notions involved behave indeed as we would expect smallness, density, reducedness and satedness to behave.

We then trace the development of the involved notions if we assume, in increasing specificity:

1. initial objects,
2. pullbacks,
3. a regular category,
4. and a coherent category.

Along the way we will introduce two basic kinds of notions that starkly exemplify the difference between smallness and density: the *smallness fibration*, which exists on every object in every category, and the *density coverage*, which exists on every regular category with a strict initial object. Smallness is strictly decreasing, so it gives rise to a means of stratifying the subobject preorder of an object into objects of the same size, whereas density by its nature expresses something that on one level varies but on another remains constant and thus forms a coverage.

So density and smallness are abstractly dual but their behavior in categories is very different. But in a topological space, one notion is inferred from the opens and one from the closeds, and all notions are dual again. We take this as a defining point for the duality of opens and closeds, which corresponds to the bifurcation of coherent categories into Heyting and co-Heyting categories. This naturally leads us to a different understanding of topology as based on a categorical duality, which in turn suggests a generalization of topology that is essentially finitistic in character. Here, our theory interfaces with the vision of a *tame topology* Grothendieck outlined in one of his undervalued later works, *Esquisse d'un Programme.*[15] We discuss this in Section 7.1.

We provide a second application by sketching out a framework for *categorical measure theory* in Section 7.2. Here we do *not* take the more direct route of defining some particular category of measure spaces, but rather use the subobject functor $\mathbf{SOb}_{\mathscr{C}}$ of a category $\mathscr{C}$ to define measures *on* categories as natural transformations of submonic hyperdoctrines. This corresponds to the "scales" of our tablets-and-scales system. We can combine these scales with the tablets from an open-closed duality to define a generalization of Borel measures, which can be extended from the opens or closeds of an open-closed duality in a way that is purely finitistic and thereby circumvents all of the difficulties of the Borel construction - but also loses the connection to σ-algebras, which are often taken as the basis of measure theory. We outline two remedies for this, the more experimental of which is to transition to a nonstandard setting and use Loeb measures instead of Lebesgue measures.

Crucially, placing our notions on a very low level not merely allows us to use them to study, say, smallness in every object in every category, or to study morphisms in terms of whether they preserve it, but also to study *categories and functors* and perhaps even natural transformations in terms of what notions of size they allow and how they preserve it, and then even 2-categorical notions in terms of the classes of categories they give rise to. This is due to the particularities of **UBP** and the $\mathbf{SOb}_{\mathscr{C}}$. So the final point we make is that these occupy a very peculiar place in the multiverse of n-categories: in any n-category we can obtain some notion of subobjects (perhaps many!) that forms a preorder through some truncation or factorization condition. Not just that, but monomorphism-preserving functors and monic natural transformations cast shadows into **UBP**, in a way that some things are forgotten while others are cast into even sharper relief - like any good projection, with the crucial difference that *both the internal structure in each category and the external structure between categories are on roughly the same levelling* - a rather unique feature. Thus, any definition or result obtained in **UBP** reverberates throughout the multiverse of categories in a prismatic fashion. The only real point of comparison for this are concrete categories, but:

- We do not need to assume any additional structure, the subobject functor $\mathbf{SOb}_{\mathscr{C}}$ exists on (the monic part of) every category.
- Upward-bounded preorders have a lot more structure than sets.

Thus, we close with a thorough investigation of the place of these two notions, which is the most technically challenging part of the paper. Through this investigation we find that the pair $(\mathbf{UBP}, \mathbf{SOb})$ can be characterized by a representation property which only really starts appearing at the 2-categorical level. The precise nature of this property is slightly hard to explain without a more thorough setup, but an attempt shall be made:

The 2-category **Cat** has a property somewhat akin to closure under taking powers: for each category $\mathscr{C}$ there exists a 2-functor $\mathscr{C}_{/(-)} : \mathscr{C} \to \mathbf{Cat}$ which maps $X : \mathscr{C}$ to $\mathscr{C}_{/X}$ and $f : X \to Y$ to the functor f_* which acts through postcomposition, so a kind of representation of $\mathscr{C}$ in **Cat**. These representations interact well with functors and natural transformations and thus lift to one **Cat**-parameterized family of representations of categories in **Cat**, and through that to a representation of **Cat** in some 2-category $\mathbf{Cat}_{\blacklozenge\mathbf{Cat}}$ of **Cat**-valued representations of objects in **Cat** (we pronounce this construction "diamond-over") and suitable morphisms and 2-morphisms between them. It is not that strange that the 2-category of categories should have such a reflection-property, but if $\mathscr{C}$ is monic, then the functor $\mathscr{C}_{/(-)}$ factors through the **UBP**-valued functor **SOb**.

Moreover, if $\mathscr{A}$ is a sub-2-category of **Cat** consisting of monic categories, then it itself parameterizes a family of representations of objects of $\mathscr{A}$ in **UBP** with natural transformations between them, meaning that there exists a 2-functor $\mathbf{SOb}_{(-)} : \mathscr{A} \to \mathscr{A}_{\blacklozenge\mathbf{UBP}}$ which assigns to each $\mathscr{C} : \mathscr{A}$ the functor $\mathbf{SOb}_{\mathscr{C}} : \mathscr{C} \to \mathbf{UBP}$. Now, note for one that $\mathbf{SOb}_{\mathscr{C}}$ takes values in the monic part $\overset{\hookrightarrow}{\mathbf{UBP}}$ of **UBP** and if fact maps each monomorphism $\iota : A \hookrightarrow X$ in $\mathscr{C}$ to an ideal embedding $\iota_* : \mathbf{SOb}(A) \hookrightarrow \mathbf{SOb}(X)$, which is principal because $\mathbf{SOb}(A)$ has a top $\mathbf{id}_A$ and so designates an element $\iota := \iota_*(\top)$ of $\mathbf{SOb}(X)$. So in $\mathbf{SOb}(X)$ the images of subobjects of X are as densely packed as possible without identifying any of them. If now $F : \mathscr{C} \to \mathbf{UBP}$ is some other representation of $\mathscr{C}$, then this maps ι to some other morphism $F(\iota)$, which also designates an element $F(\iota)(\top_{F(A)})$ of $F(X)$, so that we can define a morphism $F_{\bigstar,X} : \mathbf{SOb}(X) \to F(X) : \iota \mapsto F(\iota)(\top_{F(A)})$. Thus, $\mathbf{SOb}_{\mathscr{C}}$ is the initial representation of $\mathscr{C}$ in **UBP**. Taking this together we find that $\mathbf{SOb}_{(-)}$ is adjoint to the 2-functor $\mathbf{s} : \mathscr{A}_{\blacklozenge\mathbf{UBP}}$, which assigns to each functor $F : \mathscr{C} \to \mathbf{UBP}$ its source, and so is an initial reflection of $\mathscr{A}$ in $\mathscr{A}_{\blacklozenge\mathbf{UBP}}$.

Proving this is the easy part, the hard part is to show that **UBP**, or rather its monic part $\overset{\hookrightarrow}{\mathbf{UBP}}$, is actually universal as a 2-category allowing for this kind of reflection property, which we do in Section 8.3. What makes our proof particularly interesting is that it is based on an instance of the microcosm principle: what we use is that, if there is another 2-category $\mathscr{B}$ that can play the same role as **UBP** and is monic enough, then we can *also* define a 2-functor $\mathbf{SOb}_{\mathscr{B}} : \mathscr{B} \to \mathbf{UBP}$ that we can use to get back to **UBP**. For this, we adopt the notion of a pseudomonic and show that these do indeed for each object X of a 2-category form suitable preorders, which allow us to transition back to **UBP**. What this highlights is the "multiversal" position of **UBP**: regardless of where on the categorical ladder we are, if we fix enough structure we can find that the subobjects of an object always define a preorder with a maximal element and so hearken back to **UBP**: in fact, they will do so in increasingly many forms as the ways we can count subobjects increase. We close the paper with a conjecture that this is because **UBP** is the first of a tower of $(n+1)$-categories we can at each level of the categorical ladder provide a common home for the monic parts of n-categories.

2 Preamble

2.1 Acknowledgements

I would like to thank my advisor Tobias Fritz for his general support as well as numerous suggestions and observations, but above all for giving me the freedom to pursue the mathematics I wanted and being patient when I had to set up my writing environment. I would also like to thank my parents.

This paper was written with the help of various LLMs and the seams of their contributions might readily be visible to the keen observer. Their main contributions were

- writing out examples,
- proving pre-defined statements, often from a proof outline,
- proof checking.

2.2 Prerequisites

The prerequisites of this work vary widely:

1. All of Section 3 can be understood with only minimal category-theoretic knowledge, *except for* the examples in Section 3.1, which are grouped by area and some are more difficult to understand than the surrounding text. We provide the relevant definitions and try to introduce the context for each of these areas, but some of the significance might be lost without prior understanding of that area. These examples are built upon in the subsequent section but each "thread" of examples is largely self-contained and so can in principle be ignored, except for the category-theoretic examples, which re-appear in Section 8.3. Section 4 is largely the same.
2. Section 5 contains some tedious proofs but does not require anything but elementary order-theory, but is largely setup for Section 6 and Section 7, which don't strictly require but certainly benefit from a solid category-theoretic background. The beginning (up to the preservation and reflection statements) is also important for Section 8.
3. Section 8, and Section 8.3 in particular, is by far the hardest part of the text and requires at least an elementary familiarity with (strict) 2-categories. With that said, the difficulty of Section 8 lies not so much in its prerequisites and more in the cumulative complexity of the involved 2-cells.

2.3 Notation

Throughout this article we adhere to the following notational conventions:

- **Categories and objects.** We use calligraphic font for variables of a higher-categorical level and bold font for particular categories and functors. So we denote an arbitrary category $\mathscr{C}$ but write $\mathbf{PrO}$ for the 2-category of preorders and $\mathbf{P}$ for the powerset functor. An object X belonging to a category $\mathscr{C}$ is written $X : \mathscr{C}$ (type-theoretic style). In preorders P we generally write $X \in P$, which allows us to keep preorders notationally apart from outside categories. The category of small categories is $\mathbf{Cat}$.
- **Morphisms and composition.** A morphism f from X to Y is written $f : X \to Y$. Composition is always written in diagrammatic order: $f \mathbin{;} g$ denotes "f then g". If we vertically compose 2-cells, we use the unadorned semicolon: $\varphi ; \psi$. Identities are $\mathbf{id}_X$ or $\top_X$ if we reason internally to the subobject preorder of X.
- **Subobjects.** We denote subobjects as $\iota : A \hookrightarrow X$ and $\kappa, \lambda \ldots$. For a category $\mathscr{C}$, $\mathbf{SOb}_{\mathscr{C}}(X)$ denotes the preorder of monomorphisms into X (subobjects of X). When $\mathscr{C}$ is clear we simply write $\mathbf{SOb}(X)$.
- **Preorders and lattices.** A preorder is a category with at most one arrow between any two objects. The top element of a bounded preorder is written $\top$ (or $\top_P$ to stress the preorder), the bottom $\bot$ (or $\bot_P$). Joins are $\vee$, meets $\wedge$. If we are taking unions and intersections of sets or set-like subobjects we write $\cup$ and $\cap$. $\mathbf{UBP}$ is the 2-category of upward-bounded preorders and all monotone maps; $\mathbf{UBP}_w$ is its wide sub-2-category of top-preserving maps. Dually, $\mathbf{DBP}$ denotes the 2-category of downward-bounded preorders. The image of a preorder top $\top_P$ under a morphism $f : P \to Q$ is written as $\top_f$ as described in Section 5.
- **Miscellaneous.** The Yoneda embedding of a category $\mathscr{C}$ is $\widehat{(-)} : \mathscr{C} \hookrightarrow \widehat{\mathscr{C}}$. The set-theoretic powerset of a set X is $\mathbf{P}(X)$. The subobject classifier in $\mathbf{Set}$ is $\mathbb{T} = \{\top, \bot\}$ with $\top$ as "true". We denote the underlying function of a morphism f as $|f|$.

All other notation is either standard or introduced locally and explained at its first occurrence.

2.4 Categorical Assumptions

We are using the notion of a preorder as our starting point. In Section 8 we need to consider 2-categories other than **UBP** as well, which raises the question of weakness. Our considerations do not use unitors and associators at all, so that our proofs would be needlessly complicated if we used an explicit weak foundation, i.e. what are called bicategories, but if we were to strictify all 2-categories in question, this would create a disconnect with our use of preorders, which are only strict up to homotopy. The way we solve this is to implicitly assume univalent foundations,[1] so that a "2-category" is already strict only up to homotopy. Consequently, we use equality signs for considerations in preorders, not equivalence signs. It should be underlined that this does not matter in any way to our arguments, it merely simplifies their exposition.

2.5 Duality

The results from Section 3 up to Section 6 have pure dualizations. But in light of Section 6 some proficiency has to be built up with thinking on either side of the duality; to use Grothendieck's vocabulary, to do a yoga it does not suffice to know that the asanas can be dualized, one actually has to do the other side. To deal with this, we provide lots of examples, do some proofs on either side and provide explicit dualizations of the preservation and reflection results in Section 5.

3 Smallness & Density

In a general preorder, joins and meets don't always exist. But just as not all limits in a category need to exist so that we can talk about them, so too can we still use the notions of joins and meets in general preorders. So we don't assume they exist, but do assume that they fulfill their universal property if they do.

Remark 3.1. Note how this is similar to several category-theoretical notions, e.g. separated presheaves. On a deeper level, the elements of a monoid are only connected through the monoidal product, so if we make this operation partial, this results in a big loss. But the elements of a preorder have a categorical structure that partial operations can be described from through universal properties. This is why it is not particularly problematic to state our definitions without existence assumptions.

Definition 3.2. A preorder P is *upward-bounded* if it has a largest element $\top$. P is *downward-bounded* if it contains a smallest element $\bot$. P is *bounded* if it is upward- and downward-bounded.

Let us consider some elementary notions in this setting:

Definition 3.3. Given an upward-bounded preorder P, an element $a \in P$ is *small* if the only b such that $a \vee b = \top$ is $\top$. Dually, an element d in a downward-bounded preorder P is *dense* if the only c such that $c \wedge d = \bot$ is $\bot$.

Remark 3.4. This notion of a dense element is a generalization of a notion in lattice theory (see e.g. [19]), where we use partially defined joins instead of a globally defined pseudocomplement. That notion in turn generalizes that of density in topological spaces. But this is not why we use the name *density* instead

[1] See [31] for an introduction on univalent foundations and homotopy type theory more generally.

of the formal dual, *largeness*: in Section 6.1 we will see that the formal duality of smallness and density breaks down once the interaction with category theory is considered and the non-dual behavior of density provides a justification for its name that would not be visible in the topological, the lattice-theoretic or order-theoretic contexts.

This notion also generalizes essential modules in algebra. In particular, the definition of an essential element as given in [16] is essentially the same definition as that of a dense element, but restricted to a modular lattice.

Example 3.5. In each bounded preorder, $\top$ is dense and $\bot$ is small.

Terminology 3.6. If $a \vee b = \top$ we say that "a joins with b". More generally, if $a \vee b = c$ we say that "a joins with b to c". Dually for meets.

Proposition 3.7. *The small objects of an upward-bounded preorder are stable under downward closure and binary unions, i.e. an ideal in the lattice-theoretic sense.*

Proof. Given $a \leq b$ with b small and $c \in P$ such that $a \vee c = \top$, then by monotonicity of joins $b \vee c = \top$, so that $c = \top$ by smallness.

Now let a, b be small, c such that $(a \vee b) \vee c = \top$ and d be some common upper bound of b and c (one exists because $\top$ is one). Then each common bound e of a and d is $\geq (a \vee b) \vee c = \top$ and thus equal to $\top$, so that $a \vee d$ exists and is $\top$. So by smallness of a, $d = \top$, so $b \vee c = \top$, so $c = \top$ by smallness of b. □

3.1 Examples

Let us now begin to study how these notions play out over mathematics using examples. To avoid disturbing the exposition we will provide some proofs in lemmata after the main example, which introduces the context.

The first thing to note is that if there is too much decomposability, size becomes trivial:

Example 3.8. No non-trivial element a of any Boolean algebra is small or dense, since it has a negation $\neg a$ such that $a \vee \neg a = \top$ and $a \wedge \neg a = \bot$. In particular, smallness and density in the subobject lattice of every set are trivial.

So at this point we will not get anything interesting from **Set**. What we need is structure.

3.1.1 Combinatorial

Let us start with some simple cases:

Definition 3.9. An upward-bounded preorder P is *unreachable* if $\top$ is the only element in P that is not small.

Example 3.10. Any total order with a top element is unreachable.

Observation 3.11. Let ⬢ be any of

1. the simplex category $\blacktriangle$, given by the order-preserving functions between finite ordinals,
2. the cube category $\blacksquare$, given by the free strict monoidal category generated by an object $\blacksquare^1$ with two maps $\iota_0, \iota_1 : \blacksquare^0 \rightrightarrows \blacksquare^1$ from the monoidal unit $\blacksquare^0$, (see e.g. [14])

3. the globe category $\CIRCLE$, given by objects $\CIRCLE^n$, $n \in N$ with maps generated by $\sigma, \tau \colon \CIRCLE^n \to \CIRCLE^{n+1}$ subject to the relations $\sigma \mathbin{;} \sigma = \sigma \mathbin{;} \tau$ and $\tau \mathbin{;} \sigma = \tau \mathbin{;} \tau$,
4. the category of finite abstract polytopes and lattice-preserving maps ⬟, i.e.:
   (a) lattices L with a top $\top$ and a bottom $\bot$,
   (b) such that all total sub-orders between $\top$ and $\bot$ have the same length, meaning that there exists a unique ranking function $\mathbf{rank} \colon L \to N$,
   (c) L is *strongly connected*: for any two elements $a, b \in L$ it holds that in the full sub-lattice $[a, b]$ of elements in L between a and b there exists a sequence connecting any two proper faces.
   (d) satisfies the *diamond law*: if $\mathbf{rank}(a) = \mathbf{rank}(b) - 2$ and $a \leq b$, then there exist precisely two elements between a and b.

Then each object of ⬢ has only finitely many subobjects.

All of these are both atomic and coatomic. Remember the definition of atomic orders:

Definition 3.12. An upward-bounded preorder P is *atomic* if for each non-top element x there exists an atom $\alpha \in P$ such that $\alpha \leq x$. Dually P is *coatomic* if for each non-bottom element x there exists a co-atom $\omega \in P$ such that $x \leq \omega$. P is *atomistic* if each element of P is the join of the atoms below it and *coatomistic* dually.

Example 3.13. Each finite preorder P is atomic and coatomic.

Proposition 3.14. *Let P be a coatomic upward-bounded preorder. Then $a \in P$ is small if and only if it is under all co-atoms in P.*

Proof. Assume a is small and let ω be a co-atom. If $a \nleq \omega$, then $\omega \vee a$ is larger than ω and thus $\top$. By smallness, $\omega = \top$, contradicting $\omega < \top$. Hence $a \leq \omega$.

Conversely, suppose a lies below every co-atom and let b be such that $a \vee b = \top$. If $b \neq \top$, there exists a co-atom ω with $b \leq \omega$; then $a \vee b \leq \omega < \top$, a contradiction. Thus $b = \top$ and a is small. □

Corollary 3.15. *Let P be an upward-bounded preorder bounded by exactly one co-atom. Then P is unreachable.*

Proof. Since P has a co-atom ω, there exists an element a below $\top$. for any such a let b be such that $a \vee b = \top$. If $b < \top$ then $a, b \leq \omega$ so that $a \vee b \leq \omega < \top$. Thus, $b = \top$. □

Corollary 3.16. *Let P be atomistic with an atom. Then no non-trivial element of P is dense.*

Proof. This follows immediately from the dual of Proposition 3.14 and $a = \vee_{\alpha \leq a} \alpha$ with each α being an atom. □

So a subobject of any of the objects we introduced in Observation 3.11 is small if and only if it is contained in all faces, which only happens in trivial cases. Let's put a pin in that.

3.1.2 Topological

To see how Definition 3.3 works in a non-trivial setting we can return to topological spaces:

Example 3.17. A closed subset A is small in the lattice (more precisely co-frame) of closeds $\mathbf{C}(X)$ of a topological space X if and only if the only closed subset B such that $A \cup B = X$ is X, so if the closure of its set-theoretic complement is X itself, and thus if it is nowhere dense. Dually, an open subset is dense in our sense if and only if it is dense as a topological subset.

We can also ask for dense elements of $\mathbf{C}(X)$. If X is Hausdorff, this can only happen in trivial cases, but it is common in algebraic geometry. For instance, in the spectrum X of a local ring R with a maximal ideal m, the closed subscheme corresponding to R/m is contained in every other closed subscheme and thus "closed-dense".

A lot of examples we will be looking at are variants of this basic one, but we have to be careful: for one, this notion of size depends on the granularity of our preorder. For instance:

Example 3.18. Let us study smallness in the category **TDisk** whose objects are n-dimensional (closed) disks $\bullet^n$ and whose morphisms are closed continuous functions. The join $\iota \vee \iota' : \bullet^{n'} \hookrightarrow \bullet^n$ of two closed embeddings $\iota : \bullet^m \hookrightarrow \bullet^n, \iota' : \bullet^{m'} \hookrightarrow \bullet^n$ is the smallest n'-dimensional disk in $\bullet^n$ containing both ι and ι', if it exists. It follows from Lemma 3.19 below that the image $\mathbf{Im}(\partial \bullet^{n'})$ of the manifold-boundary of $\bullet^{n'}$ is contained in the set-theoretic union of ι and ι', so in particular, if $\iota \vee \iota' = \top_{\bullet^n}$ then $\partial \bullet^n \subseteq \iota \cup \iota'$, and conversely, $\top_{\bullet^n}$ is the only disk in $\bullet^n$ containing $\partial \bullet^n$. Thus, $\iota \vee \iota' = \top_{\bullet^n}$ if and only if $\iota \cup \iota'$ contains $\partial \bullet^n$.

Given now $\iota : \bullet^m \hookrightarrow \bullet^n$, if $\iota \cap \partial \bullet^n$ is nowhere dense in $\partial \bullet^n$, then every closed subset ι' of $\bullet^n$ such that $\iota \cup \iota' = \partial \bullet^n$ has to contain $\partial \bullet^n$, so that it has to be $\top_{\bullet^n}$. Conversely, if $\iota \cap \partial \bullet^n$ is not nowhere dense in $\partial \bullet^n$, then it contains an $(n-1)$-dimensional disk κ. Then the closure $\overline{\partial \bullet^n \setminus \kappa}$ of the complement of κ in $\partial \bullet^n$ is an $(n-1)$-dimensional disk ι' such that $\iota \vee \iota' = \top_{\bullet^n}$.

So since everything is a disk the size measure becomes one on the boundary.

Lemma 3.19. *Let $\bullet^n \subseteq R^n$ be the closed n-disk, and let $\iota : \bullet^m \hookrightarrow \bullet^n$, $\iota' : \bullet^{m'} \hookrightarrow \bullet^n$ be closed embeddings whose join $\iota \vee \iota' : \bullet^{n'} \hookrightarrow \bullet^n$ exists in $\mathbf{SOb}_{\mathbf{TDisk}}(\bullet^n)$. Then the image of the manifold boundary $\partial \bullet^{n'} \subseteq \bullet^{n'}$ under $\iota \vee \iota'$ is contained in the set-theoretic union of the images of ι and ι'.*

Proof. Write $D := (\iota \vee \iota')(\bullet^{n'})$ as above, so that D is homeomorphic to the closed disk $\bullet^{n'}$ and is a closed subset of $\bullet^n$. Let us denote

$$S := \iota(\bullet^m) \cup \iota'(\bullet^{m'}) \subseteq D.$$

Assume for contradiction that there is a boundary point x of D not in S. Since D is a manifold with boundary, there is a neighborhood basis of x in D by sets homeomorphic to a closed half-ball in $R^{n'}$. We can choose a small closed "cap" neighborhood $C \subseteq D$ such that:

1. $x \in C^\circ$, the interior of C in D;
2. $C \cap S = \emptyset$, i.e. this cap does not meet the union of the two images $\iota(\bullet^m)$ and $\iota'(\bullet^{m'})$. For this note that both $\iota(\bullet^m)$ and $\iota'(\bullet^{m'})$ are images of closed subsets and that ι is a closed map by the closed map lemma, so that S is closed.
3. C is strictly contained in D.

(Concretely: use a coordinate chart φ sending a neighborhood of x in D homeomorphically to a neighborhood of the origin in a closed half-space in $\mathbb{R}^{n'}$, and take a small closed half-ball about the origin disjoint from $\varphi(S)$; this can be done because $\mathbb{R}$ is T_4; pull it back to D).

Now consider the closed subset

$$D' := D \setminus C^\circ \subseteq D \subseteq \bullet^n.$$

Geometrically, we have "cut off" a small boundary cap from the disk D at a point that does not lie in either $\iota(\bullet^m)$ or $\iota'(\bullet^{m'})$. Thus, $S \subseteq D'$ and D' is (up to homeomorphism) still an n'-disk. So we can regard D' as the image of another closed embedding $k\colon \bullet^{n'} \hookrightarrow \bullet^n$. Now, since $S \subseteq D'$, both original images $\iota(\bullet^m)$ and $\iota'(\bullet^{m'})$ lie in D'. Hence both embeddings ι and ι' factor through k as subobjects in **TDisk**, and we obtain a new n'-dimensional disk k of $\bullet^n$ that contains both and is strictly smaller than $\iota \vee \iota'$. This contradicts the assumption that $\iota \vee \iota'$ was the smallest embedded n'-disk containing ι and ι'.

Therefore our assumption that there exists a boundary point of D outside S must be false. Consequently,

$$\partial D = (\iota \vee \iota')(\partial \bullet^{n'}) \subseteq S = \iota(\bullet^m) \cup \iota'(\bullet^{m'}),$$

as claimed. □

Remark 3.20. In [29], Schreiber uses the category $\mathbf{CartSp}_{top}$, whose objects are the Cartesian spaces $\mathbb{R}^n$ and whose morphisms are continuous functions to construct a topological model for cohesion. The objects of **TDisk** are compactifications of the spaces $\mathbb{R}^n$ and the morphisms between them are also continuous functions, which are closed since each closed disk is compact Hausdorff. So **TDisk** is the compact, i.e. universally closed, analogue of $\mathbf{CartSp}_{top}$. **TDisk** will be of importance in later texts due to its site structure. From that perspective, what Example 3.18 shows to us is that there is an inherent conflict between a rich geometry and a manageable site structure, which was known before, albeit in different contexts.

But if the subobject preorder is sufficiently rich, closed and homogeneous, smallness reduces to lower-dimensionality. This is in particular the case for manifolds with boundary. But for this we only need local homeomorphy to a real vector space. Hausdorffness and second-countability are inessential to the question we are studying so we don't assume them:

Definition 3.21 (Topological manifold with boundary)**.** Given $n \in \mathbb{N}$, a *(not necessarily Hausdorff or second-countable) topological n-manifold with boundary* is a topological space M such that for every point $x \in M$ there exists

- an open neighborhood $U \subseteq M$ of x, and
- a homeomorphism (a *chart*) $\varphi\colon U \xrightarrow{\cong} V$ onto an open subset $V \subseteq \mathbb{H}^n$,

where the *closed upper half-space* $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined by

$$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

with the induced topology. The collection of such charts is called an *atlas* for M.

A point $x \in M$ is called an *interior point* of M if there exists a neighborhood U_x of x and a homeomorphism $\varphi\colon U_x \cong \mathbb{R}^n$ and a *boundary point* otherwise.

An *embedding of topological manifolds* $\iota\colon A \hookrightarrow M$ is a regular monomorphism of topological spaces, i.e. a continuous injection such that each open subset of A is the restriction of some open of M. ι is *closed* if its image in M is.

Remark 3.22. The set of all interior points of a manifold with boundary M is denoted M°; it is an open subset of M and carries the structure of a (boundaryless) n-manifold.

Example 3.23. For a manifold with boundary X, write $\mathbf{SOb}_{\partial\mathbf{Mfld}}(X)$ for the preorder of closed embedded submanifolds of X, ordered by inclusion. Then a closed submanifold $\iota\colon A \hookrightarrow X$ is small in $\mathbf{SOb}_{\partial\mathbf{Mfld}}(X)$ precisely when $\mathbf{dim}A < \mathbf{dim}X$.

Proof. Write $n = \mathbf{dim}X$.

If $\mathbf{dim}A < n$*:* We have to show that a closed submanifold of lower dimension is nowhere dense in X. Suppose $B \hookrightarrow X$ is another closed submanifold with $A \vee B = X$. Then the set-theoretic union $A \cup B$ has to consist of all of X. To see this, assume otherwise. Then there exists a $p \in X \setminus A \cup B$. Since X is a manifold, there exists a local homeomorphism φ_p from a neighborhood U of p to D, which is either R^n or $R^n_{\geq 0}$ mapping p to 0. Since A and B are closed, $A \cup B$ is as well, so that $\varphi_p(A \cup B)$ is closed in D and does not contain 0. Since R is regular, there exists a closed disk or half disk $\bullet_p$ around 0 that is disjoint from $\varphi_p(A \cup B)$, so that $\varphi_p^{-1}(\bullet_p)$ is disjoint from $A \cup B$. Assume now that Y is a closed submanifold with boundary containing $A \cup B$ and p. Then $\overline{Y \setminus \bullet_p}$ is also a closed submanifold with boundary that contains $A \cup B$ but does not contain p. Thus, p is outside of $A \vee B$ by the minimality of the join, so no such p can exist.

Now, because A is lower-dimensional, it is nowhere dense in each coordinate chart around its points and thus nowhere dense. Thus, the interior of $A \cup B$ coincides with the interior of B. Hence B° is dense in X. As B is closed, this forces $B = X$. Thus the only subobject joining with A to give X is X itself, so A is small.

If $\mathbf{dim}A = n$*:* Then A is a closed submanifold of full dimension. Choose a closed n-disk $D \subseteq A^\circ$. Then $\overline{X \setminus D}$ is a closed submanifold with boundary and $A \vee \overline{X \setminus D} = X$, showing that A is not small. ◻

So here, unlike in Example 3.18 we have enough regularity to make our definition of smallness local.

3.1.3 Graph-Theoretic

Another decently simple but important example is that of graphs:

Example 3.24. Let G be a (directed) graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathbf{SOb}(G)$ denote the preorder of (directed) subgraphs of G, ordered by inclusion of vertices and edges. This is an upward-bounded preorder with top element $\top_G = G$ and bottom element $\bot_G = \varnothing$ (the empty graph on the empty vertex set). Joins of finite families of subgraphs are given by set-theoretic unions on both vertices and edges:

$$H_1 \vee H_2 \quad = \quad H_1 \cup H_2,$$

and meets are intersections $H_1 \wedge H_2 = H_1 \cap H_2$.

Applying Definition 3.3 to $\mathbf{SOb}(G)$ gives concrete notions of small and dense subgraphs.

1. **Small subgraphs.** A subgraph $H \hookrightarrow G$ is small in $\mathbf{SOb}(G)$ if the only subgraph $K \hookrightarrow G$ with

$$H \vee K \;=\; H \cup K \;=\; G$$

is G. Assume now that H contains no edges or isolated vertices in G. Then, since $H \cup K$ has to contain all edges and isolated vertices in G, K has to contain them as well, so that $K = G$ and H is small. Conversely, if H contains an edge or an isolated vertex a, then $H \cup (G \smallsetminus a) = G$, so that H is not small.

2. **Dense subgraphs.** It follows from the dual of Proposition 3.14 that a subgraph is dense if and only if it contains all vertices of G.

3.1.4 Algebraic

We can give general statements about smallness and density for a whole slew of algebraic theories. For this, we work in a category $\mathscr{C}$ that is monadic over **Set**. Monads provide a categorical framework for universal algebra and the categories of most algebraic structures are monadic. Let us quickly recount the definitions:

Definition 3.25 (Monad)**.** A *monad* on a category $\mathscr{C}$ consists of

- an endofunctor $T : \mathscr{C} \to \mathscr{C}$,
- a natural transformation $\eta : \mathbf{id}_{\mathscr{C}} \Rightarrow T$ (the *unit*), and
- a natural transformation $\mu : T^2 \Rightarrow T$ (the *multiplication*)

such that the following diagrams commute for every $X : \mathscr{C}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathbf{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "\eta_{T(X)}"'] \arrow[dl, "\mathbf{id}_{T(X)}"] \\ & T(X) & \end{tikzcd}$$

Equivalently, a monad is a monoid in the monoidal category $([\mathscr{C}, \mathscr{C}], ;, \mathbf{id}_{\mathscr{C}})$ of endofunctors with composition.

Definition 3.26 (Algebra of a monad)**.** Let (T, η, μ) be a monad on $\mathscr{C}$. A *T-algebra* (or *Eilenberg-Moore algebra*) is a pair (X, ξ) consisting of an object $X : \mathscr{C}$ and a morphism $\xi : T(X) \to X$ (the *structure map*) such that the diagrams

$$\begin{tikzcd} T^2(X) \arrow[r, "T(\xi)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "\xi"] \\ T(X) \arrow[r, "\xi"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathbf{id}_X"'] & T(X) \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

commute. A morphism of T-algebras from (X, ξ) to (Y, θ) is a morphism $f : X \to Y$ in $\mathscr{C}$ satisfying $\xi \mathbin{;} f = T(f) \mathbin{;} \theta$. This forms the *Eilenberg-Moore category* $\mathscr{C}^T$.

Definition 3.27 (Monadic category)**.** A category $\mathscr{A}$ is called *monadic* over $\mathscr{C}$ (or *monadic* with respect to a forgetful functor $U : \mathscr{A} \to \mathscr{C}$) if there exists a monad T on $\mathscr{C}$ and an equivalence of categories $\mathscr{A} \simeq \mathscr{C}^T$ that commutes with the forgetful functors to $\mathscr{C}$. A functor $U : \mathscr{A} \to \mathscr{C}$ is *monadic* if it has a left adjoint F and the comparison functor $\mathscr{A} \to \mathscr{C}^T$ (where T is the induced monad) is an equivalence.

Remark 3.28. The categories of many algebraic structures such as groups, rings, modules over a fixed ring, etc. are monadic over **Set**. $F(S)$ is given by the free algebra generated by S, $U(M)$ by the projection onto the underlying set and an epimorphism $F(S) \to X$ a generalized quotient by some system of algebraic relations between the elements of X. So what monads really describe from our perspective is the idea of generation, and this is what we can use to describe smallness and density.

The most crucial consequence of assuming monadicity for our purposes is that U as a right adjoint creates limits. Hence, monomorphisms are lifted from **Set** and are thus injective functions, so that subobjects are special subsets (those that are algebraically closed). Moreover, the meet in the subobject lattice is computed as the set-theoretic intersection, which is again a subalgebra. Colimits, i.e. joins, are more delicate however: suppose we have two subobjects $\iota \colon A \hookrightarrow M$ and $\kappa \colon B \hookrightarrow M$. Their join $\iota \vee \kappa$ is the smallest subobject of M that contains both A and B. In the category $\mathscr{C}$ this is obtained by taking the image of the canonical map $[\iota, \kappa] \colon A + B \to M$ from the coproduct $A + B$ to M induced by the two inclusions. For $\mathscr{C} = \mathbf{Set}$ the image of the function underlying that map is precisely the union $A \cup B$ inside M. However, $A \cup B$ need not be a subalgebra; the categorical image equips the union with the algebraic structure it inherits from M - that is, it takes the subalgebra generated by the union. Hence:

$$\iota \vee \kappa \;=\; \langle A \cup B \rangle \hookrightarrow M.$$

where $\langle A \cup B \rangle$ is the image of the canonical map $F(U(A) \cup U(B)) \to M$.

Because of this, we can make the requirement for smallness a little more concrete:

Definition 3.29. Let $A \leq M$ be a subalgebra. An element $a \in A$ is called *anchored* (in A) if there exists a proper subalgebra $B < M$ such that

$$\langle A \cup B \rangle = M \qquad \text{and} \qquad a \notin B.$$

Proposition 3.30. *$A \leq M$ is small in $\mathbf{SOb}(M)$ if and only if A contains no anchored element.*

Proof. ($\Rightarrow$): If A is small and $a \in A$ exists with $\langle A \cup B \rangle = M$, then smallness forces $B = M$ so $a \in B$. Hence no element of A is anchored.

($\Leftarrow$): Suppose A is not small. Then there exists a proper subalgebra $B < M$ with $\langle A \cup B \rangle = M$. Since $B \neq M$, A cannot be contained in B; pick $a \in A \setminus B$. This B witnesses that a is anchored. Thus A contains an anchored element. □

In the finitary and finitely generated case we can further reduce smallness to a known theory:

Definition 3.31 (Finitary monad)**.** A monad T on a category $\mathscr{C}$ is called *finitary* (or *filtered-colimit preserving*) if its underlying endofunctor $T \colon \mathscr{C} \to \mathscr{C}$ preserves filtered colimits.

Note the following standard fact about finitary monads:

Lemma 3.32. *Let $\mathscr{C}$ be a finitary monadic category and M an object in $\mathscr{C}$. Then every subalgebra in M is the join of its finitely generated subalgebras.*

Proof. Let $U \colon \mathscr{C} \to \mathbf{Set}$ be the forgetful functor, with left adjoint F. The monad $T = F \mathbin{;} U$ is finitary, hence T preserves filtered colimits. A standard result about finitary monads on **Set** states that U creates (and therefore preserves) filtered colimits (see Proposition 4.3.2 of [5]).

Let $\mathscr{F}$ be the set of all finitely generated subalgebras of M contained in some subalgebra A. Order $\mathscr{F}$ by inclusion. If $F_1, F_2 \in \mathscr{F}$ are generated by the finite sets $S_1, S_2 \subseteq U(A)$, then the subalgebra generated by $S_1 \cup S_2$ is again in A, finitely generated and contains both F_1 and F_2; hence $\mathscr{F}$ is a directed set.

Consider the directed diagram $D\colon \mathscr{F} \hookrightarrow \mathbf{SOb}_{\mathscr{C}}(M)$ that sends each finitely generated subalgebra to its inclusion in M. Since U creates filtered colimits, the colimit A of D exists and is created by the set-theoretic colimit of the underlying sets. The underlying sets of the subalgebras in $\mathscr{F}$ are subsets of $U(A)$ and the union of the underlying sets of all $F \in \mathscr{F}$ is exactly $U(A)$ (every element $a \in U(A)$ lies in the subalgebra it generates, which is finitely generated). Thus $\underrightarrow{lim}_{F\in\mathscr{F}}\, U(F) = U(A)$ with the canonical inclusions. □

By Proposition 3.7 the diagram of small subobjects of M is filtered and thus has a colimit $\mathbf{S}(M)$. This object is often the same as the Frattini subalgebra:

Definition 3.33. In an algebraic category $\mathscr{C}$ an element $g \in M$ is a *non-generator* if for every subset $S \subseteq M$

$$\langle S \cup \{g\}\rangle = M \quad\implies\quad \langle S\rangle = M.$$

The diagram of all finitely generated subalgebras of non-generators of M is finitely directed, so if $\mathscr{C}$ is finitary, then its colimit is the union of its underlying sets. This is called the *Frattini subalgebra* $\Phi(M)$ of M.

This result is standard (see for instance the classical group-theoretic treatment of the Frattini subgroup):

Proposition 3.34. *In any finitary algebraic category $\mathscr{C}$, let M be an object. An element $g \in M$ is a* non-generator *if and only if g belongs to every maximal proper subalgebra of M.*

Proof. Denote by $\mathbf{SOb}(M)$ the lattice of subalgebras of M; its meet is intersection and the join of a set of subalgebras is the subalgebra generated by their union. Because $\mathscr{C}$ is finitary, every subalgebra is the join of its finitely generated subalgebras.

Let g be a non-generator and $N < M$ a maximal proper subalgebra. If $g \notin N$, then $N \subsetneq \langle N \cup \{g\}\rangle$. By maximality the only larger subalgebra is M, hence $\langle N \cup \{g\}\rangle = M$. Since g is a non-generator, this forces $\langle N\rangle = M$, contradicting that N is proper. Thus g is in every maximal proper subalgebra.

Conversely, assume g lies in every maximal proper subalgebra of M. For every subset $S \subseteq M$ with $\langle S \cup \{g\}\rangle = M$ but $\langle S\rangle \neq M$ the subalgebra $H := \langle S\rangle$ is proper and does not contain g (otherwise $H = M$). If H were in a maximal subalgebra H' then $\langle S \cup \{g\}\rangle \subseteq H'$, so H cannot lie in any maximal subalgebra. Thus, there has to exist an infinite chain $H < H_1 < \dots < M$ such that g is not in any H_i. But by Zorn's lemma, the chain H_i has an upper bound, so any maximal such chain is bounded by a maximal subalgebra, which does not contain g since $\mathscr{C}$ is finitary and is thus proper. Consequently no such S exists, so that g must be a non-generator. The argument is uniform and does not require the *a priori* existence of any maximal proper subalgebra; if there are none, the condition "g lies in every maximal proper subalgebra" is vacuously true for every g, and the proof above shows that then every element is a non-generator - which matches the intersection over the empty family being the whole algebra. □

So the notion of a non-generator is very close to the smallness of its orbit, but for a perfect correspondence we need to assume finite generation:

Definition 3.35 (Finitely generated algebra)**.** Let T be a finitary monad on **Set**. A T-algebra (X, ξ) is called *finitely generated* if there exists a finite set K and an epimorphism of T-algebras $F(K) \twoheadrightarrow (X, \xi)$ from the free T-algebra $F(K)$ onto (X, ξ).

In many algebraic theories this reduces to the standard notion: an algebra is finitely generated if it is generated by finitely many elements in the usual sense.

Lemma 3.36. *Let T be a finitary monad on* **Set** *and (X,ξ) a finitely generated T-algebra. Then the subalgebra lattice* $\mathbf{SOb}\big((X,\xi)\big)$ *is coatomic.*

Proof. Because T is finitary, the forgetful functor $U\colon \mathbf{Set}^T \longrightarrow \mathbf{Set}$ creates filtered colimits; in particular, for any filtered family of subalgebras the underlying set of the colimit is the union of the underlying sets. Hence a chain of subalgebras of (X,ξ) has a subalgebra as its union.

Since (X,ξ) is finitely generated, there exists an epimorphism $e\colon F(K) \twoheadrightarrow (X,\xi)$. Since F is free, e is uniquely induced by a function $|e|\colon K \to |X|$ to the underlying set $|X|$, so that (X,ξ) itself is the only subalgebra that contains the set-theoretic image $|e|(K)$ of K.

Let $A \subsetneq X$ be a proper subalgebra. Consider the poset

$$\mathscr{P} \;=\; \{\, B \leq X \mid A \leq B \text{ and } B \neq X \,\},$$

ordered by inclusion. That poset is non-empty because $A \in \mathscr{P}$. Let $(B_i)_{i\in I}$ be a chain in $\mathscr{P}$ and set $B := \bigcup_i B_i$ as the set-theoretic union. As noted, B is a subalgebra of X and clearly contains A. If $B = X$ then each of the finitely many generators $k \in K$ lies in some B_i; because the chain is totally ordered and K is finite, there exists an index i_0 with $K \subseteq B_{i_0}$. Since B_{i_0} is a subalgebra containing a generating set, we obtain $B_{i_0} = X$, contradicting $B_{i_0} \in \mathscr{P}$. Therefore $B \neq X$ and $B \in \mathscr{P}$; in other words, every chain in $\mathscr{P}$ has an upper bound in $\mathscr{P}$.

By Zorn's Lemma, $\mathscr{P}$ contains a maximal element M. This M is a proper subalgebra of X that contains A and admits no proper intermediate subalgebra; hence M is a maximal proper subalgebra and so a co-atom of $\mathbf{SOb}((X,\xi))$. ◻

Proposition 3.37 (Smallness and the Frattini subalgebra)**.** *Let $\mathscr{C}$ be a finitary algebraic category and $M:\mathscr{C}$. Let A be a subalgebra of M.*

1. $\mathbf{S}(M) \subseteq \Phi(M)$.
2. *If M is coatomic (e.g. if M is finitely generated), then the converse holds:* $\Phi(M) \subseteq \mathbf{S}(M)$.

Proof. Suppose A is small. Let N be any maximal proper subalgebra. If $A \not\subseteq N$, then $A \vee N = M$ (by maximality of N), but $N \neq M$, contradicting smallness of A. Hence $A \subseteq N$ for every maximal proper subalgebra, so $A \subseteq \Phi(M)$.

Item 2 follows from Proposition 3.14. ◻

When M does not have enough co-atoms, one can have a subalgebra contained in the Frattini subalgebra that is *not* small. Here is a simple counterexample:

Counterexample 3.38. Work in the finitary algebraic category of abelian groups and let $M = \mathbb{Q}$.

- $\mathbb{Q}$ is divisible, hence so is every quotient. The only simple abelian groups are the cyclic groups $\mathbb{Z}/p\mathbb{Z}$ for primes p, and none of these is divisible (multiplication by p is the zero map). Consequently $\mathbb{Q}$ admits no maximal proper subgroup, and the Frattini subgroup $\Phi(\mathbb{Q})$ - the intersection over the empty family - is $\mathbb{Q}$ itself.

- Fix a prime p and set

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; gcd(b, p) = 1 \right\}, \qquad B = \mathbb{Z}\left[\frac{1}{p}\right] = \left\{ \frac{a}{p^k} \;\middle|\; a \in \mathbb{Z},\ k \geq 0 \right\}.$$

  Both are proper subgroups of $\mathbb{Q}$: $\frac{1}{p} \notin A$ and $\frac{1}{q} \notin B$ for any prime $q \neq p$. Yet $A \vee B = A + B = \mathbb{Q}$. Indeed, any element of $\mathbb{Q}$ can be written $\frac{a}{p^k m}$ with $gcd(m, p) = 1$, and Bézout's identity supplies integers $r,\ s$ with $rm + sp^k = 1$, so that

$$\frac{1}{p^k m} = \frac{r}{p^k} + \frac{s}{m} \in B + A.$$

- Thus A joins with the proper subgroup B to the top element $\top = \mathbb{Q}$, so A is not small - even though $A \subseteq \mathbb{Q} = \Phi(\mathbb{Q})$.

However, this difference of our theory to Frattini algebras is not necessarily a disadvantage: note that in Counterexample 3.38, our notion captures the idea of relative smallness better, since the maximal subobject should not be small relative to itself.

Example 3.39. In the lattice of submonoids of a monoid M, the join $\iota \vee \kappa$ is given by the completion of the set-theoretic union $\iota \cup \kappa$ under multiplication. By Proposition 3.30, ι is small if and only if it contains no anchored element, and if M is finitely generated, this is the same as being in the intersection of all maximal subalgebras.

On the other hand, A is dense if and only if for each non-identity element m of M a non-identity power m^n of m is in A (or M is trivial). This is because each non-trivial submonoid of M has to contain a non-identity element and thus its powers, and each submonoid B generated by elements b_i contains the submonoid generated by each b_i.

We now analyse density in subobject lattices of algebraic categories. Note that F preserves the initial object since it is a colimit, so each subobject preorder $\mathbf{SOb}(M)$ has a bottom $\perp_M$. Because the meet is just the intersection, the defining condition can be rephrased as:

$$A \text{ is dense} \iff \big(B \neq \perp \Rightarrow A \cap B \neq \perp\big).$$

Now, the same idea as in Example 3.39 generalizes as soon as the category $\mathscr{C}$ has free algebras on one generator. Recall that for a monad T on **Set**, the free T-algebra on a singleton set is written $F(1)$. A *cyclic subalgebra* of M is the image of a morphism $F(1) \to M$; equivalently it is the subalgebra $\langle x \rangle$ generated by a single element $x \in M$. In every algebraic theory that is monadic over **Set**, every non-trivial subalgebra contains a non-trivial cyclic subalgebra: pick any element of B not belonging to the minimal subalgebra and take the subalgebra it generates. Consequently we obtain a fully general "orbit" criterion:

Proposition 3.40 (Cyclic-intersection criterion for density)**.** *Let M be an algebra in a category monadic over* **Set**. *A subalgebra $A \leq M$ is dense in M if and only if for every element $x \in M$ that does not lie in the minimal subalgebra $\perp_M$, the cyclic subalgebra $\langle x \rangle$ has non-trivial intersection with A:*

$$\langle x \rangle \cap A \neq \perp_M.$$

Proof. If A is dense and $x \notin \perp_M$, then $\langle x \rangle \neq \perp_M$, so $\langle x \rangle \cap A \neq \perp_M$. Conversely, let $B \leq M$ be a non-trivial subalgebra. Choose any $x \in B \setminus \perp_M$; then $\langle x \rangle \leq B$. The hypothesis gives $\langle x \rangle \cap A \neq \perp_M$; hence $B \cap A \neq \perp_M$. Thus A is dense. □

Remark 3.41. In contrast to the smallness case (Proposition 3.37), density does not require finite generation or the existence of enough atoms. However, when the subalgebra lattice does have enough atoms, the dual of Proposition 3.14 shows that a subalgebra is dense if and only if it contains all atoms - a property that can be easier to check than the cyclic-intersection criterion.

Example 3.42. In the module theory of rings, the joins are sums of submodules and meets are set-theoretic intersections. There, smallness is also called *superfluousness* while density is called *essentiality*.

3.1.5 Algebro-Geometric

We can also do it in the algebro-geometric setting:

Example 3.43. Let X be an affine scheme represented by a ring R and $\mathbf{C}(X)$ its lattice of closed subschemes. This is dual to the lattice of quotient rings of R and thus that of ideals in R. Since these are modules, this is actually a special case of Example 3.42. So for any proper ideal $\iota\colon I \hookrightarrow R$ the corresponding subscheme $\mathbf{Sch}(\iota) : \mathbf{Sch}(R/I) \hookrightarrow \mathbf{Sch}(X)$ is small if and only if ι is dense. Nevertheless, a general formula is still hard to pin down. The most important cases are clear however:

1. If ι contains a non-zero-divisor a, then for each $\kappa\colon K \hookrightarrow R$ such that $\iota \wedge \kappa = \iota \cap \kappa = \{0\}$ and each $k \in K \setminus \{0\}$, it holds that $ak \neq 0$ since a is a non-zero-divisor, yet $ak \in \iota \cap \kappa = \{0\}$, a contradiction. So $\kappa = \{0\}$. So ι is dense.

   These ideals are of Krull height > 0 and thus their subschemes are lower-dimensional, in line with the case of manifolds. The difficulty is with the zero-divisors.

2. If R is reduced, ι consists of zero-divisors and there exists a $b \in \mathbf{Ann}(\iota)$ that is not in ι, then the intersection $(b) \cap \iota$ of the ideal generated by b with ι consists of elements $br = i$ with $r \in R, i \in \iota$. But also $bi = 0$ since i is in ι, so that $bbr = 0$ and in particular $(br)^2 = brbr = (bbr)r = 0$ by commutativity. So $br = 0$ since R is reduced, and so $i = 0$, which means that $(b) \cap \iota = (b) \cap \{0\}$. Thus, ι is not dense.

   But not all ideals of zero-divisors have a non-empty annihilator. A ring in which every finitely generated ideal of zero-divisors has a non-empty annihilator is said to have *Property A*. This includes all Noetherian rings but also many others, see [3]. Geometrically, non-nilpotent zero-divisors correspond to pieces of X with the same dimension, so at least in the finite-dimensional case the pieces of X outside of the subscheme $\mathbf{Sch}(\iota)$ can be brought into a common ideal. So an equidimensional piece is never small (if R is reduced and Noetherian, of course).

3. If on the other hand R is not reduced and ι contains its annihilator $\mathbf{Ann}(\iota)$, then for any other non-zero ideal κ, $\iota\kappa \subseteq \iota \cap \kappa$. If $\iota\kappa \neq \{0\}$ then $\iota \cap \kappa \neq \{0\}$; if $\iota\kappa = \{0\}$ then $\kappa \subseteq \mathbf{Ann}(\iota) \subseteq \iota$, so $\iota \cap \kappa = \kappa \neq \{0\}$. Either way, ι is dense. This is the case for instance for (x) in $R[x]/x^n$ (or more generally $R[x, \overline{y}]/x^n$). Here, (x) corresponds to the quotient $R[x]/x^n \to R\colon x \mapsto 0$ and thus the embedding of the reduced part of an infinitesimally thickened point.

4. We can also ask for what happens when an infinitesimally thickened point is embedded into a line. This amounts to asking whether the ideal (x^n) is dense in $R[x]$ and thus is a special case of Item 1. So even though an infinitesimal thickening appears higher-dimensional than its reduction in itself ((x) has Krull height 1 in $R[x]/x^n$), it seems lower-dimensional within $R[x]$.

So the general pattern is clear: if an embedding is lower-dimensional or (relatively) reduced in an infinitesimally thickened scheme, then it is small, if it is some equidimensional equally infinitesimally extended part it is not, except perhaps with infinite-dimensional shenanigans.

On the other hand, note that by Zorn's Lemma every ideal is bounded by some maximal ideal, so that the collection of small elements is the set of those elements that are below every maximal ideal by Proposition 3.14. Since ideals are stable under intersections, this set is an ideal, which is also known as the *Jacobson radical* (at least in the commutative case). Note that this means that the dual of the Jacobson-radical is the smallest "closed-dense" subscheme of X. By duality, this is the smallest subscheme of X that contains all points of X which are minimal in that their corresponding ideal is maximal (not the points of minimal primes!).

Remark 3.44. We can use the results of Section 6 to extend smallness and density into general schemes.

Example 3.45. Let $\mathbf{SSAlg}(R^n)$ be the lattice of semialgebraic subsets of R^n, so of finite unions of finite intersections of polynomial equalities $p = q$ and strict polynomial inequalities $p < q$. It is well-known that $\mathbf{SSAlg}(R^n)$ is closed under (Boolean) complementation. As a consequence, (absolute) size is trivial in $\mathbf{SSAlg}(R^n)$. But we can define a semialgebraic set as

1. *open* if it can be defined solely through strict inequalities $p < q$,
2. *closed* if it can be defined solely through equalities $p = q$ or inequalities $p \leq q$.

In particular, a closed semialgebraic subset ι is small in the lattice of closed semialgebraic subsets $\mathbf{CSSAlg}(R^n)$ if and only if it can be covered by a union of intersections of polynomial equalities, so if and only if it is lower-dimensional or equivalently if it can be contained in a finite number of algebraic varieties by Proposition 3.46. Density meanwhile is trivial in $\mathbf{CSSAlg}(R^n)$ since it is atomistic, but of course the considerations for smallness in $\mathbf{CSSAlg}(R^n)$ are purely dual to those for density in the lattice of open semialgebraic subsets $\mathbf{OSSAlg}(R^n)$, which are only given through strict polynomial inequalities. Thus, $\mathbf{SSAlg}(R^n)$ is kind of like a version of the topology of R^n that is small enough to be algebraically tractable. This is not a coincidence - semialgebraic subsets are a slightly smaller version of the semi-analytic subsets that motivated Grothendieck to develop a theory of tame topology in *Esquisse d'un Programme*[15]:

> It soon appeared that there could be no question of getting such an ambitious statement in the framework of topological spaces, because of the sempiternal "wild" phenomena. Already when X itself is a manifold and Y is reduced to a point, one is confronted with the difficulty that the cone over a compact space Z can be a manifold at its vertex, even if Z is not homeomorphic to a sphere, nor even a manifold. It was also clear that the contexts of the most rigid structures which existed then, such as the "piecewise linear" context were equally inadequate - one common disadvantage consisting in the fact that they do not make it possible, given a pair (U, S) of a "space" U and a closed subspace S, and a glueing map $f : S \to T$, to build the corresponding amalgamated sum. Some years later, I was told of Hironaka's theory of what he calls, I believe, (real) "semi-analytic" sets which satisfy certain essential stability conditions (actually probably all of them), which are necessary to develop a usable framework of "tame topology".

Both semialgebraic and semianalytic sets are an instance of an open-closed duality in the sense that we will develop in Section 7.1.

Proposition 3.46. *Let* $\mathbf{CSSAlg}(R^n)$ *be the preorder of closed semialgebraic subsets of* R^n *ordered by inclusion, regarded as an upward-bounded preorder with top* $\top = R^n$*. For* $A \in \mathbf{CSSAlg}(R^n)$*, the following are equivalent:*

1. A *is* small *in the sense of Definition 3.3.*
2. A *has empty interior in* R^n*.*
3. $\mathbf{dim} A < n$*.*
4. A *is covered by a finite union of (real) proper algebraic subvarieties of* R^n*, i.e. sets of the form*

$$V(p_1, \dots, p_k) = \{x \in R^n \mid p_1(x) = \cdots = p_k(x) = 0\}.$$

Proof. We proceed in steps.

(1) $\iff$ *(2).* In $\mathbf{CSSAlg}(R^n)$, finite joins are given by set-theoretic unions: if A, B are closed semialgebraic, then $A \vee B = A \cup B$, since the union of closed semialgebraic sets is again closed semialgebraic and minimal with this property. Thus (1) says that the only closed semialgebraic B with $B \supseteq R^n \setminus A$ is $B = R^n$. This is equivalent to

$$\overline{R^n \setminus A} = R^n,$$

where the closure is in the Euclidean topology. But $\overline{R^n \setminus A} = R^n$ if and only if $A^\circ = \emptyset$, since

$$A^\circ = R^n \setminus \overline{R^n \setminus A}.$$

(2) $\iff$ *(3).* A semialgebraic subset $A \subseteq R^n$ has nonempty interior if and only if $\mathbf{dim} A = n$. This is a standard fact of semialgebraic geometry. For $\Rightarrow$, take $x \in A^\circ$. Then there exists a ball $\bullet_x \subseteq A^\circ$ since R^n is regular Hausdorff, so that $\mathbf{dim}(A) = n$. Conversely, it follows immediately from the Euclidean topology that if $\mathbf{dim}(A) < n$ then A has an empty interior.

(3) $\iff$ *(4).* Suppose $\mathbf{dim} A < n$ and A is closed semialgebraic. We may write A as a finite union of semialgebraic subsets P_i each of which is of codimension at least 1. Each such piece lies inside an algebraic hypersurface, meaning the zero set of a nonzero polynomial. Thus, there exist finitely many polynomial maps $f_1, \dots, f_r$ such that

$$A \subseteq \bigcup_{j=1}^{r} V(f_j).$$

Conversely, if A is covered by a finite union of algebraic varieties in R^n, each such variety is a real algebraic set of dimension at most n, and if A had dimension n, one of its components would contain a nonempty open subset of R^n, which is impossible unless that component is all of R^n since a proper real algebraic subvariety of R^n has dimension strictly less than n. Thus if A is a proper closed subset and covered by a finite union of algebraic varieties, then $\mathbf{dim} A < n$. Hence (4) implies (3) in the nontrivial case $A \neq R^n$; for $A = R^n$ all four conditions fail.

Combining the implications, we obtain (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). ◻

In particular, the informal sentence in Example 3.45 can be read precisely as: a closed semialgebraic subset A of R^n is small in $\mathbf{CSSAlg}(R^n)$ if and only if it has empty interior, equivalently if and only if it is of strictly lower dimension than R^n, equivalently if and only if it is a finite union of algebraic varieties.

3.1.6 Order- and Category-Theoretic

We can even apply this definition to the subobject lattices of preorders themselves, where we see a mix of the algebraic and graph-theoretic cases going on:

Example 3.47. Let **PrO** be the category whose objects are preorders $(P, \leq_P)$ and morphisms $\iota : P \to Q$ are monotone (order-preserving) maps. Then for each P, the subobject preorder $\mathbf{SOb}(P)$ has as objects monomorphisms into P, so order-preserving injections. This is a lattice where, given $\iota : Q \hookrightarrow P, \kappa : R \hookrightarrow P$:

1. the meet $\iota \wedge \kappa = \iota \cap \kappa$ is given on elements by the intersection of the images of ι and κ, such that for $a, b \in \iota \wedge \kappa$ it holds that $a \leq b$ if and only if $a \leq_\iota b$ and $a \leq_\kappa b$,
2. the join $\iota \vee \kappa$ is the completion of the set-theoretic union under composition, so is given on elements by the set-theoretic union and such that for $a, b \in \iota \vee \kappa$ it holds that $a \leq b$ if and only if there exist families of elements $c_i \in \iota \wedge \kappa$ such that $a \leq_S c_0 \leq_T c_1 ... \leq_S c_n \leq_T b$ with $S = Q$ if $a \in \mathbf{Im}(\iota)$ and R otherwise, and similar for T and b.

We can describe the size of each $\iota : Q \hookrightarrow P$.

1. **Density** Since the meets in $\mathbf{SOb}(P)$ are given by set-theoretic intersections, density is similar to Example 3.24. In particular, $\perp$ is the empty embedding and $\mathbf{SOb}(P)$ has enough atoms. So by the dual of Proposition 3.14, ι is dense in $\mathbf{SOb}(P)$ if and only if it contains all atoms, i.e. one-element-embeddings into P.
2. **Smallness** Since the join is given objectwise by the set-theoretic union and sequences of inequalities, any κ such that $\iota \vee \kappa = \top$ has to generate all inequalities with ι. Thus, κ has to contain all inequalities $a \leq_P b$ with $b \in \iota$ such that there is no $c \in \iota$ such that $a \leq c <_\iota b$, so it has to contain each such b. Thus, if ι does not contain any non-trivial inequalities and P does, then ι is small. More generally, let $a < c < b$ in P be such that $a <_\iota b$ but not $c \in \iota$. Then κ has to contain all of $a < c < b$ and thus $a <_\kappa b$. Thus, if ι contains only morphisms in its image that are compositions of morphisms outside its image, then ι is small. Conversely, assume that ι contains a morphism $a < b$ such that there does not exist a factorization of this morphism into morphisms outside the image of ι and let $\kappa := \top \setminus (a < b)_\iota$ be given by removing all morphisms from $\top$ that $a < b$ factors through and that are in the image of ι. Then κ does not contain the chain of inequalities between a and b and thus $a \not<_\kappa b$ but $\iota \vee \kappa = \top$, so ι is not small.

So here we see a mixture of algebraic and geometric smallness going on, where smallness both encompasses lower-dimensionality and factorizability. This is because preorders are special categories, and we can (and have to) extend our considerations to general categories. For this let us first remember some functor types that will feature throughout this text:

Definition 3.48 (Faithful, full and fully faithful functors)**.** Let $\mathscr{C}, \mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ a functor.

1. F is *faithful* if for all objects $X, Y : \mathscr{C}$ the function

$$F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y)), \quad f \mapsto F(f)$$

is injective. Equivalently, whenever $f, g : X \to Y$ satisfy $F(f) = F(g)$, we must have $f = g$.

2. F is *full* if for all objects $X, Y : \mathscr{C}$ the function $F_{X,Y}$ is surjective. Equivalently, every morphism $h : F(X) \to F(Y)$ in $\mathscr{D}$ is of the form $h = F(f)$ for some $f : X \to Y$ in $\mathscr{C}$.

3. F is *fully faithful* if it is both full and faithful, i.e. for every pair $X, Y : \mathscr{C}$ the map

$$F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y))$$

is a bijection.

Example 3.49 (Smallness for faithful subcategories)**.** Let $\mathscr{C}$ be a category and $\mathbf{SOb}'_{|f|}(\mathscr{C})$ the preorder whose elements are faithful functors $\iota : \mathscr{D} \hookrightarrow \mathscr{C}$, ordered by $\iota \leq \kappa$ if ι factors through κ up to isomorphism. The top element is $\mathbf{id}_{\mathscr{C}}$; the bottom is the empty embedding.

Joins in this preorder exist and are generated by the set-theoretic union by closure under morphisms. It follows again from the dual of Proposition 3.14 that a subcategory is dense if and only if it contains all objects of $\mathscr{C}$.

In general, $\mathbf{SOb}'_{|f|}$ is a little pathological: for instance, the function between cardinals $[2] \to [1]$ is a faithful functor (though it is not full) and thus in $\mathbf{SOb}'_{|f|}$. But the defining condition makes it so that each faithful functor ι is equivalent to its image, literally given in terms of object and morphism sets. So for instance $\mathbf{SOb}'_{|f|}([1])$ is not even a small category but equivalent to the one-object category. We can avoid this pathology by restricting to faithful functors that are also isomorphism-reflecting, so we define $\mathbf{SOb}_{|f|}$ to consist of the isomorphism-reflecting faithful functors.

Such an isomorphism-reflecting faithful functor is what we use as the definition of a subcategory:

Definition 3.50. A*subcategory* $\iota : \mathscr{C} \hookrightarrow \mathscr{D}$ is an isomorphism-reflecting faithful.

This is also known as a pseudomonic (see [6]) and will factor in later.

We can mimic the arguments of Section 3.1.4 for smallness step-by-step by using morphisms as generators. However, note that it is rare that a category of interest is generated by a finite set of morphisms. Moreover, it is certainly not clear that a category like **Top** has maximal proper subcategories. But we can still use our anchor criterion:

Definition 3.51 (Anchored morphism)**.** Let $\mathscr{C}$ be a category and let $\mathbf{SOb}_{|f|}(\mathscr{C})$ as in Example 3.49. A morphism f of $\mathscr{C}$ is *anchored* relative to a subcategory $\iota : \mathscr{A} \hookrightarrow \mathscr{C}$ if there exists a subcategory κ such that in $\mathbf{SOb}_{|f|}(\mathscr{C})$

$$\iota \vee \kappa = \mathscr{C} \qquad \text{and} \qquad f \notin \kappa.$$

It is straightforward to show that a subcategory is small if and only if it does not contain an anchored morphism. Moreover, one simple criterion for smallness is that a subcategory ι contains only morphisms that are composites of morphisms that are not factored by morphisms in ι:

Definition 3.52. A morphism f of a category $\mathscr{C}$ is called *ι-irreducible* for a subcategory $\iota : \mathscr{A} \hookrightarrow \mathscr{C}$ if f is either an identity or cannot be written as a composite of the form $f = g \mathbin{;} h \mathbin{;} i$ where at least one of g, h, i is a non-identity in $\mathscr{A}$.

Proposition 3.53 (Categorial smallness criterion)**.** *Let $\iota : \mathscr{A} \hookrightarrow \mathscr{C}$ be a faithful subcategory. Assume that every non-identity morphism f in ι admits a factorization*

$$f \;=\; g_1 \mathbin{;} g_2 \mathbin{;} \cdots \mathbin{;} g_n$$

into finitely many morphisms g_i that are ι-irreducible and not in ι. Then ι is small in the upward-bounded preorder of faithful subcategories of $\mathscr{C}$.

Proof. We shall prove that under this hypothesis, ι cannot contain any anchored morphism.

Given f in ι and a proper faithful subcategory $\kappa \subsetneq \mathscr{C}$ such that $\iota \vee \kappa = \mathscr{C}$ we may write $f = g_1 ; g_2 ; \cdots ; g_n$ where each g_i is ι-irreducible. Hence $g_i \notin \iota$.

Now, because $\iota \vee \kappa = \mathscr{C}$, every morphism of $\mathscr{C}$ can be expressed as a composition of morphisms taken from $\iota \cup \kappa$. In particular, for each i we can choose one such representation of g_i:

$$g_i \;=\; h_{i1} ; h_{i2} ; \cdots ; h_{ik_i}, \qquad h_{ij} \in \iota \cup \kappa.$$

Since morphisms are closed under composition and g_i is ι-irreducible, its representation cannot contain a non-identity factor from ι. Therefore none of the h_{ij} belong to ι. Consequently every h_{ij} lies in κ.

Thus each g_i is a composition of morphisms of κ and therefore belongs to the subcategory κ. Finally $f = g_1 ; \cdots ; g_n$ is also a composition of morphisms in κ, whence $f \in \kappa$. Thus, no morphism of ι is anchored, so ι is small. $\square$

Remark 3.54. This works great for preorders or deloopings of monoids without inverses but does not work for deloopings of groups or categories with non-trivial automorphism categories.

Both Example 3.47 and Example 3.49 are somewhat unsatisfying because they do not consider the categorical structure of the target category at all. We can define subobject preorders that do however:

Example 3.55. Given a category $\mathscr{C}$, let $\mathbf{SOb}_f$ be the preorder whose elements are subcategories and $\iota \leq \kappa$ if and only if there exists a functor $\lambda \colon \mathscr{D} \to \mathscr{E}$ and a monic natural transformation $\Phi \colon \iota \leq \lambda ; \kappa$:

$$\begin{tikzcd}
\mathscr{E} \arrow[dr, "\kappa"] & \\
 & \mathscr{C} \\
\mathscr{D} \arrow[uu, "\lambda"] \arrow[ur, "\iota"'] \arrow[u, phantom, "\Phi\;\Nwarrow"] &
\end{tikzcd}$$

Figure 1

This is a less truncated structure than Example 3.49, but its smallness relations stay the same: $\top$ is still $\mathbf{id}_{\mathscr{C}}$, $\bot = \varnothing$ and the join is given by the closure of the union of the subcategories under composition. The meet is interesting however: $\iota \wedge \kappa$ is the full subcategory on objects that are simultaneously subobjects of something in $\mathscr{C}_\iota$ and of something in $\mathscr{C}_\kappa$ in some way. So the dense elements are different from those in Example 3.49 : ι is dense if and only if it contains for each object X in $\mathscr{C}$ some object A that is a subobject of X. In particular, if $\mathscr{C}$ contains an initial object whose universal morphisms are monic, then a subcategory is dense if and only if it contains that. Note that the dual is not true for the terminal object even if the morphisms into it are monic, which is also a much rarer assumption. This is because of the variance of the 2-cells in it (which would be necessary to define filters, for instance). $\mathbf{SOb}_f$ will be central in Section 8 but Remark 8.24 would not work for either $\mathbf{SOb}_{|f|}$ or the 2-dual of $\mathbf{SOb}_f$. So here we can already begin to see that the entire structure we are describing is of one piece and it is all *covariant* on the lowest level (the variance on higher levels will grow quite complex in Section 8.3).

We also have some specializations of this preorder:

Example 3.56. Let $\mathbf{SOb}_{ff}(\mathscr{C})$ be the quotient of $\mathbf{SOb}_f(\mathscr{C})$ by the equivalence relation $\iota \sim \kappa$ if and only if ι and κ have the same fully faithful image. Then for each ι it holds that ι is equivalent to the embedding $|\iota|$ of the discrete subcategory on the objects in the image of ι, and for $|\iota|$ and $|\kappa|$ it holds that $|\iota| \leq |\kappa|$ if and only if there exists a function $|\iota| \hookrightarrow |\kappa|$ and monomorphism $I : |\iota|(X) \hookrightarrow |\kappa|(X)$. So $\iota \leq \kappa$ in $\mathbf{SOb}_{ff}(\mathscr{C})$ holds if each X in the image of ι is the subobject of some Y in the image of κ. We can similarly define a quotient $\mathbf{SOb}_{|ff|}(\mathscr{C})$ of $\mathbf{SOb}_{|f|}(\mathscr{C})$, where $\iota \leq \kappa$ if and only if the objects of ι are contained in the objects of κ up to isomorphism. The joins and meets of $\mathbf{SOb}_{|ff|}(\mathscr{C})$ can be identified with set-theoretic unions and intersections, so that both smallness and density are trivial, but $\mathbf{SOb}_{ff}(\mathscr{C})$ has a more interesting structure. The joins $\iota \vee \kappa$ can still be identified with set-theoretic unions of the object sets and thus the only subcategory that is small in $\mathbf{SOb}_{ff}(\mathscr{C})$ is $\varnothing$. But the meet $\iota \wedge \kappa$ is again the full subcategory on objects that are simultaneously subobjects of something in $\mathscr{C}_\iota$ and of something in $\mathscr{C}_\kappa$, which is not generally the set-theoretic intersection.

Example 3.57 (Wide subcategories of a category)**.** Let $\mathscr{C}$ be a category and write $\mathbf{Wide}(\mathscr{C})$ for the preorder of wide subcategories of $\mathscr{C}$ (i.e. subcategories having all objects of $\mathscr{C}$), ordered by inclusion of morphism sets. We can construct this category as the preorder of faithful isomorphism-reflecting factorizations of the functor $\mathbf{Disc}(\mathscr{C}) : \mathbf{Ob}(\mathscr{C}) \hookrightarrow \mathscr{C}$. The top element is $\mathscr{C}$ itself; the bottom is the discrete wide subcategory $\mathbf{Disc}(\mathscr{C})$ containing only identities. A wide subcategory ι is again small in $\mathbf{Wide}(\mathscr{C})$ if and only if it does not contain an anchor. ι is dense in $\mathbf{Wide}(\mathscr{C})$ if and only if

1. every non-endomorphism of $\mathscr{C}$ belongs to ι, and
2. for each non-identity endomorphism $e : X \to X$ of $\mathscr{C}$ there exists an integer $n \geq 1$ such that e^n is in ι with e^n not an identity.

To see this, note that taking wide subcategories generated by a single non-endomorphism forces all non-endomorphisms into ι; taking those generated by an endomorphism e forces some power of e into ι. The converse is a straightforward verification using the closure properties of wide subcategories.

So on the level of wide subcategories density starts behaving as for monoids. Note that this is similar to how the periodic table for k-tuply n-monoidal categories only works when we take them as pointed.[4]

3.1.7 Topos-Theoretic

An important but perhaps somewhat surprising class of examples of unreachability arises in topos theory:

Proposition 3.58. *The subobject preorder of every representable presheaf is unreachable.*

Proof. Let $\mathscr{C}$ be a category, X an object of $\mathscr{C}$ and $F \hookrightarrow \widehat{X}$ a proper subfunctor of the functor $\widehat{X}$ represented by X. We have to show that the only $G \hookrightarrow \widehat{X}$ such that $F \vee G = \widehat{X}$ is $\widehat{X}$ itself. For this remember that joins of presheaves are given by objectwise set-theoretic unions $F(Y) \cup G(Y)$. Now $\mathbf{id}_X : X \to X$ is the only element of $\widehat{X}(X)$ whose presence in a sub-functor forces that sub-functor to be all of $\widehat{X}$. Since F is proper, it does not contain $\mathbf{id}_X$ at X. But $F(X) \cup G(X) = \widehat{X}(X)$, so $\mathbf{id}_X \in G(X)$. By Yoneda closure, this forces $G = \widehat{X}$. □

Remark 3.59. At least if $\mathscr{C}$ is monic and X is Dedekind-finite, i.e. has no proper self-embeddings, we can construct a co-atom $\partial X \hookrightarrow \widehat{X}$ as the presheaf with $(\partial X)(Y)$ given by the set of morphisms $f : Y \to X$ such that there exists a $\iota : Z \hookrightarrow X$ with $Z \not\cong X$ and a $g : Y \to Z$ with $f = g \mathbin{;} \iota$. So we can freely construct a boundary.

Remark 3.60. This behavior is in contrast to that of density on representable presheaves: in Remark 6.15 we find that for a thin subcanonical site with an initial object the Yoneda embedding is a density-equivalence. The behavior on general categories is currently unknown but the mere fact that the structure of the base category matters independently of the coverage is already different from the case of smallness, where the coverage completely mediates what is small.

Remark 3.61. Proposition 3.58 recontextualizes Observation 3.11: generally we would think that faces, being lower-dimensional, would be small. And they are - once we add enough unions to construct a boundary. It is the mark of the combinatorial that there are no subcanonical coverages on a combinatorial site ⬢ that would make some objects in the sheaf topos **Sh**(⬢) small and others not: there is only the *boundary coverage*, which identifies an object with its boundary - but this recreates the original site.

3.1.8 Convex

We also have two examples related to the work of Fritz. In [30], Stone defines an abstract convex space, which Fritz takes up in [11]. One thing that can be gleaned from the definition is the idea of a convex space as a kind of continuous category: it has generalized associativity and unitality laws. We take up that theme and make it fully explicit, but in a simplified setting where we forget the combinatorics of convexity and just remember the generated geodesics, somewhat similar to taking the path groupoid of a topological space. But for that we need to laxify the definition of a subcategory itself:

Definition 3.62 (Lax 2-functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be 2-categories. A *lax 2-functor* $F : \mathscr{C} \to \mathscr{D}$ consists of the following data:

1. A function on objects, mapping $X : \mathscr{C}$ to $F(X) : \mathscr{D}$.
2. for every pair of objects $X, Y : \mathscr{C}$, a functor on hom-categories
$$F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y)),$$
mapping a 1-cell $f : X \to Y$ to $F(f) : F(X) \to F(Y)$ and a 2-cell $\alpha : f \Rightarrow g$ to $F(\alpha) : F(f) \Rightarrow F(g)$.
3. for every object $X : \mathscr{C}$, a 2-cell (the *lax unitor*)
$$\nu_X : \mathbf{id}_{F(X)} \Rightarrow F(\mathbf{id}_X).$$
4. for every pair of composable 1-cells $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$, a 2-cell (the *lax compositor*)
$$\mu_{f,g} : F(f) \,;\, F(g) \Rightarrow F(f \,;\, g).$$

Subject to the same coherence conditions as pseudofunctors. An *oplax 2-functor* is defined in the same way, but with dual unitors and compositors.

Given a 2-category $\mathscr{B}$, a *sub-2-category* $\iota : \mathscr{A} \hookrightarrow \mathscr{B}$ is a 2-functor which is injective and isomorphism-reflecting on objects, 1-morphisms and 2-morphisms. A *lax sub-2-category* is a lax 2-functor with the same properties and dually for oplaxity.

Remark 3.63. A pseudofunctor is precisely a lax 2-functor for which the lax unitors ν_X and lax compositors $\mu_{f,g}$ are invertible 2-cells (i.e. equivalences in the hom-categories). If these 2-cells are further constrained to be identity 2-cells, F reduces to a strict 2-functor. Laxifying a pseudofunctor thus amounts to replacing the requirement that composition and identities are preserved up to coherent isomorphism with the weaker requirement that they are preserved only up to coherent morphisms pointing in a specific direction. This process is known as *laxification*. Now, consider this little-appreciated fact: laxification is evil.[2]

Let I be the 2-category with two elements X, Y, morphisms generated by $f : X \to Y, g : Y \to X$ and 2-cells generated by $\mathbf{id}_X < f ; g, \mathbf{id}_Y < g ; f$. Let $I_0 = \{X, Y\}$ be the set of objects of I, and let $\mathbf{CDisc}(I_0)$ be the codiscrete category on X, Y, consisting of two morphisms $[X, Y] : X \to Y, [Y, X] : Y \to X$ that compose to the identities. Then there exists an oplax functor $F : \mathbf{CDisc}(I_0) \to I : [X, Y] \mapsto f, [Y, X] \mapsto g, \mathbf{id}_X \mapsto \mathbf{id}_X, \mathbf{id}_Y \mapsto \mathbf{id}_Y$ that is not equivalent to any constant functor, even though $\mathbf{CDisc}(I_0)$ is equivalent to the point.

What should we make of that?

Should we use lax *anafunctors* instead of lax functors? Or does laxification detect structure that is invisible to weakness? The following exposition suggests the latter. It also has to be noted that it is known that Lawvere had a negative opinion of $(\infty, 1)$-category theory. The exact reason why is less well-known but can to some extent be reconstructed: in [20], Lawvere explicitly *denies* that homotopy types are the same as infinitesimal skeleta because to him, they are an instance of an identity of opposites. Understanding this *only* to an identity would seem like an undialectical collapse from this perspective. And we do have to ask ourselves: if, say, the codiscrete categories on different sets are just the same, *then why are they different*? Because they do have different object cardinalities and, more saliently, different lax functors out of them.

Example 3.64. Given a Lawvere metric space (X, d) with its induced topology generated by $U_{x,r} := \{y \in X | d(x, y) < r\}$ we can define a 2-categorical structure on X whose

- objects are the elements of X,
- morphisms are the contracting functions $f : [0, l] \to X$ with $l_f := l \in R_{\geq 0}$,
- $f \leq g$ if $l_f \leq l_g$,
- $f ; g$ is the morphism $[0, l_f + l_g] \to X$ given by f on $[0, l_f]$ and g on $[l_f, l_f + l_g]$,
- $\mathbf{id}_x : [0, 0] \to X$ is given by $x : * \hookrightarrow X$ (since $[0, 0]$ is canonically equivalent to the terminal set),
- associativity and identity laws follow from pointwise evaluation.

This is kind of like the path groupoid of a topological space except that we don't assume any identifications whatsoever. Note also that no object of X has a non-contractible automorphism 2-group, so each natural isomorphism between paths is equivalent to an identity.

Let us now consider the example of R^n and the embedding of the codiscrete groupoid

$$\gamma : \mathbf{CDisc}(R^n) \hookrightarrow R^n : x \mapsto x, (f : x \to y) \mapsto [x, y]$$

which maps the (unique) path from x to y to the (affine) line from x to y. Then γ maps $\mathbf{id}_X$ to $[x, x]$, which is the identity in R^n, and maps $x \xrightarrow{f} y \xrightarrow{g} z$ to $[x, z]$, which is minimal in $X(x, z)$ and is thus an oplax embedding of $\mathbf{CDisc}(R^n)$ into $(R^n)^{[0,1]}$. In other words, γ is the embedding of the geodesics between points in R^n.

[2] To the knowledge of the author, this was first noticed by Chris Schommer-Pries [28].

This is the example we want to generalize:

Definition 3.65. An oplax sub-2-category $\iota\colon\mathscr{A} \hookrightarrow \mathscr{B}$ of a 2-category $\mathscr{B}$ is *convex* if for each 2-isomorphism $\sigma\colon\iota(f) \overset{\sim}{\Rightarrow} g \,;\, h$ it holds that σ, g and h are in ι.

ι is *convexly full* if for each f in $\mathscr{B}$ such that

1. $\mathbf{s}(f)$ and $\mathbf{t}(f)$ are in (the image of) ι,
2. and for each 2-isomorphism $\sigma\colon f \simeq g \,;\, h$ it holds that $\mathbf{t}(g) = \mathbf{s}(h)$ is in ι,

then σ, g and h are in ι.

An object $\varnothing$ *is sup-initial* if it has at most one morphism to every object (so if it is the dual of a subterminal object). $\varnothing$ is *strict sup-initial* if it is sup-initial and each morphism with codomain $\varnothing$ is uniquely 2-isomorphic to the identity.

A *2-category with paths* is a pair of a 2-category $\mathscr{B}$ and a wide convex oplax subcategory $\iota\colon\mathscr{A} \hookrightarrow \mathscr{B}$. A *2-category with unique paths* is a 2-category with paths such that $\mathscr{A}$ is the codiscrete groupoid $\mathbf{CDisc}(\mathscr{B})$ on the objects of $\mathscr{B}$. A *2-category with geodesics* is a 2-category with paths ι such that each morphism $p\colon X \to Y$ in the image of ι is strict sup-initial in $\mathscr{B}(X, Y)$. A *geodesic 2-category* is a 2-category such that there exists a convex strict initial object $\gamma\colon\mathscr{G} \hookrightarrow \mathscr{B}$ in the 2-category of wide oplax sub-2-categories of $\mathscr{B}$ on $\mathbf{CDisc}(\mathscr{B})$, or equivalently if each hom-category $\mathscr{B}(X, Y)$ contains a strict initial object $\gamma_{X,Y}$ such that each 2-isomorphism $\sigma\colon\gamma_{X,Y} \simeq f \,;\, g$ is an identity (since it is in the image of $\mathbf{CDisc}(\mathscr{B})$) and $f = \gamma_{X,\mathbf{t}(f)}, g = \gamma_{\mathbf{s}(g),Y}$.

Remark 3.66. So convex fullness is like fullness but only applies to morphisms that don't just go between objects in ι but also never factor through objects outside ι - no detours allowed!

And this we can apply to Stone's convex spaces:

Example 3.67. In [11], Fritz describes a convex space as a set X with a family of operations $cc_\lambda\colon X \times X \to X$ for $\lambda \in [0, 1]$ such that $cc_0(x, y) = x$ and $cc_1(x, y) = y$ (by commutativity). This is equivalent to a family of paths $\gamma_{x,y} \in X^{[0,1]}\colon\gamma_{x,y}(\lambda) := cc_\lambda(x, y)$ where we don't assume any structure on X or $[0, 1]$. Fritz moreover introduces a commutativity axiom which ensures that the inverse of some path is literally the same path backwards, and an associativity axiom which regulates the continuous structure through combinatorial approximation. We reduce this to a discrete, i.e. categorical structure. This loses some geometry and there should be a continuous version of category theory that builds on that structure, but starting from a more basic perspective allows us to seamlessly integrate our theory with standard category theory before introducing more genuinely continuous considerations.

Each convex space in the sense of [11] has an underlying convex 2-category $|\prod|(X)$ whose

1. objects are the elements of X,
2. morphisms are generated by $\gamma_{x,y}$ (freely except for the 2-morphisms below),
3. 2-morphisms are the horizontal and vertical composites of the following two kinds:
   (a) 2-cells $\gamma_{x,y} \,;\, \gamma_{y,z} \geq \gamma_{x,z}$ for all x, y, z,
   (b) inverses $\gamma_{x,z} \sim \gamma_{x,y} \,;\, \gamma_{y,z}$ if y is in the image of $\gamma_{x,z}$.

This is a 2-category whose identities are given by constant paths, which is both more rough and finer than the standard fundamental groupoid: it does not consider any higher-categorical structure (and there isn't really any unless it is induced by paths), but it also does not identify the composite $\gamma \mathbin{;} \overline{\gamma}$ of a path $\gamma\colon[0,1] \to X$ and its reflection $\overline{\gamma}\colon[0,1] \xhookrightarrow{\overline{x}:=1-x} [0,1] \overset{\gamma}{\hookrightarrow} X$ with $\mathbf{id}(\mathbf{s}(x))$. It follows directly from the definition that each $\gamma_{x,y}$ is the minimal path from x to y, and that γ is convex (up to contractible homotopy). Thus $\gamma\colon\mathbf{CDisc}(X) \to X$ is the *underlying geodesic 2-category* of X.

These examples sure seem weird from the point of view of homotopy theory but they work as convexity should, where it is not that structures unfold along higher cells but that higher cells are added in by recursively connecting points. And a pleasant feature of geodesic 2-categories is that they allow for a notion of convex closure:

Definition 3.68. Given a convex wide oplax sub-2-category $\iota\colon\mathscr{A} \hookrightarrow \mathscr{B}$ and a family of objects $\mathscr{F}$ in $\mathscr{B}$, the *ι-closure* of $\mathscr{F}$ in $\mathscr{B}$ is the smallest convex and convexly full (weak) sub-2-category of $\mathscr{B}$ containing the restriction of ι to the objects in $\mathscr{F}$. A sub-2-category κ of $\mathscr{B}$ is *ι-convex* if it is equivalent to the ι-closure of a family of objects in $\mathscr{B}$. If $\mathscr{B}$ is geodesic, then κ is *geodesically convex* if it is γ-convex.

Remark 3.69. Note in particular that we can identify each ι-convex sub-2-category of $\mathscr{B}$ with its subset of objects. In particular, for a family $s\colon S \hookrightarrow \mathscr{B}$ the geodesic closure $\hat{s}$ has to be the smallest convex and convexly full weak subcategory of $\mathscr{B}$ containing $\gamma|_s\colon\mathbf{CDisc}(S) \hookrightarrow \mathscr{B}$.

Example 3.70. In a real vector space R^n with its usual metric, let $\mathscr{F}$ be a family of points. Applying Example 3.64 we get that $\gamma_{x,y}$ is the shortest path $[0,1] \to R^n$ between x and y, up to the points it contains. So the geodesic closure of $\mathscr{F}$ contains $\mathscr{F}$, $\gamma_{x,y}$ for each x, y in $\mathscr{F}$ and the shortest path between each pair of points in $\mathscr{F}$, so concretely can be obtained from $\mathscr{F}$ by adding each $\gamma_{x,y}$, then adding all points on each obtained line, then adding the geodesics between each pair of new points and so on, then adding all paths that only go on points in the closure. Thus, $\overline{\mathscr{F}}$ is indeed the convex closure of $\mathscr{F}$ as traditionally understood. For instance if $\mathscr{F}$ is given by function $s\colon S \hookrightarrow R^n$, with S a finite set, then $\overline{s}\colon\mathbf{CDisc}(S) \to R^n$ is a (possibly degenerated) convex simplex in R^n with vertices given by the images of the elements of S. Note in particular that $\mathbf{CDisc}([n])$ with $[n]$ a finite cardinal is the localization of the total order on the same elements, which is also known as the abstract $(n-1)$-simplex. Note also that a geodesic weak natural transformation $\eta : \iota_0 \Rightarrow \iota_1 : I \overset{\hookrightarrow}{\hookrightarrow} R^n$ between geodesic embeddings of the directed interval category, i.e. arrow category $0 \hookrightarrow 1$, is a square with edges $\iota_0, \iota_1, \eta_0, \eta_1$ such that

$$|\eta_0 \mathbin{;} \iota_1| = |\eta_0| + |\iota_1| = |\iota_0| + |\eta_1| = |\iota_0 + \eta_1|,$$

i.e. an ex-tangential quadrilateral.

The point is now that we can develop a very general theory of smallness for convexity:

Example 3.71. Let $\mathscr{B}$ be a geodesic 2-category with distinguished geodesic morphisms $\gamma_{x,y}\colon x \to y$ for all objects x, y. The geodesically convex sub-2-categories of $\mathscr{B}$ form an upward-bounded preorder $\mathbf{SOb}_{gcc}(\mathscr{B})$ under inclusion, with top element $\mathscr{B}$ and bottom element the empty subcategory. Meets are given by intersections if they exist (which they often do not), and joins are geodesic closures of unions: $\iota \vee \kappa = \overline{\mathbf{Ob}(\iota) \cup \mathbf{Ob}(\kappa)}$.

We now analyse smallness in $\mathbf{SOb}_{gcc}(\mathscr{B})$. A geodesically convex subcategory ι is not small if and only if there exists a proper geodesically convex κ such that $\overline{\iota \cup \kappa} = \mathscr{B}$. Since both ι and κ are geodesic closures of objects ι_0, κ_0, this closure is equivalent to $\overline{\iota_0 \cup \kappa_0}$. Thus, ι is not small if there exists a κ_0 such that each object of $\mathscr{B}$ can be obtained by repeatedly taking geodesics of points on geodesics... between ι_0 and κ_0, but there is a point in ι_0 that cannot be obtained from repeated geodesics between points in κ_0.

So for instance in a non-trivial convex cone in $\boldsymbol{R}^n$ with the convex 2-categorical structure from Example 3.64, the one-point subset on the tip is not small while all other points are small.

Remark 3.72. We might expect smallness in Example 3.71 to be given by “A is contained in the closure of its set-theoretic complement”, but the notion we derived is generally stronger than that. This is roughly because smallness is derived using both the fewer subobjects that are convex subspaces and their more expansive join. In Example 4.26 we will learn how to use only one of these at a time, recovering a weaker notion of smallness.

3.1.9 Categorical Probability Theoretic

The final example is also one from Fritz but in synthetic probability theory:

Example 3.73. Let us look at the category **FStoch** whose objects are finite sets and whose morphisms are stochastic matrices. Recall that stochastic matrix $M : X \to Y$ is a function $f(- \mid -) : X \times Y \to [0,1]$ such that for each $x \in X$ it holds that

$$\sum_{y \in Y} f(y \mid x) = 1.$$

f here is a synthetic descriptor of conditional probability, where the defining equation says that each conditional probability is a probability function. However, M is also the descriptor of an affine transformation that maps a unit simplex $\blacktriangle_1^X \subseteq \boldsymbol{R}^X$, which is the convex hull of the unit vectors δ_i in $\boldsymbol{R}^X$, into $\blacktriangle_1^Y \subseteq \boldsymbol{R}^Y$, given by multiplying δ_i with M. Thus, we can identify $X : \mathbf{FStoch}$ with the $(|X| - 1)$-dimensional unit simplex $\blacktriangle_1^X$ and the subobjects $\iota : A \hookrightarrow X$ with the affine sub-simplices of $\blacktriangle_1^X$.[3] Since each point is an affine sub-simplex, $\mathbf{SOb}(\blacktriangle_1^X)$ is atomistic by Corollary 3.16, so density is trivial; the question is again smallness. For joins, $\sigma \vee \tau = \blacktriangle_1^X$ holds if and only if $\blacktriangle_1^X$ is the only subsimplex of $\blacktriangle_1^X$ containing the union $\sigma \cup \tau$.

Dimension 0 is clear. In dimension 1 the only sub-simplex of $\blacktriangle^1 = [0,1]$ that contains both 0 and 1 is $\top = \mathbf{id}_{[0,1]}$ itself, so the join $\iota \vee \iota'$ of two affine sub-simplices ι, ι', i.e. sub-lines, is $\top$ if and only if both 0 and 1 are in the union $\iota \cup \iota'$. Thus, ι is small if and only if it does not contain either 0 or 1.

For dimension > 1 we claim that the only small subobject in $\mathbf{SOb}(X)$ is $\perp$. To prove this, we treat $\blacktriangle_1^X$ as embedded into $\boldsymbol{R}^X$ and construct for each $p \in \blacktriangle_1^X$ a full-dimensional, maximal proper subsimplex that does not contain p. For this, take some maximal face τ such that p is not in τ. Since each simplex is T_4 and both p and τ are closed and disjoint in $\blacktriangle_1^X$ there exists some open neighborhood U_τ of τ that does not contain p and thus, by the definition of the restricted Euclidean topology, an $\varepsilon > 0$ such that the ε_p-thickening $\varepsilon_p \tau \cap \blacktriangle_1^X$ of τ does not contain p. Let now τ_p be the sub-simplex of $\blacktriangle_1^X$ parallel to τ in $\blacktriangle_1^X$ whose orthogonally measured distance from τ is ε_p. This is a shrunken but otherwise undeformed copy of τ and thus each of its vertices is linearly independent and independent of each vertex in the interior of τ. Thus, the simplex τ_p' spanned by τ_p and the barycenter of τ is maximal (since it is equidimensional and each of its vertices in on the boundary) and proper in $\blacktriangle_1^X$ and does not contain p, so that $\{p\} \vee \tau_p' = \top$.

Remark 3.74. Example 3.73 would seem like a trivialization of our theory in a part of synthetic probability theory; and it is - on its own. But it is re-contextualized by Proposition 3.58, which shows that smallness in the sheaf topos of a site $\mathcal{S}$ only depends on the coverage on $\mathcal{S}$. So for a site, having rough smallness

[3] This geometric interpretation was pointed out to me by Fritz.

relations is in a sense welcome as long as the coverages on it are interesting. So the key point of Example 3.73 is not that smallness in **FStoch** is trivial but that its structure is very similar to that of **TDisk** from Example 3.18, which is quite important, as mentioned in Remark 3.20. We will show in later work that this similarity of site structures descends to one of topoi, which opens up a connection of synthetic probability theory to cohesion and closed geometric logic that could be of interest.

3.2 Reducedness & Satedness

We can already describe what it means for an element to not contain any small elements, except within non-small ones.

Definition 3.75. An element a of an upward-bounded preorder P is *reduced* if for any small b and arbitrary c such that $a \leq b \vee c$ it holds that $a \leq c$.

Dually, a is *sated* if for every dense b and c such that $a \geq b \wedge c$ it holds that $a \geq c$.

So a is small if the only b such that $a \vee b = \top$ is $\top$ and a is reduced if whenever $a \leq b \vee c$ with b small, then $a \leq c$. Since $\top$ is at the top, $a \vee b \geq \top$ implies $a \vee b = \top$, so for $\top$ this inequality simplifies to the defining equality for smallness. So an element is reduced if it is in the same relation to small elements that $\top$ has. Dually, an element is sated if it contains any subobject that it contains a dense subobject of.

Example 3.76. A paradigmatic example are the closed subsets $\mathbf{C}(X)$ of a topological space: each closed C is equal to the union $|C| \cup \mathbf{s}(C)$ where $|C|$ is the reduction of C, i.e. the closure of its interior, and $\mathbf{s}(C)$ is the closure in C of the complement $C \setminus |C|$. Thus, if C is reduced, then $C \leq |C|$ and hence $C = |C|$. Conversely, if $C \leq D \cup E$ with D small, then the interior C° is contained in the interior $(D \cup E)^\circ$, which is equal to E by smallness. Thus, in the lattice of closeds of a topological space an element is reduced if and only if it is the closure of an open subset, so if it is regular. Dually, an open subset is sated if and only if it is the interior of some closed subset, so also if it is regular.

Example 3.77. In Example 3.39 we saw that a submonoid C is dense if and only if it contains a power of each element in M. So A is sated if and only if whenever $A \geq B \cap C$ for some dense C, then $A \geq B$, or equivalently if and only if it contains each root of its elements.

Let us note that at least in bounded preorders smallness and reducedness are exclusive in non-trivial cases:

Proposition 3.78. *Let P be a bounded preorder. Then a is both small and reduced if and only if $a = \bot$.*

Proof. Because P is bounded, $\bot$ exists. $\bot$ is small: $\bot \vee b = b$, so $\bot \vee b = \top \Rightarrow b = \top$. $\bot$ is reduced: if $b \vee c \leq \bot$ with b small, then $b \leq \bot$ and $c \leq \bot$, hence $b = \bot = c$, giving $\bot = c$.

For the converse, assume a is both small and reduced. Choose $b = a$ (which is small) and $c = \bot$. Then $a \leq a \vee \bot$, so by reducedness $a \leq \bot$, so $a = \bot$. ◻

3.3 The Smallness Stratification

We can use smallness to stratify every upward-bounded preorder:

Definition 3.79. Given an upward-bounded preorder P, every element $a \in P$ is 0-*small*. a is n-*small* for n a successor ordinal if there is an $(n-1)$-small element $b \geq a$ such that a is small in the (restriction of P

to the) principal ideal (b) of b. a is *purely n-small* if a is n-small but not $(n+1)$-small. a is *ω-small* for ω a limit ordinal if it is n-small for all $n < \omega$.

It follows immediately from the definition of n-smallness and the monotony of smallness that this assignment forms a stratification:

Definition 3.80. Let $\mathbf{Ord}$ be the large preorder of ordinals. Given an upward-bounded preorder P the *smallness stratification* $\mathbf{small}\colon P \to \mathbf{Ord}$ is the order morphism given by mapping $a \in P$ to n if and only if a is purely n-small, and to ω if a is ω-small. The *smallness fibration* is the morphism $\coprod_{n \in \mathbf{Ord}} \mathbf{small}^{-1}(n) \to P$ classified by $\mathbf{small}$. We define the *density stratification* and *density fibration* dually.

Proposition 3.81 (Well-definedness of the smallness stratification)**.** *Let P be a (small) upward-bounded preorder. For every $a \in P$, there exists an ordinal n such that a is purely n-small.*

Proof. Assume that a is not purely n-small for any n. By definition a is 0-small. To not be purely 0-small, a must be 1-small, meaning a is small in some 0-small principal ideal. Proceeding by induction, a is $(n+1)$-small for each n. For each limit ordinal ω, if a is n-small for all $n < \omega$, then a is ω-small by definition. Thus, by transfinite induction, a is α-small for every ordinal α. But α-smallness requires a to be small within some $(\alpha - 1)$-small principal ideal, which itself must be small within some $(\alpha - 2)$-small principal ideal, and so on. This creates a strictly ascending chain of principal ideals of length α for each ordinal α, so there exists an injection $\mathbf{Ord} \hookrightarrow P$. Consequently, P cannot be small. □

So smallness stratifies each upward-bounded preorder into sections of the same size. This is very similar to how dimension stratifies various subobject lattices, and indeed these two stratifications often overlap:

Example 3.82. In Example 3.23 we saw that a (topological) sub-manifold with boundary $\iota\colon A \hookrightarrow X$ is small in a manifold with boundary X if and only if $m := \mathbf{dim}(A) < n := \mathbf{dim}(X)$. If A is non-empty then there exists $p \in A$, and thus a local isomorphism σ_p from some open U_p in X to either R^n or $R^n_{\geq 0}$, which without loss of generality we can choose so that $\iota|_{U_p} \mathbin{;} \sigma$ is isomorphic to R^m. In particular we can for each $m < n' < n$ find an n'-dimensional disk or half-disk D_p around $\iota|_U \mathbin{;} \sigma(p)$. Let us take the union $\kappa := \bigcup_{p \in A} D_p$ in X. Then the $\sigma_p|_{D_p}$ provide an atlas for κ, so that κ is an m-dimensional submanifold with boundary of X. Thus, the smallness fibration in $\mathbf{SOb}_{\partial\mathbf{Mfld}}(X)$ agrees with the dimensional fibration.

However, note that for instance in Example 3.18 the smallness stratification does not align with the dimensional one at all.

3.4 Monoidal Smallness

Building on Remark 3.1 we can extend Definition 3.3 in another way, namely by replacing joins and meets by some other monoidal product:

Definition 3.83. An element a of an ordered monoid M with a product $+$ and a top element $\top$ is *$+$-small* if and only if for each $b \in M$ such that $a + b = \top$ it holds that $b = \top$. Dually, an element a of an ordered monoid M with a contravariantly monotone monoidal product $\otimes$ and a bottom element $\bot$ is *$\otimes$-dense* if and only if for each $b \in M$ such that $a \otimes b = \bot$ it holds that $b = \bot$.

One of these notions is infinitely more useful than the other. Can you guess which one?

Correct, it is $+$-smallness. This is because for the most important such product, addition, is strictly increasing on positive quantities $a + b \geq a, a + b \geq b$. The reason for this is the deep connection between joins and addition, which allows us to use Definition 3.83 for distinguishing finitude and infinity. This connection will only be sketched here, a deeper treatment will have to wait for the introduction of tols in subsequent work. But the essential point is in how the joins of subsets of natural numbers interact with addition of their cardinalities. Consider the preorder $\mathbf{P}(N)$ of subsets of N: by Example 3.8 no non-trivial element of $\mathbf{P}(N)$ is small or dense. On the other hand, in N_∞ only ∞ is not small by Example 3.10, so if we project each finite subset of N to its cardinality and each infinite subset to ∞, then the smallness structure of $\mathbf{P}(N)$ is destroyed. This is because this projection concentrates $\mathbf{P}(N)$ into a line N_∞. In particular, if $\iota \leq_{N_\infty} \kappa$ in the induced order $\leq_{N_\infty}$ on $\mathbf{P}(N)$ given by $\iota \leq_{N_\infty} \kappa := |\iota| \leq |\kappa|$, then there is some ι' of the same cardinality that is a subset of κ. So the cardinality $|{-}|$ reflects $|\iota| \vee |\kappa|$ if $\iota \leq \kappa$ but otherwise it essentially brings ι in a position where its overlap with κ is the largest and *then* takes the join, which is the least expansive way to take a join in $\mathbf{P}(N)$. But what if we take the join in *most* expansive way before projecting to N_∞, i.e. make ι and κ disjoint before taking the join? The natural numbers are actually the smallest set for which we can coherently do this, since they have an isomorphism to their sum:

$$\gamma : N \xrightarrow{\sim} N + N = N_1 + N_2 : n \mapsto \begin{cases} (n/2)_1 \text{ if } n \textit{ mod } 2 = 0 \\ ((n-1)/2)_2 \text{ else.} \end{cases}$$

So if we have two subsets $\iota, \kappa \subseteq N$, we can apply γ^{-1} to $\iota + \kappa \subseteq N + N$, which gives us two subsets $\iota' := \gamma^{-1}(\iota + \varnothing)$ and $\kappa' := \gamma^{-1}(\varnothing + \kappa)$ such that $\iota' \cap \kappa' = \varnothing$ and thus $\iota' \cup \kappa'$ is maximally large. So we can define $\iota \oplus \kappa := \gamma^{-1}(\iota + \kappa) = \iota' \cup \kappa'$, which provides a semicocartesian laxly unital monoidal operation on $\mathbf{P}(N)$ (although we postpone proofs of semicocartesianism, lax unitality and weak associativity to future work, where we provide them in a more general setting) such that $|\iota \oplus \kappa| = |\iota| + |\kappa|$. So addition is the disjoint union of subsets in the most literal sense.

Now, for $n \in N_\infty$ consider the sub-preorder $[0, n]$ of N_∞ with the monoidal product $+_n$ given by $(a+b) \wedge n$ (note that this is still associative). Since $[0, n]$ is a total order, the only non-small element of $[0, n]$ is n, but when is $a \in [0, n]$ $+_n$-small? It is easy to see that if n is finite, then a is $+_n$-small if and only if $a = 0$ but if $n = \infty$ then a is $+_n$-small if and only if $a < \infty$. So the difference in size between finitude and infinity only becomes significant if we assume the capability of duplication, whereupon every finite quantity becomes small compared to infinity, which cannot be reached through a finite number of duplications. In particular, if we identify all infinite subsets of N (e.g. use the N_∞-valued scale from Example 4.19), then ι is $\oplus$-small if and only if it is finite.

We will not use monoidal smallness in this text but take it up in future work.

4 Tablets & Scales

Let us put our new paradigm into praxis by providing the means for different subobject preorders to interact. For now, we do this abstractly on the level of preorders, in Section 7 we will lift our constructions to the level of categories.

4.1 Tablets

We start by providing some tablets:

Definition 4.1. Let $\tau : T \to P$ be a morphism of upward-bounded preorders and a an element of P. We call τ the *tablets* we use.

a is *small under* τ if for each join $a \vee \tau(b) = \top_P$ it holds that $b = \top_T$.

a is *reduced under* τ if for each element $b \in P$ that is small under τ and each $c \in T$ it holds that

$$\text{if } a \leq b \vee \tau(c) \text{ then } a \leq \tau(c). \tag{1}$$

Remark 4.2. The use of inequalities for reducedness in Definition 4.1 allows us to bound the fuzziness between the images of elements of T. Note that the formula $a \vee \tau(b) = \top_P$ is already equivalent to $a \vee \tau(b) \geq \top_P$ due to the definition of a top, so for smallness the question of whether to use an inequality doesn't arise.

We can understand these tablets as the forms we use to infer subobject-theoretic questions in our base structure when the collection of all its subobjects is too fine. This mirrors how it is done with topological spaces:

Example 4.3. Building on Example 3.76, consider the inclusion $\iota_{\mathbf{C},X} : \mathbf{C}(X) \hookrightarrow \mathbf{P}(X)$ of the closed subsets of a topological space X into all subsets. A subset $a \subseteq X$ is small under $\iota_{\mathbf{C},X}$ if and only if X itself is the only closed subset c such that $a \cup c = X$, so if its closure is small, i.e. nowhere dense. A subset $b \subseteq X$ is reduced under $\iota_{\mathbf{C},X}$ if and only if whenever $b \leq a \cup c$ with a nowhere dense and c closed, then already $b \leq c$, so if the closure $\overline{b}$ is regular. So our tablets allow us to bound the sets in $\mathbf{P}(X)$ by those in $\mathbf{C}(X)$.

Dually, if we take $\iota_{\mathbf{O},X} : \mathbf{O}(X) \hookrightarrow \mathbf{P}(X)$ as the tablets, then a subset a of X is dense under $\iota_{\mathbf{O},X}$ if and only if its meet with every open subset is non-empty, so if it is dense in the topological sense. b is sated if and only if whenever b is locally dense in that $b \geq a \cap c$ with a open and c dense, then b just contains a.

In particular, let X be a metric space with its underlying topology and the tablets τ again given by the opens of X. Then a subset $\iota : A \hookrightarrow X$ is τ-dense if and only if for each $x \in X, r \in R_{>0}$ the intersection $\iota \cap U_{x,r}$ with the open ball around x with radius r is nonempty, i.e. if for each x there exists some Cauchy sequence converging to x. Thus, ι is sated if and only if for each union of Cauchy sequences δ that converge to all of X, if ι contains the restriction of δ to some open U, then ι contains all of U. So our notion generalizes a topological relative version of Cauchy completeness.

We can also zoom into a point x by reducing our collection of tablets to the opens of X containing x. This allows us to say that ι *is* τ*-dense at* x if it contains some sequence converging to x. But actual Cauchy completeness is not straight-forwardly described as satedness in this way, which makes sense since it is not homeomorphism-invariant.

There are some subtleties in the definition though, namely the inequality and the restriction to T for reducedness. To understand these, let us consider another example where we see how they work out:

Example 4.4 (Density and satedness for monoid actions)**.** Let M be a monoid and X a set with a left action $\lambda : M \hookrightarrow \mathbf{End}(X)$. Write $\mathbf{SOb}_\lambda(X)$ for the preorder of sub-λ-sets of X, ordered by inclusion; it is a bounded lattice with meets given by intersection and joins by union (since each union of λ-sets is a λ-set).

Consider the tablets given by the inclusion $\tau_{\mathbf{SOb}_\lambda} : \mathbf{SOb}_\lambda(X) \hookrightarrow \mathbf{P}(X)$ of sub-λ-sets. Applying the definitions of density and satedness from Remark 4.17 we obtain that:

1. A subset $D \subseteq X$ is dense under $\tau_{\mathbf{SOb}_\lambda}$ if and only if every sub-λ-set $S \subseteq X$ such that $D \cap S = \varnothing$ is equal to $\bot_\tau = \bigcap_{\iota \in \mathbf{SOb}_\lambda(X)} \iota = \varnothing$ since the empty set is closed under the action of λ. Equivalently, D meets every orbit.

If k is a ring and X a k-module, then we can further restrict the tables τ' to be sub-modules of X. Then $\bot_{\tau'}$ is no longer empty however, since each submodule of X contains a neutral element. In particular let k be a field and V a k-vector space equipped with the action of k by multiplication. The orbits are $\bot_\tau = \{0\}$ and the lines $\mathcal{O}_v = \{\lambda v \mid \lambda \in k\}$ for $v \neq 0$. Thus a subset $D \subseteq V$ is dense if and only if $D \cap \mathcal{O}_v \neq \varnothing$ for every $v \neq 0$. One simple way this can happen is when 0 is in D, otherwise D has to intersect all orbits individually.

For instance if we take $k = R$ and $V = R^n$, the nonzero orbits are the lines through the origin, which correspond bijectively to points of the real projective space $R\mathbf{P}^{n-1}$ via the direction map

$$q\colon R^n \setminus \{0\} \to R\mathbf{P}^{n-1}, \quad v \mapsto [v],$$

where $[v]$ denotes the equivalence class of v under scalar multiplication. If we instead take k as $R_{\geq 0}$ (but leave V the same), then q goes to $\mathbb{S}^{n-1}$. Hence D is dense precisely when it contains the origin or its image under q is all of $\mathbb{S}^{n-1}$; in geometric language, D is *radially dense*: it is either at the center or has at least one point in every direction from the origin.

2. A subset $A \subseteq X$ is sated under $\tau_{\mathbf{SOb}_\lambda}$ if and only if whenever $A \supseteq D \cap B$ with D dense and B an λ-subset of X, then already $A \supseteq B$. If now a is some element of A and O an orbit that contains a, then $(\{a\} \cup \bigcup_{\mathcal{O}'} \mathcal{O}') \setminus \bot_\tau$ is dense, where $\mathcal{O}'$ ranges over the orbits of X that do not contain a and the complement is set-theoretic. Moreover, if O is an orbit that contains a and whose intersection with any orbit that does not contain a factors through $\bot_\tau$, then $O \cap (\{a\} \cup \bigcup_{\mathcal{O}'} \mathcal{O}') = \{a\} \subseteq A$, so that $O \subseteq A$. In particular, if λ is a group or the action is one of a field on a vector space, then all orbits fulfill this condition , so that A has to contain every orbit it intersects.

3. $S \subseteq X$ is small under $\tau_{\mathbf{SOb}_\lambda}$ if the smallest λ-set T whose union with S is X is the largest λ-subset, which is X itself, so if the complement of S is in no proper λ-subset of X, meaning that the closure of $\neg S$ under the action of λ contains S, so if each element of $S \setminus \bot_\tau$ is a multiple of one outside it. If M is a group then its orbits are disjoint partitions, so that S is small if and only if it does not contain an orbit. So in R^n as an R-module S is small if it does not contain a (linear) line. But in a general monoid the property is looser: for instance in R^n as a Z-module no central n-disk of radius > 0 is small, since its elements cannot be generated by elements outside it.

4. $R \subseteq X$ is reduced if for each small S and λ-set Q if $R \subseteq S \cup Q$, then $R \subseteq Q$. Now, if $R \subseteq S \cup Q$, then $R \subseteq \overline{S} \cup Q$, where $\overline{S}$ is the closure of S under the action of λ, so R is reduced if any part of R that is in a small λ-subset is also in any λ-subset that contains the rest of R.

Between these two examples we can get at how Definition 4.1 works: in Example 4.3 we infer about arbitrary subsets of a topological space through its open and closed subsets. In Example 4.4 we similarly infer about arbitrary subsets of a module, but with some added difficulties and particularities: for one, the intersection of all tablets is not empty, so if, for instance, we did not use the meet of tablets $\bot_M$ but demanded that a dense subset have a non-empty intersection with each non-empty tablet, then a dense subset would have to intersect with $\bot_M$ and so over a field contain 0, which would mess up the satedness calculation. Similarly, if, in Eq. (1) we assumed that c would be in P, so if we applied that equation to all subsets in Example 4.4, then the condition for satedness would be way too stringent and trivialize. *But* if we do restrict c to T but assume equality in Eq. (1) instead of inequality, then Example 4.3 would no longer work because we could not infer anything about subsets more general than intersections of opens and dense subsets. It is only when we take inequalities and restrict c to T that we get the “obviously” right results for both examples.

But the calculus of tablets generally allows us to be creative in its usage:

Example 4.5. In Corollary 3.16 we found that if a preorder $\boldsymbol{P}$ is atomistic, then its notion of density is trivial, and similar for smallness. However, if $\boldsymbol{P}$ is bounded in both directions we can obtain an interesting notion of density by considering the embedding $\iota_{cat} : \boldsymbol{P}_{cat} \hookrightarrow \boldsymbol{P}$ of the bounded preorder on co-atoms $\boldsymbol{P}_{cat}$ into $\boldsymbol{P}$. Density with respect to these tablets is more about the position in the preorder than dimension: an element is ι_{cat}-dense if and only if its intersection with every co-atom is non-empty. Dually, an element is ι_{at}-small if and only if it is in a proper face with every other atom. For points in a polytope this is a known property and a vertex that fulfills it is called a *universal vertex* or *dominating vertex.*

If we apply this to the objects of the sites in Observation 3.11 an interesting split emerges:

1. In globes and simplices (above dimension one), every (maximal) face touches every other face and thus all are cat-dense, but no other faces are cat-dense. On the other hand, every non-empty proper face shares some non-empty non-maximal face with any vertex, so all are small, while every maximal face shares no proper face with the vertex opposite to it, and thus is not small.
2. In cubes over dimension one, every proper face has an opposite face which it does not touch and thus is not small or dense. So no non-trivial faces or points are cat-dense or at-small.
3. In abstract polytopes, any case can arise. Moreover, while the abstract duality of the two makes it so that in many cases, such as the above, the two develop analogously, this is not according to any straightforward formal duality. For instance, consider an n-ary pyramid, consisting of a tip τ, an opposite face ω, n rays ρ_i between them and the n induced faces φ_i. In this, there is a clear duality: there is one at-small vertex τ and one cat-dense maximal face ω and for $n > 3$ no other faces are cat-dense or at-small. But if we now take any ray ρ_i and put a small digon into the middle of it in such in a way that it does not touch anything except ρ_i and its two adjacent faces, then τ remains small but (again for $n > 3$) there is no dense face anymore.

This points again at the interesting geometry of abstract polytopes, which will be featured more in future work.

But we postpone further examples a little. Let us first note that this extension is rather conservative:

Proposition 4.6. *Let L be an upward-bounded preorder and $T' \xrightarrow{\tau'} T \xrightarrow{\tau} L$ be morphisms of preorders of L such that τ and τ' reflect tops . If $b \in L$ is τ-small, then it is $(\tau' \mathbin{;} \tau)$-small. Moreover, if τ' is surjective, then $(\tau' \mathbin{;} \tau)$-smallness and τ-smallness are equivalent. Thus, the smallness of b only depends on the underlying set of T. In particular, we can assume without loss of generality that τ, τ' are order-reflecting embeddings (and thus fully faithful).*

Proof. If the only $a \in T$ such that $\tau(a) \vee b = \top_L$ is $\top_T$, then if $(\tau' \mathbin{;} \tau)(a') \vee b = \top_L$ it follows from τ-smallness that $\tau'(a') = \top_T$ and since τ' reflects $\top_T$, $a' = \top_{T'}$. Moreover, since the join is taken in L, the order-theoretic structure of T does not matter. □

In particular Proposition 4.6 tells us that, the rougher our tablets, the harder it is to use them to complement some other ι without them covering the entire space, so the more things will look not small, which makes sense. More generally:

Proposition 4.7 (Lax comparison for smallness)**.** *Let L be an upward-bounded preorder, and let $\tau : T \to L$ and $\tau' : T' \to L$ be monotone maps. Suppose there exists a monotone map $f : T' \to T$ such that:*

1. *$\tau' \leq f \mathbin{;} \tau$, i.e. $\tau'(t') \leq \tau(f(t'))$ for all $t' \in T'$;*
2. *if $\tau(f(t')) = \top_L$ then $t' = \top_{T'}$.*

Then every τ-small element of L is also τ'-small.

Proof. Assume $b \in L$ is τ-small, and let $t' \in T'$ satisfy $b \vee \tau'(t') = \top_L$. From $\tau'(t') \leq \tau(f(t'))$ we obtain $b \vee \tau(f(t')) = \top_L$. By τ-smallness of b, we must have $\tau(f(t')) = \top_L$. But Item 2 then forces $t' = \top_{T'}$, which is exactly what is required for b to be τ'-small. □

Remark 4.8. In the special case where f is surjective and $\tau' = f \mathbin{;} \tau$ so the inequality in Item 1 becomes an equality Item 2 is trivial, we recover the original statement of Proposition 4.6. When τ and τ' are order-embeddings, Item 2 simply says that if τ contains the top of L, then τ' does as well.

This also allows us to recover the usual order-theoretic notion of density:

Example 4.9. We found in Example 3.24 that a subset of a graph is small if and only if contains no edges or isolated vertices. Given a preorder P, let $\iota_{\textbf{non-small}_\textbf{G}} \colon \textbf{non-small}_\textbf{G}(P) \hookrightarrow \textbf{SOb}(P)$, be the embedding of all full sub-preorders of P that are not small in the underlying graph of P (without identities in P). Then $\textbf{non-small}_\textbf{G}(P)$ contains all isolated vertices and proper intervals in P, i.e. the full sub-preorders $[a, b]$ for $a < b$ consisting of all elements c in P such that $a \leq c \leq b$. Thus, a sub-preorder of P is $\iota_{\textbf{non-small}_\textbf{G}}$-dense if and only if it meets isolated vertex and every proper interval in P non-trivially, so if it is dense in the order-theoretic sense.

The final point we want to make is that, as with scales, the power of tablet exceeds our use case. In reality they are ubiquitous in mathematics and what has to be underlined is both their elementarity and their heteromorphy, which can be displayed by an example we hinted at in the introduction: cones and hyperplanes are not common subobjects of R^n in any natural category that does not also contain many more subobjects. What connects them is that they can be embedded into the same object, R^n, which is incarnated in different categories. *So we fix the object and vary the categories* when we use hyperplanes as tablets to study cones. And while this is a somewhat abstract way to state what we are doing, the result are conic sections, one of the most venerable subjects of mathematics![4]

4.2 Scales

We can relativize this system in the other direction as well, which allows us to encompass measure-theoretic examples, as we will do in Section 7.2:

Definition 4.10. A *scale* on an upward-bounded preorder P is a map $P \xrightarrow{\sigma} S$ into some preorder S. An element a of P is *small over σ* if for each $b \in P$ and join $a \vee b = \top_P$ it holds that $\sigma(b) = \top_\sigma$ where $\top_\sigma$ is the top of the image of σ. Dually for density.

c is *reduced over σ* if for each a that is small over σ and $b \in P$ such that $a \vee b \geq c$ it holds that $\sigma(b) \geq \sigma(c)$. Dually for satedness.

[4]Note that all the geometric information of R^n is contained in $\mathbf{P}(R^n)$ and its automorphisms, so when we analyze conic sections as the subobjects of a cone $\chi \colon C \hookrightarrow R^n$ generated by the tablets that are hypersurfaces we can apply standard geometric arguments. Of course for these arguments we might need to introduce further kinds of subobjects, e.g. lines, but note how these have to be treated on a different levelling from hypersurfaces to not mess up the definition of conic sections. Note also that we are using only a tiny part of $\mathbf{P}(R^n)$, we could restrict it a lot, e.g. to semialgebraic subsets. To obtain algebraic representations we would also have to fix the coordinate system induced by the coprojections $R \hookrightarrow R^n$.

Remark 4.11. Unlike with tablets and P itself we do not necessarily assume that scales are upward-bounded. The difference is small, but conceptually a scale need not come from the subobjects of some object.

Example 4.12. Let $\sigma : \mathbf{M}(X) \to R_{\geq 0,\infty}$ be a measure on the σ-algebra of a measure space. A set S in M is small over σ if and only if for each T such that $S \cup T = X$ it holds that $\sigma(T) = \sigma(X)$. There is a minimal such choice, which is the Boolean negation $\neg X$, so S is small when $\sigma(\neg S) = \sigma(X)$. By binary additivity, $\sigma(S) + \sigma(\neg S) = \sigma(X)$, so when $\sigma(X)$ is finite we can form $\sigma(S) = \sigma(X) - \sigma(\neg S) = 0$ since $R_{\geq 0}$ is cancellative. If $\sigma(X) = \infty$, then S can be small for arbitrary $\sigma(S)$, so our notion is not *exactly* the same as negligibility, but we can derive measure-theoretic negligibility from σ-smallness by defining it through restriction to finite-measure subspaces.

Now, R is reduced over σ if for each small S and arbitrary T if $R \subseteq S \cup T$ then $\sigma(R) \leq \sigma(T)$. Since a σ-algebra is closed under complementation we can reduce to the case that S and T are disjoint, so that $\sigma(R) \leq \sigma(S) + \sigma(T)$. So if $\sigma(X)$ is finite then this holds for all elements of $\mathbf{M}(X)$, since $\sigma(S) = 0$. If $\sigma(X)$ is ∞ then we have to think some more. First, note that in that case the complement of each finite-measure subset of X has infinite measure, so each finite-measured R is small and only reduced if its measure is 0. If $\sigma(R) = \infty$ then R reducedness implies $\sigma(T) \geq \sigma(R) = \infty$, so $\sigma(T) = \sigma(R)$, so that R is infinite-measured on each infinite-measured set. This holds if and only if $\neg R$ has finite measure.

D is dense over σ if for each T such that $D \cap T = \varnothing$ it holds that $\sigma(T) = 0$, so if its complement is negligible. A is sated if for each such dense subset D it holds that if A contains $D \cap T$ then $\sigma(A) \geq \sigma(T)$. Since $\sigma(T) = \sigma(T \cap D) + \sigma(T \wedge \neg D) \leq \sigma(A) + \sigma(\neg D) = \sigma(A)$, so this holds trivially in our case.

Note in particular that we can do this for the linear extension $|{-}| : \mathbf{P}(N) \to N_\infty$ of the unit measure, i.e. cardinality for finite sets and ∞ for infinite ones. Then a subset S of N is σ-small if and only if each T such that $S \cup T = N$ has infinite cardinality, so if the complement of S is infinite, so if S is co-infinite.

So in general our framework allows us to measure smallness by introducing two optional parameters, tablets and a scale:

Definition 4.13. Let $T \xrightarrow{\tau} P \xrightarrow{\sigma} S$ be a triple of maps of preorders where we assume that T and P are upward-bounded, and let a be an element of P. Then a is *small under τ over σ* if for each join $a \vee \tau(b) = \top_P$ with b in T it holds that $(\tau \,;\, \sigma)(b) = \top_\sigma$ where $\top_\sigma$ is the top of the image of σ. Dually for density.

c is *reduced under τ over σ* if for each a that is small under τ over σ and $b \in T$ such that $a \vee \tau(b) \geq c$ it holds that $(\tau \,;\, \sigma)(b) \geq \sigma(c)$. Dually for satedness.

The first generalizes topological inferences, the second measure theoretic ones.

Notation 4.14. As a convenient shorthand, we just write the parameters next to each other separated by |-signs. So we write "$(\tau \mid \sigma)$-small" for "small under τ over σ". Since all parameters are optional we leave those out that we don't need.

We have a statement similar to Proposition 4.6 but on the scale side:

Proposition 4.15 (Lax comparison for scales)**.** *Let P be an upward-bounded preorder, let $\tau : T \to P$ be a monotone map of tablets and $\sigma : P \to S$, $\sigma' : P \to S'$ monotone maps of scales. Suppose there is a monotone map $g : S \to S'$ such that:*

1. *$\sigma' \geq \sigma \,;\, g$, i.e. for all $p \in P$,*
$$g(\sigma(p)) \leq_{S'} \sigma'(p),$$

2. *the top elements of the images of σ and σ' correspond:*

$$g(\top_\sigma) = \top_{\sigma'},$$

where $\top_\sigma$ and $\top_{\sigma'}$ denote the top elements of the images $\mathbf{Im}(\sigma)$ and $\mathbf{Im}(\sigma')$, respectively.

Then any element of P that is $(\tau \mid \sigma)$-small is also $(\tau \mid \sigma')$-small.

Proof. Let $a \in P$ be $(\tau \mid \sigma)$-small. By Definition 4.13, this means:

$$(\forall b \in T)\ \big(a \vee \tau(b) = \top_P\big) \Rightarrow \sigma\big(\tau(b)\big) = \top_\sigma.$$

Now suppose $b \in T$ satisfies $a \vee \tau(b) = \top_P$. Then by $(\tau \mid \sigma)$-smallness,

$$\sigma\big(\tau(b)\big) = \top_\sigma.$$

Applying g and using Item 2 gives

$$g\big(\sigma(\tau(b))\big) = g(\top_\sigma) = \top_{\sigma'}.$$

Using Item 1, we also have

$$g\big(\sigma(\tau(b))\big) \leq \sigma'(\tau(b)).$$

Since $\top_{\sigma'}$ is the maximum element in the image of σ', this inequality forces

$$\sigma'(\tau(b)) = \top_{\sigma'}.$$

This is exactly the condition that a is $(\tau \mid \sigma')$-small. □

Corollary 4.16. *Given $\sigma : P \to S$ and an order-preserving bijection $f : S \to S'$, smallness over σ and $\sigma \mathbin{;} f$ are equivalent.*

This is the exact scale-side analogue of Proposition 4.6 for tablets: the "rougher" the scale (in the sense of collapsing more values into the top of its image), the more elements of P appear small with respect to it. In particular, the only contribution of σ to the measurement is the filter in the preimage of its maximal image element, so one might question whether its definition might not be simplified. But there are reasons for doing things this way: for one, consider the dualization of Definition 4.13, concerning dense elements: it is identical except for using downward-bounded preorders.

Remark 4.17 (Dualization for density and satedness). By order-theoretic duality, one obtains analogous notions for downward-bounded preorders, which we call *density* and *satedness* under a tablet and over a scale. Explicitly, let $T \xrightarrow{\tau} P \xrightarrow{\sigma} S$ be morphisms of preorders, and assume that P and S are downward-bounded (with bottoms $\bot_P$ and $\bot_S$) and that the image of σ has a bottom element $\bot_\sigma$.

An element $d \in P$ is *dense under τ over σ* if for every $b \in T$ with $d \wedge \tau(b) = \bot_P$, it holds that $(\tau \mathbin{;} \sigma)(b) = \bot_\sigma$. An element $e \in P$ is *sated under τ over σ* if for every $d \in P$ that is dense under τ over σ and every $b \in T$ such that $d \wedge \tau(b) \leq e$ we have $(\tau \mathbin{;} \sigma)(b) \leq \sigma(e)$.

Thus, if the preorders in question have both an upward and a downward bound, *the same data* can be used to define them, in which case both the upper filter and the lower ideal of σ matter. Moreover, our treatment allows us to consider the smallness relations in different preorders using the same scale and/or tablet preorder by forming under/overcategories and provides a fluid transition to measure theory, which will be used in Section 7.

4.3 Examples

This system allows us to mix topological and measure-theoretic smallness:

Example 4.18. A (not necessarily open) subset ι of X is dense with respect to the morphism triple $\mathbf{O}(X) \hookrightarrow \mathbf{M}(X) \to R_{\infty,\geq 0}$, where $\mathbf{M}(X)$ are the measurable subsets of X, if and only if for each open subset κ of X it holds that if $\iota \wedge \kappa = \varnothing$ in $\mathbf{M}(X)$, then κ has measure 0. So ι is dense if and only if the only open subsets that it does not intersect with are measure-theoretically negligible.

If we take, for concreteness, $X = R^n$ with its usual measure and topology, then this is a slightly disappointing example since no non-trivial open subset of R^n has measure 0. But what if we switch out $\mathbf{O}(X)$ with $\mathbf{C}(X)$? Then ι is dense if and only if each closed subset κ of R^n that ι does not intersect with has measure 0. Now, if κ is regular, so if it is the closure of a non-trivial open, then its measure is surely not 0, so it has to intersect with ι. But there are small (and so in particular non-regular) closed subsets of R^n that do not have measure 0, such as a *fat Cantor set* (or *Smith-Volterra-Cantor set*), which is closed, nowhere dense, yet has positive Lebesgue measure.

Thus, the density condition with respect to closed tablets and the Lebesgue scale is sensitive to these "thick" nowhere dense sets while also excluding subsets that are too small such as atoms. In any case, it is a different and unexpected notion. Now, what does completeness mean? Well, if $\iota \supseteq \kappa \cap \lambda$, with κ closed-dense and λ closed, then Remark 4.17 tells us that $\sigma(\iota) \geq \sigma(\lambda)$, so completeness says that if ι is dense in a closed λ, then ι is at least as big as λ when measured.

Dually, ι is closed-σ-small if and only if each closed κ such that $\iota \cup \kappa = X$ has the same measure as X and similar for opens, and ι is closed-reduced if for each closed κ and small λ it holds that if ι is contained in $\kappa \cup \lambda$ then $\sigma(\iota) \leq \sigma(\kappa)$ and similar for opens.

It also allows us to introduce weight and rounding:

Example 4.19. Let S be a set, $w\colon S \to R_{\geq 0,\infty}$ an arbitrary weighting function and $\overline{w}\colon \mathbf{P}(S) \to R_{\geq 0,\infty}$ be defined by linear extension, i.e. for some subset $\iota\colon A \hookrightarrow S$ of S and well-ordering $\overset{>}{A}$ of A the limit $\mathbf{lim}_{a\in\overset{>}{A}} \sum_{b\in(a)} w(\iota(b))$, with that being ∞ if that series diverges. Note that since the reals are Cauchy-complete and w is positively valued, each such $\overline{w}$ on each ι either goes to infinity for every well-ordering of A or is absolutely convergent on some well-ordering, which implies that it is so on every well-ordering. In other words, this definition is independent of the chosen order. Since $\overline{w}$ is given by linear extension, it is binarily additive, so smallness and density are special cases of Example 4.12.

Example 4.20. In the situation of Example 4.12 let $\lfloor - \rfloor_c\colon R_{\geq 0,\infty} \to N_{\geq 0,\infty}$ be given on $r \in R_{\geq 0,\infty}$ by the largest $n \in N_{\geq 0,\infty}$ such that $r \geq nc$. Note that these definitions also work for $r = \infty$ and $c = \infty$. Then ι is $(\sigma \mathbin{;} \lfloor - \rfloor_c)$-small if and only if $\lfloor \sigma(X) \rfloor_c = \lfloor \sigma(\neg\iota) \rfloor_c$. So since $\sigma(X) = \sigma(\neg\iota) + \sigma(\iota)$ we have that ι is $(\sigma \mathbin{;} \lfloor - \rfloor_c)$-small if and only if

$$\lfloor \sigma(\neg\iota) + \sigma(\iota) \rfloor_c = \lfloor \sigma(\neg\iota) \rfloor_c$$

.

So basically ι is small if and only if it fits within a rounding error.

Dually, ι is dense if its intersection with each $\geq c$-large subset is non-nil, and sated if for every dense κ and λ, if ι contains κ in λ, then the measure of ι is as large or larger as the measure of λ, up to rounding downwards.

But it is generally of considerable flexibility. For instance we can say:

Example 4.21. Let $\iota\colon \mathbf{P}_{inf}(N) \hookrightarrow \mathbf{P}(N)$ be the embedding of the sub-preorder of $\mathbf{P}(N)$ whose non-trivial elements are the infinite subsets of N. Then $S \subseteq N$ is ι-dense if and only if its intersection with each infinite subset of N is infinite, i.e. if its complement is finite, so if S is co-finite. S is ι-sated if and only if whenever $S \supseteq Q \cap T$ with Q infinite and T co-finite, then $S \supseteq Q$. This holds trivially for finite sets. If S is infinite but not N, choose $x \notin S$. Set $Q = S \cup \{x\}$ (infinite) and $T = N \setminus \{x\}$ (co-finite). Then $Q \cap T = S$ but $Q \neq S$, violating satedness. For N, satedness holds again.

Related to this is the following example:

Example 4.22. Let $\iota\colon \mathbf{P}_{\not{b}}(R^n) \hookrightarrow \mathbf{P}(R^n)$ be the embedding of unbounded subsets of R^n. Then $S \subseteq R^n$ is ι-dense if and only if its intersection with every unbounded subset of R^n is non-empty, so if its complement is bounded. We can again prove as in Example 4.21 that the only subsets of R^n that are ι-sated are bounded or R^n itself.

Interestingly, our system also allows us to stack notions of smallness/density:

Example 4.23. If the tablets $\tau\colon \mathbf{SOb}_{\mathbf{Mon}}(M) \hookrightarrow \mathbf{P}(M)$ are given by the embedding of the sub-monoids of a monoid M, then a subset ι of M is τ-small if and only if its complement $\neg\iota$ is not contained in a proper submonoid. So for instance in N the even numbers are small while the odd numbers are not. ι is τ-reduced if whenever ι is contained in the union of a submonoid and a small subset, then ι is in the submonoid.

Note that $\bot_\tau$ is $\{e\}$, not $\varnothing$ (the bottom of $\mathbf{P}(M)$), so that ι is τ-dense if for each non-trivial submonoid κ the intersection of ι with κ contains an element outside $\{e\}$. It suffices to test this on principally generated submonoids, so ι is τ-dense if its intersection with each orbit $\langle a\rangle$ contains a non-zero power of a. ι is τ-sated if for each τ-dense κ and submonoid λ it holds that ι contains $\kappa \cap \lambda$ if and only if ι contains λ. Given now $a \in M$ such that ι contains a^n let κ consist of a^n and all b such that a is not in the orbit of b. Then κ is τ-dense and ι contains a^n; so that ι is again τ-sated if and only if it contains each radical of its elements and thus if it is reduced in the monoid-theoretic sense.

Let us also add some examples of a completely different variety:

Example 4.24. Let M be some kind of machine and $\mathbf{SOb}(M)$ the preorder of parts of M where $a \leq b$ if and only if b is not functional without a. Then a is small if and only if it factors through each maximal part of M by Proposition 3.14 and the finitude of real-world machines. Now let $\tau\colon \mathbf{T}(M) \hookrightarrow \mathbf{SOb}(M)$ be the embedding of those parts that can be bought stand-alone. Then a part ι is τ-dense if and only if it factors through each part of M that can be bought, so if we get it delivered each time we buy a replacement. Notice that the embedding τ is connected to the material we exposed in Section 3.4: τ embeds those parts that can be duplicated, so for which addition makes sense (and arises, for instance if b is broken and we buy a replacement b', but $a \leq b$ is fine, so we obtain an a-spare a').

But because $\mathbf{SOb}(M)$ is elementary, i.e. consists of parts and not families of parts, its smallness relations are somewhat rough. So let us endow $\mathbf{P}(\mathbf{SOb}(M))$ with the relation $F \leq G$ if and only if for each $a \in \mathbf{SOb}(M)$ such that each element of G is $\leq a$ it holds that each element of F is $\leq a$. So in particular if $a = \bigvee B$ then $a = B$ in this order up to univalence, so as a category this preorder is actually in part a weak quotient and equivalent to the ideal completion.

Now, smallness under the powerset-induced function $\mathbf{P}(\tau)\colon \mathbf{P}(\mathbf{T}(M)) \hookrightarrow \mathbf{P}(\mathbf{SOb}(M))$ gives us a more useful size measure. Then a family of parts F of M is not small if and only if we can buy a family G that contains all parts of M missing in F but not all parts of M in itself, so that we can assemble M from F and G, given the necessary skill and machinery.

But of course there is a *cost* associated with duplication. So let $\sigma : \mathbf{T}(M) \to R_{\geq 0,\infty}$ be given by the price of each part and $\overline{\sigma} : \mathbf{SOb}(M) \to R_{\geq 0,\infty}$ by $\overline{\sigma}(a) := \bigwedge_{a' \in \mathbf{T}(M) : a \leq \tau(a')} \sigma(a')$ so basically the least cost we have to pay for a. Finally, let $\hat{\sigma}$ be given by $\hat{\sigma}(F) := \bigwedge_{F \simeq G} \sum_{\iota \in G} \overline{\sigma}(\iota)$, so the least cost we have to pay for the parts of F. Then F is $(\mathbf{P}(\tau) \mid \hat{\sigma})$-small if and only if each family of parts of M that we can buy costs at least much as M itself - a frequent occurrence when one tries to repair something. If we combine this scale with the rounding we introduced in Example 4.20, we can further refine smallness so that a family is small if and only if the cheapest way to complete it is as expensive as M up to a rounding error.

Example 4.25. As noted in the introduction, our notion of density is motivated by the topological one, which is binary. Here we want to sketch out the connection to the treatments of density through continuous functions, as arise in chemistry. So let X be spacetime and γ some gas. When should we say that γ is dense in X? This question is tricky because γ has, strictly speaking, only finitely many atoms while X is continuous; and often glanced over. Arguably the most conceptually thorough way to handle such questions is through Robinson's nonstandard analysis,[27] or even more particularly through Nelson's internal set theory.[22] In these, the "standard" natural numbers N are embedded in a *nonstandard* extension $\overline{N}$ which fulfills the same algebraic axioms but contains additional numbers $N > N$, which are larger than any standard natural number (this is called *hyperfinite*). So a nonstandard number is a number that is larger than any finite number but acts in essentially the same way. As Nelson worked out, this gives a rigorous idealization of what it means that "a number beyond the standard range", as is often assumed in the natural sciences. Similarly the "standard" real numbers R are embedded into a nonstandard extension $\overline{R}$ that also contains the nonstandard natural numbers $\overline{N}$, so that there exists for each nonstandard number N an inverse $\frac{1}{N}$ and thus an *infinitesimal number* δ such that $0 < \delta < R_{>0}$. We can define a scale $\mathbf{st}_\infty : \overline{R} \to R_{\pm\infty}$, given by assigning to each extended real r its standard part, and a scale $\mathbf{infl} : \overline{R} \to \mathbf{h}(0)_{\pm\infty}$ to the *halo* $\mathbf{h}(0)$ of such infinitesimal numbers around 0, given by the identity on $\mathbf{h}(0)$ and $\pm\infty$ otherwise. These two jointly describe non-standard finitude and infinitesimality through smallness.

It is also well-known (see for instance [2]) that in such a setting each Hausdorff space admits a nonstandard extension, so if we assume that our spacetime is Hausdorff, which seems reasonable, then there exists an extension $\xi_X : X \hookrightarrow \overline{X}$ and a partial function $\mathbf{st} : \overline{X} \to X$ such that $\xi_X \,;\, \mathbf{st} = \mathbf{id}_X$. So $\overline{X}$ houses X but also for each point x of X a *monad* or *halo* $\mathbf{h}(x) := \mathbf{st}^{-1}(x)$ filled with nonstandard elements around X.

A very simple definition for "γ is dense in X" is then that each open subset of X contains some atom in γ. If we assume that the spatial extension of each atom is 0,[5], then this is impossible if γ has only a standard amount of atoms (since X is continuous), but quite possible if γ contains a nonstandard set G of them. It makes sense then to treat γ as a function $\gamma : G \hookrightarrow \overline{X}$ which assigns to each atom of γ its position. But this just says that γ is dense with respect to the collection of tablets given by the composite $\mathbf{O}(X) \hookrightarrow \mathbf{P}(X) \hookrightarrow \mathbf{P}(\overline{X})$.

The question is now whether we can derive from such a nonstandard gas a function which assigns to each point x in X a density of γ at x, which would complete the bridge from our notion to density functions. An easy approach in line with our notion of density would be to first fix a $\delta \in \mathbf{h}(0)$ that we use to scale infinitesimal lengths to normal ones. We can derive a density function $\delta'_\gamma(x) := \frac{1}{\mathbf{min}\{r \in R_\infty \mid \exists g \in G : \gamma(g) \in \bullet_x^{\delta r}\}}$ where $\bullet_x^{\delta r}$ is the ball around x of radius δr, so by finding the smallest ball around each x that does not contain any elements of γ and thus the representators of the smallness-ideal. This gives us the right result

[5] More generally if we assume that the diameter of each atom is infinitesimal.

at x in that if there is an atom of γ precisely at x then the density function maps to ∞ and the function decreases with the perimeter.

But in chemistry a density function is dependent on both the local number of atoms and their distance, and this introduces a parameter that cannot cleanly be handled just by the current framework. We could try to sum over all atoms in $\mathbf{h}(x)$, perhaps scaled by their mass and inversely scaled by their distance to x, but this sum is ill-defined in non-standard analysis. What we can do is fix a nonstandard δ and sum over all atoms in the δ-ball around x. But the result of this sum would, without further assumptions, depend on δ, and not necessarily just up to a nonstandard part. So sadly it seems like non-standard analysis cannot be used to derive a density function in a way that accounts for both the mass of atoms and their distance from x without some arbitrariness, even in an idealized way.

Example 4.26. In Definition 3.68 we defined geodesically convex sub-2-categories in such a way that each geodesically convex $\iota\colon\mathscr{A} \hookrightarrow \mathscr{B}$ is uniquely determined by the objects it contains. Thus, there exists a preorder morphism $\tau_{conv}\colon\mathbf{Conv}(\mathscr{B}) \hookrightarrow \mathbf{Ob}(\mathscr{B})$ from the convex sub-2-categories of $\mathscr{B}$ forward ordered by inclusion (which is non-evil if $\mathscr{B}$ has no non-trivial isomorphisms like the examples we looked at in Section 3.1.8). so that a subset of the objects of $\mathscr{B}$ is τ_{conv}-small if and only if the smallest geodesically convex sub-2-category of $\mathscr{B}$ that contains all objects outside ι is $\mathscr{B}$ itself.

5 Smallness & Density under Morphisms

Now let us first consider how smallness interacts with morphisms. For this we formally introduce three 2-categories of importance to us:

Definition 5.1. The *2-category of preorders* **PrO** is the 2-category whose objects are preorders, whose morphisms are preorder morphisms and whose 2-morphisms are objectwise inequalities between morphisms. The 2-category **UBP** of *upward-bounded preorders and preorder morphisms* is the full sub-2-category of **PrO** on upward-bounded preorders. The wide sub-2-category $\mathbf{UBP}_w$ of *upward-bounded preorders and top-preserving preorder morphisms* is the wide sub-2-category of **UBP** whose objects are upward-bounded preorders and whose morphisms are top-preserving preorder morphisms.

It might seem weird not to demand preservation of tops in **UBP**, but note that the local property of having a maximal element assembles into a global 2-categorical one even without strict preservation. Let us quickly explain that, which will hopefully familiarize the reader with **PrO** and **UBP** somewhat. We start by studying the natural transformations from the constant functor on the terminal object $\mathbf{K}(*)\colon\mathbf{Aut}(\mathbf{PrO})$ to the identity functor $\mathbf{id}\colon\mathbf{Aut}(\mathbf{PrO})$. A natural transformation $\eta\colon\mathbf{K}(*) \Rightarrow \mathbf{Aut}(\mathscr{C})$ on some category $\mathscr{C}$ is a family of points $\eta_X\colon * \to X$ that are preserved by all morphisms in $\mathscr{C}$, something that any pointed category like that of pointed sets and point-preserving maps or that of monoids and identity-preserving maps has. **PrO** has no such distinguished points, so unsurprisingly $\mathbf{Aut}(\mathbf{PrO})(\mathbf{K}(*), \mathbf{id})$ is empty: for each family of points in all preorders we can easily find a map that avoids them.

If we replace **PrO** with **UBP**, the same holds, with one almost-exception: the family $\top_P\colon * \hookrightarrow P$ is not preserved by all morphisms, but for each $f\colon P \to Q$ the image $\top_f := f(\top_P) = \top_P \mathbin{⨾} f$ is the source of a unique 1-cell $\top_f \leq \top_Q$ since $\top_Q$ is maximal. Moreover, if $\eta\colon\mathbf{K}(*) \to \mathbf{id}$ is some other 2-cell, and there exists some $\eta_P < \top_P$, then η_P is not laxly preserved by the constant automorphism of P on $\top_P$. Thus, the family $(\top_P, \top_f)$ is the unique lax point of **id**. Moreover, since $\top_P$ is maximal, $\top_{(-)}$ is a right adjoint to the natural transformation $*_{(-)}$ that represents $*$ as a terminal object. Note that an abstract way to state that a preorder is upward-bounded is to say that $*_P$ has a right adjoint. So **id** is itself upward-bounded in a non-trivial way even though it has only one element.

Practically speaking, this means in particular that the image of a morphism f has a unique maximal element $\top_f$, which is natural with respect to composition in that for composable f, g it holds that $\top_{f;g} \leq \top_g$. This allows us to represent morphisms f by elements $\top_f$ to some extent, which we will use a lot in Section 8.

Remark 5.2. In particular, each ideal embedding $\iota\colon a \hookrightarrow P$ in **UBP** is principal since $\top_\iota$ represents ι as an ideal.

Remark 5.3. So visually, **UBP** looks a bit like a saw: $\top_P$ is the tip of a tooth, $f\colon P \to Q$ a tooth-diagonal downward and $\top_f \leq \top_Q$ the blade going back up to the tip.

Note that **UBP**, unlike **PrO**, does not have an initial object since the empty preorder is not upward-bounded by Definition 3.2. But since $\top_P$ is for each preorder P maximal, $*$ satisfies a laxening of the universal property of an initial object. Let us give this a name:

Definition 5.4. An object O of a 2-category $\mathscr{B}$ is *contravariantly initial* if for each object X of $\mathscr{B}$ the category $\mathscr{B}(O, X)$ has a terminal object. Dually, an object $T\colon\mathscr{B}$ is *contravariantly terminal* if for each object X of $\mathscr{B}$ the category $\mathscr{B}(X, T)$ has an initial object.

Remark 5.5. This definition has a global component and a local one, and in the name we track the variance between the two.

So $*$ is contravariantly initial in **UBP**. This makes sense if we view **UBP** in analogy with pointed categories, where the point becomes the initial object. Here, we only have lax point preservation, so the initiality is lax as well.

Note that contravariantly terminal objects are not generally unique in 2-categories:

Counterexample 5.6. Let $\mathscr{B}$ be the 2-category on two elements a, b, generated by two morphisms $f\colon a \to b, g\colon b \to a$ and two 2-morphisms $\alpha\colon \mathbf{id}_a \Rightarrow f \mathbin{;} g, \beta\colon \mathbf{id}_b \Rightarrow g \mathbin{;} f$. Then each of a and b are contravariantly terminal, but they are not equivalent. Moreover, if we introduce an inverse to β, then a is (weakly) initial but b is still contravariantly initial.

So there can be more than one contravariantly initial object in a 2-category and in particular there can be an initial contravariantly initial object (which then has to be an actual initial object) and a terminal initial object, a “largest of the first” so to speak.

Example 5.7. Actually, every object P of **UBP** is contravariantly initial, with the terminal object in $\mathbf{UBP}(P, Q)$ given by $\top_Q$ for each $Q\colon\mathbf{UBP}$, but $*$ is uniquely described by being terminally contravariantly initial, since it is terminal. Moreover, $*$ is the unique contravariantly initial object in the monic part $\overset{\hookrightarrow}{\mathbf{UBP}}$ of **UBP**.

In Section 8.3 we will show that the monic part of $\overset{\hookrightarrow}{\mathbf{UBP}}$ with **SOb** is initial among the contravariantly terminal coefficient objects for a certain kind of coefficient system.

We can also isolate two kinds of morphisms $f\colon P \to Q$ in **UBP** that are extremal with respect to $\top_f$:

1. the *top-preserving morphisms*, where $\top_f = \top_Q$,
2. and the *ideal embeddings*, which are principal by Remark 5.2 and thus determined by $\top_f$.

These two actually form a factorization system:

Proposition 5.8 (Ideal-top factorization system on **UBP**). *In the* 2*-category* **UBP***, the classes*

$$\mathscr{E} \;=\; \{\textit{top-preserving maps}\}, \qquad \mathscr{M} \;=\; \{\textit{ideal embeddings}\}$$

constitute an orthogonal factorization system. Explicitly, every morphism $f : P \to Q$ factors uniquely up to unique isomorphism as

$$P \xrightarrow{f'} (\top_f) \xhookrightarrow{\iota_{\top_f}} Q,$$

where $(\top_f) = \{q \in Q \mid q \le \top_f\}$ is the principal ideal of $\top_f$ in Q, the map f' is the corestriction $f'(p) = f(p)$, and $\iota_{\top_f}$ is the inclusion.

Proof. We verify the defining properties of an orthogonal factorization system: existence of factorizations and the diagonal fill-in.

Factorization. Given $f : P \to Q$, set $\top_f := f(\top_P)$. Since f is monotone, $f(p) \le f(\top_P) = \top_f$ for every $p \in P$, so f corestricts to $f' : P \to (\top_f)$ with $f'(p) = f(p)$. This map is top-preserving:

$$f'(\top_P) \;=\; f(\top_P) \;=\; \top_f \;=\; \top_{(\top_f)}.$$

The inclusion $\iota_{\top_f} : (\top_f) \hookrightarrow Q$ is an ideal embedding whose image is the principal ideal $(\top_f)$, and $f = f' ; \iota_{\top_f}$.

Diagonal fill-in. Suppose given a strictly commuting square

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "e"'] & R \arrow[d, "m"] \\ Q \arrow[r, "k"'] & S \end{tikzcd} \qquad e ; k \;=\; h ; m,$$

where $e : P \to Q$ is top-preserving and $m : R \hookrightarrow S$ is an ideal embedding with image $(\top_m)$. We construct a unique diagonal $d : Q \to R$.

For each $q \in Q$, monotonicity of k gives $k(q) \le k(\top_Q)$. Since e is top-preserving, $\top_Q = e(\top_P)$, whence

$$k(\top_Q) \;=\; k(e(\top_P)) \;=\; m(h(\top_P)) \;\le\; m(\top_R) \;=\; \top_m,$$

so $k(q) \le \top_m$, i.e. $k(q) \in (\top_m) = \mathbf{Im}(m)$. Because m is an order-embedding, it admits an order-theoretic inverse $m^{-1} : (\top_m) \xrightarrow{\cong} R$. We may therefore define

$$d(q) \;:=\; m^{-1}(k(q)),$$

which is monotone as the composite $Q \xrightarrow{k} (\top_m) \xrightarrow{m^{-1}} R$. The two triangles commute:

- $d ; m = k$: for all $q \in Q$, $m(d(q)) = m(m^{-1}(k(q))) = k(q)$.
- $e ; d = h$: for all $p \in P$, $d(e(p)) = m^{-1}(k(e(p))) = m^{-1}(m(h(p))) = h(p)$.

Uniqueness is immediate: if $d' : Q \to R$ also satisfies $d' ; m = k$, then $m(d'(q)) = m(d(q))$ for all q, and injectivity of m forces $d' = d$.

Intersection. If $f : P \to Q$ is both top-preserving and an ideal embedding, then f is an order-embedding with image $(\top_f) = (\top_Q) = Q$, hence an order-isomorphism. Thus $\mathscr{E} \cap \mathscr{M} = \mathbf{Iso}(\mathbf{UBP})$. ◻

Remark 5.9. The diagonal fill-in admits a natural oplax 2-categorical strengthening: if the square commutes only up to a 2-cell $e \mathbin{;} k \leq h \mathbin{;} m$, the same construction yields d with $d \mathbin{;} m = k$ strictly and a 2-cell $e \mathbin{;} d \leq h$. This oplax lifting is unique, since **UBP** is preorder-enriched and there is at most one 2-cell between any pair of parallel morphisms.

Both of these will be important later on.

Remark 5.10. Building on Remark 5.3 we can understand the "saw-toothedness" of **UBP** as a correspondence between horizontal and vertical distance for ideals in that for each $a, b \in P$ the interval between a and b is in one-to-one correspondence with the principal ideals that (a) factors through that factor through (b).

Remark 5.11. Of course dual statements hold for the category **DBP** of downward-bounded preorders, with filters instead of ideals.

Now to preservation and reflection:

Definition 5.12. Let $f : P \to Q$ be a morphism in **UBP**:

1. f *preserves smallness at* a if a small in P implies $f(a)$ small in Q.
2. f *reflects smallness at* a if $f(a)$ small in Q implies a small in P.

Now, the proofs of this section are very simple in principle. Consider for instance smallness-preservation. For this we need some way of relating joins of $f(a)$ in Q to those of a in P:

Definition 5.13 (Lifting of a-joins)**.** Let $f : P \to Q$ in **UBP** and $a \in P$. We say that f is $\top$*-join-lifting on* a if for every $c \in Q$ with

$$f(a) \vee c = \top_Q$$

there exists a $b \in P$ such that $f(b) \leq c$ and $a \vee b = \top_P$.

But then the proof is simple:

Proposition 5.14 (Elementwise preservation of smallness)**.** *Given $f : P \to Q$ in* **UBP** *and $a \in P$, assume that f preserves joins of a to $\top_P$ and lifts $\top$-joins on a. Then f preserves smallness at a.*

Proof. Let $c \in Q$ satisfy $f(a) \vee c = \top_Q$. By the a-join-lifting property there exists a $b \in P$ with $f(b) \leq c$ and $a \vee b = \top_P$. If a is small, then $b = \top_P$, so that $f(a) \leq f(a) \vee f(b) = f(a) \vee \top_f = \top_f \leq c$. So $f(a) \vee c = c = \top_Q$. □

Let us also treat the reflecting case: assuming we have $f : P \to S$ and $a \in P$ with small $f(a)$. Why could a *not* be small in P? Three reasons come to mind:

1. $\top_f$ is not reduced in S.
2. there is a $b \in P$ with $a \vee b$ such that $f(a) \vee f(b)$ either doesn't exist or is smaller than $\top_f$.
3. each $f(b)$ does join with $f(a)$ to $\top_f$ and thus $f(b) = \top_f$ *but* b is something smaller than $\top_P$ that is still mapped to $\top_f$.

Example 5.15. Let $f^* : \mathbf{C}([0,1]) \to \mathbf{C}([0,2])$ from Example 3.17 be induced through pullback along the continuous function $f : [0,2] \twoheadrightarrow [0,1]$ that is the identity on $[0,1] \subseteq [0,2]$ and maps $[1,2]$ to 1. Then f^*

is not smallness-preserving because the fiber over 1 is not small: $[0,1] \vee f^*(\{1\}) = [0,1] \vee [1,2] = [0,2]$, so $f^*(\{1\})$ is not small even though $\{1\}$ is.

Proposition 5.16. *Let $f : P \to Q$ be a top-join preserving morphism of preorders that is injective on $\top_f$. If $\top_f$ is reduced then f is smallness-reflecting.*

Proof. Let $a \in P$ be an element such that $f(a)$ is small in Q. Suppose $b \in P$ is such that $a \vee b = \top_P$. Since f preserves top-joins, we have:

$$f(a) \vee f(b) \;=\; f(\top_P) \;=\; \top_f.$$

By the hypothesis that $f(a)$ is small in Q and $\top_f$ is reduced in Q, it follows from Definition 3.75 that:

$$f(b) \;=\; \top_f.$$

Using the assumption that f is injective on $\top_f$, this implies $b = \top_P$. Thus, the only element b in P that joins with a to give $\top_P$ is $\top_P$ itself. Therefore, a is small in P. □

Corollary 5.17. *Let $\imath$ be an ideal embedding. If $\top_\imath$ is reduced then $\imath$ reflects smallness.*

What complicates general proofs a little is that the use of tablets and scales decomposes the functions of smallness-measurement in a way that has to be kept track of. In principle this is more fine-grained and will allow the reader to apply the resulting calculus in the topological and measure-theoretic contexts we discuss in Section 7. But we need to extend our considerations to a commutative diagram of upward-bounded preorders and monotone maps:

$$\begin{tikzcd}
T \arrow[r, "\tau"] \arrow[d, "h"] & P \arrow[r, "\sigma"] \arrow[d, "g"] & S \arrow[d, "k"] \\
T' \arrow[r, "\tau'"] & P' \arrow[r, "\sigma'"] & S'
\end{tikzcd}$$

Figure 2

Definition 5.18 (Relative preservation of top-joins)**.** With $f = (h, g, k)$ as in Fig. 2 and $a \in P$, we say that (h, g, k) *preserves* (τ, a)*-top-joins* or *preserves* τ*-top-joins on* a if for every $b \in T$ such that $a \vee \tau(b) = \top_P$ one has

$$g\big(a \vee \tau(b)\big) \;=\; g(a) \vee g(\tau(b)) = g(a) \vee \tau'(h(b)) \;=\; \top_g.$$

On the other hand we say that f *reflects* τ'*-top-joins on* a if for each $b \in T$ with $g(a) \vee (\tau \,;\, g)(b) = \top_g$ it holds that $a \vee \tau(b) = \top_P$.

Proposition 5.19 (Elementwise reflection of relative smallness)**.** *Assume:*

1. *(h, g, k) preserves τ-top-joins on a;*
2. *the top $\top_g$ is reduced under τ' over σ',*
3. *k reflects $\top_{\sigma;k}$,*

Under these hypotheses (h, g, k) reflects relative smallness at $a : P$.

Proof. Suppose $a' := g(a)$ is $(\tau' \mid \sigma')$-small. To show a is $(\tau \mid \sigma)$-small, take any $b \in \boldsymbol{T}$ such that

$$a \vee \tau(b) = \top_{\boldsymbol{P}}.$$

Let now b' be given by $h(b)$. Applying g and using preservation of τ-top-joins on a-joining families:

$$\top_g \;=\; g(a \vee \tau(b)) \;=\; g(a) \vee (\tau \mathbin{;} g)(b) \;=\; a' \vee \tau'(b'). \tag{2}$$

So b' joins with a' to contain $\top_g$ under τ' in $\boldsymbol{P}'$, and since a' is $(\tau' \mid \sigma')$-small in $\boldsymbol{P}'$ and $\top_g$ is $(\tau' \mid \sigma')$-reduced, it follows from Definition 4.13 that

$$(\tau \mathbin{;} \sigma \mathbin{;} k)(b) = (\tau' \mathbin{;} \sigma')(b') \geq \sigma'(\top_g) = (g \mathbin{;} \sigma')(\top_{\boldsymbol{P}}) = (\sigma \mathbin{;} k)(\top_{\boldsymbol{P}}) = \top_{\sigma;k}, \tag{3}$$

and since $\tau(b) \leq \top_{\boldsymbol{P}}$ and thus $\tau'(b') \leq g(\top_{\boldsymbol{P}})$, the reverse inequality also holds and thus $(\tau' \mathbin{;} \sigma')(b') = \top_{\sigma;k}$.

But since k reflects $\top_{\sigma;k}$ and $(\tau \mathbin{;} \sigma \mathbin{;} k)(b) = \top_{\sigma;k}$, it follows that $(\tau \mathbin{;} \sigma)(b) = \top_\sigma$. Since this holds for every $b \in \boldsymbol{T}$, a is small. □

Remark 5.20. It seems like not much (or nothing) can be gained from laxifying the squares of Fig. 2: for instance to apply $(\tau' \mid \sigma')$-reducedness in the proof of Proposition 5.19 we would have to assume that the left square is lax so that the last equality in Eq. (2) became $g(a) \vee (\tau \mathbin{;} g)(b) \leq a' \vee \tau'(b')$. But then the first equality of Eq. (3) would become an inequality in the wrong direction $(\tau \mathbin{;} \sigma \mathbin{;} k)(b) \leq (\tau' \mathbin{;} \sigma')(b')$, so that the proof would fall apart.

Let us now turn to preservation of smallness at a:

Definition 5.21 (Relative lifting of top-joins)**.** With (h, g, k) as in Fig. 2 and $a : \boldsymbol{P}$, g and h are $\top$*-join-lifting on* a if for each $b' \in \boldsymbol{T}'$ such that $g(a) \vee \tau'(b') = \top_{\boldsymbol{P}'}$ there exists a $b \in \boldsymbol{T}$ such that $h(b) \leq b'$ and $a \vee \tau(b) = \top_{\boldsymbol{P}}$.

Proposition 5.22 (Elementwise preservation of relative smallness)**.** *Given a diagram of preorders as in Fig. 2 and a in $\boldsymbol{P}$. Suppose:*

1. *(h, g, k) preserves τ-top-joins on a-joining families;*
2. *g and h are $\top$-join-lifting on a;*
3. *k maps $\top_\sigma$ to $\top_{\sigma'}$.*

Then (h, g, k) preserves relative smallness at a.

Proof. Assume a is $(\tau \mid \sigma)$-small. Let b' be such that

$$g(a) \vee \tau'(b') = \top_{\boldsymbol{P}'}.$$

Using the (τ, a)-join-lifting property, there exists a $b \in \boldsymbol{T}$ with $h(b) \leq b'$ and

$$a \vee \tau(b) = \top_{\boldsymbol{P}}.$$

By smallness of a in the unprimed context we have

$$(\tau \mathbin{;} \sigma)(b) = \top_\sigma.$$

Since k maps $\top_\sigma$ to $\top_{\sigma'}$ and $h(b) \leq b'$, $(\tau' \mathbin{;} \sigma')(b') \geq (\tau \mathbin{;} \sigma \mathbin{;} k)(b) = \top_{\sigma'}$, hence $(\tau' \mathbin{;} \sigma')(b') = \top_{\sigma'}$, so $g(a)$ is $(\tau' \mid \sigma')$-small, hence (h, g, k) preserves relative smallness at a. □

Now we universalize over a:

Definition 5.23. In the above diagram, we say that the triple (h, g, k):

- *reflects* $(\tau' \mid \sigma')$*-smallness* if it reflects $(\tau' \mid \sigma')$-smallness at every $a : \boldsymbol{P}$;
- *preserves* $(\tau \mid \sigma)$*-smallness* if it preserves $(\tau \mid \sigma)$-smallness at every $a : \boldsymbol{P}$.

Now we can describe the global behavior of smallness with respect to morphisms in one proposition:

Proposition 5.24 (Global relative reflection/preservation of smallness)**.** *In the situation of Fig. 2:*

1. *Assume that:*
   (a) (h, g, k) *preserves* τ*-top-joins,*
   (b) $\top_g$ *is* $(\tau' \mid \sigma')$*-reduced in* $\boldsymbol{P}'$ *in the sense of Definition 4.13;*
   (c) k *reflects* $\top_{\sigma;k}$.

   Then (h, g, k) *globally reflects* $(\tau' \mid \sigma')$*-smallness.*
2. *Assume that for every* $a : \boldsymbol{P}$*:*
   (a) (h, g, k) *preserves* τ*-top-joins;*
   (b) (h, g, k) *are* $\top$*-join-lifting;*
   (c) k *maps* $\top_\sigma$ *to* $\top_{\sigma'}$.

   Then (h, g, k) *globally preserves* $(\tau \mid \sigma)$*-smallness.*

Proof. The preservation and reflection statements follow directly from Proposition 5.19 and Proposition 5.22 □

Remark 5.25. Conceptually:

- The *reflection* part says that if the comparison triple (h, g, k) preserves joins and reflects $\top_{\sigma;k}$, then smallness-reflection follows from the reducedness of $\top_g$.
- The *preservation* part says that, if the upstairs context (τ, σ) sees a as negligible and (h, g, k) can systematically lift downstairs joining situations back upstairs without changing the measurements, then the downstairs image $g(a)$ remains small for (τ', σ').

Let us also note that reducedness and smallness-reflection are equivalent for ideals:

Corollary 5.26. *Let* $\iota : \boldsymbol{P} \to Q$ *be an ideal embedding. If* ι *reflects smallness, then* $\top_\iota$ *is reduced.*

Proof. Let $\top_\iota \leq a \vee b$ in Q be such that $\iota^*(b)$ is small in P. Since ideals are downward-closed, ι both preserves and reflects joins in its image. Thus, $\iota^{-1}(a) \vee \iota^{-1}(b) = \top_\iota$. Since ι reflects smallness, $\iota^{-1}(b)$ is small in P, so that $\iota^{-1}(a) = \top_P$. So $\iota(\iota^*(a)) = \top_\iota$, so $\top_\iota \leq a$. Since this holds for each $a, b : Q$, $\top_\iota$ is reduced in Q. □

Let us also note the dual case:

Remark 5.27 (Dualization for density and satedness). Given a commutative diagram as above but in the category of downward-bounded preorders, the following hold:

1. (Global reflection of $(\tau' \mid \sigma')$-density) Assume that:
   (a) (h, g, k) preserves τ-bottom-meets;
   (b) $\bot_g$ is sated in P' relative to $(\tau' \mid \sigma')$;
   (c) k reflects $\bot_{\sigma;k}$.

   Then (h, g, k) globally reflects $(\tau' \mid \sigma')$-density.
2. (Global preservation of $(\tau \mid \sigma)$-density) Assume that:
   (a) (h, g, k) lifts τ-bottom-meets;
   (b) (h, g, k) preserves τ-bottom-meets;
   (c) k maps $\bot_\sigma$ to $\bot_{\sigma'}$.

   Then (h, g, k) globally preserves $(\tau \mid \sigma)$-density.

The proofs are obtained from those of Proposition 5.24 by systematically reversing all orders, replacing joins by meets, $\top$ by $\bot$, smallness by density, and reducedness by satedness.

Corollary 5.28. *Let ι be a downward-closed embedding $\iota\colon A \hookrightarrow P$ with P a downward-bounded preorder. Then ι is density-reflecting.*

Proof. Since ι is downward-closed it in particular contains $\bot : P$ and thus by the dual of Proposition 5.16 density-reflection holds if and only if ι is injective on $\bot$. But this holds because ι is an embedding. □

But also:

Corollary 5.29. *Let $f : P \to Q$ be a monotone map between downward-bounded preorders. Assume that:*

1. *f reflects bottoms;*
2. *f is surjective on elements.*

Then if $d \in P$ is dense, then $f(d)$ is dense in Q.

Proof. Let $c \in Q$ with $c \wedge f(d) = \bot_Q$. Since f is surjective, choose $b \in P$ with $f(b) = c$. Because f is monotone we have
$$f(b \wedge d) \;\leq\; f(b) \wedge f(d) \;=\; c \wedge f(d) \;=\; \bot_Q.$$
Hence $f(b \wedge d) = \bot_Q = f(\bot_P)$. Thus, since f reflects bottoms, $b \wedge d = \bot_P$. Since d is dense in P, this yields $b = \bot_P$, hence $c = f(b) = \bot_Q$, which is exactly the density of $f(d)$ in Q. □

Finally we have to describe how smallness acts with regard to the 2-categorical structure in **UBP**. This is easy though:

Proposition 5.30. *Given* $\leq : f \Rightarrow g : P \to S$ *then* $\forall a \in P$*:*

1. *if* g *is smallness-preserving at* a*, then* f *is smallness-preserving at* a*,*
2. *if* f *is smallness-reflecting at* a*, then* g *is smallness-reflecting at* a*.*

Proof. For Item 1 first assume that a is small. Since g is smallness-preserving at a, $g(a)$ is small. But since the small elements of S form an ideal and $f(a) \leq g(a)$, so is $f(a)$. If a is not small, then the statement is void.

For Item 2, assume that $g(a)$ is small, else the statement is void. Since $f(a) \leq g(a)$, $f(a)$ is small, so a is small. □

Let us also clarify the relation between pseudocomplements, tablets and scales, which we'll use in Proposition 5.46:

Definition 5.31. Given an element a of a bounded preorder P, an *open pseudocomplement* $\mathring{\neg} a$ of a is a largest b such that $a \wedge b = \bot_P$. More generally, given an embedding $\tau : T \hookrightarrow P$, a $(\tau \mid \sigma)$*-relative open pseudocomplement* of a is a largest $b \in T$ such that $\sigma(a \wedge \tau(b)) = \bot_\sigma$. Dually, a has a *closed pseudocomplement* $\bar{\neg} a$ if there exists a smallest b such that $a \vee b = \top_P$ and similar for a $(\tau \mid \sigma)$*-relative closed pseudocomplement.*

Example 5.32. For the tablets $\tau : \mathbf{O}(X) \hookrightarrow \mathbf{P}(X)$ given by the opens of a topological space, a τ-open pseudocomplement $\mathring{\neg}\iota$ is a largest open that does not intersect with a given subspace ι, so the interior of the set-theoretic complement of ι. Dually for closeds.

Remark 5.33. We will generally refrain from studying closed pseudocomplements in this paper, as we will do that more in an upcoming paper anyway. So when we say "pseudocomplement" without a qualifier then we mean open pseudocomplements.

Proposition 5.34 (Pseudocomplements and density)**.** *Let* $T \xrightarrow{\tau} P \xrightarrow{\sigma} S$ *be morphisms of downward-bounded preorders with bottom* $\bot_P$*, and let* $a \in P$ *have a* τ*-relative pseudocomplement* $\neg a$ *. Then the following are equivalent:*

1. a *is* $(\tau \mid \sigma)$*-dense in the sense of Remark 4.17;*
2. $(\tau \mathbin{;} \sigma)(\neg a) = \bot_\sigma$*.*

Proof. Assume a is dense. Since $a \wedge \tau(\neg a) = \bot_P$, the definition of $(\tau \mid \sigma)$-density forces $(\tau \mathbin{;} \sigma)(\neg a) = \bot_\sigma$. Conversely, if $(\tau \mathbin{;} \sigma)(\neg a) = \bot_\sigma$ then for any $c \in T$ with $a \wedge \tau(c) = \bot_P$ we have $c \leq \neg a$, hence $(\tau \mathbin{;} \sigma)(c) = \bot_\sigma$. Thus a is $(\tau \mid \sigma)$-dense. □

Remark 5.35. In a Heyting algebra, where the pseudocomplement is defined by the Heyting implication $\neg a' := a' \to \bot$, the condition $\neg a' = \bot$ is exactly the usual definition of a dense element.

5.1 Pullback and Subobject Tablets and Scales

Let us note that tablets and scales can be transported along pullback:

Definition 5.36. Let $f : Q \to P, T \xrightarrow{\tau} P \xrightarrow{\sigma} S$ be preorder morphisms between upward-bounded preorders. We define the *T-pullback tablets and scales* on Q as the sequence $T_f \xrightarrow{\tau_f} Q \xrightarrow{\sigma_f} S$ with $\tau_f := f^*(\tau), \sigma_f := f \mathbin{;} \sigma$.

We say that *the pullback smallness* under f *reflects* $(\tau \mid \sigma)$*-smallness* if

$$\forall a \in Q, \quad f(a) \text{ is } (\tau \mid \sigma)\text{-small in } P \;\Rightarrow\; a\ (\tau_f \mid \sigma_f)\text{-small in } Q$$

.

We say that *the pullback smallness* under f *preserves* $(\tau \mid \sigma)$*-smallness* if

$$\forall a \in Q, \quad a \text{ is } (\tau_f \mid \sigma_f)\text{-small in } Q \;\Rightarrow\; f(a)\ (\tau \mid \sigma)\text{-small in } P.$$

If f is an ideal embedding we call this the *subobject tablets and scale.*

Remark 5.37. If the tablets $\tau : \mathbf{O}(X) \hookrightarrow \mathbf{P}(X)$ are the opens of a topological space and $\iota : U \hookrightarrow X$ is a subset of X, then the pullback tablets $(\iota_*)^*(\tau)$, where $\iota_* : \mathbf{P}(U) \hookrightarrow \mathbf{P}(X)$ is the morphism induced by postcomposition, are the opens contained in ι. If ι is open in X then this is the same as the subspace topology on U but if ι is not closed in X, then the opens of the subspace topology of U are the *intersections* of opens in X with U and thus *not* the same as the opens contained in ι, so the pullback tablets of do not describe the subspace topology of non-opens. In fact, there is not even a canonical morphism from the opens of the subspace topology of U to the opens of X if ι is not open. Dually for $\mathbf{C}(X)$.

So pullback-smallness describes e.g. restrictions of nowhere dense subsets to subspaces.

Proposition 5.38. *In a situation as in Definition 5.36 assume that f preserves joins to $\top_Q$. Then:*

1. *if $\top_f$ is reduced, then the pullback smallness under f reflects $(\tau \mid \sigma)$-smallness.*
2. *If f is a full epimorphism, f preserves $f^*(\tau)$-top-joins and f reflects τ-top-joins, then the pullback smallness under f preserves $(\tau \mid \sigma)$-smallness.*

Proof. Item 1 follows from Corollary 5.17. For Item 2 assume f is a full epimorphism and preserves joins with $\top$. To show global preservation of smallness, we use Proposition 5.24(Item 2) in our setting, with h the pullback morphism, $g = f$, $k = \mathbf{id}_S$. The three conditions required there reduce to:

1. f preserves top-joins on τ-joining families: this is our hypothesis that f is top-join-preserving.
2. (h, g, k) lift $\top$-joins: for any $f(a) \vee \tau(b) = \top_P$ there exists a $c \in f^*(T)$ with $\tau^*(f)(c) = b$. Then by τ-top-join-preservation

$$f(a \vee f^*(\tau)(c)) = f(a) \vee f(f^*(\tau)(c)) = f(a) \vee \tau(b) = \top_P = \top_f.$$

   Because f reflects τ-top-joins, we must have $a \vee f^*(\tau)(c) = \top_P$.

3. $k = \mathbf{id}_S$ maps $\top_{\sigma_f}$ to $\top_\sigma$, which is trivial

Thus all hypotheses of Proposition 5.24(Item 2) are met, so f globally preserves $(\tau \mid \sigma)$-smallness. This proves Item 2. □

Note that it is not the case that each ideal embedding $\iota\colon A \hookrightarrow P$ preserves smallness. Rather, there has to be at least some homogeneity to ensure that joins always act the same way:

Counterexample 5.39. Let P be the order given by elements $v_1, v_2, v_3, l_{1|2}, l_{2|3}, l_{3|1}, v_1 \cup v_2, \top, \bot$ such that $v_2 \vee v_3 = l_{2|3}$ and $v_3 \vee v_1 = l_{3|1}$ but $v_1 \vee v_2 = v_1 \cup v_2 < l_{1|2}$. Geometrically, this is a triangle simplex except that the two vertices v_1 and v_2 join to their set-theoretic union $v_1 \cup v_2$ while the other vertices join to their containing line. Let $\iota\colon (l_{1|2}) \hookrightarrow P$ be the principal ideal embedding of $l_{1|2}$. Since $v_1 \vee v_2 = v_1 \cup v_2 < l_{1|2}$, v_1 is small in $(l_{1|2})$, but $v_1 \vee l_{2|3} = \top$, so v_1 is not small in $\mathbf{SOb}(\top)$. So not every ideal embedding preserves smallness.

Modularity suffices however to ensure smallness is preserved:

Lemma 5.40 (Preservation of smallness in modular lattices.)**.** *Let L be a modular lattice and $\iota\colon(a) \hookrightarrow L$ a principal ideal embedding. Then ι preserves smallness.*

Proof. Since principal ideal embeddings are in particular join-preserving, we can use Proposition 5.14. So we just have to check that for each $b \leq a$ and c such that $b \vee c = \top$ there is a $d \leq c$ such that $b \vee d = a$. But it follows from modularity that

$$b \vee (c \wedge a) = (b \vee c) \wedge a = \top \wedge a = a.$$

$\square$

Remark 5.41. Geometrically the failure of general smallness-preservation can be understood through the tablets we introduced in Section 4.1: the image of an element a of (ι) under ι is still small according to the tablets from (ι), but P can contain wholly new tablets that are fine enough to make a seem big. Modularity then says that the subobjects of Y restrict to X cleanly enough to guarantee fine tablets in each ideal.

5.2 Reducedness & Satedness under Morphisms

Let us also record general conditions for reducedness under morphisms:

Proposition 5.42 (Reducedness under morphisms in **UBP**)**.** *Consider a commuting diagram in* **UBP***:*

$$\begin{tikzcd} T \arrow[r, "\tau"] \arrow[d, "h"] & P \arrow[r, "\sigma"] \arrow[d, "g"] & S \arrow[d, "k"] \\ T' \arrow[r, "\tau'"] & P' \arrow[r, "\sigma'"] & S' \end{tikzcd}$$

Figure 3

1. ***(Preservation)*** *Assume that:*
   (a) *(h, g, k) reflects $(\tau' \mid \sigma')$-smallness (as in Proposition 5.24 Item 1);*
   (b) *g and h are ideal embeddings.*

   Then for any $a \in P$ that is $(\tau \mid \sigma)$-reduced, $g(a)$ is $(\tau' \mid \sigma')$-reduced in P'.
2. ***(Reflection)*** *Assume that:*

(a) (h, g, k) *preserves* $(\tau' \mid \sigma')$*-smallness;*

(b) g *preserves binary joins;*

(c) k *is order-reflecting.*

Then for any $a \in \boldsymbol{P}$*, if* $g(a)$ *is* $(\tau' \mid \sigma')$*-reduced in* $\boldsymbol{P}'$*, then* a *is* $(\tau \mid \sigma)$*-reduced in* $\boldsymbol{P}$*.*

Proof. We treat the two parts separately.

1. *Preservation.* Let a be $(\tau \mid \sigma)$-reduced. Suppose $g(a) \leq b' \vee \tau'(c')$ in $\boldsymbol{P}'$ with b' being $(\tau' \mid \sigma')$-small. Since g and h are ideal embeddings and thus order-reflecting and bijective on its image,

$$a = g^*(a) \leq g^*(b' \vee c') = g^*(b') \vee g^*(c') = g^*(b') \vee g^*(c') = g^*(b') \vee \tau(h^*(c')).$$

   thus there exist $b \in \boldsymbol{P}, c \in \boldsymbol{T}$ with $g(b) = b',\ h(c) = c'$ and $a \leq b \vee \tau(c)$, so that $\sigma(a) \leq \sigma(b \vee \tau(c))$. By Item 1a the condition that b' is $(\tau' \mid \sigma')$-small implies b is $(\tau \mid \sigma)$-small. Since a is $(\tau \mid \sigma)$-reduced, $\sigma(a) = \sigma(\tau(c))$, hence $\sigma'(g(a)) = k(\sigma(\tau(c))) = \sigma'(\tau'(c'))$. Thus $\sigma'(g(a)) \leq \sigma'(\tau'(c'))$, showing that $g(a)$ is $(\tau' \mid \sigma')$-reduced.

2. *Reflection.* Let $g(a)$ be $(\tau' \mid \sigma')$-reduced in $\boldsymbol{P}'$. Suppose $a \leq b \vee \tau(c)$ in $\boldsymbol{P}$ with b being $(\tau \mid \sigma)$-small. Then we have $g(a) \leq g(b) \vee g(\tau(c))$.

   By Item 2a the fact that b is $(\tau \mid \sigma)$-small implies $g(b)$ is $(\tau' \mid \sigma')$-small. Because $g(a)$ is $(\tau' \mid \sigma')$-reduced, we obtain $k(\sigma(a)) = \sigma'(g(a)) \leq \sigma'(g(\tau(c))) = k(\sigma(\tau(c)))$. Order-reflectivity of k then forces $\sigma(a) \leq \sigma(\tau(c))$, so a is $(\tau \mid \sigma)$-reduced.

□

We also include the dual case:

Remark 5.43 (Dualization for satedness under morphisms). Consider the diagram Fig. 3 but in the category of downward-bounded preorders.

1. **(Preservation of satedness)** Assume that:

   (a) (h, g, k) reflects $(\tau' \mid \sigma')$-density;

   (b) g and h are filter embeddings.

   Then for any $a \in \boldsymbol{P}$ that is $(\tau \mid \sigma)$-sated, $g(a)$ is $(\tau' \mid \sigma')$-sated in $\boldsymbol{P}'$.

2. **(Reflection of satedness)** Assume that:

   (a) (h, g, k) preserves $(\tau' \mid \sigma')$-density;

   (b) g preserves binary meets;

   (c) k is order-reflecting.

   Then for any $a \in \boldsymbol{P}$, if $g(a)$ is $(\tau' \mid \sigma')$-sated in $\boldsymbol{P}'$, then a is $(\tau \mid \sigma)$-sated in $\boldsymbol{P}$.

Moreover, we have the following effect:

Proposition 5.44. *Let a be an element of a downward-bounded preorder P and $b \leq a$ such that b is sated in the ideal (a) of a. Then for each dense c and arbitrary d in P it holds that:*

$$\text{if } b \geq c \wedge d, \ \text{then } b \geq a \wedge d$$

.

Proof. Since $b \leq a$ and meets commute and are idempotent,

$$b \geq c \wedge d = a \wedge c \wedge d = (a \wedge c) \wedge (a \wedge d)$$

.

These are all in (a), so since b is sated in (a) and $(a \wedge c)$ is dense in (a),

$$b \geq (a \wedge c) \wedge (a \wedge d) = a \wedge d$$

. □

Conceptually, a subobject $\kappa \in \mathbf{SOb}(X)$ is sated if it is "saturated" with respect to dense subobjects: whenever κ contains the meet of some dense subobject μ and another subobject λ, the dense part can be replaced by the ambient part $\iota \colon A \hookrightarrow X$ without changing the intersection - i.e.

$$\kappa = \mu \wedge \lambda \quad \Longrightarrow \quad \kappa = \lambda \wedge \iota.$$

Thus a sated subobject is one that "absorbs" dense subobjects: it cannot be obtained by meeting with a dense part without actually being the whole of the other factor. In short, satedness is the order-theoretic expression of being so solid that all dense subobjects in it are locally filled.

Curiously, it does not seem to follow that in the situation of Proposition 5.44 it holds that if a is sated in P then b is sated in P without further assumptions.

Counterexample 5.45. We present a counterexample in a meet-semilattice (with bottom) that is not a Heyting algebra. The example shows that the conditions

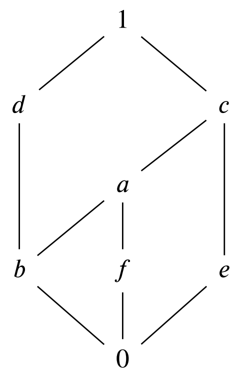
1. a is sated in P,
2. $b \leq a$ and b is sated in the ideal (a),

do not force b to be sated in P.

Let P be the partially ordered set consisting of the eight elements

$$\{0, \ a, \ b, \ c, \ d, \ e, \ f, \ 1\}$$

with the order generated by the following covering relations (reflexive closure is understood):



More explicitly:

$$0 < b, f < a < c < 1, \quad 0 < e < c, \quad b < d < 1,$$

Only c and 1 are dense:

- c meets every nonzero element nontrivially;
- 1 is trivially dense;
- a is not dense because $a \wedge e = 0$ with $e \neq 0$;
- d is not dense because $d \wedge f = 0$ with $f \neq 0$;
- b, e, f are not dense for similar reasons.

a is sated:

- $a = 1 \wedge y$ if and only if $a = y$ is trivially true.
- For c, the equation $a = c \wedge y$ forces $y \geq a$. Among the elements $\geq a$ we have $a, c, 1$. Only $y = a$ gives $c \wedge a = a$; $y = c$ gives $c \wedge c = c \neq a$; $y = 1$ gives $c \neq a$. Hence the condition "$a = c \wedge y \Rightarrow a = y$" holds vacuously except for the valid instance $y = a$.

In (a) the only dense element is a. Thus, every element is sated in (a), so in particular b is.

Take the dense element c and the element d. We have

$$b = c \wedge d \qquad \text{(by construction)},$$

but $d > b$ (so $d \neq b$). A sated element would force $b = d$; hence b is not sated in $\boldsymbol{P}$.

Our counterexample evades Proposition 5.46 because $\boldsymbol{P}$ is not distributive; the non-distributivity causes condition Item 2 of the proposition to fail.

We have one more result, which is inspired by and will be applied to topology: satedness stabilizes with pseudocomplements.

Proposition 5.46 (Reflection of satedness via dense envelopes)**.** *Consider a commuting diagram in* **DBP***:*

$$\begin{tikzcd}
T \arrow[r, "\tau"] \arrow[d, "h"] & P \arrow[r, "\sigma"] \arrow[d, "g"] & S \arrow[d, "k"] \\
T' \arrow[r, "\tau'"] & P' \arrow[r, "\sigma'"] & S'
\end{tikzcd}$$

Assume the following:

1. *g is an ideal embedding;*
2. *P' is distributive;*
3. *There exists a (τ', σ')-pseudocomplement for each element of P' in T' and pseudocomplements distribute over joins;*
4. *k is order- and bottom-reflecting;*

Then (h, g, k) is satedness-reflecting.

Proof. Let $d \in P$ be $(\tau \mid \sigma)$-dense and $t \in T$ such that $d \wedge \tau(t) \leq a$. We must show $(\tau \mathbin{;} \sigma)(t) \leq \sigma(a)$. Set

$$d' \;:=\; g(d) \vee \tau'(\neg\top_g) \;\in\; P'.$$

Step 1: $d' \wedge \tau'(h(t)) \leq g(a)$. Since $\tau' \circ h = g \circ \tau$ and P' is distributive,

$$\begin{aligned}
d' \wedge \tau'(h(t)) &= \big(g(d) \vee \tau'(\neg\top_g)\big) \wedge g(\tau(t)) \\
&= \big(g(d) \wedge g(\tau(t))\big) \vee \big(\tau'(\neg\top_g) \wedge g(\tau(t))\big) \\
&= g(d \wedge \tau(t)) \vee \big(\tau'(\neg\top_g) \wedge g(\tau(t))\big).
\end{aligned}$$

Because g is an ideal embedding, $g(\tau(t)) \leq \top_g$. By Definition 5.31, the pseudocomplement $\neg\top_g$ satisfies $\top_g \wedge \tau'(\neg\top_g) = \bot_{P'}$, so the second term vanishes. This leaves $d' \wedge \tau'(h(t)) = g(d \wedge \tau(t)) \leq g(a)$.

Step 2: d' is $(\tau' \mid \sigma')$-dense. By Proposition 5.34, it suffices to show that $(\tau' \mathbin{;} \sigma')(\neg d') = \bot_{\sigma'}$. By Item 3, pseudocomplements distribute over joins, so

$$\neg d' \;=\; \neg\big(g(d) \vee \tau'(\neg\top_g)\big) \;=\; \neg g(d) \wedge \neg\tau'(\neg\top_g).$$

Since d is $(\tau \mid \sigma)$-dense in P, Proposition 5.34 yields $(\tau \mathbin{;} \sigma)(\neg d) = \bot_\sigma$. As $g(d) \leq \top_g$, any element of T' meeting $\top_g$ trivially also meets $g(d)$ trivially, so $\neg\top_g \leq \neg g(d)$. Consequently, evaluating the scale on the complement gives

$$(\tau' \mathbin{;} \sigma')(\neg d') \;=\; \sigma'\big(\tau'(\neg g(d)) \wedge \tau'(\neg\tau'(\neg\top_g))\big) \;\leq\; \sigma'\big(\tau'(\neg\top_g) \wedge \tau'(\neg\tau'(\neg\top_g))\big) \;=\; \bot_{\sigma'}.$$

Thus d' is $(\tau' \mid \sigma')$-dense.

Step 3: Apply satedness. Since $g(a)$ is $(\tau' \mid \sigma')$-sated in P', and we have established that d' is $(\tau' \mid \sigma')$-dense with $d' \wedge \tau'(h(t)) \leq g(a)$, the definition of satedness implies

$$(\tau' \mathbin{;} \sigma')(h(t)) \;\leq\; \sigma'(g(a)).$$

By commutativity of the diagram, $(\tau' \mathbin{;} \sigma')(h(t)) = (\tau \mathbin{;} \sigma \mathbin{;} k)(t)$ and $\sigma'(g(a)) = (\sigma \mathbin{;} k)(a)$. Thus $(\tau \mathbin{;} \sigma \mathbin{;} k)(t) \leq (\sigma \mathbin{;} k)(a)$, and order-reflection of k (Item 4) yields $(\tau \mathbin{;} \sigma)(t) \leq \sigma(a)$. Since t was arbitrary, a is $(\tau \mid \sigma)$-sated. ◻

Corollary 5.47. *If P is a Heyting algebra, then the ideal embedding (a) of an element $a \in P$ is satedness-reflecting.*

Proof. Item 4 is trivial if we don't use a non-trivial scale, Item 1 is true by assumption and Item 2 and Item 3 are true because P is a Heyting algebra. □

So like smallness depends on the size of the tablets, so does satedness depend on the number of dense subobjects. With smallness, it is modularity that ensures larger tablets restrict cleanly so smallness is preserved; with satedness, Heyting complements ensure that each locally dense subobject can be made globally dense, so that satedness is reflected.

6 The Inter-Categorical Development of the Calculus

What gives our analysis its true power lies in how it interacts with categorical structure through the **SOb**-functor, so let us take stock of that. For this, we have to avoid some size issues:

Definition 6.1. A category $\mathscr{C}$ is *essentially subobject-small* if for each object $X : \mathscr{C}$ the subobject preorder $\mathbf{SOb}_{\mathscr{C}}(X)$ is essentially small, i.e. a set (up to equivalence).

Remark 6.2. Subobject-smallness is weaker than smallness but incomparable to local smallness: for instance, take the category $\mathscr{C}$ whose object class is given by $C+1$ with C a proper class and whose non-trivial morphisms are given for each element $c \in C$ by $\iota : c \hookrightarrow 1$. Then $\mathscr{C}$ is locally small but not subobject-small. We will see the converse also doesn't hold in Counterexample 6.4.

Example 6.3. **Set** is subobject-small but not small.

Given now a subobject-small category $\mathscr{C}$ we have a functor $\mathbf{SOb}_{\mathscr{C}} : \overset{\hookrightarrow}{\mathscr{C}} \to \mathbf{UBP}$ on the subcategory $\overset{\hookrightarrow}{\mathscr{C}}$ of monomorphisms in $\mathscr{C}$ given by mapping $X : \mathscr{C}$ to the preorder $\mathbf{SOb}_{\mathscr{C}}(X)$ of monomorphisms into X. It would be tempting to think that $\mathbf{SOb}_{\mathscr{C}}$ is faithful on $\overset{\hookrightarrow}{\mathscr{C}}$, but this is actually not always true:

Counterexample 6.4. If $\mathscr{C}$ is the delooping of a group, then all monics into its point $*$ are isomorphisms and thus isomorphic to the identity in $\mathscr{C}_{/*}$. So $\mathbf{SOb}_{\mathscr{C}}(*)$ is the one-element preorder, and thus the representation $\mathbf{SOb}_{\mathscr{C}} : \mathscr{C} \to \mathbf{UBP}$ is trivial.

This could be avoided by restricting to monic categories with no non-trivial isomorphisms, but that would be excessive: the problem is not that there are isomorphisms, but that they are not sufficiently supported by non-isomorphic monics:

Definition 6.5. A category $\mathscr{C}$ is *subobject-separated* if for each pair of morphisms $f, g : X \rightrightarrows Y$ there exists a subobject $\iota : A \hookrightarrow X$ such that for each $\sigma : \mathbf{Aut}(A)$:

$$\text{if } \iota \mathbin{;} f = \sigma \mathbin{;} \iota \mathbin{;} g \text{ then } f = g. \tag{4}$$

Example 6.6. Suppose $\mathscr{C}$ has a separating family of subterminal objects $\mathcal{S}$. Remember that an object S is subterminal if and only if for every Y, the hom-category $\mathscr{C}(Y, S)$ has at most one element and a family of morphisms $\mathcal{S}$ is separating if and only if for any two distinct parallel morphisms $f, g : X \rightrightarrows Y$ there exists $S \in \mathcal{S}$ and a morphism $h : S \to X$ with $h \mathbin{;} f \neq h \mathbin{;} g$.

Given such f, g, pick an $h : S \to X$ such that $h \mathbin{;} f \neq h \mathbin{;} g$. Because S is subterminal, any morphism out of S is trivially monic. Therefore h is a monomorphism. Is it proper? If h is an isomorphism, then $X \cong S$ is subterminal, so both h and σ in Eq. (4) necessarily equal $\mathbf{id}_X$, so $\mathbf{id}_X$ is a suitable ι. If h is a proper monomorphism, then, σ in Eq. (4) still has to be an identity, so that ι subobject-separates f and g by the separativeness of $\mathscr{S}$.

This applies in particular to all localic topoi (see C1.4.7. of [18]).

Proposition 6.7 (**SOb**$_{\mathscr{C}}$ is faithful when $\mathscr{C}$ is subobject-separated)**.** *Let $\mathscr{C}$ be a monic subobject-small category and let $\overset{\hookrightarrow}{\mathscr{C}}$ be its wide subcategory of monomorphisms. If $\mathscr{C}$ is subobject-separated (in the sense of Definition 6.5), then the subobject functor*

$$\mathbf{SOb}_{\mathscr{C}} : \overset{\hookrightarrow}{\mathscr{C}} \longrightarrow \mathbf{UBP}$$

is faithful.

Proof. Let $f, g : X \rightrightarrows Y$ be two monomorphisms and assume $\mathbf{SOb}_{\mathscr{C}}(f) = \mathbf{SOb}_{\mathscr{C}}(g)$. Concretely, this means that for every subobject $\iota : A \hookrightarrow X$, the two monomorphisms $\iota \mathbin{;} f$ and $\iota \mathbin{;} g$ become equal in the subobject preorder of Y; i.e. there exists an isomorphism $\sigma : A \to A$ (depending on ι) such that

$$\iota \mathbin{;} f \;=\; \sigma \mathbin{;} \iota \mathbin{;} g.$$

Now apply the hypothesis that $\mathscr{C}$ is subobject-separated: then for the pair (f, g) there exists a subobject $\iota : A \hookrightarrow X$ with the property that if there exists such an isomorphism $\sigma : A \to A$ for ι, then $f = g$. Hence $\mathbf{SOb}_{\mathscr{C}}$ is faithful. ◻

Remark 6.8. Note that this means that **SOb** induces an equivalence relation on $\overset{\hookrightarrow}{\mathbf{Cat}}$, given by $\mathscr{C} \sim D$ if and only if the image of $\mathbf{SOb}_{\mathscr{C}}$ is isomorphic to that of $\mathbf{SOb}_{\mathscr{D}}$, which is like Morita-equivalence for **UBP**. We can call $\mathscr{C}$ *subobject-equivalent* to $\mathscr{D}$ in this case.

Example 6.9. The delooping of every group is subobject-equivalent to the terminal category.

Definition 6.10. A functor $F : \mathscr{C} \to \mathscr{D}$ is *monomorphism-preserving* if for each monomorphism $\iota : X \hookrightarrow Y$ in $\mathscr{C}$, $F(\iota)$ is a monomorphism.

Example 6.11. Every pullback-preserving functor is monomorphism-preserving because every monomorphism is a pullback along itself. In particular, every right adjoint R is monomorphism-preserving since it preserves limits.

Given now a monomorphism-preserving functor $F : \mathscr{C} \to \mathscr{D}$, we can to each monomorphism $\iota : X \to Y$ in $\mathscr{C}$ associate a monomorphism $F(\iota)$ in $\mathscr{D}$, and thus obtain a natural transformation $F_* : \mathbf{SOb}_{\mathscr{C}} \Rightarrow \mathbf{SOb}_{\mathscr{D}}$, whose component at X is the map $F_{*,X} : \mathbf{SOb}_{\mathscr{C}}(X) \to \mathbf{SOb}_{\mathscr{D}}(F(X))$ sending each monomorphism ι to $F(\iota)$. So $F(\iota)$, viewed as an element of $\mathbf{SOb}_{\mathscr{C}}(F(X))$, is precisely the top $\top_{F(\iota)_*}$ of the ideal embedding $F(\iota)_* : \mathbf{SOb}(F(A)) \hookrightarrow \mathbf{SOb}_{\mathscr{C}}(F(X))$. This same construction - extracting the element designated by the image of a monomorphism - will reappear throughout this text in several different categorical settings, and we shall uniformly denote it by $F_{*,X}$ in each case. In particular they will be important in Section 8.

Note that this means that both parts of the top-preserving/ideal-embedding factorization system serve complementary roles for displaying a shadow of $\overset{\hookrightarrow}{\mathbf{UBP}}$: the top-preserving morphisms are the shadows of functors, the ideal embeddings are shadows of morphisms between objects. And, as we will learn later, the 2-cells, i.e. inequalities, between morphisms in **UBP** display, in some sense, natural transformations.

6.1 The Duality Breaking Point

This is where the duality between smallness and density finally breaks: first of all, the subobject preorder $\mathbf{SOb}_{\mathscr{C}}(X)$ of every object X of every category $\mathscr{C}$ is *upward*-bounded but not necessarily *downward*-bounded, so in a general category this notion of density makes no sense. It is only when $\mathscr{C}$ has an initial object that $\mathbf{SOb}_{\mathscr{C}}(X)$ is also downward-bounded. But this loss of generality is only the tip of the iceberg.

6.1.1 Density & Satedness

For each $\iota : X \hookrightarrow Y$ the morphism $\iota_* : \mathbf{SOb}_{\mathscr{C}}(X) \hookrightarrow \mathbf{SOb}_{\mathscr{C}}(Y)$ is an *ideal embedding*, which is principal by Remark 5.2, so it preserves $\bot$ if it exists (but not $\top$), and it preserves and reflects both meets and joins. Thus it fulfills the requirements of Corollary 5.28, so that ι_* is density-reflecting. And this is why density deserves its name: because it is reflected by *all* subobject inclusions in *all* categories with initial objects.

Moreover, if a functor $F : \mathscr{C} \to \mathscr{D}$ preserves pullbacks and the initial object $\varnothing$, then it preserves monomorphisms and the induced natural transformation F_* preserves meets to $\bot$. So if F also reflects the initial object, then F_* reflects density by Proposition 5.19.

The Yoneda Embedding Let us also study how density behaves with respect to the Yoneda embedding:

Example 6.12. Let $\widehat{(-)} : \mathscr{S} \hookrightarrow \widehat{\mathscr{S}}$ be the Yoneda embedding of a site with a subcanonical coverage and a strict initial object $\varnothing$ such that $\varnothing_{\widehat{\mathscr{S}}} \cong \widehat{\varnothing_{\mathscr{S}}}$ (i.e. the empty sheaf is representable) and consider the induced map on subobject preorders $\top_{\widehat{(-)},X} : \mathbf{SOb}_{\mathscr{S}}(X) \to \mathbf{SOb}_{\widehat{\mathscr{S}}}(\widehat{X}) : \iota \mapsto \hat{\iota}$. Since $\mathscr{S}$ is subcanonical, $\widehat{(-)}$ preserves finite limits and by assumption it preserves the initial object $\varnothing$. Since $\top_{\widehat{(-)},X}$ is also fully faithful (by the Yoneda lemma and subcanonicity), it reflects $\varnothing$. Thus, $\widehat{(-)}$ reflects density.

Not just that, but for preorders the Yoneda embedding also *preserves* density. For this, we need the following lemma, which was probably recorded somewhere else before:

Lemma 6.13. *Let $\mathscr{S}$ be a thin site with a subcanonical coverage. Then every nonzero sheaf F on $\mathscr{S}$ receives a monomorphism from some representable.*

Proof. Consider presheaves $F : \mathscr{S}^{op} \to \mathbf{Set}$. Recall that a natural transformation $\hat{x} : \widehat{X} \Rightarrow F$ from a representable corresponds, by Yoneda, to a choice of element $x \in F(X)$. Its component at Y is

$$\overline{x}_Y : \mathscr{S}(Y, X) \to F(Y), \quad f \mapsto F(f)(x).$$

In a presheaf category, $\hat{x}$ is monic if and only if every component $\hat{x}_Y$ is injective, and since $\mathscr{S}$ is subcanonical, finite limits in $\widehat{\mathscr{S}}$ reduce to those in the surrounding presheaf topos $\mathbf{PSh}(\mathscr{S})$. So we have to find a componentwise monic into F. If F is non-trivial, then there exists some $x \in F(X)$ for some object X. But since for each object Y, the hom-set $\mathscr{S}(Y, X)$ has size 0 or 1 and any function with domain of size ≤ 1 is trivially injective. Thus, $\hat{x}$ is componentwise injective. So any element of any sheaf in $\widehat{\mathscr{S}}$ is monic. $\square$

Proposition 6.14. *Given a thin site $\mathscr{S}$ with a subcanonical coverage and an initial object, the natural transformation $\top_{\widehat{(-)}} : \mathbf{SOb}_{\mathscr{S}} \hookrightarrow \widehat{(-)} \,;\, \mathbf{SOb}_{\widehat{\mathscr{S}}}$ induced by the Yoneda embedding $\widehat{(-)} : \mathscr{S} \hookrightarrow \mathbf{Set}^{\mathscr{S}^{op}}$ is density-preserving.*

Proof. We need to show that if $\imath$ is dense in $\mathbf{SOb}_{\mathcal{S}}(X)$, then $\hat{\imath}$ is dense in $\mathbf{SOb}_{\widehat{\mathcal{S}}}(\widehat{X})$. Let $\kappa\colon K \hookrightarrow \widehat{X}$ be any subsheaf such that

$$\kappa \wedge \hat{\imath} = \bot$$

in $\mathbf{SOb}_{\widehat{\mathcal{S}}}(\widehat{X})$. We must show $\kappa = \bot$. Since $\mathcal{S}$ is subcanonical, the sheafification functor preserves meets, so we can take the meet of $\kappa \wedge \hat{\imath}$ as sub-presheaves, which is given by objectwise pullback. If κ is not initial, then by Lemma 6.13 there is some $X : \mathcal{S}$ such that $K(X)$ contains an element y, so that $\kappa \wedge \hat{\imath} \geq \hat{y} \mathbin{;} \kappa$, where $\hat{y}$ is the morphism $\widehat{X} \to K$ induced by the Yoneda lemma. But since each cover of $\mathcal{S}$ is jointly epimorphic and thus in particular non-empty, the empty presheaf on $\mathcal{S}$ is a sheaf and thus initial. So in particular $\hat{y} \mathbin{;} \kappa > \bot$. □

Remark 6.15. So the embedding of a thin subcanonical site with an initial object into its sheaf topos is both density-preserving and density-reflecting, so a *density-equivalence.*

Coproducts Another aspect where the duality between smallness and density breaks is with respect to coproducts, whose subobject structure in good cases is controlled by the summands:

Definition 6.16 (Disjoint coproduct, monic coprojections)**.** Let $\mathcal{C}$ be a category with an initial object $\varnothing$. A coproduct $X + Y$ with coprojections

$$\tau_X : X \to X + Y, \qquad \tau_Y : Y \to X + Y$$

is called a *disjoint coproduct with monic coprojections* if τ_X and τ_Y are monic and the pullback of τ_X and τ_Y is initial.

Equivalently, the images of τ_X and τ_Y are subobjects of $X + Y$ with intersection equal to the initial subobject.

Proposition 6.17 (Density in a disjoint coproduct)**.** *Let $\mathcal{C}$ be a category with an initial object $\varnothing$ and disjoint coproducts with monic coprojections. Then the following equivalent statements hold:*

1. *If for some monomorphism $\imath\colon A \hookrightarrow X$ the composite*

$$\imath \mathbin{;} \tau_X : A \longrightarrow X + Y$$

   is dense in $\mathbf{SOb}(X + Y)$, then $Y \simeq \varnothing$.

2. *The coprojection τ_X itself is dense in $\mathbf{SOb}(X + Y)$ if and only if $Y \simeq \varnothing$.*

Proof. For Item 1 assume $\imath \mathbin{;} \tau_X$ is dense. Consider the subobject $\tau_Y : Y \hookrightarrow X + Y$. Because the coproduct is disjoint, the pullback of τ_X and τ_Y is initial. Since $\imath \mathbin{;} \tau_X$ factors through τ_X, the pullback of τ_Y and $\imath \mathbin{;} \tau_X$ is also initial; in symbols,

$$\tau_Y \wedge (\imath \mathbin{;} \tau_X) = \bot.$$

By definition of density, this forces $\tau_Y = \bot$, i.e. the monomorphism τ_Y is isomorphic to the initial subobject. Hence $Y \simeq \varnothing$.

The first direction of Item 2 follows from Item 1 by taking $\imath = \mathbf{id}_X$. For the converse, note that if $Y \simeq \varnothing$, then the coproduct $X + \varnothing$ is isomorphic to X via τ_X. □

Satedness This regularity of density does not extend to satedness however without further assumptions: by Proposition 5.42 the reflectiveness of density is useful for morphisms that induce a surjection on subobject preorders, but this is never the case for non-trivial embeddings. Of course, it is the case for quotients, which we will turn to in a moment. Moreover, if $\mathscr{C}$ is a Heyting category, then each $\mathbf{SOb}_{\mathscr{C}}(X)$ fulfills the requirements of Corollary 5.47. Thus, satedness is reflected along embeddings in a Heyting category, which explains why every open subset of a sated open subset of a topological space is sated in every open neighborhood. We also have the effect of Proposition 5.44: that, for any $\iota\colon A \hookrightarrow X$ in any category with an initial object, if $\kappa = \kappa' \mathbin{;} \iota$ factors through ι, $\kappa'\colon B \hookrightarrow A$ is sated in $\mathbf{SOb}(A)$ and $\kappa \geq \lambda \wedge \mu$ with μ dense, then $\kappa \geq \lambda \wedge \iota$, so we can switch out any dense subobject with ι in such meet inequalities.

6.1.2 Smallness & Reducedness

Smallness on the other hand is a lot more whimsical. For instance, it is not the case that each ι_* preserves smallness for $\iota\colon X \hookrightarrow Y$ in some category $\mathscr{C}$. We have however the following consequence of Lemma 5.40:

Corollary 6.18. *Let $\mathscr{C}$ be a category such that each subobject preorder of objects in $\mathscr{C}$ is a modular lattice. Then each monomorphism in $\mathscr{C}$ is smallness-preserving.*

Smallness is also respectively more densely interwoven with reducedness: by Corollary 5.26 a subobject $\iota\colon A \hookrightarrow X$ is reduced if and only if it reflects smallness.

Another deep difference between smallness and density regards the Yoneda embedding: we saw in Section 6.1.1 that the density of sheaves on a thin subcanonical site is completely controlled by that in $\mathscr{C}$, while we saw in Proposition 3.58 that *every* non-trivial representable subobject is small in a representable presheaf. So Yoneda just makes everything small. Sheafification then makes the smallness relation non-trivial again by identifying unions of representable subobjects with the entirety of the sheaf. So what is small in a sheaf topos depends solely on the coverage.

Finally, another consequence of the ideality of subobject preorder embeddings is that the smallness stratification from Section 3.3 is *way* more useful than its dense counterpart: in categorical language, "ι is n-small" just means "ι is $(n-1)$-small in a small subobject" while "ι is n-dense" means "ι is $(n-1)$-dense in a dense principal filter" *which does not correspond to anything category-theoretically*, at least on a basic level. Thus, the smallness stratification can be found *everywhere* while the density stratification is much more obtuse.

6.1.3 Comparison

So while the order-theoretic definitions are perfectly dual, the categorical realisations are not: density imposes much stricter constraints on how an object can be decomposed. Since density is preserved by a much wider class of functors (those preserving pullbacks and preserving and reflecting the initial object) and reflected by all monomorphisms while smallness is preserved by a more restricted class and reflected only by those monomorphisms that are reduced, the structural properties of density are more tightly interwoven with the way objects interact. So density becomes a global notion, which is highly sensitive to disturbances, while smallness becomes local - but local along reducedness, which itself is defined through smallness, leading to a something of a chicken-and-egg situation when trying to infer global smallness from local data.

Reducedness and satedness are interesting - since reducedness is essentially equivalent to (covariant) smallness-reflection, it means that size-wise the image of a morphism looks relatively the same as the containing object: at least in coherent categories smallness is then both preserved and reflected along reduced monomorphisms. Satedness doesn't work that way at all. This is because for a proper principal ideal embedding $\iota_a : (a) \hookrightarrow P$ it trivially holds that $\iota(\top_{(a)}) < \top_P$, so reducedness is all about how joins to a look compared to $\top_P$; but since bottoms are preserved by f_* for each morphism f in each category, the criterion for completion is both in P and in (a) based on how meets to $\bot$ act, So the question is: "How much is under a?" and the answer has to be "enough to make up the difference between $\top$ and a dense subobject."

6.2 Extension to Non-Monic Categories

6.2.1 Extension along Pullbacks

If a category $\mathscr{C}$ has pullbacks along monics, then there also exists a functor $\mathbf{SOb}^*_{\mathscr{C}} : \mathscr{C}^{op} \to \mathbf{UBP}$, which maps a morphism $f : Y \to X$ to the pullback map $f^* : \mathbf{SOb}_{\mathscr{C}}(X) \to \mathbf{SOb}_{\mathscr{C}}(Y)$. This is the archetypal hyperdoctrine.

Moreover, since pullbacks always preserve identities, f^* preserves tops. As before, $\mathbf{SOb}^*_{\mathscr{C}}$ is not faithful for instance when $\mathscr{C}$ is the delooping of a groupoid. Note however that $\mathbf{SOb}^*$ is still faithful on (perhaps surprisingly) many categories:

Example 6.19. In the internal language of any topos $\mathscr{T}$, for any $f : X \to Y, \iota : A \hookrightarrow Y$, $f^*(\iota) = \{x \in X | \exists a \in A : f(x) \in \iota\}$, so that it holds that $x \in f^*(\iota) \Leftrightarrow f(x) \in \iota$, so if for all ι in Y $f^*(\iota) = g^*(\iota)$, then $\forall x \in X : f(x) = g(x)$. Since elements in the internal language of a topos correspond to $\kappa : B \hookrightarrow X$ such that B is subterminal, it follows that $f = g$ if $\mathscr{T}$ is separated by its subterminal objects. So again $\mathbf{SOb}^*$ is faithful on all topoi that are separated by subterminal objects.

On the other hand, for any monic $\iota : A \hookrightarrow Y, \kappa : Y \hookrightarrow X$ in any category, the square

$$\begin{tikzcd} Y \arrow[r, hook, "\kappa"] & X \\ A \arrow[u, hook, "\iota"] \arrow[r, "\mathbf{id}_A"] & A \arrow[u, hook, "\iota ; \kappa"'] \end{tikzcd}$$

is a pullback square, so that the restriction $\mathbf{SOb}^*_{\mathscr{C}}|_{\overset{\hookrightarrow}{\mathscr{C}}}$ of $\mathbf{SOb}^*_{\mathscr{C}}$ to the monic part $\overset{\hookrightarrow}{\mathscr{C}}$ of $\mathscr{C}$ maps ι^* to the right adjoint of ι_*. Thus, $\mathbf{SOb}^*_{\mathscr{C}}$ lifts with the covariant subobject functor from the last section to a functor $\mathscr{C}^{op}|_{\overset{\hookrightarrow}{\mathscr{C}}} \hookrightarrow \mathbf{UBP}_{l\text{-}adj}$ into the category on upward-bounded preorders whose morphisms $(l, r) : P \to Q$ are pairs of left-right adjoints (going in the left direction) in **UBP** , or equivalently the subcategory of **UBP** on those morphisms that are left adjoints. Thus, we can compare smallness and density along all morphisms both covariantly (through postcomposition) and contravariantly (through pullback).

Density and Smallness of Coproducts In the presence of pullbacks, we can sharpen the behavior of density and smallness in coproducts. The following standard result characterizes the subobject structure of a coproduct in an extensive category.

Proposition 6.20. *Let $\mathscr{C}$ be an extensive category. For any coproduct $X + Y$ with coprojections*

$$\tau_X : X \to X + Y, \qquad \tau_Y : Y \to X + Y,$$

the canonical map

$$(\tau_X^*, \tau_Y^*)\colon \mathbf{SOb}_{\mathscr{C}}(X+Y) \longrightarrow \mathbf{SOb}_{\mathscr{C}}(X) \times \mathbf{SOb}_{\mathscr{C}}(Y)$$

is an isomorphism of preorders.

Proof. Let $m\colon A \hookrightarrow X+Y$ be a subobject. By extensivity, the pullbacks $m_X\colon A_X \hookrightarrow X$ and $m_Y\colon A_Y \hookrightarrow Y$ along the coprojections form a coproduct $A \cong A_X + A_Y$. This establishes a bijection between subobjects of $X+Y$ and pairs of subobjects of X and Y. Conversely, given a pair $(a\colon A \hookrightarrow X, b\colon B \hookrightarrow Y)$, the universal property of the coproduct induces a monomorphism $a+b\colon A+B \hookrightarrow X+Y$ (the coproduct of monomorphisms in an extensive category is monic). These constructions are clearly order-preserving and mutually inverse, yielding the isomorphism. □

Remark 6.21. Thus, in an extensive category, the subobject functor converts coproducts into products, behaving much like a powerset. This preservation of limits allows us to analyze smallness and density component-wise.

We can now describe the behavior of smallness under coproducts:

Corollary 6.22. *Let $\mathscr{C}$ be extensive. For a monomorphism $\iota\colon A \hookrightarrow X+Y$ with associated decomposition $A \cong A_X + A_Y$ under Proposition 6.20, the following are equivalent:*

1. *ι is small in $\mathbf{SOb}(X+Y)$;*
2. *Both pullbacks $\iota_X\colon A_X \hookrightarrow X$ and $\iota_Y\colon A_Y \hookrightarrow Y$ are small in $\mathbf{SOb}_{\mathscr{C}}(X)$ and $\mathbf{SOb}_{\mathscr{C}}(Y)$ respectively.*

Proof. Under the isomorphism $\mathbf{SOb}(X+Y) \cong \mathbf{SOb}_{\mathscr{C}}(X) \times \mathbf{SOb}_{\mathscr{C}}(Y)$, joins are computed componentwise. Thus, $\iota \vee \kappa = \top_{X+Y}$ holds if and only if $\iota_X \vee \kappa_X = \top_X$ and $\iota_Y \vee \kappa_Y = \top_Y$. ι is small precisely when the only pair (κ_X, κ_Y) satisfying this condition is $(\top_X, \top_Y)$. This is equivalent to the requirement that there is no proper κ_X in X joining with ι_X to $\top_X$ (and similarly for Y); otherwise, taking $(\kappa_X, \top_Y)$ would contradict the smallness of ι. Hence, both components must be small. □

Consequently, in an extensive category, the smallness of a subobject of a coproduct is determined entirely by its restrictions to the summands. The behavior of density is then the abstractly same:

Corollary 6.23. *Let $\mathscr{C}$ be an extensive category. Then each subobject $\iota\colon A \hookrightarrow X+Y$ of a coproduct $X+Y$ is dense if and only if both pullbacks $\tau_X^*(\iota)$ and $\tau_Y^*(\iota)$ are dense in $\mathbf{SOb}_{\mathscr{C}}(X)$ and $\mathbf{SOb}_{\mathscr{C}}(Y)$ respectively.*

Proof. By Proposition 6.20, we identify $\mathbf{SOb}(X+Y)$ with the product $\mathbf{SOb}_{\mathscr{C}}(X) \times \mathbf{SOb}_{\mathscr{C}}(Y)$, where the bottom element is $(\bot_X, \bot_Y)$ and meets are computed pointwise. An element (d_X, d_Y) in a product of downward-bounded preorders is dense if and only if for any (a, b), the condition $(a \wedge d_X, b \wedge d_Y) = (\bot_X, \bot_Y)$ implies $(a, b) = (\bot_X, \bot_Y)$.

This condition holds precisely when d_X is dense in $\mathbf{SOb}_{\mathscr{C}}(X)$ and d_Y is dense in $\mathbf{SOb}_{\mathscr{C}}(Y)$: testing with $b = \bot_Y$ shows that $a \wedge d_X = \bot_X$ forces $a = \bot_X$, and symmetrically for b. The claim follows by transporting this characterization back along the isomorphism. □

but different with respect to component inclusions:

Corollary 6.24. *If $\iota\colon A \hookrightarrow X$ is small in X and $\mathscr{C}$ is extensive, then $\iota \,;\, \tau_X\colon A \hookrightarrow X+Y$ is small for each Y.*

6.2.2 Extension to Regular Categories

Now, since each subobject meet is a limit, it is preserved by pullback, so that f^* preserves meets. Thus, if $\mathscr{C}$ has an initial object $\varnothing$ and f^* reflects it, then it fulfills the requirements of Proposition 5.19 and thus *also* reflects density. In particular, if f is left orthogonal in a factorization system $\mathfrak{f}$ whose right orthogonal morphisms includes all initial morphisms and whose left-orthogonal cells are stable under pullback, then for each $\iota\colon A \hookrightarrow X$ such that $f^*(\iota) = \iota_\varnothing\colon \varnothing \hookrightarrow Y$, the left square in the diagram

$$\begin{tikzcd} A \arrow[r, "\mathbf{id}_A"] & A \arrow[r, hook, "\iota"] & X \\ \varnothing \arrow[u, "\iota_\varnothing"] \arrow[r, "\mathbf{id}_\varnothing"] & \varnothing \arrow[u, "\iota^*(f)"] \arrow[r, hook, "f^*(\iota)"] & Y \arrow[u, "f"] \end{tikzcd}$$

is a factorization square and thus there exists a $k\colon A \to \varnothing$. Thus:

Corollary 6.25. *In a category $\mathscr{C}$ with a strict initial object, let $\mathfrak{f}$ be a factorization system whose right orthogonal part includes all initial morphisms. Then for each morphism f in the left part of $\mathfrak{f}$ the induced pullback morphism f^* is density-reflecting.*

There is an obvious candidate for such a factorization, which is the regular-epi/mono-factorization that exists in every regular category [13]. Remember that a regular category is, roughly speaking, a setting where we can take images of maps in a well-behaved, pullback-stable way:

Definition 6.26 (Regular category)**.** A category $\mathscr{C}$ is *regular* if:

1. $\mathscr{C}$ has all finite limits.
2. Every morphism $f\colon X \to Y$ admits a factorization

   $$X \xrightarrow{e_f} I_f \xrightarrow{\mathbf{Im}(f)} Y$$

   where

   - e_f is a *regular epimorphism* (i.e. a coequalizer of some parallel pair),
   - the *image* $\mathbf{Im}(f)$ of f is a monomorphism,

   and this factorization is *stable under pullback*: for any morphism $g\colon Z \to Y$, if we form the pullback

   $$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

   then the induced map $X' \to Z$ again factors as a regular epi followed by a mono, and this factorization is the pullback of e_f and $\mathbf{Im}(f)$.

So given a regular category $\mathscr{C}$, for each $f\colon X \to Y$ the morphism $\mathbf{SOb}^*_{\mathscr{C}}(f)\colon \mathbf{SOb}^*_{\mathscr{C}}(Y) \to \mathbf{SOb}^*_{\mathscr{C}}(X)$ has a left adjoint given by mapping ι for $\iota\colon A \hookrightarrow X$ to the element $\mathbf{Im}(\iota \mathbin{;} f)$ of $\mathbf{SOb}_{\mathscr{C}}(Y)$.

The Density Coverage Let now $f : X \to Y$ be a morphism in a regular category $\mathscr{C}$ and let $\iota : A \hookrightarrow Y, \kappa : B \hookrightarrow Y$ be such that $\iota \wedge \kappa = \varnothing$. If the initial object of $\mathscr{C}$ is strict, then $(\iota \wedge \kappa)^*(f)$ is $\mathbf{id}_\varnothing$ (up to equivalence). Since pullbacks commute with each other,

$$f^*(\iota) \wedge f^*(\kappa) = f^*(\iota \wedge \kappa) = f^*(\varnothing).$$

Since the source of $(\iota \wedge \kappa)^*(f)$ is the same as that of $f^*(\varnothing)$, meaning $\varnothing$, we have $f^*(\iota) \wedge f^*(\kappa) = \mathbf{id}_\varnothing$.

So if ι is dense and $f^*(\iota) \wedge \lambda = \varnothing$ for some $\lambda : C \hookrightarrow X$ then $\iota^*(\lambda \mathbin{;} f)$ factors through $\varnothing$ and thus is initial if $\varnothing$ is strict. Since ι is dense it follows that $\mathbf{Im}(\lambda \mathbin{;} f) = \varnothing$, so $\lambda \leq f^*(\mathbf{Im}(f(\lambda))) = f^*(\varnothing) = \varnothing$. Thus:

Corollary 6.27. *Given a morphism $f : X \to Y$ in a regular category $\mathscr{C}$ with a strict initial object, the induced morphism $f^* : \mathbf{SOb}^*_{\mathscr{C}}(Y) \to \mathbf{SOb}^*_{\mathscr{C}}(X)$ is density-preserving. In particular, if g has a dense image m, then since the regular-epi/mono factorization $g = e \mathbin{;} m$ is functorial, the pullback $f^*(g)$ of g has a dense image $\mathbf{Im}(f^*(g)) = f^*(m)$.*

Thus, and by Corollary 6.25, the contravariant subobject morphism e^* of a regular epimorphism e is a density-equivalence.

Remark 6.28. When $\iota : A \hookrightarrow X$ is a monomorphism in a regular category with a strict initial object then the density-reflection along ι is a special case of Corollary 6.27 but it is important to note that although f^* is density-preserving, f_* is not always density-reflecting.

Thus, we can make density into a coverage:

Definition 6.29. Given a regular category $\mathscr{C}$ with a strict initial object, the *density coverage* is given by the morphisms in $\mathscr{C}$ whose image is dense.

Note that the density coverage is decidedly *not* subcanonical (since each dense morphism in $\mathscr{C}$ is a cover for this coverage). So by itself, sheafification of this site is more about inferring stuff from dense subobjects:

Example 6.30. Let k be a field, $\mathbf{Var}_k$ the category of varieties over k and for each $V : \mathbf{Var}_k$ let $\mathbf{O}(V)$ be given by the preorder of open subvarieties of V. Fix some variety Y over k and let $\mathbf{Var}_{k,V}(-, Y) : \mathbf{O}(V)^{op} \to \mathbf{Set}$ be the composition of the forgetful functor $\mathbf{s} : \mathbf{O}(V) \to \mathbf{Var}_k$ with the Yoneda functor of Y, which assigns to each V the set of algebraic maps from V to Y. The sheafification $\widehat{\mathbf{Var}_k(-, Y)}$ of $\mathbf{Var}_k(-, Y)$ with respect to the density coverage on $\mathbf{O}(V)$ freely adds for each open dense $d : V' \hookrightarrow V$ and algebraic map $f : V' \to Y$ an element $\hat{f}$ to $\widehat{\mathbf{Var}_k(V, Y)}$ such that if $e : V'' \hookrightarrow V'$ is dense in V', then $\widehat{e \mathbin{;} f} = \hat{f}$. It follows that $\hat{f}$ represents a rational map from V to Y.

However, the density coverage can also be made subcanonical by intersecting it with the canonical coverage.

Density is Preserved by Regular Epimorphisms But density is *also* covariantly preserved by regular epimorphisms. For this, fix a regular epi $\varepsilon : X \twoheadrightarrow Y$. For each subobject $\iota : A \hookrightarrow X$ in $\mathbf{SOb}_{\mathscr{C}}(X)$, define

$$\mathbf{SOb}_{\mathscr{C}}(\varepsilon)(\iota) \; := \; \mathbf{Im}(\iota \mathbin{;} \varepsilon) \in \mathbf{SOb}_{\mathscr{C}}(Y).$$

Because $\mathscr{C}$ is regular, images exist and are stable under pullback; $\mathbf{SOb}_{\mathscr{C}}(\varepsilon)$ is monotone and preserves finite joins, and since $\mathscr{C}$ is strict initial, it also preserves $\perp$. We need:

Lemma 6.31. *Let $\mathscr{C}$ be a regular category and $\varepsilon\colon X \twoheadrightarrow Y$ a regular epimorphism. Then the induced monotone map*

$$\mathbf{SOb}_{\mathscr{C}}(\varepsilon)\colon \mathbf{SOb}_{\mathscr{C}}(X) \to \mathbf{SOb}_{\mathscr{C}}(Y), \qquad \iota \mapsto \mathbf{Im}(\iota \mathbin{;} \varepsilon)$$

is surjective on elements.

Proof. To see this, take any subobject $\kappa\colon B \hookrightarrow Y$ and form its pullback along ε, yielding a subobject $\iota\colon A \hookrightarrow X$ and a pullback square where $\varepsilon'\colon A \to B$ is a regular epimorphism. Since $\varepsilon' \mathbin{;} \kappa = \iota \mathbin{;} \varepsilon$, the image of $\iota \mathbin{;} \varepsilon$ is isomorphic to κ. Thus, $\mathbf{SOb}(\varepsilon)(\iota) = \kappa$, proving surjectivity. ◻

Proposition 6.32 (Density is covariantly preserved by regular epimorphisms)**.** *Let $\mathscr{C}$ be a regular category with strict initial object, and $\varepsilon\colon X \twoheadrightarrow Y$ a regular epimorphism. If $\iota\colon A \hookrightarrow X$ is dense in $\mathbf{SOb}_{\mathscr{C}}(X)$, then its image $\mathbf{Im}(\iota \mathbin{;} \varepsilon) \hookrightarrow Y$ is dense in $\mathbf{SOb}_{\mathscr{C}}(Y)$.*

Proof. Let $\iota \in \mathbf{SOb}_{\mathscr{C}}(X)$ be dense. Then $\mathbf{SOb}_{\mathscr{C}}(\varepsilon)(\iota) = \mathbf{Im}(\iota \mathbin{;} \varepsilon)$ is dense in $\mathbf{SOb}_{\mathscr{C}}(Y)$ by Corollary 5.29, specialized to $f = \mathbf{SOb}_*(\varepsilon)$, using the regular-epi properties noted above and the fact that regular epimorphisms in a strict initial regular category reflect bottoms, as we saw at the section beginning. ◻

Remark 6.33. So density is stable under composition with regular epimorphisms and thus e^* is a density-equivalence and e_* preserves density. The only thing e does not do is induce a density-reflecting e_*.

Corollary 6.34. *Let $e\colon X \to Y$ be a regular epimorphism in a regular category. Then the pullback morphism $e^*(\mathbf{SOb}_{\mathscr{C}}(Y)) \to \mathbf{SOb}_{\mathscr{C}}(X)$ reflects satedness.*

Proof. By Remark 5.43 we have to show that e^* preserves meets, preserves density and is injective. Meet-preservation is clear because limits commute with each other. Density-preservation follows from Corollary 6.27. Finally, order-reflection is proved below. ◻

This lemma may well have been stated somewhere else already in one form or another:

Lemma 6.35. *Let $\mathscr{C}$ be a regular category and $e\colon X \twoheadrightarrow Y$ a regular epimorphism. Then the pullback functor*

$$e^*\colon \mathbf{SOb}_{\mathscr{C}}(Y) \longrightarrow \mathbf{SOb}_{\mathscr{C}}(X)$$

is order-reflecting: for subobjects ι, ι' of Y, if $e^*(\iota) \le e^*(\iota')$ in $\mathbf{SOb}_{\mathscr{C}}(X)$, then $\iota \le \iota'$ in $\mathbf{SOb}_{\mathscr{C}}(Y)$.

Proof. Let $\iota\colon A \hookrightarrow Y$ and $\iota'\colon A' \hookrightarrow Y$ be subobjects of Y, and form the pullbacks of e along each:

$$\begin{tikzcd} P \arrow[r, "e^*(\iota)"] \arrow[d, "\iota^*(e)"'] & X \arrow[d, "e"] & & P' \arrow[r, "e^*(\iota')"] \arrow[d, "\iota'^*(e)"'] & X \arrow[d, "e"] \\ A \arrow[r, "\iota"'] & Y & & A' \arrow[r, "\iota'"'] & Y \end{tikzcd}$$

Since pullbacks of monomorphisms are monomorphisms, $e^*(\iota)$ and $e^*(\iota')$ are monic; and since regular epimorphisms are stable under pullback in a regular category, $\iota^*(e)\colon P \to A$ and $\iota'^*(e)\colon P' \to A'$ are regular epimorphisms. The pullback squares give

$$\iota^*(e) \mathbin{;} \iota \;=\; e^*(\iota) \mathbin{;} e, \qquad \iota'^*(e) \mathbin{;} \iota' \;=\; e^*(\iota') \mathbin{;} e.$$

Suppose $e^*(\iota) \le e^*(\iota')$, so there exists $h\colon P \to P'$ with $h \mathbin{;} e^*(\iota') = e^*(\iota)$. Then:

$$\iota^*(e) \mathbin{;} \iota \;=\; e^*(\iota) \mathbin{;} e \;=\; (h \mathbin{;} e^*(\iota')) \mathbin{;} e \;=\; h \mathbin{;} (e^*(\iota') \mathbin{;} e) \;=\; h \mathbin{;} (\iota'^*(e) \mathbin{;} \iota') \;=\; (h \mathbin{;} \iota'^*(e)) \mathbin{;} \iota'. \qquad (\star)$$

Since $\iota^*(e)$ is a regular epimorphism, it is the coequalizer of some pair $r, s\colon R \rightrightarrows P$. By definition, $r \mathbin{;} \iota^*(e) = s \mathbin{;} \iota^*(e)$, hence $r \mathbin{;} \iota^*(e) \mathbin{;} \iota = s \mathbin{;} \iota^*(e) \mathbin{;} \iota$. Applying $(\star)$ to both sides:

$$r \mathbin{;} (h \mathbin{;} \iota'^*(e)) \mathbin{;} \iota' \;=\; s \mathbin{;} (h \mathbin{;} \iota'^*(e)) \mathbin{;} \iota'.$$

Since ι' is monic, we may cancel it on the right to obtain $r \mathbin{;} (h \mathbin{;} \iota'^*(e)) = s \mathbin{;} (h \mathbin{;} \iota'^*(e))$. By the universal property of the coequalizer, there exists a unique $g\colon A \to A'$ with $\iota^*(e) \mathbin{;} g = h \mathbin{;} \iota'^*(e)$. Composing both sides with ι' and using $(\star)$:

$$\iota^*(e) \mathbin{;} (g \mathbin{;} \iota') \;=\; (h \mathbin{;} \iota'^*(e)) \mathbin{;} \iota' \;=\; \iota^*(e) \mathbin{;} \iota.$$

Since $\iota^*(e)$ is an epimorphism, we conclude $g \mathbin{;} \iota' = \iota$, which is precisely $\iota \le \iota'$.

The equality-reflection statement follows by applying the argument in both directions. $\square$

Monic Quotients There are two basic kinds of regular epimorphisms: those that glue together two different parts of the same objects and those that make some part of an object smaller. In a regular category we can say that one of these kinds is unproblematic:

Definition 6.36. A *regular quotient* (or effective quotient) in a regular category $\mathscr{C}$ is the coequalizer $e\colon X \twoheadrightarrow Y$ of its kernel pair $(\iota, \kappa)\colon R \rightrightarrows X$.

Proposition 6.37 (Exactness of the kernel pair square)**.** *Let $\mathscr{C}$ be a regular category, and let*

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[d, "\kappa"'] & X \arrow[d, two heads, "e"] \\ X \arrow[r, two heads, "e"'] & Y \end{tikzcd}$$

be the commutative square formed by a regular epimorphism e and its kernel pair (ι, κ). Then this square is both a pullback and a pushout.

Proof. This is a standard fact in regular categories: every regular epimorphism is the coequalizer of its kernel pair. By definition, a kernel pair is a pullback of e along itself, so the square is a pullback. Because e is the coequalizer of ι and κ, the square is also a pushout. $\square$

Proposition 6.38 (Subobjects of a regular quotient)**.** *If $\mathscr{C}$ is a regular category, then the subobject functor $\mathbf{SOb}_{\mathscr{C}}$ preserves regular quotients. Specifically, in the situation of Proposition 6.37, the induced diagram*

$$\mathbf{SOb}_{\mathscr{C}}(R) \;\overset{\iota_*}{\underset{\kappa_*}{\rightrightarrows}}\; \mathbf{SOb}_{\mathscr{C}}(X) \xrightarrow{e_*}\!\!\!\!\rightarrow \mathbf{SOb}_{\mathscr{C}}(Y)$$

is a coequalizer diagram in **UBP**.

Proof. Since $\iota \,;\, e = \kappa \,;\, e$, we have $\iota_* \,;\, e_* = \kappa_* \,;\, e_*$.

Suppose $h\colon \mathbf{SOb}_{\mathscr{C}}(X) \to P$ is a monotone map satisfying $\iota_* \,;\, h = \kappa_* \,;\, h$. By Lemma 6.31, e_* is surjective on elements, so every $y \in \mathbf{SOb}_{\mathscr{C}}(Y)$ admits some preimage x with $e_*(x) = y$. We set $k(y) := h(x)$ and must verify independence of the chosen preimage.

Well-definedness. Suppose $x\colon A \hookrightarrow X$ and $x'\colon A' \hookrightarrow X$ satisfy $e_*(x) = e_*(x') = y\colon B \hookrightarrow Y$. Factor $x \,;\, e$ and $x' \,;\, e$ through their common image y:

$$A \xrightarrow{\bar{x}} B \xrightarrow{y} Y, \qquad A' \xrightarrow{\bar{x}'} B \xrightarrow{y} Y,$$

where $\bar{x}$ and $\bar{x}'$ are regular epimorphisms. Form the pullback $P = A \times_B A'$ with projections $p_1\colon P \to A$ and $p_2\colon P \to A'$; by stability of regular epimorphisms under pullback in a regular category, both p_1 and p_2 are regular epimorphisms. The two composites $p_1 \,;\, x$ and $p_2 \,;\, x'$ from P to X become equal after postcomposition with e, since both factor as $P \to B \xrightarrow{y} Y$. Hence they induce a map $u\colon P \to X \times_Y X = R$. Let $\bar{r}\colon \bar{P} \hookrightarrow R$ be the monomorphism in the image factorization of u. Then

$$\iota_*(\bar{r}) \;=\; \mathbf{Im}(\bar{P} \hookrightarrow R \xrightarrow{\iota} X) \;=\; \mathbf{Im}(P \xrightarrow{p_1 ; x} X) \;=\; x,$$

where the last equality holds because $P \xrightarrow{p_1} A \xrightarrow{x} X$ is already a regular-epi/mono factorization: p_1 is a regular epimorphism and x is monic. By the same token $\kappa_*(\bar{r}) = x'$. The hypothesis $\iota_* \,;\, h = \kappa_* \,;\, h$ now gives $h(x) = h(x')$.

Monotonicity. Let $y \leq y'$ in $\mathbf{SOb}_{\mathscr{C}}(Y)$, and choose preimages x, x' with $e_*(x) = y$ and $e_*(x') = y'$. Set $z = e^*(y)$ and $z' = e^*(y')$, the pullbacks along e. Since e is a regular epimorphism, the map $e^*(y) \to y$ is a regular epimorphism (stability under pullback), so $e_*(z) = y$ and similarly $e_*(z') = y'$. By well-definedness, $h(x) = h(z)$ and $h(x') = h(z')$. Now $y \leq y'$ implies $z = e^*(y) \leq e^*(y') = z'$, so monotonicity of h yields $h(x) = h(z) \leq h(z') = h(x')$.

By construction $h = e_* \,;\, k$, and uniqueness of k follows from the surjectivity of e_* (Lemma 6.31). Thus $\mathbf{SOb}_{\mathscr{C}}(Y)$ satisfies the universal property of the coequalizer in **UBP**. $\square$

So at this point we have a fairly good picture of how density behaves in a decently well-behaved category: reflected by all morphisms, preserved by regular epimorphisms and given componentwise in (disjoint, monic) coproducts. We have far fewer results for smallness - that is both its weakness and strength. But it is also to be expected because joins in subobject preorders are more category-theoretically involved than meets: meets align completely with categorical pullbacks, while joins in subobject preorders do not correspond to *any* universal constructs in the ambient category $\mathscr{C}$, they have to be added by definition.

6.2.3 Extension to Coherent Categories

Finally let us take a look at coherent categories. Remember that a coherent category is one in which we have well-behaved unions of subobjects:

Definition 6.39 (Coherent category)**.** A category $\mathscr{C}$ is *coherent* if:

1. $\mathscr{C}$ is regular;
2. for every object X, the subobject poset $\mathbf{SOb}_{\mathscr{C}}(X)$ has finite joins and for every arrow $f\colon Y \to X$, the pullback map

$$f^*\colon \mathbf{SOb}_{\mathscr{C}}(X) \to \mathbf{SOb}_{\mathscr{C}}(Y)$$

preserves finite joins.

Remark 6.40. Since f^* commutes with other pullbacks, it also preserves finite meets, which exist since $\mathscr{C}$ is regular, so that f^* is a lattice-morphism.

So in a coherent category $\mathscr{C}$, each subobject preorder has per definition all joins and they are preserved by pullbacks. So how does this change the behavior of smallness under morphisms? Well, for one, it fulfills the preconditions of Lemma 5.40, thus:

Corollary 6.41. *Smallness is covariantly preserved and reducedness reflected under monomorphisms in a coherent category.*

But preservation of joins under pullbacks also means that for each morphism f the induced contravariant subobject preorder morphism f^* fulfills Item 1a. Moreover:

Lemma 6.42. *Let $\mathscr{C}$ be a regular category and $f : X \to Y$ a morphism. Then the following are equivalent:*

1. *f is a regular epimorphism;*
2. *the pullback functor on subobjects*

$$f^* : \mathbf{SOb}_{\mathscr{C}}(Y) \to \mathbf{SOb}_{\mathscr{C}}(X)$$

reflects tops, *i.e. if $\iota : A \hookrightarrow Y$ satisfies $f^*(\iota) = \top_X$ in $\mathbf{SOb}_{\mathscr{C}}(X)$, then $\iota = \top_Y$.*

Proof. $\Rightarrow$: If $f^*(\iota) = \top_X$, then f factors through the subobject $\iota : A \hookrightarrow Y$. Since f is a regular epimorphism and ι is a monomorphism, the orthogonality property of the (regular epi, mono) factorization system supplies a morphism $d : Y \to A$ with $d \mathbin{;} \iota = \mathbf{id}_Y$, so ι is a split epimorphism and hence an isomorphism, so $\iota = \top_Y$.

$\Leftarrow$: Factor f into a regular epimorphism followed by a monomorphism, $f = e \mathbin{;} m$. By the universal property of pullbacks, pulling back the subobject m along f yields $f^*(m) = \top_X$. Because f^* reflects tops by hypothesis, we must have $m = \top_Y$, meaning m is an isomorphism. Thus, f is a regular epimorphism. □

Thus, a regular epimorphism in a coherent category fulfills all requirements of Item 1 and thus

Corollary 6.43. *For each regular epimorphism $e : X \to Y$ in a coherent category $\mathscr{C}$ the pullback morphism $e^* : \mathbf{SOb}_{\mathscr{C}}(Y) \to \mathbf{SOb}_{\mathscr{C}}(X)$ is smallness-reflecting.*

Note that some care has to be taken in parsing this statement: taking the pullback reverses direction, then the reflection reverses it again. So the statement says that if the preimage of $\iota : A \hookrightarrow Y$ is small, then ι is small, so taking the preimages along monomorphisms is size-increasing, which makes sense since the universal property of the pullback makes everything as large as possible.

Coherent functors We also have a nice result about many coherent functors:

Definition 6.44 (Coherent functor)**.** Let $\mathscr{C}, \mathscr{D}$ be coherent categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is *coherent* if:

1. It preserves finite limits.
2. for every object $X : \mathscr{C}$, the induced map on subobjects

$$\mathbf{SOb}_{\mathscr{C}}(X) \xrightarrow{\mathbf{SOb}(F)_X} \mathbf{SOb}_{\mathscr{D}}(F(X)), \quad \iota \mapsto F(\iota)$$

preserves finite joins (including the initial subobject).

A functor $F : \mathscr{C} \to \mathscr{D}$ *preserves pure monics* if for each non-isomorphic monomorphism $\iota : A \hookrightarrow X$ in $\mathscr{C}$ the image $F(\iota)$ is a non-isomorphic monomorphism. F is *purely monomorphism-preserving* if it preserves monomorphisms and pure monics.

Lemma 6.45. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor that*

1. *preserves monomorphisms;*
2. *preserves pure monics.*

Fix $X : \mathscr{C}$. Then the fiber

$$(\mathbf{SOb}(F)_X)^{-1}(\top_{F(X)}) \subseteq \mathbf{SOb}_{\mathscr{C}}(X)$$

consists only of $\top_X$; i.e. the only subobject $\iota : A \hookrightarrow X$ with $F(\iota) = \top_{F(X)}$ is $\iota = \top_X = \mathbf{id}_X$.

Proof. Let $\iota : A \hookrightarrow X$ be a subobject such that $F(\iota) = \top_{F(X)}$. This means that $F(\iota) : F(A) \hookrightarrow F(X)$ is an isomorphism. But since F preserves pure monics, ι must itself be an isomorphism in $\mathscr{C}$ and thus $\iota = \top_X$ in $\mathbf{SOb}_{\mathscr{C}}(X)$ (up to univalence). ◻

Proposition 6.46. *Let $F : \mathscr{C} \to \mathscr{D}$ be a coherent functor that is purely monomorphism-preserving. Then for each $X : \mathscr{C}$ the induced morphism $F_{*,X}$ is smallness-reflecting.*

Proof. By Lemma 6.45, $F_{*,X}$ reflects $\top_{F(X)}$ and by coherence it preserves joins, so smallness-reflection follows from Proposition 5.19. ◻

6.3 Summary

So we have four different notions: smallness, density, reducedness and satedness, which all develop differently in either variance, and we have traced them from general categories to coherent ones. The four notions of smallness, density, reducedness, and satedness unfold in a richly asymmetric pattern across categorical settings:

In a bare category with an initial object, density is *reflected by every monomorphism* Corollary 5.28, while smallness is not generally preserved. A subobject is reduced precisely when its corresponding ideal embedding reflects smallness (Corollary 5.17 and Corollary 5.26).

Adding pullbacks yields a contravariant subobject functor; if then a category $\mathscr{C}$ has a strict initial object, then density is stable under pullback, so that dense subobjects form a coverage. Moreover, if $\mathscr{C}$ is extensive, then coproducts are mapped to products under the $\mathbf{SOb}^*$-functor, which implies a behavior of smallness and density with respect to coproducts that is again abstractly the same but concretely different (Proposition 6.17, Corollary 6.22, Corollary 6.23, Corollary 6.24).

In general, regular categories sharpen the picture: regular epimorphisms *preserve density* (images of dense subobjects are dense, Proposition 6.32) and their pullbacks are density-equivalences (Corollary 6.25 and Corollary 6.27) and *reflect satedness* (Corollary 6.34).

Coherent categories bring smallness into better control: in a modular subobject lattice, monomorphisms *preserve smallness* (Lemma 5.40 and Corollary 6.18) and *reflect reducedness* (by Proposition 5.42 together with Corollary 6.18). Regular epimorphisms now also *reflect smallness* via pullback, thanks to the join-preservation (Corollary 6.43). Moreover, coherent functors reflect smallness pointwise (Proposition 6.46).

Heyting categories also allow a pseudocomplement characterization of density (Proposition 5.34), which shows that *satedness is reflected by monomorphisms* (Corollary 5.47). Moreover, we will see in upcoming work that co-Heyting categories decompose every object into a small and a reduced part.

Throughout, density emerges as the more rigid, globally behaved notion-reflected by all monos, preserved by pullbacks and regular epis, and forming a coverage-while smallness is more local, requiring modularity or co-Heyting structure to really come into itself.

Let us summarize these results in a comprehensive table:

Type / Context	Smallness	Density	Reducedness	Satedness
reduced subobject	**refl**$_*$	-	-	-
Initial object				
monomorphism	-	**refl**$_*$	-	-
Pullbacks and strict ∅				
complete functor that reflects ∅	-	**refl** along F_*	-	-
Regular with strict ∅				
arbitrary morphism	-	**prv***	-	-
regular epi	-	**prv**$_*$	-	-
regular epi	-	**refl***	-	**refl***
Coherent				
monomorphism	**prv**$_*$	-	**refl**$_*$	-
regular epi	**refl***	-	-	-
purely-monic-preserving coherent functor	**refl** along F_*	-	-	-
Heyting				
monomorphism	-	-	-	**refl**$_*$
Co-Heyting				
Arbitrary object	small+red-dcomp.	-	small+red-dcomp	-

Table 1: *Preservation and reflection of size notions in various categorical contexts.*

Key:

- "prv" = the property is preserved.
- "refl" = the property is reflected.
- $(-)_*$ preservation/reflection happens along the induced covariant morphism in **UBP**, i.e. through postcomposition.
- $(-)^*$ preservation/reflection happens along the induced contravariant morphism in **UBP**, i.e. through pullback.
- $\varnothing$ is the initial object.
- "-" = no general result stated in the text.

Notes:

1. All density statements assume a strict initial object so that $\mathbf{SOb}_{\mathscr{C}}(X)$ is downward-bounded.
2. The decomposition in co-Heyting categories is mentioned in the text as an upcoming result.
3. Behavior on disjoint coproducts with monic coprojections is induced by the isomorphism $\mathbf{SOb}_{\mathscr{C}}(X+Y) \cong \mathbf{SOb}_{\mathscr{C}}(X) \times \mathbf{SOb}_{\mathscr{C}}(Y)$.

7 Applications

We provide two applications, which jointly show the power of the "tablets and scales" approach.

7.1 Open-Closed Dualities

The nature of regular categories is somewhat ironic: they provide us with the incredibly convenient regular-epi/mono-factorization, but it is actually the notion of a regular monomorphism that most closely describes an embedding. In-between the two - monomorphisms and regular monomorphisms - is missing local structure. We saw an example of this in **PrO** already, where the regular monomorphisms are fully faithful functors i.e. order-reflecting, but the monomorphisms are merely injective functions. Another important example is **Top**, where a monomorphism $\iota : A \hookrightarrow X$ is an injective continuous function while a regular monomorphism is a subspace embedding - in-between them are all the opens A might have but its image might not.

So certainly to generalize the open/closed-duality of topological spaces but also in general we need to abstract what we are doing from the **SOb**-functor to others:

Definition 7.1. A *submonic hyperdoctrine* $\eta : \mathscr{H} \hookrightarrow \mathbf{SOb}^* : \mathscr{C}^{op} \to \mathbf{UBP}$ is a monic natural transformation of functors into **UBP** such that for each $X : \mathscr{C}$ the top $\top_{\mathbf{SOb}(X)} = \mathbf{id}_X$ is in the image of η_X.

Example 7.2. Since regular monomorphisms are stable under pullback (since for a regular ι the pullback $f^*(\iota)$ equalizes the composite of f and the cokernel pair of ι), the functor $\iota_{reg} : \mathbf{SOb}^*_{reg} \hookrightarrow \mathbf{SOb}^* : \mathscr{C}^{op} \to \mathbf{UBP}$ is a submonic hyperdoctrine.

In particular, the subspaces of a topological space X correspond to the regular monomorphisms into X. This we take as the starting point of our next definition:

Definition 7.3. An *open-closed duality* $\mathscr{D}$ on a category $\mathscr{C}$ is a triple $\mathbf{o} \overset{\omega}{\hookrightarrow} \mathbf{d} \overset{\chi}{\hookleftarrow} \mathbf{c}$ of submonic hyperdoctrines such that for each $X : \mathscr{C}$ and element $o \in \mathbf{O}(X)$ of the underlying preorder of $\mathbf{o}(X) : \mathbf{O}(X) \hookrightarrow \mathbf{SOb}(X)$ there exists an element $c \in \mathbf{C}(X)$ for which $\omega(o)$ is the pseudocomplement of $\chi(c)$ in the source $\mathbf{D}(X)$ of $\mathbf{d}(X)$. An open-closed duality $\mathscr{D}$ is *regular* if $\mathbf{d}$ is ι_{reg}. $\mathscr{D}$ is *reduced* if also for each c in each $\mathbf{C}(X)$ there exists an o such that $\chi(c)$ is the pseudocomplement of $\omega(o)$. $\mathscr{D}$ is *Heyting* if

1. for each X it holds that $\mathbf{D}(X)$ is a Boolean lattice, $\mathbf{O}(X)$ a Heyting algebra and $\mathbf{C}(X)$ a co-Heyting algebra and ω and χ are morphisms of bounded lattices,
2. and for $f : X \to Y$ in $\mathscr{C}$ the morphisms $\mathbf{O}(f) : \mathbf{O}(Y) \to \mathbf{O}(X)$ and $\mathbf{C}(f) : \mathbf{C}(Y) \to \mathbf{C}(X)$ are lattice morphisms and $\mathbf{D}(f) : \mathbf{D}(Y) \to \mathbf{D}(X)$ is a Boolean algebra morphism.

Remark 7.4. We take the direction of pseudocomplementation in Definition 7.3 such that it aligns with the case of subschemes of a scheme.

Definition 7.5. An *open-closed space* X is an open-closed duality whose middle submonic preorder $\mathbf{D}(X)$ is also its underlying category. Thus the data are preorder embeddings

$$\mathbf{O}(X) \overset{\omega}{\hookrightarrow} \mathbf{D}(X) \overset{\chi}{\hookleftarrow} \mathbf{C}(X)$$

satisfying the same pseudocomplement condition as in Definition 7.3, but with $\mathbf{D}$ itself playing the role of the ambient category of subobjects.

A *morphism of open-closed spaces* $f : X \to Y$ is the dual of a morphism of diagrams

$$f^*_{\mathbf{O}} : \mathbf{O}(Y) \to \mathbf{O}(X), f^*_{\mathbf{D}} : \mathbf{D}(Y) \to \mathbf{D}(X), f^*_{\mathbf{C}} : \mathbf{C}(Y) \to \mathbf{C}(X)$$

from Y to X such that $f^*_{\mathbf{D}}$ has a right adjoint $f_{\mathbf{D},*}$.

Example 7.6. Every object $X : \mathscr{C}$ in the base category of an open-closed duality $\mathbf{o} \overset{\omega}{\hookrightarrow} \mathbf{d} \overset{\chi}{\hookleftarrow} \mathbf{c}$ gives rise to an open-closed space

$$\mathbf{O}(X) \xhookrightarrow{\omega_X} \mathbf{D}(X) \xhookleftarrow{\chi_X} \mathbf{C}(X),$$

whose underlying category is the middle preorder $\mathbf{D}(X)$. Thus, open-closed spaces are the fibrewise form of an open-closed duality. Properties such as regularity and Heytingness are inherited fibrewise whenever they are present in the ambient open-closed duality; reducedness is fulfilled by definition.

Of course the paradigmatic example is the topological one:

Example 7.7. Let $\mathbf{O}, \mathbf{C}, \mathbf{D} : \mathbf{Top}^{op} \to \mathbf{UBP}$ be given by mapping each topological space X to its lattices of opens and closeds and subsets. Then $\mathbf{O}$, $\mathbf{C}$ and $\mathbf{D}$ with their embeddings into $\mathbf{SOb}_{\mathbf{Top}}$ form the components of a regular Heyting open-closed duality.

But an advantage of this more restricted notion is that it can be applied in settings of a different variety:

Example 7.8. Let X be a scheme, $\mathbf{O}$ and $\mathbf{C}$ the open and closed immersions into X and $\mathbf{R}$ the preorder of all immersions into X. As per [26] an open immersion $\iota_{\mathbf{O}}$ of schemes is given by an open subspace $|\iota_{\mathbf{O}}| : |A| \hookrightarrow |X|$ of the underlying topological space $|X|$ of X with the structure sheaf induced by restriction. A closed immersion is locally in each affine neighborhood given by an embedding of an affine neighborhood, so dual to a ring quotient. An immersion is a closed immersion into an open immersion. By the Zariski topology, for each open immersion $\iota_{\mathbf{O}}$ the complement $|X| \setminus |\iota_{\mathbf{O}}|$ in the underlying topological space of X

is the underlying subset of some reduced closed subscheme $\iota_{\mathbf{C}}$. Then $\iota_{\mathbf{O}}$ is the pseudocomplement of $\iota_{\mathbf{C}}$: since the two don't share any points and every non-trivial scheme has a point (since every non-trivial ring has a maximal ideal by the axiom of choice), no non-trivial subscheme factors through their intersection. On the other hand, every other subscheme κ that does not factor through $\iota_{\mathbf{O}}$ shares some point p with $|\iota_{\mathbf{C}}|$ and thus there exists a subscheme of X under ι and κ (the maximal subscheme corresponding to p), so that $\iota_{\mathbf{O}}$ is in fact maximal with respect to being disjoint with $\iota_{\mathbf{C}}$.

These are not particularly well-behaved preorders because the closed part is both too encompassing and too strict: it is too encompassing because many closed subschemes can correspond to one open subscheme and too strict because there are too few closed subobjects for many topological constructions. But we can avoid this by proceeding along the lines of [8]. But with one crucial difference: the authors introduce, rather *ad-hoc*, the notion of a *restricted topology* as a set with a collection of subsets which is closed under finite unions and intersections. We instead use Heyting algebras:

Definition 7.9. An *abstract Heyting space* is a Heyting open-closed space

$$\mathbf{O}(X) \xhookrightarrow{\omega_X} \mathbf{D}(X) \xhookleftarrow{\chi_X} \mathbf{C}(X)$$

in the sense of Definition 7.5: thus $\mathbf{D}(X)$ is a Boolean lattice, $\mathbf{O}(X)$ is a Heyting algebra, $\mathbf{C}(X)$ is a co-Heyting algebra, and the open and closed parts are related by complementation inside $\mathbf{D}(X)$.

A *morphism of abstract Heyting spaces* $f : X \to Y$ is a morphism of open-closed spaces such that each component map of f is a lattice morphism and $f_{\mathbf{D}} : \mathbf{D}(Y) \to \mathbf{D}(X)$ preserves Boolean negations.

A *concrete Heyting space* is a set $|X|$ with an abstract Heyting space X for which the middle Boolean lattice $\mathbf{D}(X)$ is the subset preorder $\mathbf{P}(X) := \mathbf{P}(|X|)$ of $|X|$, with ω_X and χ_X given by subset embeddings. Equivalently, it is a set $|X|$ equipped with a lattice embedding $\iota_{\mathbf{O}} : \mathbf{O}(X) \hookrightarrow \mathbf{P}(X)$, where $\mathbf{O}(X)$ is a Heyting algebra; so that the *closed subsets* are the set-theoretic complements of opens, with embedding $\iota_{\mathbf{C}} : \mathbf{C}(X) \hookrightarrow \mathbf{P}(X)$.

A *morphism of concrete Heyting spaces* $f : X \to Y$ is a function $|X| \to |Y|$ such that the pullback along f induces a morphism of abstract Heyting algebras.

Remark 7.10. Note that we do *not* assume that $\mathbf{O}(f)$ and $\mathbf{C}(f)$ preserve the Heyting and co-Heyting implications, which is in line with the topological case, but we *do* assume that $\mathbf{D}(f)$ preserves complements, as it does in the case of pullbacks of sets (and thus regular subobjects of a topological space).

Remark 7.11. In pointless topology the assumption is made that what is important about a topological space is its locale of opens. Here we challenge this assumption: not just is it that the closeds matter as well, but in particular also that these two are jointly embedded into some larger algebra, which allows us to extend the notions we derive from opens and closeds to other elements.

Example 7.12. It follows almost immediately from Definition 7.9 that both the category of concrete Heyting spaces **CHS** and that of abstract Heyting spaces **AHS** house a Heyting open-closed duality.

Example 7.13. Every topological space X is a concrete Heyting space.

Example 7.14. For each real closed field R and finite-dimensional R^n, the embedding $\omega_{salg} : \mathbf{OSSAlg}(R^n) \hookrightarrow \mathbf{P}(R^n)$ of the open real semialgebraic subsets of R^n from Example 3.45 into the powerset lattice of R^n is a concrete Heyting space.

Proof. We saw in Example 3.45 already that the category of semialgebraic subsets of R^n has a distinguished collection $\mathbf{O}(V)$ of *open semialgebraic* subsets of V, closed under finite unions and finite intersections; Thus, almost all requirements of Definition 7.9 are already known. We must still show that $\mathbf{O}(V)$ is in fact a Heyting algebra; equivalently, that it is closed under *Heyting implication.* But for this we can just use the Heyting implication in the normal opens

$$U \Rightarrow W \;:=\; \big((V \setminus U) \cup W\big)^{\circ}.$$

First, we verify that this is indeed an element of $\mathbf{O}(V)$:

- $(V \setminus U) \cup W$ is a semialgebraic subset of V, being a Boolean combination of semialgebraic sets.
- By o-minimality of the semialgebraic structure, the interior of a semialgebraic set in V is again semialgebraic and open in V. Thus $U \Rightarrow W \in \mathbf{O}(V)$.

Moreover, since the structure sheaf of V is defined on all opens, it restricts cleanly to $U \Rightarrow W$. The argument that this has the required universal property proceeds as with the Heyting implication in a locale. □

Every concrete Heyting space has an underlying abstract Heyting space obtained by replacing $\mathbf{P}(X)$ with the Boolean subalgebra generated by its opens and closeds. This is a generalization of the notion of a constructible subset:

Definition 7.15. Given an open-closed space X such that $\mathbf{D}(X)$ is a lattice with (open) pseudocomplements, the *constructible closure* in $\underline{X}$ is the open-closed space with the same opens and closeds such that the middle $\underline{\mathbf{D}}(X)$ of $\underline{X}$ is the joint closure of ω and χ in $\mathbf{D}(X)$ under joins, meets and pseudocomplements.

Remark 7.16. If $\mathbf{D}(X)$ does not have joins, meets or pseudocomplements, then it does still make sense to close ω and χ under whatever $\mathbf{D}(X)$ does have and refer to that as the constructible closure.

So each concrete Heyting space determines an abstract Heyting space by its constructible closure.

Example 7.17. The constructible closure $\underline{\omega}_{salg}$ of ω_{salg} from Example 7.14 is the embedding $\mathbf{OSSAlg}(R^n) \hookrightarrow \mathbf{SSAlg}(R^n)$ of $\mathbf{OSSAlg}(R^n)$ into the preorder of all semialgebraic subsets of R^n.

Now, for concrete Heyting spaces we can often adopt standard topological arguments almost verbatim:

Proposition 7.18. *Let $f : X \to Y$ be a morphism in* **CHS**.

1. *f is a monomorphism in* **CHS** *if and only if the underlying function $|X| \to |Y|$ is injective.*
2. *f is a regular monomorphism in* **CHS** *if and only if the Heyting structure on X is the pullback structure induced from Y, i.e.*
$$\mathbf{O}(X) \;\cong\; \{\, f^*(U) \mid U \in \mathbf{O}(Y) \,\}$$
via the lattice embedding $\iota_{\mathbf{O},X}$, and similarly for closed sets.

Proof. We establish the two claims in turn.

Monomorphisms. If f is monic, distinct points $x_1 \neq x_2$ in X cannot map to the same point in Y; otherwise, the two maps from the terminal object 1 (a single point) picking x_1 and x_2 would compose with f to give the same morphism, violating monicity. Thus, f is injective. Conversely, if $|f|$ is injective and

$f \mathbin{;} g = f \mathbin{;} h$, then for every z we have $|f|(g(z)) = |f|(h(z))$, which forces $g(z) = h(z)$. Since the underlying functions agree, their inverse images of opens must also agree, so $g = h$ as morphisms of **CHS**.

Regular monomorphisms. Recall that regular monomorphisms in concrete categories are equalizers. Suppose f is regular. As equalizers in **CHS** are computed pointwise (set-theoretically), f represents an inclusion of a subset $X \subseteq Y$. The equalizer carries the coarsest Heyting structure making the inclusion continuous; consequently, every open of X is the pullback of an open from Y, establishing the isomorphism.

For the converse, assume $\mathbf{O}(X) \cong f^*(\mathbf{O}(Y))$. Let Z be the codiscrete space on the disjoint union $Y + \{*\}_1 + \{*\}_2$. Define two morphisms $g, h\colon Y \to Z$ which are equal on X but map the complement of X to the two distinct points $\{*\}_1$ and $\{*\}_2$ respectively. (Continuity is guaranteed because Z is codiscrete.) One verifies immediately that X is the set of points where g and h agree and that f is the equalizer of g and h. ◻

But the constructible closure is the greatest extent we can safely and finitistically reason about. In particular, they are what we have interior and closure operators on:

Definition 7.19. An *adjoint open-closed duality* is an open-closed duality such that

1. ω has an *interior* right adjoint $(-)^\circ\colon \mathbf{D} \to \mathbf{O}$,
2. χ has a *closure* left adjoint $\overline{(-)}\colon \mathbf{D} \to \mathbf{C}$.

Remark 7.20. In the Heyting case it follows from duality that if either adjoint to the embeddings of an open-closed duality exists then the other exists. For instance, if $(-)^\circ$ is given then for $r \in \mathbf{D}(X)$ of some X the closure $\bar{r}$ is just the complement of $(\neg r)^\circ$. For non-Heyting open-closed dualities it makes sense to consider variants of Definition 7.19 for which only one adjoint is given.

Proposition 7.21. *Let X be a Heyting space with constructible closure $\underline{X}$ with middle $\underline{\mathbf{D}}(X)$. Then the restricted embeddings form an adjoint open-closed duality.*

Proof. Since $\underline{\mathbf{D}}(X)$ is the Boolean subalgebra of $\mathbf{D}(X)$ generated by the image of ω, every element $k \in \underline{\mathbf{D}}(X)$ decomposes as a finite join of locally closed elements $L = \omega(o) \wedge \neg_{\mathbf{D}}\omega(v)$ with $o, v \in \mathbf{O}(X)$. (Here $\neg_{\mathbf{D}}$ denotes Boolean complementation in $\mathbf{D}(X)$, so that $\neg_{\mathbf{D}}\omega(v) = \chi(c_v)$ for the closed element c_v complementary to the open v.)

Closure on locally closed elements. For $L = \omega(o) \wedge \neg_{\mathbf{D}}\omega(v)$, define

$$\overline{L} \;:=\; \neg_{\mathbf{D}}\, \omega(o \Rightarrow v),$$

where $\Rightarrow$ is the Heyting implication in $\mathbf{O}(X)$. This is a closed element, and we claim it is the least closed element above L.

First, $L \leq \overline{L}$: the identity $o \wedge (o \Rightarrow v) \leq v$ gives $\omega\big(o \wedge (o \Rightarrow v)\big) \leq \omega(v)$, hence

$$L \wedge \omega(o \Rightarrow v) \;=\; \omega(o) \wedge \omega(o \Rightarrow v) \wedge \neg_{\mathbf{D}}\omega(v) \;\leq\; \omega(v) \wedge \neg_{\mathbf{D}}\omega(v) \;=\; \bot,$$

so $L \leq \neg_{\mathbf{D}}\omega(o \Rightarrow v) = \overline{L}$.

For minimality, write any closed element as $\chi(c) = \neg_{\mathbf{D}}\omega(w)$ for some $w \in \mathbf{O}(X)$. Then the following are equivalent:

$$\begin{aligned}
L \leq \chi(c) &\iff L \wedge \omega(w) = \bot \\
&\iff \omega(o \wedge w) \wedge \neg_{\mathbf{D}}\omega(v) = \bot \\
&\iff \omega(o \wedge w) \leq \omega(v) \\
&\iff o \wedge w \leq v \\
&\iff w \leq o \Rightarrow v \\
&\iff \neg_{\mathbf{D}}\, \omega(o \Rightarrow v) \leq \neg_{\mathbf{D}}\, \omega(w) \\
&\iff \overline{L} \leq \chi(c).
\end{aligned}$$

Thus $L \leq \chi(c)$ holds if and only if $\overline{L} \leq \chi(c)$. This universal property characterizes $\overline{L}$ as the least closed element containing L, so the definition is independent of the chosen presentation of L as $\omega(o) \wedge \neg_{\mathbf{D}}\omega(v)$.

Extension over finite joins. For a general element $k = \bigvee_i L_i$, set $\overline{k} := \bigvee_i \overline{L_i}$. This is the least closed element above k: for any $c \in \mathbf{C}(X)$,

$$k \leq \chi(c) \iff L_i \leq \chi(c) \text{ for every } i \iff \overline{L_i} \leq c \text{ for every } i \iff \overline{k} \leq c,$$

where the middle equivalence uses the locally closed case. Hence $\overline{k}$ depends only on k and not on the decomposition, and the assignment $k \mapsto \overline{k}$ is a left adjoint to $\chi : \mathbf{C}(X) \hookrightarrow \underline{\mathbf{D}}(X)$.

Interior by duality. Define $k^\circ := \neg_{\mathbf{D}} \overline{\neg_{\mathbf{D}} k}$. This is an open element (the Boolean complement of a closed). For every $o \in \mathbf{O}(X)$, writing c_o for the closed complement of o so that $\chi(c_o) = \neg_{\mathbf{D}}\omega(o)$, we have

$$\omega(o) \leq k \iff \neg_{\mathbf{D}} k \leq \chi(c_o) \iff \overline{\neg_{\mathbf{D}} k} \leq c_o \iff \omega(o) \leq \neg_{\mathbf{D}} \overline{\neg_{\mathbf{D}} k} = k^\circ,$$

where the second equivalence applies the left-adjoint property of $\overline{(-)}$. Thus k° is the right adjoint to ω. $\square$

One nice consequence of Heyting spaces is that satedness becomes stable under restriction to subobjects:

Definition 7.22. Given an abstract Heyting space X and $\iota : \mathbf{O}(X)$, the *induced Heyting space* $X|_\iota$ on ι is the Heyting space:

- such that $\mathbf{D}(X)|_\iota$ is given by the ideal (ι) in $\mathbf{D}(X)$,
- $\omega|_\iota := \omega_{X|_\iota}$ is given by the embedding into (ι) of the image of ω_X under the morphism $(- \wedge \iota)$ given by taking meets with ι.
- $\chi|_\iota$ is also given like that and that opens and closeds are complementary follows from the fact that if $\chi_X(C) = \neg\omega_X(O)$ then for each $\kappa : \mathbf{D}(X)$ it holds that $\kappa \wedge \chi_X(C) = \kappa \wedge \neg\omega_X(O)$ since $\mathbf{D}(X)$ is Boolean.

This space comes with a morphism of Heyting spaces $\iota_* : X|_\iota \hookrightarrow X$ where

- $(\iota_*)^*_{\mathbf{D}}$, $(\iota_*)^*_{\mathbf{O}}$ and $(\iota_*)^*_{\mathbf{C}}$ are given by $(- \wedge \iota)$,
- $(\iota_*)_{*,\mathbf{D}}$ (which we'll just denote ι_*) is given by the ideal embedding $(\iota) \hookrightarrow \mathbf{D}(X)$.

Proposition 7.23 (Satedness restricts to Heyting subspaces)**.** *Given a concrete Heyting space X, $\kappa : \mathbf{O}(X)$ and $\iota \leq \omega_X(\kappa)$, if ι is ω_X-sated in X, then ι is $\omega|_\iota$-sated in $X|_\iota$.*

Proof. It follows from Definition 7.22 and the closedness opens under meets that $\omega|_\iota \,;\, \iota_*$ factors through ω_X, so that there exists a diagram

$$\begin{tikzcd}
\mathbf{O}(X|_\iota) \arrow[r, "\omega|_\iota"] \arrow[d, "\iota_*|_{\mathbf{O}(X|_\iota)}"] & \mathbf{P}(X) \arrow[d, "\iota_*"] \\
\mathbf{O}(X) \arrow[r, "\omega_X"] & X
\end{tikzcd}$$

that fulfills the preconditions of Proposition 5.46 for a trivial scale. □

7.1.1 Semialgebraic Heyting Spaces

We can now extend the definition of semialgebraic spaces in [8]. For this we first add rings in the usual way:

Definition 7.24. Given a ring R, an *R-ringed abstract Heyting space* X is an abstract Heyting space $[X]$ equipped with a *structure sheaf* $\mathcal{S}_X : \mathbf{O}(X)^{op} \to R - \mathbf{Alg}$ into the category of R-algebras $R - \mathbf{Alg}$ that is a sheaf with respect to the finite union coverage. A *morphism of R-ringed abstract Heyting spaces* $f : X \to Y$ is a tuple $([f], f_{\mathcal{S}})$ where $[f]$ is a morphism of abstract Heyting spaces $[X] \to [Y]$ and $f_{\mathcal{S}} : \mathcal{S}(Y) \to [f]_* (\mathcal{S}(X))$ is a natural transformation from $\mathcal{S}(Y)$ to the pushforward of $\mathcal{S}(X)$ as a sheaf of R-algebras.

Similar for concrete Heyting spaces.

Remark 7.25. Note that we do not assume that our space is *locally ringed*, i.e. that the stalks of the structure sheaf are local rings. This is in line with [8].

Proposition 7.26. *Let $\iota : V \hookrightarrow R^n$ be a semialgebraic subset of R^n. Then V is an R-ringed concrete Heyting space.*

Proof. We fix a real closed field R, an integer $n \geq 0$, and a semialgebraic subset $V \subseteq R^n$. We saw in Example 7.14 that V under its usual semialgebraic topology carries the structure of a concrete Heyting space in the sense of Definitions 7.9 and 7.24. We can endow this with the structure of the usual semialgebraic function sheaf $\mathcal{S}_V : \mathbf{O}(V)^{op} \to R\text{-}\mathbf{Alg}$ of R-algebras, which assigns to each open semialgebraic $U \subseteq V$ the R-algebra of semialgebraic maps $U \to R$, and restriction maps along inclusions. Delfs-Knebusch show that this is a sheaf for the finite-union coverage and that the resulting ringed space structure matches the "usual" semialgebraic geometry. □

So we call a semialgebraic subset of some R^n an *affine semivariety*. We can thus define:

Example 7.27. Let R be a real closed field. We define the category $\mathbf{CSAS}_R$ of concrete semialgebraic spaces over R as the category whose objects S are R-ringed concrete Heyting spaces such that there exists an open cover of $|S|$ by affine semivarieties. Then it follows from the definition of concrete Heyting spaces that $\mathbf{CSAS}_R$ features a Heyting open-closed duality. Passing fibrewise to the Boolean algebra generated by opens and closeds gives the corresponding abstract constructible closures.

$\mathbf{CSAS}_R$ is by no means unique or even minimal with respect to this property. We for instance can do similar constructions for semianalytic spaces. But the main point is that the assumption of a Heyting open-closed duality allows us to transfer all the basic notions from topology: density, regularity, boundaries, closure and interior, but also the logical structure inherent in a (co)-Heyting algebra, while the use of a semialgebraic structure sheaf keeps all notions tame, like Grothendieck wanted.

7.2 Categorical Measure Theory

UBP also allows us to define a notion of measure theory directly on categories using the **SOb**-functor. For this we want to highlight the following excerpt from Fremlin's *Measure Theory*, [10] found on the nlab: [23]

> One of the striking features of measure theory, compared with other comparably abstract branches of pure mathematics, is the relative unimportance of any notion of 'morphism'. The theory of groups, for instance, is dominated by the concept of 'homomorphism', and general topology gives a similar place to 'continuous function'. In my view, the nearest equivalent in measure theory is the idea of 'inverse-measure-preserving function'.

Inverse-measure-preservation is precisely what is implied by a weak natural transformation of hyperdoctrines $\sigma : \mathscr{H} \Rightarrow \mathbf{K}(P)$ to the constant functor $\mathbf{K}(P)$ on some preorder P. So we can define:

Definition 7.28. Given a preorder P and a submonic hyperdoctrine $\mathscr{H}$, a *weak P-scale* on $\mathscr{H}$ is a weak natural transformation $\sigma : \mathscr{H} \Rightarrow \mathbf{K}(P)$ to the constant functor $\mathbf{K}(P)$ on P.

However, let us consider Section 7.2 more carefully: there might be a reason for this relative morphism-scarcity that goes beyond the incidental, and one natural candidate is that the preservation of measures is not straightforward. Consider a very simple kind of measure space: N with the cardinality measure $|-| : \mathbf{P}(N) \to N_\infty$ on its subobject lattice. This is an N_∞-scale on the subobject lattice of N as a set, so on N as a discrete open-closed space and in particular a concrete Heyting space. Let $\iota : X \hookrightarrow N$ be some finite subset of N and consider the induced measure $\iota_* \mathbin{;} |-| : \mathbf{P}(X) \to N_\infty$. It would seem very reasonable that ι would be inverse-measure-preserving, but it is not: $|\iota_*(\iota^*(N))| = |\iota_*(X)| < |N|$ by assumption. And if try to further extend $|-|$ to the category **CHS** of concrete Heyting spaces, we encounter yet more problems, as even though cardinality is well-defined for each concrete Heyting space in **CHS**, its behavior on regular epimorphisms is measure-increasing. So while inverse-measure-preservation makes sense for fully local measures, such a probability measures, which always map the entirety of a space to 1, this is not a property of all measures, and if the global structure of a space itself is measured, a bifurcation in variance becomes apparent, roughly between monomorphisms and epimorphisms, but more often between their regular variants. Thus, it makes sense to extend the definition somewhat:

Definition 7.29. Given a preorder P and a submonic hyperdoctrine $\mathscr{H} : \mathscr{C}^{op} \to \mathbf{PrO}$, a *$P$-prescale* is a natural transformation $\sigma : \mathbf{Core}_{\mathscr{C}} \mathbin{;} \mathscr{H} \Rightarrow \mathbf{K}(P)$, where $\mathbf{Core}_{\mathscr{C}}$ is the embedding of the core groupoid on the objects of $\mathscr{C}$ into $\mathscr{C}$ (so that naturality reduces to isomorphism-invariance). σ is a *lax P-scale* if it descends to a lax natural transformation of hyperdoctrines $\mathscr{H} \Rightarrow \mathbf{K}(P)$ and dually an *oplax P-scale.* σ is a *regular P-scale* if it is a lax scale on the monic part of $\mathscr{C}$ and an oplax scale on the regular epimorphisms in $\mathscr{C}$.

Example 7.30. Cardinality is a regular scale on each concrete category whose monomorphisms are injections and whose regular epimorphisms are surjections, e.g. in **Top**, valued in the category of cardinals.

Depending on the situation we can also demand other things:

1. if $\mathscr{C}$ and P have an initial object, then it makes sense to demand that σ preserves bottoms,
2. as defined, σ is both a local measure, which assigns to each subobject $\iota : A \hookrightarrow X$ in $\mathscr{C}$ a measure of how large ι is, and a global one, as it assigns a size to X itself. Depending on the application this can

be desired or not. If we want a purely local measure, it makes sense to demand that $\sigma(\mathbf{id}_X) = \top_P$ for each $X : \mathscr{C}$, so that each space is maximal in itself.

3. We can also add whatever kind of additivity conditions we want if we simply assume that P has an internal monoid structure.

More particularly, we can combine the usage of such a scale with the open-closed dualities we described in Section 7.1 to generalize Borel measures:

Definition 7.31. Given an ordered monoid M and an open-closed duality $\mathscr{D} = (\mathbf{o}, \mathbf{d}, \mathbf{c})$ on a category $\mathscr{C}$, an *open Borel-Heyting measure* μ on $\mathscr{D}$ is an M-scale on the underlying functor $\mathbf{O}$ of $\mathbf{o}$ such that if $\iota \vee_{\mathbf{O}} \kappa$ and $\iota \wedge_{\mathbf{O}} \kappa$ exist and $\iota \wedge_{\mathbf{O}} \kappa = \bot_{\mathbf{O}}$, then $\mu(\iota \vee_{\mathbf{O}} \kappa) = \mu(\iota) + \mu(\kappa)$. A *closed Borel-Heyting measure* is the same but defined on $\mathbf{C}$.

This goes to the core of the intersection of topology and measure theory in Borel measures: we use "simple" objects like opens or closeds to infer the size of more complex ones. A key point is now that we can do this for all categories with a sufficiently rich subobject structure, e.g. Heyting spaces, and often it does not matter whether we define and measure on the opens or closeds. For this first note a standard subtlety of measure theory on lattices: because the open (or closed) subobjects do not generally form a Boolean algebra, disjoint additivity is not sufficient to uniquely extend a measure to the generated Boolean algebra, as we cannot decompose overlapping opens into disjoint ones. We must instead require the measure to be *modular* (also known as a *valuation*).

Definition 7.32. An open (resp. closed) Borel-Heyting measure μ on $\mathscr{D} = (\mathbf{o}, \mathbf{d}, \mathbf{c})$ with values in a commutative monoid M is *modular* if for all subobjects ι, κ,

$$\mu(\iota \vee \kappa) + \mu(\iota \wedge \kappa) = \mu(\iota) + \mu(\kappa).$$

μ is *standard* if $\mathscr{D}$ is defined on a category $\mathscr{C}$ with an initial object $\varnothing$, $\mathbf{O}$ resp. $\mathbf{C}$ contains each initial morphism $0_X : \varnothing \to X$ and μ maps each 0_X to the neutral element $0 \in M$.

Assuming this, let us first remember (or learn about) the *Smiley-Horn-Tarski Theorem*:

Theorem 7.33 (Smiley-Horn-Tarski Theorem for Left-Cancellative Monoids)**.** *[12](IV-9.3., IV-9.4.) A modular standard measure $\mu : L \to R_{\geq 0,\infty}$ from a sub-lattice L of a Boolean algebra B to a left-cancellative monoid extends to a finitely additive measure on the Boolean closure $\overline{L}$ of L in B. Moreover, if μ is finitely valued, then this extension is unique.*

Proposition 7.34 (Extension of Borel-Heyting Measures)**.** *Let $\mathscr{D} = (\mathbf{o}, \mathbf{d}, \mathbf{c})$ be an abstract Heyting open-closed duality. Then any standard modular open Borel-Heyting measure $\mu : \mathbf{O} \Rightarrow \mathbf{K}(R_{\geq 0,\infty})$ extends to a finitely additive measure $\hat{\mu}$ on the constructible closure $\underline{\mathbf{D}}$ of $\mathbf{D}$. The same holds for any standard modular closed Borel-Heyting measure $\nu : \mathbf{C} \Rightarrow \mathbf{K}(R_{\geq 0,\infty})$. Moreover, if μ is finitely valued, then $\hat{\mu}$ is unique. Consequently, there is a natural bijection between finite open and closed modular Borel-Heyting measures.*

Proof. Fix an object $X : \mathscr{C}$. By the definition of a Heyting open-closed duality, $\mathbf{O}(X)$ is a distributive sublattice of the Boolean algebra $\mathbf{D}(X)$, and $\mathbf{C}(X)$ consists precisely of the Boolean complements of elements of $\mathbf{O}(X)$.

Let $\underline{\mathbf{D}}(X)$ be the constructible closure. By the Smiley-Horn-Tarski extension theorem μ_X uniquely extends to an additive measure $\hat{\mu}_X : \underline{\mathbf{D}}(X) \to R_{\geq 0,\infty}$, which is unique if μ is finite. Because the closed subobjects $\mathbf{C}(X)$ are the Boolean complements of the opens, the Boolean algebra generated by $\mathbf{C}(X)$ is precisely

$\underline{\mathbf{D}}(X)$. Restricting $\hat{\mu}_X$ to $\mathbf{C}(X)$ yields a function $\nu_X : \mathbf{C}(X) \to R_{\geq 0,\infty}$ which is trivially modular, hence a closed Borel-Heyting measure.

For any open $U \in \mathbf{O}(X)$ with closed complement $\neg U \in \mathbf{C}(X)$, finite additivity on $\underline{\mathbf{D}}(X)$ implies:

$$\mu_X(U) + \nu_X(\neg U) = \hat{\mu}_X(U) + \hat{\mu}_X(\neg U) = \hat{\mu}_X(\top_X) = \mu_X(\top_X).$$

Conversely, starting with a finite modular closed Borel-Heyting measure ν, it uniquely extends to $\underline{\mathbf{D}}(X)$, and restricting this extension to $\mathbf{O}(X)$ recovers an open measure. These two extension-restriction procedures are mutually inverse.

Finally, we must check that the extension is natural (i.e. it acts as a hyperdoctrine scale). For any morphism of Heyting spaces $f : Y \to X$, the pullback f^* on $\mathbf{D}$ is a Boolean homomorphism that restricts to the given pullbacks on $\mathbf{O}$ and $\mathbf{C}$. Because f^* preserves the Boolean operations and $\hat{\mu}_X$ is the unique additive extension of μ_X, it follows that $f^* \mathbin{;} \hat{\mu}_Y = \hat{\mu}_X$ on $\underline{\mathbf{D}}(X)$. Thus, naturality holds for the extensions, and the bijection is complete. $\square$

So as long as our measure is modular (which is standard for measures on lattices that are not Boolean algebras), we can cleanly translate between measuring the open parts of a space, measuring the closed parts, and measuring the constructible parts. This is analogous to the extension of a Borel measure to the Borel subsets of a topological space, except necessarily everything is kept finite and thus more restricted, which is a mixed blessing: everything is calculable, but we have access to fewer sets. In particular, we cannot define σ-additive measures without assuming countable joins in $\mathbf{O}$. But this runs counter to our paradigmatic example of semialgebraic sets because the union of countably many semialgebraic sets is not semialgebraic - in particular, no polynomial has infinitely many zeroes. To remedy this we could either:

1. use semianalytic sets and introduce countable joins to $\mathbf{O}$,
2. or, what is perhaps even better, transition into the nonstandard-realm and use Loeb measures - in this setting the notion of a lattice becomes much stronger, since a nonstandard lattice contains hyperfinite joins and meets. This way, we can dispense of the requirement of σ-additivity altogether and do categorical measure theory in what is internally a purely finite setting.[6]

For now we leave these as suggestions however.

8 The Representability Theorems

Throughout this text we have been using the $\mathbf{SOb}_{\mathscr{C}}$-functor. But what is it? Where does it come from? Here we will derive a universal property of $\mathbf{SOb}_{\mathscr{C}}$ and **UBP** that shows that **UBP** is, in some sense, the largest context in which something like the reflection of **Cat** in itself exists for monics. First of all, we will extend $\mathbf{SOb}_{\mathscr{C}}$ to a 2-functor **SOb** on the 2-category $\overset{\hookrightarrow}{\mathbf{Cat}}$ of monic categories.

8.1 Preliminary Definitions

Let us recapitulate the definitions of the basic objects involved in this section.

[6] See [7] for an introduction to nonstandard measure theory.

Remark 8.1. As mentioned in Section 2.4 we take everything strict (up to contractible homotopy). This might seem evil but a weak framework would further complicate the already strained exposition for little gain: the material is orthogonal to questions of weakness, the only significant adaptation would be to have to consider the naturality 2-cells of weak natural transformations in 2-categories other than **UBP** (which is locally thin up to homotopy) in the latter part of Section 8.3.

Definition 8.2 (Strict 2-category)**.** A *(strict) 2-category* $\mathscr{C}$ consists of:

1. A class of *objects* (or *0-cells*) $X, Y, Z, \ldots$
2. For each pair of objects X, Y, a category $\mathscr{C}(X, Y)$ whose
   - objects are called *1-cells* (or *morphisms*) $f, g : X \to Y$,
   - morphisms are called *2-cells* (or *2-morphisms*) $\alpha : f \Rightarrow g : X \to Y$.
3. A *horizontal composition* functor

$$⨟_{X,Y,Z} : \mathscr{C}(X, Y) \times \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$$

   for each triple of objects, written simply as $f ⨟ g$ on 1-cells and $\beta ⨟ \alpha$ on 2-cells.
4. For each object X, a distinguished 1-cell $\mathbf{id}_X : X \to X$ serving as the identity for horizontal composition.

Subject to the *strict* associativity and unit axioms:

$$f ⨟ (g ⨟ h) = (f ⨟ g) ⨟ h, \qquad\qquad f ⨟ \mathbf{id}_Y = f = \mathbf{id}_X ⨟ f$$

for all composable 1-cells (and similarly for 2-cells). Thus horizontal composition is strictly associative and unital.

Vertical composition of 2-cells (i.e. composition within each hom-category) is written with a semicolon: given $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, the composite is $\alpha; \beta : f \Rightarrow h$. The *interchange law*

$$(\alpha; \beta) ⨟ (\alpha'; \beta') = (\alpha ⨟ \alpha'); (\beta ⨟ \beta')$$

holds automatically from the functoriality of $⨟$.

Definition 8.3 (Strict 2-functor)**.** Let $\mathscr{C}, \mathscr{D}$ be strict 2-categories. A *(strict) 2-functor* $F : \mathscr{C} \to \mathscr{D}$ consists of:

1. A function on objects: $X \mapsto F(X)$.
2. For each pair X, Y, a functor $F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y))$ on hom-categories, mapping 1-cells $f \mapsto F(f)$ and 2-cells $\alpha \mapsto F(\alpha)$.

Subject to *strict* preservation of identities and horizontal composition:

$$F(\mathbf{id}_X) = \mathbf{id}_{F(X)}, \qquad F(f ⨟ g) = F(f) ⨟ F(g), \qquad F(\alpha; \beta) = F(\alpha); F(\beta).$$

Definition 8.4 (Strict 2-natural transformation)**.** Let $F, G : \mathscr{C} \to \mathscr{D}$ be strict 2-functors. A *(strict) 2-natural transformation* $\theta : F \Rightarrow G$ consists of:

1. For each object $X : \mathscr{C}$, a 1-cell $\theta_X : F(X) \to G(X)$ in $\mathscr{D}$ (the *component* at X),

such that for every 1-cell $f : X \to Y$ in $\mathscr{C}$ the following square of 1-cells in $\mathscr{D}$ *commutes strictly*:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\theta_X"'] & F(Y) \arrow[d, "\theta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

i.e. $\theta_X \mathbin{;} G(f) = F(f) \mathbin{;} \theta_Y$, and for every 2-cell $\alpha : f \Rightarrow g : X \to Y$:

$$F(\alpha) \mathbin{;} \theta_Y = \theta_X \mathbin{;} G(\alpha) \qquad \text{(equality of 2-cells).}$$

If only the 1-cell condition holds and the 2-cell condition is replaced by an inequality $\leq$ (when $\mathscr{D}$ is locally preorder-enriched), the transformation is called *lax*; if the inequality is reversed, *oplax*.

Definition 8.5 (Strict 2-adjunction)**.** Let $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ be strict 2-functors. A *(strict) 2-adjunction* $F \dashv U$ consists of strict 2-natural transformations

$$\eta : \mathbf{id}_{\mathscr{C}} \Rightarrow F \mathbin{;} U \qquad \text{(the } \textit{unit}\text{)}, \qquad \varepsilon : U \mathbin{;} F \Rightarrow \mathbf{id}_{\mathscr{D}} \qquad \text{(the } \textit{counit}\text{)},$$

satisfying the *triangle identities* strictly:

$$F \xrightarrow{\eta;F} F \mathbin{;} U \mathbin{;} F \xrightarrow{F;\varepsilon} F \;=\; \mathbf{id}_F, \qquad U \xrightarrow{U;\eta} U \mathbin{;} F \mathbin{;} U \xrightarrow{\varepsilon;U} U \;=\; \mathbf{id}_U.$$

Equivalently, for each $X : \mathscr{C}$ and $Y : \mathscr{D}$ there is an isomorphism of hom-categories

$$\mathscr{D}(F(X), Y) \cong \mathscr{C}(X, U(Y))$$

that is strictly natural in X and Y and strictly 2-natural in the 2-cells.

8.2 Initiality of SOb

Now let us begin. For each (necessarily monomorphism-preserving) functor $F : \mathscr{C} \to \mathscr{D}$ we can use the natural transformations $F_* : \mathbf{SOb}_{\mathscr{C}} \Rightarrow F;\mathbf{SOb}_{\mathscr{D}}$ that we introduced in Section 6.1.1, which map $\iota : X \hookrightarrow Y$ to $F(\iota) : F(A) \hookrightarrow F(X)$. To describe what $\mathbf{SOb}$ does on 2-cells let $\eta : F \Rightarrow G$ be a natural transformation. Again we have to assume η is monic (which is trivial on $\overset{\hookrightarrow}{\mathbf{Cat}}$) to say anything about how it acts on subobjects. Then we can say that the monomorphisms η_X induce for each X an ideal embedding $\eta_{*,X} : \mathbf{SOb}(F(X)) \hookrightarrow \mathbf{SOb}(G(X)) : F(\iota) \mapsto G(\iota)$, which is necessarily principal by Remark 5.2. Since η is natural between F and G, the diagram

$$\begin{tikzcd} G(A) \arrow[r, hook, "G(\iota)"] & G(X) \\ F(A) \arrow[r, hook, "F(\iota)"] \arrow[u, hook, "\eta_A"] & F(X) \arrow[u, hook, "\eta_X"] \end{tikzcd}$$

commutes, so for each $\iota : A \hookrightarrow X$ it holds that $F(\iota)_* \mathbin{;} \eta_{*,X} \leq G(\iota)_*$. Thus, η induces for each X a triangle

$$\begin{tikzcd}
& \mathbf{SOb}(G(X)) \arrow[d, phantom, "\geq"] & \\
\mathbf{SOb}(X) \arrow[ur, "G_{*,X}"] \arrow[rr, "F_{*,X}"] & & \mathbf{SOb}\,(F(X)) \arrow[ul, "\eta_{*,X}"']
\end{tikzcd}$$

so that $G_{*,X} \geq F_{*,X} \mathbin{;} \eta_{*,X}$. For every morphism $\iota\colon X \hookrightarrow Y$ in $\mathscr{C}$ the image ι_* is an ideal embedding, so this extends to a modification

$$G_* \geq F_*; (\eta \mathbin{;} \mathbf{SOb}).$$

Thus,

$$\mathbf{SOb} : X \mapsto (X, \mathbf{SOb}(X)), F : \mapsto (F, F_*), \eta \mapsto (\eta, \mathbf{SOb}_\eta)$$

extends to a 2-functor $\overset{\hookrightarrow}{\mathbf{Cat}} \to \overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$:

1. from the (2-large) 2-category $\overset{\hookrightarrow}{\mathbf{Cat}}$ whose objects are subobject-small purely monic categories, whose morphisms are (necessarily monomorphism-preserving) functors and whose 2-morphisms are (necessarily monic) natural transformations,
2. to the (3 -large) 2-category $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$
   (a) whose objects are functors $\mathscr{C} \to \mathbf{UBP}$ into $\mathbf{UBP}$,
   (b) morphisms are triangles

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "E"'] & \arrow[d, phantom, "\Rightarrow F_*"] & \mathscr{D} \arrow[dl, "E'"] \\
& \mathbf{UBP} &
\end{tikzcd}$$

   where F_* is componentwise top-preserving.

   (c) 2-morphisms are given by (necessarily monic) natural transformations $\eta\colon F \Rightarrow G\colon \mathscr{C} \to \mathscr{D}$ over $\mathbf{UBP}$ and the induced triangles

$$\begin{tikzcd}
F \mathbin{;} E' \arrow[rr, Rightarrow, "\eta;E'"] & \arrow[d, phantom, "\Leftarrow \leq"] & G \mathbin{;} E' \\
& E \arrow[ul, Rightarrow, "F_*"] \arrow[ur, Rightarrow, "G_*"'] &
\end{tikzcd}$$

To show well-definedness, the only non-trivial part is the defining inequality for 2-cells:

$$F_*;(\eta\,;\mathbf{SOb}_{\mathscr{C}'}) \leq F'_*.$$

This holds because for any $X \in \mathscr{C}$ and $\iota\colon A \hookrightarrow X$,

$$(F_*;(\eta\,;\mathbf{SOb}_{\mathscr{C}'}))_X(\iota) = (\eta\,;\mathbf{SOb}_{\mathscr{C}'})_X(\top_{F(\iota)}) = F(\iota)\,;\eta_X = \eta_A\,;F'(\iota) \leq F'(\iota) = F'_{*,X}(\iota).$$

Remark 8.6. So $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$ is a particular variant of a slice-2-category with morphisms and 2-morphisms given by two lax kinds of triangles. We pronounce this construction as "diamond-over".

$\mathbf{SOb}_{(-)}$ maps a subobject-small category $\mathscr{C}$ to the functor $\mathbf{SOb}_{\mathscr{C}}$, a functor F to (F, F_*) and a natural transformation $\eta\colon F \Rightarrow G$ to $(\eta_*, \leq)$ as described above. Moreover, since for each monomorphism $\iota\colon X \hookrightarrow Y$ the induced map $\iota_*\colon\mathbf{SOb}(X) \hookrightarrow \mathbf{SOb}(Y)$ is an ideal-embedding, the functor $\mathbf{SOb}_{\mathscr{C}}\colon\mathscr{C} \to \mathbf{UBP}$ factors through the category $\downarrow\!\mathbf{UBP}\!\downarrow$ of upward-bounded-preorders and ideal embeddings.

Remark 8.7. We can describe the image of $\mathbf{SOb}_{(-)}$ more precisely: since each $\iota \in \mathbf{SOb}_{\mathscr{C}}(X)$ corresponds to a monomorphism $\iota\colon A \hookrightarrow X$ and every principal ideal of $\mathbf{SOb}_{\mathscr{C}}(X)$ corresponds to an element $\mathbf{SOb}_{\mathscr{C}}(X)$ by definition, every principal ideal embedding $\kappa\colon(a) \hookrightarrow \mathbf{SOb}_{\mathscr{C}}(X)$ with a an element of $\mathbf{SOb}_{\mathscr{C}}(X)$ is in the image of $\mathbf{SOb}_{\mathscr{C}}$. Thus, $\mathbf{Im}(\mathbf{SOb}_{\mathscr{C}})$ is closed under precomposition and thus an ideal[7] in $\downarrow\!\mathbf{UBP}\!\downarrow$, and, conversely, every 1-categorical ideal in $\downarrow\!\mathbf{UBP}\!\downarrow$ is a subcategory of $\downarrow\!\mathbf{UBP}\!\downarrow$ that maps to itself.

It is important that the ideal that is the image of $\mathbf{SOb}_{\mathscr{C}}$ is not generally a principal ideal in $\mathbf{UBP}$, i.e. not upward-bounded: each object in $\mathscr{C}$ is composed of its parts, but $\mathscr{C}$ itself is generally only a context in which these objects interact, they don't assemble into one whole.

Thus, $\mathbf{SOb}_{(-)}\colon\overset{\hookrightarrow}{\mathbf{Cat}} \hookrightarrow \overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$ factors through the full sub-2-category $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}|_{\downarrow}}$ of $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$ on those functors to $\mathbf{UBP}$ that factor through $\downarrow\!\mathbf{UBP}\!\downarrow$. Moreover, these F_* are unique in their position:

Lemma 8.8. *Given a functor of monic categories $F\colon\mathscr{C} \hookrightarrow \mathscr{D}$, F_* is the only top-preserving natural transformation $\varepsilon\colon\mathbf{SOb}_{\mathscr{C}} \Rightarrow F\,;\mathbf{SOb}_{\mathscr{D}}\colon\mathscr{C} \to \overset{\hookrightarrow}{\mathbf{UBP}}$.*

$$\begin{tikzcd}
\mathbf{SOb}_{\mathscr{C}}(A) \arrow[r, hook, "\varepsilon_A"] \arrow[d, hook, "\iota_*"'] & \mathbf{SOb}_{\mathscr{D}}(F(A)) \arrow[d, hook, "F(\iota)_*"] \\
\mathbf{SOb}_{\mathscr{C}}(X) \arrow[r, hook, "\varepsilon_X"'] & \mathbf{SOb}_{\mathscr{D}}(F(X))
\end{tikzcd}$$

Proof. Evaluate at $\mathbf{id}_A$. The left-down leg gives $\mathbf{id}_A\,;\iota = \iota$; the top-right leg gives $\varepsilon_A(\mathbf{id}_A) = \mathbf{id}_{F(A)}$ (by top-preservation) and then $\mathbf{id}_{F(A)}\,;F(\iota) = F(\iota)$. Thus, naturality forces $\varepsilon_X(\iota) = F(\iota)$. Since every element of $\mathbf{SOb}_{\mathscr{C}}(X)$ is some monomorphism in $\mathscr{C}$ we have $\varepsilon_X = F_{*,X}$ for all X; hence $\varepsilon = F_*$. □

Given now some monic category $\mathscr{C}$, functor $G\colon\mathscr{C} \to \mathbf{UBP}$ and $X\colon\mathscr{C}$ we can define a map $G_{\bigstar,X}\colon\mathbf{SOb}(X) \hookrightarrow G(X)$ which maps each $\iota\colon\mathbf{SOb}(X)$ to the top $\top_{G(\iota)}$ of the order-morphism $G(\iota)\colon G(A) \hookrightarrow G(X)$, which is an embedding since ι is monic. If $\iota \leq \kappa$ for some $\kappa\colon B \hookrightarrow X$, then there exists a $\lambda\colon A \hookrightarrow B$ in $\mathscr{C}$ such that $\lambda\,;\kappa = \iota$, so that $G(\lambda)\,;G(\kappa) = G(\iota)$ and thus $G_{\bigstar,X}(\iota) \leq G_{\bigstar,X}(\kappa)$.

[7]Note that we mean ideal in the category-theoretic sense, i.e. closed under precomposition but without binary upward bounds.

So $G_\bigstar$ is the component of a reflector at G. This is how we can tell that **SOb** is actually an embedding of a reflective sub-2-category.

Note that this is the basically same construction as the F_* from Section 6.1.1: when $F : \mathscr{C} \to \mathbf{UBP}$ (instead of $F : \mathscr{C} \to \mathscr{D}$ between general categories), the map $F_{*,X}$ sends ι to $\top_{F(\iota)} \in F(X)$, which is the element of $F(X)$ that corresponds to the ideal embedding ι, whereas in the general case it sends ι to $F(\iota) \in \mathbf{SOb}(F(X))$, so in the second case we take the subobject-functor from a general category while in the first we don't need to since we're already in **UBP** anyway.

Lemma 8.9. *For each functor $F : \mathscr{C} \to \mathbf{UBP}$ and morphism $f : X \hookrightarrow Y$ in $\mathscr{C}$ it holds that $f_* \mathbin{;} F_{\bigstar,Y} = F_{\bigstar,X} \mathbin{;} F(f)$. Thus $F_\bigstar$ is a natural transformation*

$$F_\bigstar : \mathbf{SOb}_{\mathscr{C}} \Rightarrow F.$$

Proof. Consider the diagram:

$$\begin{tikzcd}
\mathbf{SOb}_{\mathscr{C}}(X) \arrow[r, "f_*"] \arrow[d, "F_{\bigstar,X}"] & \mathbf{SOb}_{\mathscr{C}}(Y) \arrow[d, "F_{\bigstar,Y}"] \\
F(X) \arrow[r, "F(f)"] & F(Y)
\end{tikzcd}$$

Given $\iota : A \hookrightarrow X$, we have:

$$f_* \mathbin{;} F_{\bigstar,Y}(\iota \mathbin{;} f) = \top_{F(\iota;f)} = \top_{F(\iota);F(f)} = F(f)(\top_{F(\iota)}) = F_{\bigstar,X} \mathbin{;} F(f).$$

Since this holds for all $\iota \in \mathbf{SOb}(X)$ for all $X : \mathscr{C}$ we have that $F_\bigstar$ is natural. □

So $\mathbf{SOb}_{\mathscr{C}}$ is the *universal representation of* $\mathscr{C}$ *in* **UBP**: every other representation, i.e. monomorphism-preserving functor F from $\mathscr{C}$ to **UBP**, comes with a unique natural transformation $F_\bigstar : \mathbf{SOb}_{\mathscr{C}} \to \mathbf{UBP}$ that maps out precisely the structure of the image of $\mathscr{C}$ under F. The reason it factors through $\downarrow\!\mathbf{UBP}\!\downarrow$ is that this is the densest possible stacking of all monomorphisms in $\mathscr{C}$ without equalizations, while the $F_\bigstar$ for some functor F maps that structure into some larger preorder.

Note that the cells induced by functors and natural transformations in $\overset{\hookrightarrow}{\mathbf{Cat}}$ according to Item 2b do *not* factor through $\downarrow\!\mathbf{UBP}\!\downarrow$. Rather, each F_* and $F_\bigstar$ is a natural transformation that preserves tops (since functors preserve identities), so they factor through the wide sub-2-category $\mathbf{UBP}_w$ of **UBP** on top-preserving maps.

So to summarize:

Theorem 8.10. *The 2-functor $\mathbf{SOb}_{(-)} : \overset{\hookrightarrow}{\mathbf{Cat}} \to \overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$ is left adjoint to the 2-functor $\mathbf{s} : \overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}} \to \overset{\hookrightarrow}{\mathbf{Cat}} : (F : \mathscr{C} \to \mathbf{UBP}) \mapsto \mathscr{C}$.*

Proof. We establish the adjunction by explicitly constructing the unit and counit. First, we verify that $\mathbf{SOb}_{(-)}$ is a well-defined 2-functor. For the unit of the adjunction $\mathbf{SOb}_{(-)} \dashv \mathbf{s}$, since $\mathbf{s}(\mathbf{SOb}_{\mathscr{C}}) = \mathscr{C}$, we may simply take the identity functor:

$$\mathbf{id}_{\mathscr{C}} : \mathscr{C} \to \mathbf{s}(\mathbf{SOb}_{\mathscr{C}}).$$

For the counit, given an object $E : \mathscr{C} \to \mathbf{UBP}$ in $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$, we need a 1-cell $\mathbf{SOb}_{\mathscr{C}} \to E$ over $\mathbf{id}_{\mathscr{C}}$. This 1-cell $\mathbf{id}_{\mathscr{C}}, E_{\bigstar}$ consists of $\mathbf{id}_{\mathscr{C}}$ with the natural transformation $E_{\bigstar} : \mathbf{SOb}_{\mathscr{C}} \Rightarrow E$, defined componentwise for each $X \in \mathscr{C}$ by

$$E_{\bigstar,X} : \mathbf{SOb}_{\mathscr{C}}(X) \to E(X), \qquad \iota \mapsto \top_{E(\iota)}.$$

This is order-preserving since $E(\iota)$ is a monotone map and naturality follows from Lemma 8.9.

It remains to verify the triangle identities. The first identity requires $(\mathbf{id}_{\mathscr{C}})_* \,;\, (\mathbf{SOb}_{\mathscr{C}})_{\bigstar} = \mathbf{id}_{\mathbf{SOb}_{\mathscr{C}}}$. Indeed, for any $X : \mathscr{C}$ this evaluates to

$$\mathbf{id}_{\mathbf{SOb}_{\mathscr{C}}}(X) \,;\, \mathbf{id}_{\mathbf{SOb}_{\mathscr{C}}}(X) = \mathbf{id}_{\mathbf{SOb}_{\mathscr{C}}}(X).$$

The second identity requires $\mathbf{s}(E_{\bigstar}) \,;\, \mathbf{id}_{\mathbf{s}(E)} = \mathbf{id}_{\mathbf{s}(E)}$. Since the underlying functor of the counit $E_{\bigstar}$ is $\mathbf{id}_{\mathscr{C}}$ and $\varepsilon_{\mathscr{C}} = \mathbf{id}_{\mathscr{C}}$, this identity holds trivially.

Having verified the triangle identities, we conclude that $\mathbf{SOb}_{(-)} \dashv \mathbf{s}$. □

Remark 8.11. So the basic distinguishing feature of **UBP** that we use in this proof and the entire section is that each preorder in it has a top, which ensures that each monic $\iota : A \hookrightarrow X$ in a category $\mathscr{C}$ is mapped to an ideal embedding ι_* under $\mathbf{SOb}_{\mathscr{C}}$, gives us shadows of morphisms in categories and functors and natural transformations between categories ensures that many problematic functors from $\mathscr{C}$ to **PrO** don't factor through **UBP**.

Remark 8.12. If we don't assume that the natural transformations of the 1-cells in $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UPB}}$ are top-preserving, then we still get a kind of laxification of an adjunction, where, given a category $\mathscr{C}$ and functor $F : \mathscr{C} \to \mathbf{UBP}$, the morphisms from $(\mathscr{C}, \mathbf{SOb}_{\mathscr{C}})$ to $(\mathscr{C}, F)$ are bounded by $F_{\bigstar}$ but no longer weakly unique.

Remark 8.13. Note that there exists a 2-functor $\mathbf{Cat}_{\hookrightarrow-prv} \to \overset{\hookrightarrow}{\mathbf{Cat}}$ from the 2-category of small categories, monomorphism-preserving functors and natural transformations between them, which we can postcompose with **SOb**. This is what we are implicitly using when we extend smallness from monic categories to all categories. Since every monic category is a category, $\mathbf{Cat}_{\hookrightarrow-prv} \to \overset{\hookrightarrow}{\mathbf{Cat}}$ has a section. Moreover, for each category $\mathscr{C}$ there exists a functor $\overset{\hookrightarrow}{\mathscr{C}} \hookrightarrow \mathscr{C}$ embedding the monic part of a category. All in all this sure looks like a coreflective embedding - but it is very far from that. The universal property of an adjunction would force that for each category $\mathscr{D}$ and functor $F : \overset{\hookrightarrow}{\mathscr{C}} \to \mathscr{D}$ there would exist a unique functor $\mathscr{C} \to \mathscr{D}$ over F, we have nothing to determine how to map the non-monomorphisms in $\mathscr{C}$. If we restrict to regular categories we have a decomposition $f = e \,;\, m$ for each f with m monic and e regular epi, and F fixes objects and monomorphisms, so in this case we would only have to determine how to map e. Now, e is a coequalizer of its kernel pair k_1, k_2 which is monic and F thus mapped by F, so might try to map e to the coequalizer $\mathbf{CEq}(F(k_1), F(k_2))$, but there is no reason to even assume that $(F(k_1), F(k_2))$ is a kernel pair in $\mathscr{D}$, it can be a monic pair without a co-equalizer. And even if we assume $\mathbf{CEq}(F(k_1), F(k_2))$ exists, there is no reason to assume that its codomain is $\mathbf{s}(F(m))$, and one can construct counterexamples.

8.3 Universal Characterization of $\overset{\hookrightarrow}{\mathbf{UBP}}$

But we are not done yet: we have universally described **SOb** as an adjoint to the 2-functor assigning to each representation $F : \mathscr{C} \to \mathbf{UBP}$ its source. But the hardest part is still before us: to provide a universal description of **UBP** as a 2-category. This is a slightly dizzying task somewhat akin to staring into two reflecting mirrors, due to one crucial question: we have described the category $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$, which is a kind of like a slice category in **Cat**, *but how can we do that if* **UBP** *is not a category?* It is, after all, a 2-category itself, and a sub-2-category of **Cat**. So we're treating something like a subset like an element. Why can we do that?

The answer is that the hierachy of n-categories is closed under taking powers analogous to how **Set** is under taking power sets, just with the categorical level increasing each time. This is a triviality: each set is a special category, each category a special 2-category and so on. But this just means that there exist fully faithful embeddings $\mathbf{Set} \hookrightarrow \mathbf{Cat} \hookrightarrow \mathbf{2}\text{-}\mathbf{Cat}$..., so that **Cat** is both an object and a sub-2-category of **2**-**Cat**. So we can indeed define $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$ as a slice-category-like particular kind of comma-2-category from the image $\mathbf{Im}(\overset{\hookrightarrow}{\mathbf{Cat}})$ of $\overset{\hookrightarrow}{\mathbf{Cat}}$ in $\mathbf{2}\text{-}\mathbf{Cat}$ to the constant functor on **UBP**. We might ask whether we could extend this structure further from $\mathbf{Im}(\overset{\hookrightarrow}{\mathbf{Cat}})$ but it seems that we are touching on something universal: that not just **Cat** but every **n**-**Cat** is reflected in **UBP**, in increasingly many ways as the possibilities for sensible subobject notions increase - we got a taste of this in Section 3.1.6.

And this is also what we have to use to derive a universal characterization of **UBP**: let us assume we have a similar diamond D going from $\overset{\hookrightarrow}{\mathbf{Cat}}$ to some other 2-category $\mathscr{B}$. If we have a subobject-2-functor $\mathbf{SOb}_{\mathscr{B}} : \mathscr{B} \to \mathbf{UBP}$, then we can take that functor right back to **UBP**, and then proceed along the lines of the proof of Theorem 8.10 to get a natural transformation from $\mathbf{SOb}_{(-)} \to D \,;\, \mathbf{SOb}_{\mathscr{B}}$. So to derive a universal characterization of **UBP** we have to essentially re-do Theorem 8.10, but on a 2-categorical level. We're going to need some patience though: while most of the following pieces are simple to comprehend on their own, those pieces do sum up!

The trouble starts with the subobject structures in 2-categories: in Section 3.1.6 we found several, so which one should we use? Presumably the most general. But to express this we need to upgrade the notion of a monomorphism to 2-categories. The standard way to do this is to assume either faithfulness or fully faithfulness:

Definition 8.14. [24] Let $\mathscr{B}$ be a 2-category. A 1-cell $\iota : A \to X$ in $\mathscr{B}$ is called *faithful* if for every object C of $\mathscr{B}$, the induced functor of hom-categories

$$\iota_{*,\mathscr{C}} \;:=\; (- \,;\, \iota) : \mathscr{B}(C, A) \to \mathscr{B}(C, X)$$

is faithful. Similarly, ι is *fully faithful* if each $\iota_{*,\mathscr{C}}$ is fully faithful.

However, mere faithful functors are not good subobject-notions to us: we already saw in Example 3.55 that this results in pathologies such as every set being a subcategory of $*$. This would be tolerable, but what really shows that an additional condition is necessary is that the composition of such cells does not work right, as we will see in Remark 8.22. What is missing is isomorphism-reflectivity. Let us remember the notion of a pseudomonic morphism from [6]:

Definition 8.15. [25] A morphism $\iota : X \to Y$ in a 2-category $\mathscr{B}$ is *pseudomonic* if ι is faithful and for each pair f, g and 2-isomorphism $\sigma : f \,;\, \iota \overset{\cong}{\Rightarrow} g \,;\, \iota$ there exists a 2-isomorphism $\overline{\sigma} : f \Rightarrow g$ such that $\sigma = \overline{\sigma} \,;\, \iota$. $\mathscr{B}$ itself is *pseudomonic* if all its morphisms are pseudomonics. $\mathscr{B}$ is *2-pseudomonic* if it is pseudomonic and all its 2-cells are monic in their hom-categories.

A morphism $f : X \to Y$ is *monomorphism-preserving* if for every $A : B$ the functor $f_* : \mathscr{B}(A, X) \to \mathscr{B}(A, Y)$ is monomorphism-preserving.

A 2-functor $F : \mathscr{B} \to \mathscr{B}'$ is *2-pseudomonic* if

1. for each $X, Y : \mathscr{B}$ the induced functor $F_{X,Y} : \mathscr{B}(X, Y) \to \mathscr{B}'(F(X), F(Y))$ is pseudomonic,
2. F preserves pseudomonics and reflects equivalences.

Example 8.16. Every monic category is a 2-pseudomonic 2-category. This follows because 2-cell preservation and isomorphism-reflection are trivially true for morphisms in a category.

We do need one further definition, which is slightly technical: note that. in **Cat** a natural transformation is monic if and only if it consists of monomorphisms. Thus, monic natural transformations are stable under precomposition with functors but not postcomposition. This is what allows Example 3.49 and its variants to work, so we need to abstract it:

Definition 8.17. A 2-category is *monically left-ideal* if for each monic 2-cell $\varphi : f \Rightarrow g : X \to Y$ and $h : A \to X$ it holds that the left whiskering $h \,;\, \varphi$ is monic.

Example 8.18. Each 2-pseudomonic 2-category is monically left-ideal, since all its 2-cells are monic.

On these we can nicely define subobject preorders:

Definition 8.19. Given an object X of a monically left-ideal 2-category $\mathscr{B}$, the *(2-covariant) pseudomonic subobject preorder* $\mathbf{SOb}_{f,\mathscr{B}}(X)$ is the preorder:

1. whose objects are pseudomonics $\iota : A \hookrightarrow X$ into X,
2. such that $\iota \le \kappa : B \hookrightarrow X$ if and only if there exists a morphism $\mu : A \to B$ and a monic 2-cell $\varphi : \iota \Rightarrow \mu \,;\, \kappa$.

Lemma 8.20. $\mathbf{SOb}_{f,\mathscr{B}}(X)$ *is in fact a preorder.*

Proof. We must show that the relation $\le$ on the class of pseudomonic 1-cells $\iota : A \to X$ into a fixed object $X : \mathscr{B}$ is reflexive and transitive.

Let $\iota : A \hookrightarrow X$ be a pseudomonic. The identity 2-cell is monic in any 2-category, so this witnesses $\iota \le \iota$. For transitivity, suppose we have pseudomonics $\iota : A \hookrightarrow X$, $\kappa : B \hookrightarrow X$, and $\lambda : C \hookrightarrow X$ with $\iota \le \kappa$ and $\kappa \le \lambda$. By definition, there exists a monomorphism-preserving $f : A \to B$ and a monic 2-cell $\alpha : \iota \Rightarrow f \,;\, \kappa$ in $\mathscr{B}(A, X)$ and a pseudomonic $g : B \to C$ and a monic 2-cell $\beta : \kappa \Rightarrow g \,;\, \lambda$ in $\mathscr{B}(B, X)$. Define $h := f \,;\, g : A \to C$ and set

$$\gamma \;:=\; \alpha; (f \,;\, \beta) : \iota \Rightarrow h \,;\, \lambda.$$

Since $\mathscr{B}$ is monically left-ideal, whiskering preserves monics. Moreover, vertical composition of monics is monic and γ is monic, so $\iota \le \lambda$. Thus $\le$ is reflexive and transitive. ◻

Remark 8.21. The slice category of an object in a category is constructed as a comma category, i.e. a special kind of 2-limit. In 2-categories, this definition trifurcates into a lax, an oplax and a weak variant. Accordingly, Definition 8.19 has a 2-contravariant variant $\mathbf{SOb}_{f^{op},\mathscr{B}}(X)$ with inverse 2-cells, and a weak variant $\mathbf{SOb}_{|f|,\mathscr{B}}(X)$, where the 2-cells have to be 2-isomorphisms. But since we don't use these other functors in this subsection, we will write $\mathbf{SOb}_{\mathscr{B}}(X)$ for $\mathbf{SOb}_{f,\mathscr{B}}(X)$ to simplify notation.

Remark 8.22. For $\iota : X \hookrightarrow Y$ faithful in a 2-category $\mathscr{B}$ whose morphisms are faithful but not isomorphism-reflecting, we *do* get for each $\kappa : A \hookrightarrow X, \kappa' : B \hookrightarrow X$ and pair $(f, \alpha) : (A \hookrightarrow B, \kappa \hookrightarrow f \,;\kappa')$ displaying $\kappa \le \kappa'$ a pair $(f, \alpha \,;\, \iota) : (A \hookrightarrow B, \kappa \,;\, \iota \hookrightarrow f \,;\, \kappa \,;\, \iota)$, so that $\kappa \,;\, \iota \le \kappa' \,;\, \iota$. Thus, we get an order-preserving morphism

$\iota_*\colon \mathbf{SOb}_{\mathscr{B}}(X) \hookrightarrow \mathbf{SOb}_{\mathscr{B}}(Y)$, which is essentially injective since ι is faithful. But, given $f\colon A \hookrightarrow B$ and $\beta\colon f \mathbin{;} \kappa \mathbin{;} \iota \hookrightarrow \kappa' \mathbin{;} \iota$ there is no reason to assume that β is in the image of $\iota_*\colon \mathscr{B}(A,X) \to \mathscr{B}(A,Y)$, which means that ι_* is *not* generally a monomorphism because β might be an isomorphism. So we need to at least assume that ι is isomorphism-reflecting i.e. each induced functor ι_* is isomorphism-reflecting. So we can understand isomorphism-reflectivity as a weakening of injectivity. This is why it excludes pathologies as in Example 3.55, which go against our intuition on what a subcategory should be and why we have to use pseudomonics, not just faithful morphisms.

Proposition 8.23. *Let $\mathscr{B}$ be a 2-pseudomonic 2-category. Then the assignment $X \mapsto \mathbf{SOb}_{\mathscr{B}}(X)$ from Definition 8.19 defines a 2-functor $\mathbf{SOb}_{\mathscr{B}}\colon \mathscr{B} \to \overset{\hookrightarrow}{\mathbf{UBP}}$.*

Proof. For a 1-cell $f\colon X \to Y$ in $\mathscr{B}$, define the action of $\mathbf{SOb}_{\mathscr{B}}$ on f by postcomposition:

$$\mathbf{SOb}_{\mathscr{B}}(f)\colon \mathbf{SOb}_{\mathscr{B}}(X) \longrightarrow \mathbf{SOb}_{\mathscr{B}}(Y), \qquad \iota \mapsto \iota \mathbin{;} f.$$

Since $\mathscr{B}$ is 2-pseudomonic, the composite $\iota \mathbin{;} f$ is pseudomonic, hence an object of $\mathbf{SOb}_{\mathscr{B}}(Y)$. To verify monotonicity, let $\iota \leq \kappa$ in $\mathbf{SOb}_{\mathscr{B}}(X)$. Then there exist a pseudomonic $g\colon A \hookrightarrow B$ and a monic 2-cell $\alpha\colon \iota \Rightarrow g \mathbin{;} \kappa$. Whiskering with f yields a monic 2-cell $\alpha \mathbin{;} f\colon \iota \mathbin{;} f \Rightarrow g \mathbin{;} (\kappa \mathbin{;} f)$, which witnesses $\iota \mathbin{;} f \leq \kappa \mathbin{;} f$.

For a 2-cell $\theta\colon f \Rightarrow g\colon X \to Y$, define $\mathbf{SOb}_{\mathscr{B}}(\theta)\colon \mathbf{SOb}_{\mathscr{B}}(f) \Rightarrow \mathbf{SOb}_{\mathscr{B}}(g)$ pointwise. For each $\iota\colon A \hookrightarrow X$, whiskering yields $\iota \mathbin{;} \theta\colon \iota \mathbin{;} f \Rightarrow \iota \mathbin{;} g$. Because $\mathscr{B}$ is 2-pseudomonic, $\iota \mathbin{;} \theta$ is a monic 2-cell, corresponding to the relation $\iota \mathbin{;} f \leq \iota \mathbin{;} g$ in $\mathbf{SOb}_{\mathscr{B}}(Y)$. The interchange law in $\mathscr{B}$ ensures that this assignment is natural.

The 2-functor axioms follow directly from the unitality and associativity of composition in $\mathscr{B}$. Specifically, $\mathbf{SOb}_{\mathscr{B}}(\mathbf{id}_X)$ is the identity on $\mathbf{SOb}_{\mathscr{B}}(X)$ since $\iota \mathbin{;} \mathbf{id}_X = \iota$, and $\mathbf{SOb}_{\mathscr{B}}$ preserves composition of 1-cells since $\iota \mathbin{;} (f \mathbin{;} g) = (\iota \mathbin{;} f) \mathbin{;} g$. Preservation of vertical and horizontal composition of 2-cells follows analogously from the functoriality of whiskering. ◻

Remark 8.24. Let us take a look at the action of $\mathbf{SOb}$ on $\overset{\hookrightarrow}{\mathbf{UBP}}$ itself, which will become crucial in a moment: it sends a preorder P to the preorder $\mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}(P)$ we discussed in Example 3.47 and acts on morphisms by postcomposition. Thus there exist morphisms

1. $\{-\}\colon P \to \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}(P)$ which maps $a \in P$ to the one-element sub-preorder $\{a\}$,
2. $(-)\colon P \to \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}(P)$ whose image is the ideal (a),

It follows almost immediately from the definitions that $\{-\}$ forms a strict natural 2-transformation and from Proposition 5.8 that $(-)$ forms an oplax one. But because each preorder in $\mathbf{UBP}$ has a top, we can define a morphism $\top_{(-)}\colon \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}(P) \twoheadrightarrow P\colon \iota \mapsto \top_\iota$ such that both $\{-\}$ and $(-)$ are sections of $\top_{(-)}$. Moreover, it also follows almost immediately from the definitions that $\{-\}$ is a minimal section of $\top_{(-)}$ and $(-)$ a maximal one, so that the three form a unity of opposites in the sense of [21]. Note that the same does not hold for filters since we would need downward-boundedness.

Now let us define what it means for something to be diamond-over:

Definition 8.25. An *oplaxly pointed 2-category* $(\mathscr{B}, \bigstar_{\mathscr{B}}) := \mathscr{B}_\bigstar$ is a 2-category with terminal object $*$ and an oplax natural transformation $\bigstar_{\mathscr{B}}\colon \mathbf{K}(*) \to \mathbf{id}_{\mathscr{B}}$. We call a morphism $f\colon X \to Y$ in $\mathscr{B}$ *point-preserving* if $\bigstar_{\mathscr{B},X} \mathbin{;} f = \bigstar_{\mathscr{B},Y}$ and a natural transformation *point-preserving* if each of its component-morphisms is.

Definition 8.26 (Diamond-over 2-category $\mathscr{A}_{\blacklozenge\mathscr{B}}$)**.** Let $\mathscr{A}$ be a sub-2-category of $\mathbf{Cat}$, and let $\mathscr{B}_\bigstar$ be an oplaxly pointed 2-category. The *diamond-over 2-category $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ of $\mathscr{A}$ over $\mathscr{B}_\bigstar$* is defined as follows.

An object of $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ is a pair $(\mathscr{C}, E)$ where:

- $\mathscr{C}$ is an object of $\mathscr{A}$;
- $E : \mathscr{C} \to \mathscr{B}$ is a 2-functor (necessarily trivial on 2-cells since $\mathscr{C}$ is a category).

A 1-cell

$$(F, \varepsilon) : (\mathscr{C}, E) \longrightarrow (\mathscr{C}', E')$$

in $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ is an oplaxly (or, less confusingly, covariantly) commuting triangle, so:

- a functor $F : \mathscr{C} \to \mathscr{C}'$ in $\mathscr{A}$;
- a point-preserving 2-natural transformation

  $$\varepsilon : E \Rightarrow F \mathbin{;} E'$$

  in $\mathscr{B}_\bigstar$,

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "E"'] & \arrow[d, phantom, "\overset{\varepsilon}{\Rightarrow}"] & \mathscr{C}' \arrow[dl, "E'"] \\
& \mathscr{B} &
\end{tikzcd}$$

A 2-cell

$$(\alpha, \Phi) : (F, \varepsilon) \Rightarrow (F', \varepsilon')$$

between 1-cells $(F, \varepsilon), (F', \varepsilon') : (\mathscr{C}, E) \to (\mathscr{C}', E')$ consists of:

- a natural transformation $\alpha : F \Rightarrow F'$ in $\mathscr{A}$;
- a modification $\Phi : \varepsilon ; (\alpha \mathbin{;} E') \Rrightarrow \varepsilon'$.

$$\begin{tikzcd}
F \mathbin{;} E' \arrow[rr, Rightarrow, "\alpha;E'"] & \arrow[d, phantom, "\overset{\Phi}{\Rrightarrow}"] & F' \mathbin{;} E' \\
& E \arrow[ul, Rightarrow, "\varepsilon"] \arrow[ur, Rightarrow, "\varepsilon'"'] &
\end{tikzcd}$$

Composition in $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ is defined by horizontal/vertical composition in $\mathscr{A}$ and $\mathscr{B}$ in the obvious way. Identities are given by $(\mathbf{id}_{\mathscr{C}}, \mathbf{id}_E)$ on objects, and $(\mathbf{id}_F, \mathbf{id}_\varepsilon)$ on 1-cells. The 2-category axioms follow from those of $\mathscr{A}$ and $\mathscr{B}$.

Remark 8.27. This reduces to our previous definition: for $\mathscr{A} = \overset{\hookrightarrow}{\mathbf{Cat}}$ and $\mathscr{B} = \mathbf{UBP}$, the objects of $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ are representations $F : \mathscr{C} \to \mathbf{UBP}$; 1-cells are triangles (H, ε) as in Item 2b; 2-cells are natural transformations between the 1-cell functors over $\mathbf{UBP}$ with a corresponding modifier (which has to be unique since $\mathbf{UBP}$ consists of preorders). Thus, $\mathscr{A}_{\blacklozenge\mathscr{B}_\bigstar}$ is exactly $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge\mathbf{UBP}}$:

Remark 8.28. There is a variant of this diamond-over category for the contravariant subobject-functor $\mathbf{SOb}^*$. Take some finite-limit-preserving functor $F : \mathscr{C} \to \mathscr{D}$. This induces for each $X : \mathscr{C}$ a morphism $F_* : \mathbf{SOb}(X) \to \mathbf{SOb}(F(X))$, which preserves $\top$ and meets if they exist in $\mathscr{C}$. Given now a natural transformation $\eta : F \Rightarrow G : \mathscr{C} \to \mathscr{D}$, we get a natural transformation $\eta_X^* : G \mathbin{;} \mathbf{SOb}_{\mathscr{D}} \to F(X) \mathbin{;} \mathbf{SOb}_{\mathscr{D}}$ and for each $\iota : A \hookrightarrow X$ a diagram

$$\begin{tikzcd}
& G(A) \arrow[r, hook, "G(\iota)"] & G(X) \\
& \eta^*(G(\iota)) \arrow[u] \arrow[r, hook, "\eta_X^*(G(\iota))"] & F(X) \arrow[u, "\eta_X"'] \\
F(A) \arrow[uur, "\eta_A"] \arrow[ur, hook, "(\eta_A, F(\iota))"'] \arrow[urr, hook, "F(\iota)"'] & &
\end{tikzcd}$$

Figure 4

where $\leq$ is the morphism to $\eta^*(G(\iota))$ induced by the universal property of the pullback, which is a monomorphism since $F(\iota)$ and $\eta_X^*(G(\iota))$ are. Thus, we have for each such ι a triangle

$$\begin{tikzcd}
& \mathbf{SOb}(G(X)) \arrow[dr, "\mathbf{SOb}(\eta_X^*)"] & \\
\mathbf{SOb}(X) \arrow[ur, "G_{*,X}"] \arrow[rr, "F_{*,X}"] \arrow[r, phantom, shift left=4, "\rotatebox{90}{$\leq$}"] & & \mathbf{SOb}(F(X))
\end{tikzcd}$$

Figure 5

so the inequality goes precisely in the other direction as before. Moreover, if η is Cartesian, then by definition each square as in Fig. 4 is a pullback square, so that this inequality is an equivalence. Thus, we can lift $\mathbf{SOb}^*_{\mathscr{C}}$ to a 2-functor

$$\mathbf{SOb}^*_{(-)} : \mathbf{Cat}_{pb} \to (\mathbf{Cat}_{pb})_{\blacklozenge^{op}\mathbf{UBP}}$$

from the 2-category $\mathbf{Cat}_{pb}$ of subobject-small categories with pullbacks along monomorphisms, pullback-preserving functors and natural transformations to the 2-category $(\mathbf{Cat}_{pb})_{\blacklozenge^{op}\mathbf{UBP}}$ which is defined as in Definition 8.26 but with the 2-cell modifications as in Fig. 5 instead of as in Item 2b.

Note however that the contravariant $\mathbf{SOb}^*$-functor does not have the same universal property as the covariant one in the contravariant diamond category over $\mathbf{UBP}$: since for some $\iota : A \hookrightarrow X$ the morphism $\iota^* : \mathbf{SOb}(X) \to \mathbf{SOb}(A)$ preserves tops, the entire calculus we're developing cannot be applied and we're left without a way to compare $\mathbf{SOb}_{\mathscr{C}}$ to some other functor $F : \mathscr{C} \to \mathbf{UBP}$.

Now, a key point is:

Proposition 8.29. *Let $\mathscr{A}$ be a sub-2-category of* $\mathbf{Cat}$*, $\mathscr{B}_\star, \mathscr{B}'_\star$, be oplaxly pointed 2-categories, and $F : \mathscr{B} \to \mathscr{B}'$ a 2-functor preserving the oplax point. Then F induces a 2-functor $(- \mathbin{;} F) \colon \mathscr{A}_{\blacklozenge \mathscr{B}_\star} \longrightarrow \mathscr{A}_{\blacklozenge \mathscr{B}'_\star}$.*

Proof. We define $(- \mathbin{;} F)$ by postcomposition with F. On objects, we set

$$(- \mathbin{;} F)(\mathscr{C}, H) := (\mathscr{C}, H \mathbin{;} F).$$

On a 1-cell $(G, \varepsilon) \colon (\mathscr{C}, H) \to (\mathscr{D}, K)$, where $G : \mathscr{C} \to \mathscr{D}$ and $\varepsilon : H \Rightarrow G \mathbin{;} K$, we define

$$(- \mathbin{;} F)(G, \varepsilon) := (G, F(\varepsilon)) \colon (\mathscr{C}, H \mathbin{;} F) \to (\mathscr{D}, K \mathbin{;} F),$$

where $F(\varepsilon)$ denotes the whiskering of the 2-natural transformation ε with F, which is point-preserving since ε and F are. On a 2-cell $(\alpha, \Phi) \colon (G, \varepsilon) \Rightarrow (G', \delta)$, consisting of a 2-natural transformation $\alpha : G \Rightarrow G'$ and a modification $\Phi : \varepsilon ; (\alpha \mathbin{;} K) \Rightarrow \delta$, we set

$$(- \mathbin{;} F)(\alpha, \Phi) := (\alpha, \Phi \mathbin{;} F),$$

where $\Phi \mathbin{;} F$ is the whiskering of Φ with F.

To verify that $(- \mathbin{;} F)$ is a 2-functor, consider composable 1-cells $(G, \varepsilon) \colon (\mathscr{C}, H) \to (\mathscr{D}, K)$ and $(L, \eta) \colon (\mathscr{D}, K) \to (\mathscr{E}, J)$. Their composite in $\mathscr{A}_{\blacklozenge \mathscr{B}_\star}$ is given by $(G \mathbin{;} L, \gamma)$, where γ is the horizontal composite

$$\gamma : H \overset{\varepsilon}{\Rightarrow} G \mathbin{;} K \overset{G;\eta}{\Longrightarrow} G \mathbin{;} L \mathbin{;} J.$$

Applying F, we obtain

$$\gamma \mathbin{;} F : H \mathbin{;} F \overset{\varepsilon;F}{\Longrightarrow} G \mathbin{;} K \mathbin{;} F \overset{G;\eta;F}{\Longrightarrow} G \mathbin{;} L \mathbin{;} J \mathbin{;} F.$$

Because F is a 2-functor, this $F(\gamma)$ is exactly the structure map of the composite $(G, F(\varepsilon)) \mathbin{;} (L, F(\eta))$ in $\mathscr{A}_{\blacklozenge \mathscr{B}'_\star}$. The preservation of identity 1-cells follows identically from $F(\mathbf{id}) = \mathbf{id}$.

Finally, vertical and horizontal composition of 2-cells in $\mathscr{A}_{\blacklozenge \mathscr{B}_\star}$ are computed componentwise in $\mathscr{A}$ and $\mathscr{B}$. Since F strictly preserves all compositions and identities of 2-cells, the assignment $(\alpha, \Phi) \mapsto (\alpha, \Phi \mathbin{;} F)$ is strictly functorial with respect to both vertical and horizontal composition. Thus, $(- \mathbin{;} F)$ is a strict 2-functor. ◻

Thus, we get for each oplaxly pointed 2-category $\mathscr{B}_\star$ and sub-2-category $\mathscr{A}$ of $\overset{\hookrightarrow}{\mathbf{Cat}}$ a 2-functor $(- \mathbin{;} \mathbf{SOb}_{\mathscr{B}}) \colon \mathscr{A}_{\blacklozenge \mathscr{B}_\star} \to \mathscr{A}_{\blacklozenge \mathbf{UBP}}$. Now, if $\mathscr{A}$ consists of monic categories and monomorphism-preserving functors then all we really need to do is find a comparison with $\mathbf{SOb} : \mathscr{A}_{\blacklozenge \mathbf{UBP}}$ and verify it checks out as a universal property, and we're done. But if $\mathscr{B}$ is 2-pseudomonic, then for each $\iota : A \hookrightarrow X$ in any $\mathscr{C} : \mathscr{A}$ and any 2-functor $F : \mathscr{C} \to \mathscr{B}$ there exists an element $F(\iota) \in \mathbf{SOb}(F(X))$, so that we can define for each $X : \mathscr{C}$ an order-preserving function $F_{*,X} : \mathbf{SOb}_{\mathscr{C}} \to \mathbf{SOb}_{\mathscr{B}}, \iota \mapsto \top_{F(\iota)}$, which is unique since it is determined by its values on subobjects ι of X.

Lemma 8.30. *For each $\mathscr{C} : \mathscr{A}$ and $F : \mathscr{C} \to \mathscr{B}_\star$ with $\mathscr{B}_\star$ 2-pseudomonic and oplaxly pointed, F_* is a point-preserving 2-natural transformation* $\mathbf{SOb}_{\mathscr{C}} \Rightarrow (F \mathbin{;} \mathbf{SOb}_{\mathscr{B}}) \colon \mathscr{C} \to \mathbf{UBP}$.

Proof. Since $\mathscr{C}$ is a 1-category, 2-naturality holds trivially. It suffices to verify that each $F_{*,X}$ is monotone and natural in 1-cells.

For monotonicity, let $\iota \leq \kappa$ in $\mathbf{SOb}_{\mathscr{C}}(X)$, with representatives $\iota : A \hookrightarrow X$ and $\kappa : B \hookrightarrow X$. By definition, there exists a (necessarily monic) 1-cell $f : A \to B$ such that $\iota = f \mathbin{;} \kappa$. Applying F, we obtain a 1-cell $F(f) : F(A) \to F(B)$ in $\mathscr{B}$. Since $\mathscr{B}$ is pseudomonic, $F(f)$ is pseudomonic, and thus $F(\iota) \leq F(\kappa)$ in $\mathbf{SOb}_{\mathscr{B}}(F(X))$. Hence $F_{*,X}$ is order-preserving. Point-preservation holds since $\top_{F(\iota)} = \mathbf{id}_{F(\iota)}$.

For naturality, let $g : X \to Y$ be a 1-cell in $\mathscr{C}$. We must show that the maps $F_{*,X} \mathbin{;} F(g)_*$ and $g_* \mathbin{;} F_{*,Y}$ from $\mathbf{SOb}_{\mathscr{C}}(X)$ to $\mathbf{SOb}_{\mathscr{B}}(F(Y))$ are equal. Evaluating both on an element $\iota : A \hookrightarrow X$, the left-hand side yields

$$(F_{*,X} \mathbin{;} F(g)_*)(\iota) = F_*(g)(F(\iota)) = F(\iota) \mathbin{;} F(g),$$

while the right-hand side yields

$$(g_* \mathbin{;} F_{*,Y})(\iota) = F(\iota \mathbin{;} g).$$

By the functoriality of F, we have $F(\iota) \mathbin{;} F(g) = F(\iota \mathbin{;} g)$, so the two composites agree pointwise. Thus, F_* is a natural transformation $\mathbf{SOb}_{\mathscr{C}} \Rightarrow F \mathbin{;} \mathbf{SOb}_{\mathscr{B}}$. □

Now, let us note that for each $\mathscr{A} \hookrightarrow \mathbf{Cat}$ and each oplaxly pointed 2-category $\mathscr{B}_\star$, the 2-category $\mathscr{A}_{\blacklozenge \mathscr{B}_\star}$ is fibered over $\mathscr{A}$ through the 2-functor $\mathbf{s}_{\mathscr{A}_{\blacklozenge \mathscr{B}_\star}}$, which maps $F : \mathscr{C} \to \mathscr{B}$ to $\mathscr{C}$, and similar for higher cells.

Definition 8.31. Given an oplaxly pointed 2-category $\mathscr{B}_\star$ and a sub-2-category $\mathscr{A} \hookrightarrow \mathbf{Cat}$, *an $\mathscr{A}$-diamond in $\mathscr{B}_\star$* is a section $\mathscr{A} \to \mathscr{A}_{\blacklozenge \mathscr{B}_\star}$ of $\mathbf{s}_{\mathscr{A}_{\blacklozenge \mathscr{B}_\star}}$. We denote the 2-category of sections of $\mathbf{s}_{\mathscr{A}_{\blacklozenge \mathscr{B}_\star}}$ by $\mathscr{A} \blacklozenge \mathscr{B}_\star$.

Remark 8.32. So a diamond is a particular kind of reflection of $\mathscr{A}$ in $\mathscr{B}_\star$, which assigns to each $\mathscr{C} : \mathscr{A}$ a representation of $\mathscr{A}$ in $\mathscr{B}$ and to each functor and natural transformation in $\mathscr{A}$ parameterized diagrams in $\mathscr{B}$.

Remark 8.33. By Proposition 8.29 we can postcompose a diamond with a 2-functor as if it was a functor. Since we use the $(-)_*$ notation quite a lot, we will just denote this as $D \mathbin{;} F$.

Example 8.34. Let $\mathbf{Cat}_{term}$ be the full sub-2-category of $\mathbf{Cat}$ whose objects are small categories with a terminal object. Then slicing constitutes a $\mathbf{Cat}_{term}$-valued diamond (we spell out a proof below, but the procedure is the same as for $\overset{\hookrightarrow}{\mathbf{Cat}}_{\blacklozenge \mathbf{UBP}}$). Moreover, it seems like the proof of Theorem 8.10 can be straightforwardly adapted to $\mathbf{Cat}_{\blacklozenge \mathbf{Cat}_{term}}$. Whether the currently going-on proof of Theorem 8.38 can also be adapted is less clear, but an analogue of it for non-monics should be true.

Proof. We construct a 2-functor $D : \mathbf{Cat}_{term} \to (\mathbf{Cat}_{term})_{\blacklozenge \mathbf{Cat}_{term}}$ that is a section of the source 2-functor $\mathbf{s}$. Let $\mathscr{A} = \mathbf{Cat}_{term}$.

Action on objects. For each category $\mathscr{C} : \mathscr{A}$, we assign $D(\mathscr{C}) = (\mathscr{C}, E_{\mathscr{C}})$, where $E_{\mathscr{C}} : \mathscr{C} \to \mathbf{Cat}_{term}$ is the strict 2-functor given by taking slice categories:

$$E_{\mathscr{C}}(X) \;=\; \mathscr{C}_{/X}.$$

For a morphism $f : X \to Y$ in $\mathscr{C}$, the functor $E_{\mathscr{C}}(f) = f_* : \mathscr{C}_{/X} \to \mathscr{C}_{/Y}$ acts by postcomposition, sending an object $a : A \to X$ to $a \mathbin{;} f : A \to Y$. Because $\mathscr{C}_{/X}$ has a terminal object $\mathbf{id}_X$, it belongs to $\mathbf{Cat}_{term}$. The oplax point $\star_{\mathscr{C}_{/X}} : \mathbf{1} \to \mathscr{C}_{/X}$ selects this terminal object.

Action on 1-cells. For a functor $F : \mathscr{C} \to \mathscr{D}$ in $\mathscr{A}$, we define $D(F) = (F, \varepsilon_F)$, where $\varepsilon_F : E_{\mathscr{C}} \Rightarrow F \mathbin{;} E_{\mathscr{D}}$ is a point-preserving 2-natural transformation. For each $X : \mathscr{C}$, the component $(\varepsilon_F)_X : \mathscr{C}_{/X} \to \mathscr{D}_{/F(X)}$ is the functor $F_{/X}$ induced by postcomposition with F:

$$(A \xrightarrow{a} X) \quad \mapsto \quad (F(A) \xrightarrow{F(a)} F(X)).$$

To see that ε_F is point-preserving, evaluate it on the point of $\mathscr{C}_{/X}$, which is $\mathbf{id}_X$:

$$(\varepsilon_F)_X(\mathbf{id}_X) \; = \; F(\mathbf{id}_X) \; = \; \mathbf{id}_{F(X)},$$

which is exactly the point of $\mathscr{D}_{/F(X)}$. To verify 2-naturality, let $f : X \to Y$ be a morphism in $\mathscr{C}$. The following square of categories and functors must strictly commute:

$$\begin{tikzcd}
\mathscr{C}_{/X} \arrow[r, "F_{/X}"] \arrow[d, "f_*"'] & \mathscr{D}_{/F(X)} \arrow[d, "F(f)_*"] \\
\mathscr{C}_{/Y} \arrow[r, "F_{/Y}"'] & \mathscr{D}_{/F(Y)}
\end{tikzcd}$$

For any object $a : A \to X$ in $\mathscr{C}_{/X}$, tracing right-then-down yields $F(a) \,;\, F(f) = F(a \,;\, f)$, while down-then-right yields $F(a \,;\, f)$. Thus, the square strictly commutes.

Action on 2-cells. For a natural transformation $\alpha : F \Rightarrow G : \mathscr{C} \to \mathscr{D}$ in $\mathscr{A}$, we define $D(\alpha) = (\alpha, \Phi)$. By Definition 8.26, we must provide a modification $\Phi : \varepsilon_F ; (\alpha \,;\, E_{\mathscr{D}}) \Rrightarrow \varepsilon_G$. For each $X : \mathscr{C}$, the transformation $(\alpha ; E_{\mathscr{D}})_X : \mathscr{D}_{/F(X)} \Rightarrow \mathscr{D}_{/G(X)}$ is postcomposition with $\alpha_X : F(X) \to G(X)$. We require a natural transformation $\Phi_X : F_{/X} \,;\, (\alpha_X)_* \Rightarrow G_{/X}$ in $\mathbf{Cat}_{term}$. For an object $a : A \to X$ in $\mathscr{C}_{/X}$, the source is the object $F(a) \,;\, \alpha_X$ in $\mathscr{D}_{/G(X)}$, and the target is $G(a)$. By the naturality of α in $\mathscr{D}$, we have a morphism $\alpha_A : F(A) \to G(A)$ satisfying $\alpha_A \,;\, G(a) = F(a) \,;\, \alpha_X$. This makes α_A a morphism in $\mathscr{D}_{/G(X)}$ from $F(a) \,;\, \alpha_X$ to $G(a)$. We define $(\Phi_X)_a := \alpha_A$. The naturality of Φ_X with respect to morphisms in $\mathscr{C}_{/X}$, as well as the modification axioms for Φ, follow directly from the naturality of α.

Section and 2-functor axioms. By construction, the underlying object, 1-cell, and 2-cell of $D(\mathscr{C})$, $D(F)$, and $D(\alpha)$ are $\mathscr{C}$, F, and α respectively. Thus, D is a section of the source 2-functor $\mathbf{s} : (\mathbf{Cat}_{term})_{\blacklozenge \mathbf{Cat}_{term}} \to \mathbf{Cat}_{term}$. Identities and compositions are strictly preserved because the slice and postcomposition operations define a strict 2-functor $\mathbf{Cat} \to \mathbf{Cat}$ componentwise. Therefore, D constitutes a $\mathbf{Cat}_{term}$-valued diamond. ◻

Now we just have to put all diamond-structures into one nice package. However, there are still some pitfalls that have to be avoided. For one, note that if $\mathscr{A}$ contains non-faithful functors $F : \mathscr{C} \to \mathscr{C}'$, then the natural transformation $F_* : \mathbf{SOb}_{\mathscr{C}} \to \mathbf{SOb}_{\mathscr{C}'}$ will not be a monomorphism and thus not in $\overset{\hookrightarrow}{\mathbf{UBP}}$ but in $\mathbf{UBP}$. If we only do comparisons over $\mathbf{UBP}$ this is not a problem, but, of course, not every morphism in $\mathbf{UBP}$ is monic, so $\mathbf{UBP}$ is not a pseudomonic 2-category; but we really need to assume that $\mathscr{B}$ is monic to get a $\mathbf{SOb}$-functor on it. We could *try* to treat $\mathscr{B}$ and $\mathbf{UBP}$ on a different levelling, but it seems more prudent to just assume that $\mathscr{A}$ is pseudomonic itself i.e. consists of faithful isomorphism-reflecting functors, so that its subobject-functor factors through $\overset{\hookrightarrow}{\mathbf{UBP}}$. Note that this means $\mathscr{A}$ is 2-pseudomonic since it consists of monic categories.

Remark 8.35. This restriction means that our aimed-for theorem will sadly not just be a direct generalization of Theorem 8.10, and such a thing is probably impossible.

Definition 8.36. Given a sub-2-category $\mathscr{A} \hookrightarrow \overset{\hookrightarrow}{\mathbf{Cat}}$, the *2-category of diamonds* $\blacklozenge(\mathscr{A})$ *over* $\mathscr{A}$ is the 2-category:

1. whose objects are pairs $(\mathscr{B}_\bigstar, D\colon \mathscr{A} \blacklozenge \mathscr{B}_\bigstar)$ with $\mathscr{B}_\bigstar$ an oplaxly pointed 2-category,
2. whose morphisms $\Delta\colon (\mathscr{B}_\bigstar, D\colon \mathscr{A} \blacklozenge \mathscr{B}_\bigstar) \to (\mathscr{B}'_\bigstar, D'\colon \mathscr{A} \blacklozenge \mathscr{B}'_\bigstar)$ are given by pairs of
   (a) a point-preserving 2-functor $\overline{\Delta}\colon \mathscr{B}_\bigstar \to \mathscr{B}'_\bigstar$ (i.e. $\bigstar_{\mathscr{B}_\bigstar} \mathbin{;} \overline{\Delta} = \bigstar_{\mathscr{B}'_\bigstar}$),
   (b) a cell $\underline{\Delta}\colon D' \to D \mathbin{;} \overline{\Delta}\colon \mathscr{A} \blacklozenge \mathscr{B}'_\bigstar$.
3. whose 2-morphisms $I\colon \Delta \Rightarrow \Delta^\circ$ are given by pairs of
   (a) a point-preserving 2-natural transformation $\overline{I}\colon \overline{\Delta} \Rightarrow \overline{\Delta}^\circ\colon \mathscr{B}_\bigstar \to \mathscr{B}'_\bigstar$ and
   (b) a 2-cell $\underline{I}\colon \underline{\Delta}^\circ \Rightarrow \underline{\Delta}; (D \mathbin{;} \overline{I})\colon \mathscr{A} \blacklozenge \mathscr{B}'_\bigstar$.

The *2-category of diamonds* $\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A})$ *over* $\mathscr{A}$ *on monics* is the (on all levels) full sub-2-category of $\blacklozenge(\mathscr{A})$ on 2-pseudomonic 2-categories $\mathscr{B}_\bigstar$.

This definition is fairly complex, so let us disentangle it a bit: all cells go out of $\mathscr{A}$. Morphisms $\Delta\colon (\mathscr{B}_\bigstar, D) \to (\mathscr{B}'_\bigstar, D')$ between objects consist of one covariant "external" functor $\overline{\Delta}$ and one covariant "internal" morphism (which is a parameterized point-preserving natural transformation) $\underline{\Delta}$, which "closes the square in $\mathscr{B}$".

The more mysterious part are the 2-morphisms: so given Δ, Δ°, a 2-morphism $I\colon \Delta \Rightarrow \Delta^\circ$ between them consists of a covariant part $\overline{I}\colon \overline{\Delta} \Rightarrow \overline{\Delta}^\circ$ "on the outside" and a contravariant part $\underline{I}\colon \underline{\Delta}^\circ \Rightarrow \underline{\Delta}; (D \mathbin{;} \overline{I})$: "on the inside". But let us consider how this looks in the particular case we are interested in, which is ensuring that the morphism from $(\mathscr{B}_\bigstar, D)$ to $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ given by $(\mathbf{SOb}_{\mathscr{B}}, D_*)$ is terminal, where $D_*(\mathscr{C})\colon \mathbf{SOb}(\mathscr{C}) \Rightarrow D \mathbin{;} \mathbf{SOb}_{\mathscr{B}}$ is given by $D(\mathscr{C})_*$ on each $\mathscr{C}\colon \mathscr{A}$ and similar for higher cells. However, we have to interpret "terminal" in some lax 2-categorical sense because it is not true on the nose. So the question is: *how* is it true? Well, given any 2-functor $\overline{\Delta}\colon \mathscr{B} \to \overset{\hookrightarrow}{\mathbf{UBP}}$ and 2-cell $\underline{\Delta}\colon \mathbf{SOb} \Rightarrow D \mathbin{;} \overline{\Delta}\colon \mathscr{A} \blacklozenge \mathbf{UBP}$ we have a top-preserving natural transformation $\overline{\Delta}_\bigstar\colon \mathbf{SOb}_{\mathscr{B}} \Rightarrow \overline{\Delta}$ given by mapping $\iota\colon \mathbf{SOb}_{\mathscr{B}}(X)$ to the image top $\top_{\overline{\Delta}(\iota)}$ of $\overline{\Delta}(\iota)$ so that $D_*; (D \mathbin{;} \overline{\Delta}_\bigstar)$ goes from $\mathbf{SOb}$ to $D \mathbin{;} \overline{\Delta}$ and maps $\kappa\colon A \hookrightarrow X$ in $\mathscr{C}\colon \mathscr{A}$ to $\top_{\overline{\Delta}(\kappa)}$. Now, since $\underline{\Delta}$ is natural in each $\mathscr{C}\colon \mathscr{A}$, we have a commuting diagram:

$$\begin{tikzcd}
(D \mathbin{;} \overline{\Delta})(A) \arrow[r, "(D;\overline{\Delta})(\kappa)"] & (D \mathbin{;} \overline{\Delta})(X) \\
\mathbf{SOb}_{\mathscr{C}}(A) \arrow[u, "\underline{\Delta}(\mathscr{C})_A"] \arrow[r, "\kappa_*"] & \mathbf{SOb}_{\mathscr{C}}(X) \arrow[u, "\underline{\Delta}(\mathscr{C})_X"']
\end{tikzcd}$$

And thus, $D_*; (D \mathbin{;} \overline{\Delta}_\bigstar)(\kappa) = \top_{\overline{\Delta}(\kappa)} \geq \underline{\Delta}(\kappa)$.

Proposition 8.37. *For any subcategory $\mathscr{A} \hookrightarrow$ **Cat** of monic categories and monomorphism-preserving functors,* $\mathbf{SOb}\colon \mathscr{A} \to \overset{\hookrightarrow}{\mathbf{UBP}}$ *is contravariantly terminal in* $\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A})$ *(see Definition 5.4).*

Proof. The constructions of Proposition 8.23, Proposition 8.29, and Lemma 8.30 together furnish the morphism $(\mathbf{SOb}_{\mathscr{B}}, D_*)\colon (\mathscr{B}_\bigstar, D) \to (\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$, as was discussed in the preceding paragraphs. It remains to show that this is an initial object in the hom-category $\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A})\big((\mathscr{B}_\bigstar, D), (\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})\big)$. We have also for each other morphism $(\overline{\Delta}, \underline{\Delta})\colon (\mathscr{B}_\bigstar, D) \to (\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ constructed a 2-morphism $(\overline{\Delta}_\bigstar, \leq)$. For uniqueness, note that for any top-preserving natural transformation $\mathscr{I}:\mathbf{SOb}_{\mathscr{B}} \Rightarrow \overline{\Delta}$ and $\iota : A \hookrightarrow X$ in $\mathscr{B}_\bigstar$ it holds that the square

$$\begin{tikzcd}
\overline{\Delta}(A) \arrow[r, "\overline{\Delta}(\iota)"] & \overline{\Delta}(X) \\
\mathbf{SOb}_{\mathscr{B}}(A) \arrow[u, "\mathscr{I}_A"] \arrow[r, "\iota_*"] & \mathbf{SOb}_{\mathscr{B}}(X) \arrow[u, "\mathscr{I}_X"]
\end{tikzcd}$$

commutes, so that $\mathscr{I}_X(\iota) = \overline{\Delta}_{\bigstar,X}(\iota)$, since $\mathscr{I}_A$ preserves tops. Thus, $\mathscr{I} = \overline{\Delta}_\bigstar$, so that $\mathscr{I}$ is unique since 2-cells in **UBP** are unique. □

Since contravariantly terminal objects are not universal, Proposition 8.37 is still not a universal characterization of $\overset{\hookrightarrow}{\mathbf{UBP}}$. But let us assume that $(\mathscr{B}_\bigstar, D)$ is some other contravariantly terminal diamond.

1. The contravariant terminality of $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ toward $(\mathscr{B}_\bigstar, D)$ is witnessed by the pair $(\mathbf{SOb}_{\mathscr{B}}, D_*)$ with $D_* : \mathbf{SOb} \Rightarrow (D \mathbin{;} \mathbf{SOb}_{\mathscr{B}}) : \iota \mapsto D_{\mathscr{C}}(\iota)$.
2. The contravariant terminality of $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ toward $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb} \mathbin{;} \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}})$ is witnessed by a morphism $(\mathbf{id}_{\overset{\hookrightarrow}{\mathbf{UBP}}}, \{-\}) : \mathbf{SOb} \Rightarrow \mathbf{SOb} \mathbin{;} \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}$.
3. The contravariant terminality of $(\mathscr{B}_\bigstar, D)$ towards $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ is witnessed by a pair $(\overline{\mathbb{I}}, \underline{\mathbb{I}})$ where $\overline{\mathbb{I}} : \overset{\hookrightarrow}{\mathbf{UBP}} \to \mathscr{B}$ is initial and $\underline{\mathbb{I}} : D \Rightarrow \mathbf{SOb} \mathbin{;} \overline{\mathbb{I}}$ point-preserving and monic since $\mathscr{B}_\bigstar$ is 2-pseudomonic.
4. The initiality of $(\mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}, \{-\})$ towards $(\overline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}}, D_* \mathbin{;} (\underline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}}))$ is witnessed by a 2-cell $(-)^{\mathbb{I}} : \mathbf{SOb} \mathbin{;} \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}} \to \mathbf{SOb} \mathbin{;} \overline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}}$ which maps $\kappa : \mathbf{SOb}_{\mathscr{C}}(X)$ to $\overline{\mathbb{I}}(\kappa) \in \mathbf{SOb}_{\mathscr{B}}(\overline{\mathbb{I}}(\mathbf{SOb}_{\mathscr{C}}(X)))$.

All in all, this gives us a diagram of 2-cells of diamonds (so basically natural transformations in $\overset{\hookrightarrow}{\mathbf{UBP}}$):

$$\begin{tikzcd}
D \mathbin{;} \mathbf{SOb}_{\mathscr{B}} \arrow[r, "\underline{\mathbb{I}}_*"] & \mathbf{SOb} \mathbin{;} \overline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}} \\
\mathbf{SOb} \arrow[u, "\iota \mapsto D_{\mathscr{C}}(\iota)"] \arrow[r, "\iota \mapsto \{\iota\}"'] & \mathbf{SOb} \mathbin{;} \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}. \arrow[u, "\kappa \mapsto \overline{\mathbb{I}}(\kappa)"']
\end{tikzcd}$$

Figure 6

It follows from the definitions that this square commutes laxly, but a key step of our proof is to show that it commutes (weakly): from Remark 8.24 we know that $\{-\}$ has a retraction $\top_{(-)} : \kappa \mapsto \top_\kappa$, so that the bottom edge in Fig. 6 can also be transitioned along backwards, and so we also have a 2-cell $(-)^D$ from $\mathbf{SOb} \mathbin{;} \mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}$ to $\mathbf{SOb} \mathbin{;} \overline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}}$ given by $\kappa \mapsto \underline{\mathbb{I}}_*(\iota_\kappa)$, where ι_κ is the subobject of X corresponding to $\top_\kappa \in \mathbf{SOb}_{\mathscr{C}}(X)$ for some $\mathscr{C} : \mathscr{A}$. This gives us two 2-cells $(-)^{\mathbb{I}}, (-)^D$ from $(\mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}, \{-\})$ to $(\overline{\mathbb{I}} \mathbin{;} \mathbf{SOb}_{\mathscr{B}}, \underline{\mathbb{I}})$ where the modifications of both are pointwise inequalities in the diagram. But since $(\mathbf{SOb}_{\overset{\hookrightarrow}{\mathbf{UBP}}}, \{-\})$ is initial, these two have to be (weakly) the same, so that for each $\mathscr{C}, \iota : X \hookrightarrow Y$ in $\mathscr{C}$ it holds that $\underline{\mathbb{I}}_*(\iota) = \overline{\mathbb{I}}(\iota)$

in $\mathbf{SOb}_{\mathscr{B}}(\bar{\mathbb{I}}(\mathbf{SOb}_{\mathscr{C}}(X)))$. And since $\overset{\hookrightarrow}{\mathbf{UBP}}$ is locally a preorder, the hom-category of diamonds between these objects has at most one 2-cell, so the two 2-cells $(-)^{\mathbb{I}}$ and $(-)^{D}$ coincide.

Now, given $X : \mathscr{C}$ we have that $D_{\mathscr{C}}(\mathbf{id}_X) = \mathbf{id}_{D_{\mathscr{C}}(X)}$ and $\bar{\mathbb{I}}(\mathbf{id}_X) = \mathbf{id}_{\bar{\mathbb{I}}}(X)$, so that by definition of the covariant subobject preorder $\underline{\mathbb{I}}_*(\iota)$ has to be an isomorphism for $\underline{\mathbb{I}}_*(\mathbf{id}_X) = \bar{\mathbb{I}}(\mathbf{id}_X)$ to hold in $\mathbf{SOb}_{\mathscr{B}}(\bar{\mathbb{I}}(\mathbf{SOb}_{\mathscr{C}}(X)))$. Since this holds for each $X : \mathscr{C}$ we have $D = \mathbf{SOb} \mathbin{;} \bar{\mathbb{I}}$ up to the canonical isomorphism $\underline{\mathbb{I}}$. So each contravariantly terminal diamond is induced by some 2-functor $F : \overset{\hookrightarrow}{\mathbf{UBP}} \to \mathscr{B}_\bigstar$ to some 2-category $\mathscr{B}_\bigstar$. So to summarize:

Theorem 8.38. *Given a sub-2-category $\mathscr{A} \hookrightarrow \overset{\hookrightarrow}{\mathbf{Cat}}$, let $\mathbf{CT}(\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A}))$ be the sub-2-category of $\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A})$ on the contravariantly terminal objects, whose morphisms Δ are the pairs from Item 2 such that $\underline{\Delta}$ is a 2-isomorphism. Then $(\overset{\hookrightarrow}{\mathbf{UBP}}, \mathbf{SOb})$ is initial in $\mathbf{CT}(\overset{\hookrightarrow}{\blacklozenge}(\mathscr{A}))$.*

And thus we have finally derived a universal characterization of $\overset{\hookrightarrow}{\mathbf{UBP}}$.

Remark 8.39. It is surprisingly hard to tell which 2-functors out of $\overset{\hookrightarrow}{\mathbf{UBP}}$ induce a contravariantly terminal diamond. Each point-preserving 2-functor $F : \overset{\rightarrow}{\mathbf{UBP}} \to \mathscr{B}_\bigstar$ introduces a diamond $(\mathscr{B}_\bigstar, \mathbf{SOb} \mathbin{;} F)$ by Proposition 8.29 but to be contravariantly terminal, for each diamond $(\mathscr{D}, D)$ there has to exist an initial diamond-morphism from $(\mathscr{D}, D)$ to $(\mathscr{B}_\bigstar, \mathbf{SOb} \mathbin{;} F)$. Postcomposition with F induces a morphism given by $(\mathbf{SOb}_{\mathscr{D}} \mathbin{;} F, D_* \mathbin{;} F)$, but the question is: when is this morphism initial? For this, each diamond-morphism $\Delta : (\mathscr{D}, D) \to (\mathscr{B}_\bigstar, \mathbf{SOb} \mathbin{;} F)$ has to induce a unique 2-morphism $\mathbb{I} : (\mathbf{SOb}_{\mathscr{D}} \mathbin{;} F, D_* \mathbin{;} F) \Rightarrow \Delta$. Unpacking the definitions we find that Δ consists of a 2-functor $\overline{\Delta} : \mathscr{D} \to \mathscr{B}$ and a cell $\underline{\Delta} : (\mathscr{B}_\bigstar, \mathbf{SOb} \mathbin{;} F) \to (\mathscr{B}_\bigstar, D \mathbin{;} \overline{\Delta})$ in $\mathscr{A} \blacklozenge \mathscr{B}_\bigstar$. Unpacking further we find that $\underline{\Delta}$ consists for each $\mathscr{C} : \mathscr{A}$ of a cell $\underline{\Delta}_{\mathscr{C}} : (\mathscr{C}, \mathbf{SOb}_{\mathscr{C}} \mathbin{;} F) \to (\mathscr{C}, D(\mathscr{C}) \mathbin{;} \overline{\Delta})$ over $(\mathscr{C}, \mathbf{id}_{\mathscr{C}})$ and thus a 2-natural transformation $\mathbf{SOb}_{\mathscr{C}} \mathbin{;} F \Rightarrow \mathbf{id}_{\mathscr{C}} \mathbin{;} D(\mathscr{C}) \mathbin{;} \overline{\Delta} = D(\mathscr{C}) \mathbin{;} \overline{\Delta}$ in $\mathscr{B}^{\mathscr{C}}$. So $\underline{\Delta}$ has to be a 2-cell making this diagram commute:

$$\begin{tikzcd}
\overset{\hookrightarrow}{\mathbf{Cat}} \arrow[r, "D"] \arrow[dr, "\mathbf{SOb}"'] \arrow[r, phantom, shift right=3ex, "\Uparrow D_*"] & \mathscr{D} \arrow[d, "\mathbf{SOb}_{\mathscr{D}}"] \arrow[dr, "\overline{\Delta}"] & \\
 & \overset{\hookrightarrow}{\mathbf{UBP}} \arrow[r, "F"] & \mathscr{B}
\end{tikzcd}$$

On the other hand, $\mathbb{I}$ has to consist of a 2-natural transformation $\bar{\mathbb{I}} : \mathbf{SOb}_{\mathscr{D}} \mathbin{;} F \Rightarrow \overline{\Delta}$ and a modification $\underline{\mathbb{I}} : \underline{\Delta} \Rrightarrow (D_* \mathbin{;} F); (D \mathbin{;} \bar{\mathbb{I}})$. So for each $X : \mathscr{C} : \mathscr{A}$ the horn

$$\begin{tikzcd}
(\mathbf{SOb} \mathbin{;} F)(X) \arrow[d, "(D_* ; F)(X)"'] \arrow[dr, "\underline{\Delta}_X"] & \\
(D \mathbin{;} \mathbf{SOb}_{\mathscr{B}} \mathbin{;} F)(X) & (D \mathbin{;} \overline{\Delta})(X)
\end{tikzcd}$$

has to have a unique filling:

$$
\begin{tikzcd}
(\mathbf{SOb} \mathbin{;} F)(X) \arrow[d, "(D_*;F)(X)"'] \arrow[dr, "\Delta_X"] \arrow[dr, phantom, "\Leftarrow \mathbb{I}"'] & \\
(D \mathbin{;} \mathbf{SOb}_{\mathscr{B}} \mathbin{;} F)(X) \arrow[r, "(D;\mathbb{I})(X)"'] & (D \mathbin{;} \overline{\Delta})(X)
\end{tikzcd}
$$

8.4 The n-Diamond Conjecture

Let us close by making the proof of Theorem 8.38 into a conjecture. For this, note that in Example 3.49 and Definition 8.19 we use truncations for organizing the subobjects of 2-categories into a preorder. Accordingly, there is more than one minimal 2-functor $\mathscr{B} \to \mathbf{UBP}$. More precisely there are three of note: $\mathbf{SOb}_{f,\mathscr{B}}$, the preorder with dual 2-cells $\mathbf{SOb}_{f^{op},\mathscr{B}}$ in Item 2 of Definition 8.19, and the containment preorder $\mathbf{SOb}_{|f|,\mathscr{B}}$ where the 2-cells of Item 2 have to be 2-isomorphisms. But it stands to reason that if we don't truncate, i.e. use a "there exists"-clause in Item 2, that these lift to 3-functors into the 3-category of a 2-upward-bounded 2-preorders 2-$\mathbf{UBP}$. Conversely, the right definition of a 2-upward-bounded 2-preorder, or similar for other n, should become clear if we inspect the subobject structure of a 2-category or n-category, and the $n+1$-functors out of n-$\overset{\hookrightarrow}{\mathbf{UBP}}$ should again generate all contravariantly terminal n-diamonds. So it seems that the pattern goes:

1. 0-$\mathbf{SOb}\colon \mathbf{Set} \to \mathbf{Set}_{\blacklozenge\{\varnothing \hookrightarrow *\}}$: each part of an element of a set is either trivial or whole, so the functor 0-$\mathbf{SOb}_S$ just maps an element $s \in S$ to $*$ and the empty subobject to $\iota_\varnothing\colon \varnothing \hookrightarrow *$;
2. 1-$\mathbf{SOb} = \mathbf{SOb}$ is the case we studied;
3. n-$\mathbf{SOb}$ works recursively like $\mathbf{SOb}$ but trifurcated due to laxity and can be projected downward to (n-1)-$\mathbf{UBP}$;
4. n-$\mathbf{UBP}$ receives initial representations from all sufficiently monic n-categories;
5. n-$\overset{\hookrightarrow}{\mathbf{UBP}}$ generates contravariantly terminal diamonds.

9 Conclusion

Oh how the plainest germ can blossom into the most beautiful flower and bear the richest fruit! Definition 3.3 is of such elementarity that it might be mistaken for centipede mathematics, but this elementarity gives it a universality that makes it interact with notions of arbitrary complexity, in a way that requires 2-categorical language to capture. It has hopefully at this point been sufficiently established that **UBP** occupies a nexus within category theory and that the broken duality of smallness and density runs through all of mathematics. A natural question to ask is whether there are other notions we can similarly extend through preorders. Compactness comes to mind: using the notion of a compact object, [1][8], we can extend the notion from the opens of a topological space to arbitrary preorders, and using our calculus of tablets we can relativize how compactness is inferred from opens. This will become important in the next upcoming work, *Closed Geometric Logic*, where we investigate the, somewhat neglected, closed part of the bifurcation at the end of Section 6 specifically with regard to its logical properties and introduce *subdivisions* and the smallness-reducedness decomposition hinted at in the introduction. We will also study the geometry of abstract polytopes in closely related work, where we provide a universal characterization of them. Finally we will provide a general theory of composition.

A lot remains to be written.

> How can you see into my eyes like open doors? Leading you down into my core where I've become so numb. Without a soul my spirit's sleeping somewhere cold, until you find it there and lead it back home.
>
> Wake me up inside, wake me up inside; call my name and save me from the dark. Bid my blood to run, before I come undone, save me from the nothing I've become:
>
> **Bring me to life!**

- *Bring Me To Life*, Evanescence, shortened, with added punctuation.[9]

[8] Note that compact objects were called "finitely presentable" in that text.

Notes

[1]The unshortened quote is:

> Der Verstand bestimmt und hält die Bestimmungen fest; die Vernunft ist negativ und dialektisch, weil sie die Bestimmungen des Verstands in nichts auflöst; sie ist positiv, weil sie das Allgemeine erzeugt und das Besondere darin begreift. Wie der Verstand als etwas Getrenntes von der Vernunft überhaupt, so pflegt auch die dialektische Vernunft als etwas Getrenntes von der positiven Vernunft genommen zu werden. Aber in ihrer Wahrheit ist die Vernunft Geist, der höher als beides, verständige Vernunft oder vernünftiger Verstand ist. Er ist das Negative, dasjenige, welches die Qualität sowohl der dialektischen Vernunft als des Verstandes ausmacht; - er negiert das Einfache, so setzt er den bestimmten Unterschied des Verstandes; er löst ihn ebensosehr auf, so ist er dialektisch. Er hält sich aber nicht im Nichts dieses Resultates, sondern ist darin ebenso positiv und hat so das erste Einfache damit hergestellt, aber als Allgemeines, das in sich konkret ist; unter dieses wird nicht ein gegebenes Besonderes subsumiert, sondern in jenem Bestimmen und in der Auflösung desselben hat sich das Besondere schon mit bestimmt.

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