W*-category theory: a reappraisal

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Abstract

W*-categories were introduced by Ghez, Lima and Roberts as many-object versions of W*-algebras (von Neumann algebras). They appear for example in abstract harmonic analysis as categories of unitary representations and in algebraic quantum field theory. Here, we provide a systematic exposition of W*-category theory which serves both as a review and introduces many new results. Our approach is centred around the notion of *Hilbert presheaf*, which is the W*-categorical version of a presheaf and many-object version of a Hilbert module. We also follow a strategy of making comparisons with ordinary category theory. This highlights the surprising simplicity of W*-category theory (e.g. very W*-functor preserves all W*-limits that exist) and shows that W*-categories are highly rigid objects.

Finally, we use these results to study various bicategories, and we prove that the following bicategories are all equivalent:

(a) W*-algebras together with self-dual Hilbert bimodules and bounded bimodule morphisms.
(b) W*-algebras together with Connes correspondences and bounded intertwiners.
(c) Small W*-categories together with self-dual Hilbert profunctors and Hilbert transformations (both introduced here).
(d) W*-categories having direct sums, projection splittings and generators, together with W*-functors and bounded natural transformations.

We furthermore prove that they are compact closed bicategories.

Contents

1 Introduction

this is expected to be relevant to the representation theory of 2-groups [1], where certain W*-categories have been proposed as a definition of 2-Hilbert space [2]. In fact, it has been speculated that the bicategory just like the one that we study in this paper should be the natural setting for a workable definition of representation of 2-group beyond the finite-dimensional case [2, Section 5]. There are a priori (at least) three natural choices for how to define such a bicategory, and our main result can be seen as stating that all three natural choices are equivalent.

Our exposition of W*-category theory is not intended to be fully comprehensive. Most notable among the topics that we do not cover are the following:

- ▷ GNS representations for W*-categories,
- ▷ Modular theory [3, Section 3],

- ▷ Multiplicity theory, quasicontainment and quasiequivalence [3, Section 7].
- ▷ Operator space methods could provide additional insight, as convincingly argued by Blecher [4].
- ▷ A W*-categorical Deligne tensor product.
- ▷ Applications to abstract harmonic analysis or quantum field theory.

Other topics that could be developed by analogy with ordinary category theory, but which we will not touch upon, are the following:

- ▷ Monoidal W*-categories.
- ▷ Adjunctions.
- ▷ Kan extensions.

Rieffel showed in [5] how to recover a W*-algebra from its category of normal representations and the forgetful functor to Hilb and how to obtain a bimodule from a cocontinuous functor between categories of normal representations (Rieffel's Eilenberg–Watts theorem). In the following, we give an alternative proof of the Rieffel-Roberts version of the theorem, and we also characterize categories of normal representations of W*-algebras without assuming a forgetful functor.

We then give a version of the C*-algebraic theorem, differently formulated from Woronowicz's [6], which we then compare it to. We apply this to give an alternative definition of group C*-algebra in this setting. If there is time, we will approach the Eilenberg–Watts theorem in this setting, looking for a characterization of *strong* Morita equivalence.

Summary

Required background for reading

We assume textbook-level familiarity with the basic theory of W*-algebras (more commonly known as von Neumann algebras) and similarly with category theory up to and including bicategories.

Conventions and notation

Our C*-algebras are not assumed to be unital. Our sesquilinear inner products are conjugate linear in the first argument and linear in the second argument.

We frequently omit universal quantification over the objects in a category. For example, we may say that a certain statement "holds for all morphisms $f \in \mathsf{C}(X, Y)$" without saying "for all objects X and Y" at the same time.

- ▷ For a morphism f in a W*-category, $|f| := \sqrt{f^* f}$.
- ▷ Given a category C, we write $|\mathsf{C}|$ for its collection of objects.

Ban is the category of (small) Banach spaces and bounded linear maps of norm ≤ 1. Throughout, our ground field is $\mathbb{C}$.

Size issues

We adopt the standard solution of size issues given by Grothendieck universes.

All our W*-categories will be locally small by default, but we nevertheless emphasize local smallness separately for clarity. A general W*-category is assumed to be large; we do not consider cardinalities beyond large with the exception of the strict bicategory of W*-categories $\mathbb{W}^*\mathsf{CAT}$ from Definition 2.3.2, which is "very large".

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2 W*-categories: basic theory

This section provides a recap of the basic theory of W*-categories as developed in the original paper of Ghez, Lima and Roberts [3] as well as a detailed list of examples. Readers familiar with that paper should be able to skip this section and refer back to it if needed.

2.1 Definition and basic properties

W*-categories are many-object versions of W*-algebras. Before we state the definition, recall that a **category enriched in Banach spaces**, or Ban-category for short [7], is a category C in which every hom-set $\mathsf{C}(X, Y)$ comes equipped with a Banach space structure such that composition of morphisms is a bilinear operation of norm ≤ 1, meaning that for all $f : X \to Y$ and $g : Y \to Z$, we have

$$\|gf\| \leq \|g\| \, \|f\|.$$

C*-categories and W*-categories are then Ban-category with an extra involution suitably compatible with the Banach space structure, as follows.

Definition 2.1.1. *A **C*-category** is a* Ban*-enriched category* C *together with an identity-on-objects functor* $^* : \mathsf{C}^{\mathrm{op}} \to \mathsf{C}$ *that is conjugate linear on every hom-space, and such that for every* $f : X \to Y$*, we have:*

(i) $f^{**} = f$.
(ii) There is $g : X \to X$ *such that* $f^* f = g^* g$.
(iii) The C-identity*

$$\|f^* f\| \;=\; \|f\|^2$$

holds.[1]

[1] As with C*-algebras, it is enough to postulate this in inequality form $\|f^* f\| \geq \|f\|^2$, and the C*-identity follows by combining this inequality with $\|f^* f\| \leq \|f^*\| \, \|f\|$, since then we obtain $\|f^*\| \leq \|f\|$ and hence $\|f^*\| = \|f\|$.

*A **W*-category** is a C*-category* C *such that each hom-space* $\mathsf{C}(X,Y)$ *has a Banach space predual: there is a Banach space* $\mathsf{C}(X,Y)_*$ *together with an isometric isomorphism*

$$\mathsf{C}(X,Y) \cong (\mathsf{C}(X,Y)_*)^*. \tag{1}$$

Remark 2.1.2. This definition corresponds to the original definition in [3, Definitions 1.1 and 2.1] as follows. Axioms A1, A4 and A5 of [3] amount to a Ban-category. Our description of the involution * including (i) is their A2, our (ii) is their A3, our (iii) is their A6 and our (1) is their Definition 2.1. The second half of their A3, which states that $f = 0$ if and only if $f^* f = 0$, is redundant in the presence of the other axioms: $f^* f = 0$ implies $\|f\|^2 = \|f^* f\| = 0$ and therefore $f = 0$.

Remark 2.1.3. C*-categories can also be meaningfully considered and may be of interest without assuming the existence of identities [8, Section 3]. But since mere C*-categories are not of interest to us on this paper, we will not dwell on this further.

Remark 2.1.4. While we only work with Banach space enrichment in this paper, it is worth noting that enrichment in *operator spaces* would also be very natural to consider and could allow for better results in some respects [4].

Remark 2.1.5. If $\mathcal{A}$ is a unital C*-algebra, then the involution on $\mathcal{A}$ is uniquely determined by the Banach algebra structure of $\mathcal{A}$, since the unitary elements are exactly those elements of norm 1 that have an inverse also of norm 1; and this determines the anti-self-adjoint elements as exactly those whose one-parameter group of exponentials is unitary. As we will see in Remark 3.2.13, the analogous statement is true for W*-categories as well: the involution $*$ on a W*-category C is uniquely determined by its structure of Ban-enriched category.

Conversely, it is a standard fact that the norm on C*-algebra is uniquely determined by the $*$-algebra structure, and the C*-identity shows that this statement also holds for C*-categories. So overall, we can say that the norm and the involution on a W*-category uniquely determine each other.

Remark 2.1.6. The definition above is such that a W*-category can be **large** in the sense of having a proper class of objects, while it must be **locally small** in the sense that its hom-spaces are objects of Ban, which implies that they must be honest sets rather than proper classes.

As is the case in category theory quite generally, it can be convenient to have more flexibility with regards to the treatment of size issues [9, Section 1.1]. This is typically done by fixing a **Grothendieck universe** $\mathscr{U}$, or equivalently a set in the cumulative hierarchy $V_\kappa = \mathscr{U}$ for an inaccessible cardinal κ, assuming the existence of such [10, Section 8]. Then if $\mathsf{Ban}_\mathscr{U}$ denotes the locally $\mathscr{U}$-small category of Banach spaces with underlying sets in the universe $\mathscr{U}$, then a **locally $\mathscr{U}$-small W*-category** is defined as in Definition 2.1.1, but with $\mathsf{Ban}_\mathscr{U}$ in place of Ban. By universe enlargement, it therefore becomes possible to consider W*-categories that are not locally small (or not even locally $\mathscr{U}$-small).

However, we will not consider such extensions any further in this paper. For us, the relevant distinction is only between *small* and *large* W*-categories.

Example 2.1.7. By comparison of definitions, we see that a C*-category with one object is the same thing as a C*-algebra, given by the C*-algebra of endomorphisms of the unique object. Likewise, a W*-category with one object is the same as a W*-algebra (von Neumann algebra). For a W*-algebra N, we denote the associated single-object W*-category by $\mathfrak{B}N$.[2]

Example 2.1.8 ([3, Example 2.2])**.** Just as how Set is perhaps the most paradigmatic example of a category, the paradigmatic and most basic example of a W*-category is Hilb, the category of Hilbert spaces. Let us briefly describe its structure as W*-category.

Hilb has Hilbert spaces as objects and bounded linear maps as morphisms. If $f : \mathcal{H} \to \mathcal{K}$ is such a morphism, then its adjoint is the uniquely defined morphism $f^* : \mathcal{K} \to \mathcal{H}$ such that

$$\langle \psi, f\phi \rangle = \langle f^*\psi, \phi \rangle \qquad \forall \phi \in \mathcal{H}, \psi \in \mathcal{K}.$$

Furthermore, the norm of a morphism f is defined as its operator norm,

$$\|f\| \;\coloneqq\; \sup_{\phi \in \mathcal{H}\,:\,\|\phi\| \leq 1} \|f(\phi)\|.$$

By the elementary theory of Hilbert spaces, it is straightforward to show that this makes Hilb into a C*-category.

To conclude that Hilb is a W*-category, it thus remains to be shown that each hom-space $\mathsf{Hilb}(\mathcal{H}, \mathcal{K})$ has a predual. Indeed such a predual is given by $\mathcal{T}(\mathcal{K}, \mathcal{H})$, the space of trace class operators $\eta : \mathcal{K} \to \mathcal{H}$ considered as a Banach space with respect to the trace norm (Schatten 1-norm). To construct an isometric isomorphism $\mathsf{Hilb}(\mathcal{H}, \mathcal{K})^* \cong \mathcal{T}(\mathcal{K}, \mathcal{H})$, consider the pairing

$$\mathsf{Hilb}(\mathcal{H}, \mathcal{K}) \times \mathcal{T}(\mathcal{K}, \mathcal{H}) \longrightarrow \mathbb{C}$$

defined on bounded $f : \mathcal{H} \to \mathcal{K}$ and trace class $\eta : \mathcal{K} \to \mathcal{H}$ as

$$\mathrm{tr}_{\mathcal{H}}(\eta f) = \mathrm{tr}_{\mathcal{K}}(f\eta).$$

Due to the Hölder inequality for Schatten norms with exponents $p = 1$ and $q = \infty$, this pairing can equivalently be considered as a bounded map $\mathsf{Hilb}(\mathcal{H}, \mathcal{K}) \to \mathcal{T}(\mathcal{K}, \mathcal{H})^*$. It is a standard fact that this map is indeed an isometric isomorphism [11, Theorem 3.4.4(ii)].

Of course, also full subcategories of Hilb, such as the category of finite-dimensional Hilbert spaces, are W*-categories with respect to the induced structure. Further examples of W*-categories will be presented later on in Section 2.4.

Remark 2.1.9. To see how one can work with the involution in a C*-category C, let us show that two objects X and Y are unitarily isomorphic if and only if they are isomorphic.

[2] See e.g. also ncatlab.org/nlab/show/delooping for the algebraic topology origins of this notation.

Indeed if there is an isomorphism $f : X \to Y$, then also $f^* : Y \to X$ is an isomorphism with inverse $(f^*)^{-1} = (f^{-1})^*$. Then $|f| \coloneqq \sqrt{f^*f}$ is a positive invertible element of the C*-algebra $\mathsf{C}(X, X)$, and we can define $u \coloneqq f|f|^{-1} : X \to Y$. This is a unitary isomorphism between X and Y since

$$u^*u = |f|^{-1}f^*f|f|^{-1} = |f|^{-1}|f|^2|f|^{-1} = \mathrm{id}_X,$$
$$uu^* = f|f|^{-1}|f|^{-1}f^* = f(f^*f)^{-1}f^* = \mathrm{id}_Y,$$

as was to be shown.

As for W*-algebras, the canonical bilinear pairing between a hom-space $\mathsf{C}(X, Y)$ and a predual $\mathsf{C}(X, Y)_*$ equips the former with the **ultraweak topology**: it is defined as the weakest topology which makes all of the evaluation maps

$$\begin{aligned} \mathsf{C}(X, Y) &\longrightarrow \mathbb{C} \\ f &\longmapsto \eta(f) \end{aligned}$$

for $\eta \in \mathsf{C}(X, Y)_*$ continuous; in other words, it is exactly the weak-* topology on $\mathsf{C}(X, Y)$. By the standard duality theory of locally convex spaces [12, Section IV.1.2], the elements of $\mathsf{C}(X, Y)_*$ are in natural bijection with the ultraweakly continuous functionals $\mathsf{C}(X, Y) \to \mathbb{C}$, and in this way we identify the predual with a subspace of the dual,

$$\mathsf{C}(X, Y)_* \subseteq \mathsf{C}(X, Y)^*.$$

Ultraweakly continuous maps are also often called *normal*, although we will largely avoid this term.

The following construction is very useful for the development of basic properties of W*-categories, as it lets us reduce problems on W*-categories to problems on W*-algebras with standard solutions. This will let us in particular understand the ultraweak topology better. The following definition appears in slightly different form at [3, p. 86], while the terminology is ours.

Definition 2.1.10. *Let C be a W*-category with finitely many objects. Then the* ***linking W*-algebra***[3] *$L(\mathsf{C})$ is the $*$-algebra whose elements the matrices*

$$f_{--} = (f_{X,Y} : Y \to X)_{X,Y \in \mathsf{C}}$$

where multiplication is given by matrix multiplication,

$$(f_{--}g_{--})_{X,Z} = \sum_{Y \in \mathsf{C}} f_{X,Y}\, g_{Y,Z},$$

and with involution

$$(f^*_{--})_{X,Y} = f^*_{Y,X}.$$

[3] We used this term because of the closely related linking C*-algebras in the theory of Hilbert C*-modules [13].

We will explain the underscore notation formally and in general in Sections 3.1 and 3.6.

Lemma 2.1.11. *$L(\mathsf{C})$ is indeed a W*-algebra, and is such that for all $X, Y \in \mathsf{C}$, the canonical inclusion and projection maps*

$$\mathsf{C}(X,Y) \hookrightarrow L(\mathsf{C}) \twoheadrightarrow \mathsf{C}(X,Y)$$

are ultraweakly continuous.

Proof. It was shown in [3, p. 86] that the linking W*-algebra is indeed a C*-algebra by constructing a faithful representation (and this works already if C is merely a C*-category). For the first claim it thus remains to establish the existence of a predual.[4]

A functional in $L(\mathsf{C})^*$ is uniquely determined by how it acts on the subspaces $\mathsf{C}(X,Y)$, and therefore it can itself be represented as a matrix of functionals in $\mathsf{C}(X,Y)^*$. Let us show that the subspace of all such matrices with entries in the preduals $\mathsf{C}(X,Y)_*$, for which we write $L(\mathsf{C})_*$, is a predual of $L(\mathsf{C})$. Once we have shown this, the second statement is clear by construction. It thus only remains to be proven that the canonical map

$$L(\mathsf{C}) \longrightarrow (L(\mathsf{C})_*)^*$$

is an isometric isomorphism. To see that it is isometric on a given matrix $f_{\_\_} \in L(\mathsf{C})$, let $\eta \in L(\mathsf{C})^*$ be such that $\|\eta\| \leq 1$ and $|\eta(f_{\_\_})| = \|f_{\_\_}\|$. Then η can be written as a matrix of elements of the dual spaces $\mathsf{C}(X,Y)^*$, and the isometry claim follows since the $\mathsf{C}(X,Y)_*$ are weak-* dense in the $\mathsf{C}(X,Y)^*$. For surjectivity, it is enough to note that every element of $(L(\mathsf{C})_*)^*$ can also be represented as a matrix of functionals on the $\mathsf{C}(X,Y)_*$, and hence as an element of $L(\mathsf{C})$. □

For example, the linking W*-algebra lets us work with isometries and partial isometries as usual: a morphism $u : X \to Y$ is a **partial isometry** if it satisfies either of the following equivalent conditions:

- ▷ u^*u is a projection;
- ▷ uu^* is a projection;
- ▷ $uu^*u = u$;
- ▷ $u^*uu^* = u^*$.

Indeed in order to see that these are equivalent, simply apply the standard equivalence between these conditions for C*-algebras to the linking W*-algebra of the full W*-subcategory $\mathsf{C}|_{\{X,Y\}}$ of C on the two objects X and Y.

[4] The argument given at [3, Lemma 2.6] does not look correct to us. The attempted reasoning is that $L(\mathsf{C})$ is (non-isometrically) isomorphic to the Banach space direct sum of the $\mathsf{C}(X,Y)$, and each of these has a predual, which is apparently thought to imply the existence of a predual for $L(\mathsf{C})$. The problem with this line of argument is that the existence of a predual for an isomorphic Banach space does *not* imply the existence of a predual for the original space, not even for a commutative unital C*-algebra [14, Example 6.9.10].

We will define linking W*-algebras for (not necessarily finite) small W*-categories C in Example 3.6.22. In the current form with finite C, the linking W*-algebra is useful for the general theory of a W*-category C when applied to full W*-subcategories on finitely many objects $X_1, \ldots, X_n$, which we denote by $\mathsf{C}|_{X_1,\ldots,X_n}$.

We now apply this machinery to prove some further general properties of W*-categories of an analytic nature. The first one is the following W*-categorical generalization of Sakai's theorem on the uniqueness of the predual of a W*-algebra.

Proposition 2.1.12 ([3, p. 88])**.** *If C is a W*-category, then every hom-space $\mathsf{C}(X, Y)$ has a predual that is unique as a subspace of $\mathsf{C}(X, Y)^*$.*

Proof. Arguing in terms of $L(\mathsf{C}|_{X,Y})$, Lemma 2.1.11 shows that the elements of $\mathsf{C}(X, Y)_*$ correspond to those functionals whose canonical extension to $L(\mathsf{C}|_{X,Y})$ is ultraweakly continuous. Therefore the claim follows by Sakai's theorem. □

This uniqueness result helps illustrate why Definition 2.1.1 is natural despite not requiring any kind of compatibility between the preduals of different hom-spaces. In fact, the following Proposition 2.1.15 provides such compatibility statements, and the fact that these hold automatically can be thought of as explained by the uniqueness of the preduals.

Example 2.1.13. For a Hilbert space $\mathcal{H}$, the ultraweak topology on $\mathsf{Hilb}(\mathbb{C}, \mathcal{H}) \cong \mathcal{H}$ coincides with the usual weak topology on $\mathcal{H}$.

Example 2.1.14. For a single-object W*-category $\mathfrak{B}N$ (Example 2.1.7), the ultraweak topology on the single hom-space coincides with the ultraweak topology from W*-algebra theory, which in turn coincides with the weak operator topology (of any faithful representation) on norm-bounded subsets [15, Lemma 2.5].

Proposition 2.1.15. *For every W*-category C and objects $X, Y, Z \in \mathsf{C}$:*

(i) The involution map

$$\mathsf{C}(X, Y) \longrightarrow \mathsf{C}(Y, X)$$

is ultraweakly continuous.

(ii) The composition maps

$$\mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \longrightarrow \mathsf{C}(X, Z)$$

are ultraweakly continuous in each variable separately (but generally not jointly).

Proof. These follow by Lemma 2.1.11 and the corresponding statements for W*-algebras, which are standard. □

Corollary 2.1.16. *If C is a W*-category, then the preduals form a functor*

$$\mathsf{C}(-,-)_* : \mathsf{C} \times \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Ban}.$$

Note that the variance in each argument is exactly opposite to that of the hom-functor $\mathsf{C}(-,-) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{Ban}$.

Proof. In order to define the action on morphisms, let $f : X \to X'$ and $g : Y' \to Y$. Then we obtain a map

$$\begin{aligned} \mathsf{C}(X,Y)_* &\longrightarrow \mathsf{C}(X',Y')_* \\ \eta &\longmapsto \eta(g \circ - \circ f), \end{aligned}$$

where the right-hand side is indeed in $\mathsf{C}(X',Y')_*$ since it is an ultraweakly continuous functional on $\mathsf{C}(X',Y')$ by Proposition 2.1.15(ii). It is straightforward to see that this satisfies functoriality in both arguments. ☐

We will often characterize preduals by describing dense subspaces thereof. The following observation shows that this is enough as far as characterizing the ultraweak topology is concerned. As before, we consider the predual $\mathsf{C}(X,Y)_*$ as a subspace of $\mathsf{C}(X,Y)^*$.

Lemma 2.1.17. *Let $V \subseteq \mathsf{C}(X,Y)_*$ be a subspace that is weak-* dense. Then it is also norm dense, and the weak topology on $\mathsf{C}(X,Y)$ induced by V is exactly the ultraweak topology.*

Proof. Let $\overline{V}$ denote the norm closure. Then evaluation defines a linear map

$$\mathsf{C}(X,Y) \longrightarrow \overline{V}^*.$$

This map is an isometry by the assumed weak-* density. Surjectivity follows since the Hahn-Banach theorem lets us extend every norm continuous functional $\overline{V} \to \mathbb{C}$ to a norm continuous functional $\mathsf{C}(X,Y)_* \to \mathbb{C}$, and each functional of the latter type is given by evaluation on an element of $\mathsf{C}(X,Y)$ by assumption.

Hence $\overline{V}$ is a predual of $\mathsf{C}(X,Y)$, and we conclude the claimed $\overline{V} = \mathsf{C}(X,Y)_*$ by the uniqueness of the predual from Proposition 2.1.12. For the second claim, it is enough to note that the weak topology induced from V coincides with the weak topology induced from $\overline{V}$, which is straightforward. ☐

We will often consider infinite sums of parallel morphisms in a W*-category. For these, the following auxiliary statement will be used frequently.

Lemma 2.1.18. *Let $(f_i : X \to Y)_{i \in I}$ be a family of morphisms in a W*-category. If the partial sums $\sum_{i \in F} f_i$ for finite $F \subseteq I$ are uniformly bounded in norm, then $\sum_{i \in I} f_i$ converges absolutely ultraweakly.*

As the proof shows, this is actually a general fact about the weak-* topology on dual Banach spaces.

Proof. By the Banach-Alaoglu theorem, the net consisting of all finite partial sums has an ultraweak cluster point. To prove ultraweak convergence, it is therefore enough to show that this net is ultraweakly Cauchy. But this amounts to showing that for every $\eta \in \mathsf{C}(X,Y)_*$, the net of partial sums

$$\sum_{i\in F} \eta(f_i)$$

is Cauchy. Since this net is a net of numbers that is uniformly bounded, the claim follows from the elementary fact that a series of numbers with uniformly bounded partial sums converges, e.g. as a consequence of the Riemann rearrangement theorem. The latter also shows the absolute convergence. □

2.2 W*-functors

The natural notion of homomorphism between C*-algebras is the notion of $*$-homomorphism, while for W*-algebras it is usually desirable to require ultraweak continuity in addition. There is an entirely analogous distinction between $*$-functors and W*-functors.

When C and D are C*-categories, then a $*$-functor $F : \mathsf{C} \to \mathsf{D}$ in the sense of [3] is a functor that is linear on the hom-spaces and commutes with the involution, meaning that $F(f^*) = F(f)^*$ for all morphisms f in C. As with $*$-homomorphisms between C*-algebras, every $*$-functor between C*-categories is contractive, or equivalently Ban-enriched: for every morphism f in C, we have $\|F(f)\| \leq \|f\|$ by the C*-identity and the fact that this contractivity holds for $*$-homomorphisms.

Definition 2.2.1. *Given W*-categories* C *and* D*, a* ***W*-functor*** $\mathsf{C} \to \mathsf{D}$ *is a $*$-functor whose action on morphisms*

$$\mathsf{C}(X,Y) \longrightarrow \mathsf{D}(FX,FY), \qquad f \longmapsto F(f)$$

is ultraweakly continuous for all $X, Y \in \mathsf{C}$.

Remark 2.2.2. The original definition of Ghez, Lima and Roberts only required ultraweak continuity on endomorphism the hom-spaces $\mathsf{C}(X,X)$, but they proved this to be equivalent to Definition 2.2.1 [3, Proposition 2.12(a)⇔(b)]. Since we have not found their weakening to be useful in practice, we have adopted the more intuitive requirement of ultraweak continuity on all hom-spaces.

A **W*-subcategory** is a subcategory $\mathsf{D} \subseteq \mathsf{C}$ that is closed under the involution and such that its hom-sets $\mathsf{D}(X,Y) \subseteq \mathsf{C}(X,Y)$ are ultraweakly closed linear subspaces. The fact that this is a W*-category as well is not obvious, since it needs to be shown that the normed spaces $\mathsf{D}(X,Y)$ have preduals. The following result, of which we will make frequent use, shows that this is indeed the case.

Lemma 2.2.3. *Let* C *be a W*-category and* D *any category. Let* $F : \mathsf{D} \to \mathsf{C}$ *be any faithful functor such that the sets*

$$F(\mathsf{D}(X,Y)) \subseteq \mathsf{C}(F(X),F(Y)) \tag{2}$$

are ultraweakly closed linear subspaces that are also closed under $*$. Then D *is a W*-category with respect to the structure induced from* C*, and the ultraweak topology on every hom-space* $\mathsf{D}(X, Y)$ *is the one induced from* $\mathsf{C}(X, Y)$*.*

So in this situation, F is a W*-functor by construction.

Proof. Since the norm topology refines the ultraweak topology, it is clear that the images (2) are norm-closed. Therefore the induced structures make D into a C*-category. The existence of a predual and the statement on the ultraweak topology follow by the general duality theory of Banach spaces, which shows that $\mathsf{D}(X, Y)_*$ is the quotient Banach space of $\mathsf{C}(X, Y)_*$ with respect to the subspace of those predual elements that vanish on $\mathsf{D}(X, Y)$. □

The notion of faithful functor in W*-category theory is played by W*-functors which are *isometric* embeddings on hom-spaces. Due to the following basic observation, this is actually no stronger than faithfulness.

Lemma 2.2.4. *A W*-functor* $F : \mathsf{C} \to \mathsf{D}$ *is faithful if and only if it is isometric on hom-spaces.*

As the following proof shows, this actually holds for $*$-functors between C*-categories in general.

Proof. The nontrivial direction is from faithful to isometric. For $f : X \to Y$ in C and F faithful, the equation $\|F(f)\| = \|f\|$ follows by the C*-identity $\|f\|^2 = \|f^* f\|$ and the standard fact that an injective $*$-homomorphism is isometric, which applies to the action of F on endomorphisms given by $\mathsf{C}(X, X) \longrightarrow \mathsf{D}(FX, FX)$. □

It is straightforward to see that the composition of two W*-functors is a W*-functor again. Therefore we can consider the category of (small) W*-categories and W*-functors.

Notation 2.2.5. *We write* $\mathbb{W}^*$ *for the category of small W*-categories and W*-functors.*

Of course, category theorists will want to consider a *2-category* of W*-categories and W*-functors. We will turn to 2-morphisms in the subsequent subsection.

Especially in Section 4, we will encounter the following variation on W*-functors.

Definition 2.2.6. *Given W*-categories* B*,* C *and* D*, a **W*-bifunctor***

$$F : \mathsf{B} \times \mathsf{C} \longrightarrow \mathsf{D}$$

is a functor which is a W-functor in each argument separately.*

In particular, a W*-bifunctor acts bilinearly on hom-spaces. So although $\mathsf{B} \times \mathsf{C}$ can be considered as a W*-category in its own right (Example 2.4.9), a W*-bifunctor is not a W*-functor except in degenerate cases.

Example 2.2.7. The formation of tensor products of Hilbert spaces is a W*-bifunctor

$$\mathsf{Hilb} \times \mathsf{Hilb} \longrightarrow \mathsf{Hilb}.$$

We will construct some vast generalizations of this construction in Sections 4.3 and 4.4.

2.3 Bounded natural transformations

The notion of a natural transformation between functors is a fundamental concept in category theory. Here is the W*-categorical version (which makes sense for C*-categories just the same).

Definition 2.3.1. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be W*-functors. Then a* ***bounded natural transformation*** *is a natural transformation $\alpha : F \to G$ such that*

$$\sup_{X \in \mathsf{C}} \|\alpha_X\| \ < \ \infty.$$

It is easy to see that bounded natural transformations $F \to G$ and $G \to H$ can be composed, and this composition defines a functor category

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}).$$

As we will see in Example 2.4.11, this is a W*-category again if C is small. Moreover, given W*-categories C, D and E, we also have a horizontal composition functor

$$\mathsf{Fun}(\mathsf{D}, \mathsf{E}) \times \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \longrightarrow \mathsf{Fun}(\mathsf{C}, \mathsf{E})$$

defined in the obvious way analogous to ordinary category theory, and the analogous arguments as used there show that this defines a strict 2-category.

Definition 2.3.2. *The* ***strict bicategory of W*-categories*** $\mathbb{W}^*\mathsf{CAT}$ *has:*

- ▷ *Locally small W*-categories as objects;*
- ▷ *W*-functors as morphisms;*
- ▷ *Bounded natural transformations as 2-morphisms;*
- ▷ *The obvious composition operations.*

Its full sub-2-category of small W-categories is denoted* $\mathbb{W}^*\mathsf{cat}$.

Note that $\mathbb{W}^*\mathsf{CAT}$ is a very large 2-category. As in every 2-category, it has an internal notion of *equivalence* of objects, defined as a morphism $F : \mathsf{C} \to \mathsf{D}$ such that there exists a morphism $G : \mathsf{D} \to \mathsf{C}$ with $GF \cong \mathrm{id}_{\mathsf{C}}$ and $FG \cong \mathrm{id}_{\mathsf{D}}$. As in ordinary category theory, these **W*-equivalences** can be characterized in more concrete terms, and the characterization surprisingly carries over.

Proposition 2.3.3. *For a W*-functor $F : \mathsf{C} \to \mathsf{D}$, the following are equivalent:*

(i) F is a W-equivalence;*
(ii) F is an equivalence;
(iii) F is fully faithful and essentially surjective.

Proof. The downward implications are clear. Assuming that F is fully faithful and essentially surjective, we construct an essential inverse $G : \mathsf{D} \to \mathsf{C}$ as follows. For $X \in \mathsf{D}$, let $GX \in \mathsf{C}$ be any object with $FGX \cong X$. Then by Remark 2.1.9, there even is a unitary isomorphism $\alpha_X : FGX \cong X$. Following the proof in ordinary category theory [16, p. 94] shows that G extends in a unique way to a functor $\mathsf{D} \to \mathsf{C}$ that is an inverse equivalence to F. The fact that G is a W*-functor as well now follows from the fact that F is fully faithful together with the uniqueness of the preduals. Finally, since the natural isomorphism $\alpha : FG \cong \mathrm{id}_{\mathsf{D}}$ has unitary components, the same applies to the induced natural isomorphism $GF \cong \mathrm{id}_{\mathsf{C}}$, which is therefore bounded as well. □

2.4 Further examples of W*-categories

Here, we present all further examples of W*-categories that will appear in this paper. Let us start with a trivial examples.

Example 2.4.1. The **discrete W*-category** on a set I has I as its collection of objects, all with endomorphism W*-algebra $\mathbb{C}$, and between any two distinct objects only the zero morphism.

The next few examples are rather classical, and these might be the examples that most readers will be interested in.

Example 2.4.2 (Normal representations of a W*-algebra)**.** If C is a W*-category and N is a W*-algebra, then a **normal representation** of N in C is a pair (X, π) consisting of an object $X \in \mathsf{C}$ and a normal $*$-homomorphism $\pi : N \to \mathsf{C}(X, X)$. An **intertwiner** between two normal representations (X_1, π_1) and (X_2, π_2) is a morphism $f : X_1 \to X_2$ in C such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1(a)}\downarrow & & \downarrow{\scriptstyle \pi_2(a)} \\ X_1 & \xrightarrow{f} & X_2 \end{array} \tag{3}$$

commutes in C for every $a \in N$. It is straightforward to see that this defines a category $\mathsf{NRep}(N, \mathsf{C})$ with normal representations as objects and intertwiners as morphisms. By construction, we have a faithful functor $\mathsf{NRep}(N, \mathsf{C}) \to \mathsf{C}$. Its hom-set images are clearly linear subspaces of the hom-spaces in C. They are also closed under $*$, since if f is an intertwiner as above, then we also have the diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{f^*} & X_2 \\ {\scriptstyle \pi_1(a^*)}\uparrow & & \uparrow{\scriptstyle \pi_2(a^*)} \\ X_1 & \xleftarrow{f^*} & X_2 \end{array}$$

which implies that f^* is also an intertwiner upon replacing every a by a^*. By Lemma 2.2.3, $\mathsf{NRep}(N, \mathsf{C})$ therefore becomes a W*-category if we can show that the spaces of intertwiners are ultraweakly closed. To this end, it is enough to show that the subspace of morphisms f which make (3) commute for a particular $a \in N$ is ultraweakly closed. But this is a consequence of the respective ultraweak continuity of composing with $\pi_1(a)$ and $\pi_2(a)$. Therefore $\mathsf{NRep}(N, \mathsf{C})$ is indeed a W*-category.

Since the case $\mathsf{C} = \mathsf{Hilb}$ is of particular significance, we also use the shorthand notation $\mathsf{NRep}(N) := \mathsf{NRep}(N, \mathsf{Hilb})$.

Clearly the same argument as above goes through if instead of N we merely have C*-algebra A and we consider representations $\rho : A \to \mathsf{C}(X, X)$. But this is already covered by the above, since the double dual A^{**} is a W*-algebra such that $*$-homomorphisms $A \to \mathsf{C}(X, X)$ are in natural bijection with ultraweakly continuous $*$-homomorphisms $A^{**} \to \mathsf{C}(X, X)$; and the fact that $A \subseteq A^{**}$ is ultraweakly dense also implies that the intertwiners correspond exactly. Thus considering the W*-category $\mathsf{Rep}(A, \mathsf{C})$ of representations of a C*-algebra A in any W*-category C is already covered by the above: there is a canonical isomorphism of W*-categories $\mathsf{NRep}(A^{**}, \mathsf{C}) \cong \mathsf{Rep}(A, \mathsf{C})$.

Example 2.4.3 (Representations of a topological group)**.** Let G be a topological group and C a W*-category. A **unitary representation** of G in C then consists of $X \in \mathsf{C}$ together with a group homomorphism $\rho : G \to \mathcal{U}(X)$ that is continuous with respect to the ultraweak topology, where $\mathcal{U}(X) \subseteq \mathsf{C}(X, X)$ is the group of unitary endomorphisms of X. For example if $\mathsf{C} = \mathsf{Hilb}$, then this specializes to the usual notion of continuous unitary representation on a Hilbert space $\mathcal{H}$, since then $\rho : G \to \mathcal{U}(\mathcal{H})$ is ultraweakly continuous if and only if it is strongly continuous [17, Section 13.1].

Even for general C, these representations form a W*-category with respect to the obvious notion of intertwiner analogous to Example 2.4.2, where the existence of preduals follows in the same way from Lemma 2.2.3. In fact, this W*-category can equivalently be defined as $\mathsf{NRep}(W^*(G), \mathsf{C})$, where $W^*(G)$ denotes the universal enveloping W*-algebra of G [18].

The possibility of considering W*-categories of representations in any W*-category C applies generally to arbitrary $*$-algebras equipped with a suitable family of seminorms, and these W*-categories then coincide with the W*-categories of normal representations of the universal enveloping W*-algebra [18, Theorem 3.9]. We do not consider this further since it does not produce any genuinely new examples of W*-categories.

There are many other types of mathematical objects for which representations on Hilbert spaces can be meaningfully considered and for which one therefore obtains a W*-category of representations. Let us consider operator systems as one further example, for which one again can consider representations in an arbitrary W*-category, and then proceed to other types of examples.

Example 2.4.4 (Representations of an operator system)**.** If C is a W*-category and V is an operator system [19], then a **representation** of V in C is a pair (X, π) consisting of an object $X \in \mathsf{C}$ and a completely positive unital map $\pi : V \to \mathsf{C}(X, X)$. An intertwiner of representations is again defined in the same way as in Example 2.4.2. Another application of Lemma 2.2.3 then shows that representations of V in C together with their intertwiners also form a W*-category in the obvious way.

For the following two examples, we refer to Appendix A for background on Hilbert modules.

Example 2.4.5 (Hilbert modules)**.** For a C*-algebra A, let us write $\mathsf{HilbMod}(A)$ for the category of self-dual A-Hilbert modules with bounded A-linear maps, or equivalently adjointable maps, as morphisms. This is a C*-category: the involution is the obvious one given by taking adjoints, and the norm on hom-spaces is the operator norm as usual, which is easily seen to be complete. The C*-identity for a morphism $t : X \to Y$ can be verified in the obvious manner [20, p. 8]: the definition of the norm and the Cauchy-Schwarz inequality in the form (85) give

$$\|tx\|^2 \;=\; \|\langle tx, tx\rangle\| \;=\; \|\langle t^*tx, x\rangle\| \;\leq\; \|t^*t\| \cdot \|x\|^2,$$

which is enough. The fact that t^*t is positive in $L(X,X)$ follows from [20, Lemma 4.1]. Hence we are dealing with a C*-category. In general, this C*-category is not a W*-category. For example if A is a unital C*-algebra, then it is an object in the category itself with endomorphism algebra $L(A,A) = A$, which need not have a predual.

If N is a W*-algebra, then $\mathsf{HilbMod}(N)$ is even a W*-category. To prove this, it remains to be shown that every $L(X,Y)$ has a predual; for $X = Y$, this result is due to Paschke [21, Proposition 3.10], and we now adapt his proof to the general case. So let $L(X,Y)_*$ be the closed subspace of $L(X,Y)^*$ spanned by all functionals of the form

$$\mathrm{ev}^{\eta}_{x,y} \;:\; \begin{aligned} L(X,Y) &\longrightarrow \mathbb{C} \\ t &\longmapsto \eta(\langle y, tx\rangle) \end{aligned}$$

for $x \in X$ and $y \in Y$ as well as $\eta \in N_*$. Then the canonical map

$$L(X,Y) \longrightarrow (L(X,Y)_*)^* \tag{4}$$

is an isometry. Indeed it having norm ≤ 1 is clear. Conversely, for given $t \in L(X,Y)$, choose $x \in X$ with $\|x\| \leq 1$ such that $\|tx\| \geq \|t\| - \varepsilon$, put $y := tx$ and also choose $\eta \in N_*$ with $\|\eta\| \leq 1$ such that the norm of $\langle tx, tx\rangle \in N$ is also attained up to ε. This means that

$$|\eta(\langle y, tx\rangle)| \;=\; |\eta(\langle tx, tx\rangle)| \;\geq\; \|\langle tx, tx\rangle\| - \varepsilon \;=\; \|tx\|^2 - \varepsilon \;\geq\; \|y\|\,(\|t\| - \varepsilon) - \varepsilon.$$

The isometry claim now follows since ε was arbitrary.

It remains to be shown that the map (4) is surjective. Every element of $(L(X,Y)_*)^*$ defines a map

$$\mathcal{T} \;:\; X \times Y \times N_* \longrightarrow \mathbb{C}$$

by restriction to the $\mathrm{ev}^{\eta}_{x,y} \in L(X,Y)^*$. This map is N-linear in its first argument, N-conjugate linear in its second argument, $\mathbb{C}$-linear in its third argument, and jointly bounded. To finish the argument, it is enough to find $t \in L(X,Y)$ such that

$$\mathcal{T}(x,y,\eta) \;=\; \eta(\langle y, tx\rangle) \qquad \forall x, y, \eta.$$

Indeed for given x and y, varying η allows us to consider $\mathcal{T}$ as a map $X \times Y \to (N_*)^* = N$, which we also denote by $\mathcal{T}$ for simplicity. Since for fixed x, the map $y \mapsto \mathcal{T}(x,y)$ is

N-conjugate linear and bounded, the assumed self-duality of Y means that there is a unique $tx \in Y$ such that $\mathcal{T}(x, y) = \langle y, tx \rangle$ for all y.

To see that the thus defined t belongs to $L(X, Y)$, it is enough to show that t is adjointable. But this holds because, by the same construction applied with X and Y interchanged, we can also construct a map $t^* : Y \to X$ satisfying the relevant equation

$$\langle t^* y, x \rangle \;=\; \mathcal{T}(x, y) \;=\; \langle y, tx \rangle.$$

Therefore $L(X, Y)_*$ is indeed a predual for $L(X, Y)$, and the category of self-dual N-Hilbert modules is a W*-category. By Proposition 2.1.12, the theory of W*-categories also let us conclude that the predual $L(X, Y)_*$ is unique as a subspace of $L(X, Y)^*$.

In particular, the canonical isometric isomorphism $L(N, X) \cong X$ shows that every self-dual N-Hilbert module itself has a predual, which again is a result of Paschke [21, Proposition 3.8]. Setting $Y = X$ shows that every endomorphism C*-algebra $L(X, X)$ is a W*-algebra [21, Proposition 3.10].

Example 2.4.6 (Hilbert bimodules)**.** If M and N are W*-algebra, then we also have a notion of N-Hilbert M-module (Definition A.1.8). These form a W*-category $\mathsf{HilbBiMod}(M, N)$ that we now turn to. Its objects are normal self-dual N-Hilbert M-modules, and the morphisms $X \to Y$ are bounded bimodule maps, or equivalently maps that are adjointable as maps of N-Hilbert modules and in addition preserve the left action by M.

To see in which way $\mathsf{HilbBiMod}(M, N)$ is a W*-category, it is easiest to note that its definition amounts to defining it as a category of normal representations,

$$\boxed{\mathsf{HilbBiMod}(M, N) \;:=\; \mathsf{NRep}(M, \mathsf{HilbMod}(N)).}$$

Therefore the W*-category structure is implied by Examples 2.4.2 and 2.4.5.

Definition 2.4.7. *Let M and N be W*-algebras. A* ***Connes correspondence*** *from M to N is a triple $(\mathcal{H}, \ell, \rho)$ consisting of a Hilbert space $\mathcal{H}$ and normal representations*

$$\alpha : M \to \mathcal{B}(\mathcal{H}), \qquad \beta : N \to \mathcal{B}(\mathcal{H})$$

such that $[\alpha(a), \beta(b)] = 0$ for all $a \in M$ and $b \in N$.

Example 2.4.8 (Connes correspondences)**.** Let again M and N be W*-algebras. Then there is a W*-category with Connes correspondences as objects and bounded bimodule maps as morphisms. Similar to the previous example, this amounts to the definition

$$\boxed{\mathsf{Connes}(M, N) \;:=\; \mathsf{NRep}(M, \mathsf{NRep}(N^{\mathrm{op}})).}$$

Recall that the **Guichardet–Dauns tensor product** $M \underline{\otimes} N$ defines a new W*-algebra which satisfies the obvious universal property of a tensor product, in that it classifies pairs of normal unital $*$-homomorphisms out of M and N into another

W*-algebra with commuting images [22, 23].[5] This[6] shows that there is a canonical isomorphism

$$\boxed{\mathsf{Connes}(M,N) \;\cong\; \mathsf{NRep}(M\underline{\otimes}N).} \tag{5}$$

Let us now present some ways to construct new W*-categories from given ones. For the following example, recall that we write $\mathbb{W}^*$ for the category of small W*-categories and W*-functors.

Example 2.4.9 (Products of W*-categories)**.** Let I be a set and $(\mathsf{C}_i)_{i\in I}$ a family of W*-categories and I a set. Then we construct a W*-category $\prod_{i\in i} \mathsf{C}_i$ in which the objects are families $(X_i)_{i\in I}$ of objects $X_i \in \mathsf{C}_i$, and a morphism $(X_i)_{i\in I} \to (Y_i)_{i\in I}$ is a family of morphisms $(f_i : X_i \to Y_i)_{i\in I}$ that is bounded,

$$\|(f_i)_{i\in I}\| \;\coloneqq\; \sup_{i\in I} \|f_i\| \;<\; \infty.$$

These morphisms compose componentwise in the obvious way, and this defines a Ban-enriched category with respect to the componentwise vector space structure. Using also the componentwise involution $(f_i)^* \coloneqq (f_i^*)$, the C*-identity clearly holds. So to get a W*-category, it only remains to be shown that the hom-spaces have preduals. Indeed the predual of $\big(\prod_{i\in I} \mathsf{C}_i\big)((X_i)_{i\in I}, (Y_i)_{i\in I})$ can be constructed as the ℓ^1-direct sum of the preduals,

$$\left(\prod_{i\in I} \mathsf{C}_i\right)((X_i)_{i\in I}, (Y_i)_{i\in I})_* \;\coloneqq\; \left\{ (\eta_i)_{i\in I} : \prod_{i\in I} \mathsf{C}_i(X_i, Y_i)_* \;\middle|\; \sum_{i\in I} \|\eta_i\| < \infty \right\}, \tag{6}$$

turned into a Banach space with respect to the ℓ^1-norm $\|\eta\| = \sum_{i\in I} \|\eta_i\|$. This is indeed the predual by the general fact that the dual of an ℓ^1-direct sum of Banach spaces is the ℓ^∞-direct sum of their duals. Therefore $\prod_{i\in I} \mathsf{C}_i$ is indeed a W*-category with respect to the componentwise operations introduced above.

For every $j \in I$, the canonical projection functor $\prod_{i\in I} \mathsf{C}_i \to \mathsf{C}_j$ is ultraweakly continuous on hom-spaces due to the canonical inclusion of preduals going the other way. These projections turn $\prod_{i\in I} \mathsf{C}_i$ into the **product W*-category** of the family $(\mathsf{C}_i)_{i\in I}$, since the relevant universal property holds: if D is any other W*-category D, the W*-functors $F : \mathsf{D} \to \prod_{i\in I} \mathsf{C}_i$ are in canonical bijection with the families of W*-functors $(F_i : \mathsf{D} \to \mathsf{C}_i)_{i\in I}$ through composition with the projections, since for a given such family, the induced $*$-functor $\mathsf{D} \to \prod_{i\in I} \mathsf{C}_i$ is also ultraweakly continuous on hom-spaces as a consequence of the fact that the finitely supported families $(\eta_i)_{i\in I}$ are dense in (6). In particular if the C_i are small, then $\prod_{i\in I} \mathsf{C}_i$ is the categorical product in $\mathbb{W}^*$.

[5] Although Guichardet's version of the universal property works again without assuming unitality of the $*$-homomorphisms, it restricts to the relevant universal property on unital $*$-homomorphisms [24].

[6] … together with the fact that the elementary tensors span an ultraweakly dense subspace of $M\underline{\otimes}N$ [23, Corollary 4.9].

In the special case where all the C_i contain just a single object, this product W*-category specializes to the usual product of W*-algebras [22, Proposition 3.1] with the corresponding universal property.[7]

Another special case is that the family of categories is constant and consists of a single category $\mathsf{C}_i = \mathsf{C}$. In this situation we also write

$$\ell^\infty(I, \mathsf{C}) \coloneqq \prod_{i \in I} \mathsf{C}$$

for the resulting product category. We then have a canonical "diagonal" W*-functor

$$\mathsf{C} \longrightarrow \ell^\infty(I, \mathsf{C}) \tag{7}$$

given by mapping every object and morphism to the associated constant family.

Example 2.4.10 (Coproducts of W*-categories)**.** Let $(\mathsf{C}_i)_{i \in I}$ be a family of W*-categories as in Example 2.4.9. Then their **coproduct W*-category**

$$\coprod_{i \in I} \mathsf{C}_i$$

is defined to have the disjoint union of the objects of all the C_i as its class of objects. The hom-space between objects X and Y from the same C_i is defined to be just $\mathsf{C}_i(X, Y)$, while it is defined to be the zero vector space otherwise. The composition and involution are then inherited from the C_i in the obvious way, and the existence of preduals is obvious. Therefore $\coprod_{i \in I} \mathsf{C}_i$ is clearly a W*-category again. If the C_i are small, then it is clearly the coproduct in the category of small W*-categories $\mathbb{W}^*$.

For example, the discrete W*-category on a set I from Example 2.4.1 is the coproduct of I copies of the trivial single-object W*-category $\mathfrak{B}\mathbb{C}$.

Example 2.4.11 (Functor W*-categories)**.** We can now generalize the W*-categories of normal representations of a W*-algebra (Example 2.4.2) to W*-categories of W*-functors. To this end, let C and D be arbitrary W*-categories where D is small.

Together with the bounded natural transformations, we already saw that the W*-functors $\mathsf{D} \to \mathsf{C}$ form a category $\mathsf{Fun}(|\mathsf{D}|, \mathsf{C})$ in the obvious way. There is a canonical faithful functor to the product W*-category $\ell^\infty(\mathsf{D}, \mathsf{C})$ from Example 2.4.9 which forgets the action on morphisms of D, or equivalently amounts to restricting along the inclusion of the discrete W*-category on the objects of D into D. As in Example 2.4.2, this facilitates the application of Lemma 2.2.3 in order to equip $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ with the structure of W*-category. The norm on the hom-spaces is given exactly by

$$\|\alpha\| \; \coloneqq \; \sup_{X \in \mathsf{D}} \|\alpha_X\|.$$

In fact when D is the discrete W*-category on a set I as in Example 2.4.1, then $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ is exactly $\ell^\infty(I, \mathsf{C})$.

[7]A minor difference to Guichardet's setting in [22] is that he does not require his $*$-homomorphisms to be unital, but the universal property holds either way with and without unitality.

When D only has a single object with endomorphism W*-algebra N, then we clearly have $\mathsf{Fun}(\mathsf{D},\mathsf{C}) = \mathsf{NRep}(N,\mathsf{C})$. When C *also* just has a single object with endomorphism W*-algebra M, then $\mathsf{Fun}(\mathsf{D},\mathsf{C})$ has normal $*$-homomorphisms $N \to M$ as objects and homomorphism-intertwining elements of M as morphisms. This kind of W*-category plays an important role in algebraic quantum field theory [25].

Functor application now defines a functor

$$\mathsf{C} \times \mathsf{Fun}(\mathsf{C},\mathsf{D}) \longrightarrow \mathsf{D}, \tag{8}$$

and it is straightforward to see that this is a W*-bifunctor.

Example 2.4.12. More generally, let C and D be small W*-categories and E an arbitrary W*-category. Then we write

$$\mathsf{BiFun}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$$

for the category of W*-bifunctors $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ together with bounded natural transformations (which are defined between W*-bifunctors in the exact same way as between W*-functors). Once again the same definitions of norm and involution can be applied, and the analogous argument as in the previous example shows that this is a W*-category as well.

Example 2.4.13 (Arrow W*-categories)**.** In ordinary category theory, arrow categories are occasionally relevant. We expect the same to be the case in W*-category theory.

For a W*-category C, its **arrow W*-category** $\mathsf{C}^{\to}$ is defined as follows: the objects are the morphisms of C, and the morphisms from $f : A \to B$ to $g : C \to D$ are pairs of morphisms

$$(\ell : A \to C, r : B \to D)$$

making *both* the diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle r} \\ C & \xrightarrow[g]{} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xleftarrow{f^*} & B \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle r} \\ C & \xleftarrow[g^*]{} & D \end{array}$$

commute, where the second one is needed in order for $\mathsf{C}^{\to}$ to inherit the involution from C. These morphisms compose in the obvious way and make $\mathsf{C}^{\to}$ into a W*-category with respect to the norm $\|(\ell, r)\| := \max(\|\ell\|, \|r\|)$. The existence of preduals is once again a consequence of Lemma 2.2.3, now applied to the obvious faithful functor $\mathsf{C}^{\to} \to \mathsf{C} \times \mathsf{C}$.

There are a few canonical W*-functors associated to an arrow W*-category:

- ▷ Taking the domain and codomain of an arrow defines two W*-functors $\mathsf{C}^{\to} \to \mathsf{C}$.
- ▷ Mapping every object to its identity morphism defines a fully faithful W*-functor $\mathsf{C} \to \mathsf{C}^{\to}$.

Especially the first two are helpful in analyzing properties of $\mathsf{C}^{\to}$; see in particular Example 3.6.12. Also worth considering is the **contractive arrow W*-category** $\mathsf{C}^{\to,1} \subseteq \mathsf{C}^{\to}$, by which we mean the full W*-subcategory on those arrows that have norm at most 1.

One should expect that such diagram W*-categories make sense also for more general diagram shapes, but we will not consider this here. It also seems natural to expect that $\mathsf{C}^{\to}$ is a special cases of the functor W*-categories as in Example 2.4.11, in the sense that there would be a W*-category D together with a W*-equivalence $\mathsf{Fun}(\mathsf{D}, \mathsf{C}) \cong \mathsf{C}^{\to}$ for every C, but the details of this are unclear: there is no such thing as a "free W*-category generated by an arrow" without giving an upper bound on the norm of that arrow.

Example 2.4.14. If C is a W*-category, then its **complex conjugate W*-category** $\overline{\mathsf{C}}$ has the same objects as C, but the hom-spaces are the complex conjugate Banach spaces,

$$\overline{\mathsf{C}}(X, Y) \coloneqq \overline{\mathsf{C}(X, Y)},$$

while all other structure coincides with the one of C, apart from the preduals which are also given by the corresponding complex conjugate Banach spaces. If $f : X \to Y$ in C, then we write $\overline{f} : X \to Y$ for the associated morphism in $\overline{\mathsf{C}}$.

We also have the **opposite W*-category** C^{op} defined as the opposite category in the standard way with the induced W*-category structure. There then is a canonical isomorphism

$$\boxed{\overline{\mathsf{C}} \cong \mathsf{C}^{\mathrm{op}}} \tag{9}$$

given by the identity on objects, and mapping every $\overline{f} : X \to Y$ to the formal dual of $f^* : Y \to X$. Furthermore, it is clear that for a W*-functor $F : \mathsf{C} \to \mathsf{D}$, we obtain W*-functors $\overline{F} : \overline{\mathsf{C}} \to \overline{\mathsf{D}}$ and $F^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$. In this way, taking the complex conjugate and opposite W*-categories are functors $\mathbb{W}^* \to \mathbb{W}^*$ that are naturally isomorphic.

For W*-categories C and D with D small, we have a canonical isomorphism[8]

$$\mathsf{Fun}(\mathsf{D}, \mathsf{C})^{\mathrm{op}} \cong \mathsf{Fun}(\mathsf{D}^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}}) \tag{10}$$

which is most easily described by applying (9) to reduce it to

$$\overline{\mathsf{Fun}(\mathsf{D}, \mathsf{C})} \cong \mathsf{Fun}(\overline{\mathsf{D}}, \overline{\mathsf{C}}).$$

This is now easily constructed by sending every $F : \mathsf{D} \to \mathsf{C}$ to $\overline{F} : \overline{\mathsf{D}} \to \overline{\mathsf{C}}$ and every bounded natural transformation $F \to G$ to the natural transformation $\overline{F} \to \overline{G}$ with the same components.

Taking $\mathsf{D} = \mathfrak{B}N$ for a W*-algebra N and $\mathsf{C} = \mathsf{Hilb}$ in (10) shows in particular that we have canonical isomorphisms

$$\boxed{\mathsf{NRep}(N^{\mathrm{op}}) \cong \mathsf{NRep}(N)^{\mathrm{op}}.} \tag{11}$$

[8]Note that there is no such thing in ordinary category theory, and what makes this work is the involution on hom-sets.

Several general constructions that we introduce in upcoming sections will provide further examples, including W*-categories of Hilbert presheaves (Corollary 2.5.22), the direct sum completion (Theorem 3.6.25) and the projection completion (Theorem 3.7.15) of a W*-category.

2.5 Hilbert presheaves

Following work of Mitchener [8, Section 8] and Henry [26, Section 2.2] in the C*-setting, we now generalize Hilbert modules over a W*-algebra to Hilbert modules over a small W*-category. As we will see, these are the W*-categorical analogue of presheaves in ordinary category theory, and so we adopt that terminology. The next section will show that these structures play a fundamental and simplifying role in W*-category theory.

Definition 2.5.1. *Let C be a W*-category. Then a **pre-Hilbert presheaf** on C is a functor*

$$H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Vect}$$

which is linear on hom-spaces together with a sesquilinear form

$$\langle -, - \rangle \; : \; HY \times HX \longrightarrow \mathsf{C}(X, Y) \tag{12}$$

for every $X, Y \in \mathsf{C}$ such that the following hold:

(i) Symmetry: for all $\alpha \in HX$ and $\beta \in HY$,

$$\langle \beta, \alpha \rangle^* = \langle \alpha, \beta \rangle.$$

(ii) Naturality: for all $\alpha \in HX$ and $\beta \in HY$ and $f : X' \to X$,

$$\langle \beta, \alpha f \rangle = \langle \beta, \alpha \rangle f, \tag{13}$$

where we use the shorthand notation $\alpha f \coloneqq (Hf)(\alpha)$.

(iii) Positive definiteness: for all $\alpha \in HX$, we have

$$\langle \alpha, \alpha \rangle \geq 0,$$

and $\langle \alpha, \alpha \rangle = 0$ implies $\alpha = 0$.

*A pre-Hilbert presheaf H is a **Hilbert presheaf** if every HX is a Banach space with respect to the norm*

$$\|\alpha\| \coloneqq \|\langle \alpha, \alpha \rangle\|^{\frac{1}{2}}$$

for every $X \in \mathsf{C}$.

As already done in condition (ii), we will generally denote the action of H on morphisms by juxtaposition on the right: for $f : X' \to X$ and $\alpha \in HX$, we therefore write

$$\alpha f \coloneqq (Hf)(\alpha).$$

In this notation, the contravariant functoriality simply amounts to associativity in the form $\alpha(gf) = (\alpha g)f$. Combining the symmetry and naturality properties also shows "antinaturality" of a pre-Hilbert presheaf in the first argument: for all $\alpha \in HX$ and $\beta \in HY$ as above and all $g : Y' \to Y$,

$$\langle \beta g, \alpha \rangle = g^* \langle \beta, \alpha \rangle.$$

As one might expect, pre-Hilbert presheaves play a minor auxiliary role for us, while Hilbert presheaves are the main objects of interest. It is straightforward to see that every pre-Hilbert presheaf has an objectwise completion to a Hilbert presheaf.

Example 2.5.2. For every $A \in \mathsf{C}$, the hom-functor $\mathsf{C}(-, A)$ is a C-Hilbert presheaf in a canonical way: for $f : X \to A$ and $g : Y \to A$, we have

$$\langle g, f \rangle := g^* f \quad \in \quad \mathsf{C}(X, Y),$$

and all relevant properties are straightforward to verify.

Example 2.5.3. More generally, let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful W*-functor and $A \in \mathsf{D}$. Then the restriction of the hom-functor $\mathsf{D}(-, A)$ along F is a Hilbert presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ with respect to the inner products defined as in Example 2.5.2 and pulled back to F. More explicitly, the inner product of $f : FX \to A$ and $g : FY \to A$ for $X, Y \in \mathsf{C}$ is the unique morphism $X \to Y$ in C such that

$$F(\langle g, f \rangle) = g^* f.$$

Example 2.5.4. If $\mathsf{C} = \mathfrak{B}N$ for a W*-algebra N is a single-object W*-category, then a C-Hilbert presheaf is manifestly the same thing as an N-Hilbert module.

Example 2.5.5. As a generalization of Example 2.5.2, let $k : A \to A$ be positive and invertible. Then $\mathsf{C}(-, A)$ is also a Hilbert presheaf on C with respect to the inner product *weighted* by the kernel k,

$$\langle g, f \rangle := g^* k f \quad \in \quad \mathsf{C}(X, Y).$$

For example, If $A = \mathcal{H}$ in Hilb is any Hilbert space, then this presheaf is represented by $\mathcal{H}$ itself with the modified inner product $\langle -, k- \rangle$.

Remark 2.5.6. If $f : X \to X$ is positive, then we also have

$$\langle \alpha, \alpha f \rangle = \left\langle \sqrt{f}\alpha, \sqrt{f}\alpha \right\rangle \geq 0$$

by the positive semidefiniteness of the inner product.

We can use this to show that every Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ is automatically a Ban-enriched functor, which means that the action on hom-spaces has norm ≤ 1, or more explicitly that for any $\alpha \in HX$ and $f : Y \to X$ we have

$$\|\alpha f\| \leq \|\alpha\| \, \|f\|.$$

Indeed this follows from

$$\|\alpha f\|^2 = \langle \alpha f, \alpha f\rangle = \langle \alpha, \alpha f f^*\rangle \leq \|f\|^2 \langle \alpha, \alpha\rangle,$$

where the inequality step uses the above together with the fact that both summands on the right-hand side of

$$\|f\|^2 \,\mathrm{id}_X = f f^* + (\|f\|^2 \,\mathrm{id}_X - f f^*)$$

are positive. In conclusion, a Hilbert presheaf is automatically a Ban-enriched functor, and we will occasionally use this terminology.

Essentially the same Cauchy-Schwarz inequality as for pre-Hilbert modules also holds for pre-Hilbert presheaves, as follows.

Lemma 2.5.7. *For every $\alpha \in HX$ and $\beta \in HY$ in a pre-Hilbert presheaf H, we have*

$$\langle \beta, \alpha\rangle\langle \alpha, \beta\rangle \leq \|\alpha\|^2 \langle \beta, \beta\rangle. \tag{14}$$

Proof. The proof for the Hilbert module case [20, Proposition 1.1] works verbatim the same: for every $f : Y \to X$, and assuming $\|\alpha\| = 1$ without loss of generality,

$$\begin{aligned} 0 &\leq \langle \alpha f - \beta, \alpha f - \beta\rangle \\ &= f^*\langle \alpha, \alpha\rangle f - \langle \beta, \alpha\rangle f - f^*\langle \alpha, \beta\rangle + \langle \beta, \beta\rangle \\ &\leq f^* f - \langle \beta, \alpha\rangle f - f^*\langle \alpha, \beta\rangle + \langle \beta, \beta\rangle. \end{aligned}$$

Putting $f := \langle \alpha, \beta\rangle$ now proves the claim. □

Definition 2.5.8. *Given pre-Hilbert presheaves $H, K : \mathsf{C}^{\mathrm{op}} \to \mathsf{Norm}$, a transformation*[9] *$t : H \to K$ is* ***adjointable*** *if there is a transformation $t^* : K \to H$ such that*

$$\langle \beta, t\alpha\rangle = \langle t^*\beta, \alpha\rangle \qquad \forall \alpha \in HX,\ \beta \in KY.$$

It is clear that t^* is unique if it exists. For example for every $f : X \to Y$, the induced transformation

$$f \circ - \ : \ \mathsf{C}(-, X) \longrightarrow \mathsf{C}(-, Y)$$

is adjointable, and its adjoint is given by $f^* \circ -$. Just as in the case of Hilbert modules [20, p. 8], adjointability for transformations between pre-Hilbert presheaves has important consequences.

Lemma 2.5.9. *An adjointable transformation $t : H \to K$ is automatically natural, has linear components and satisfies $t^{**} = t$.*

Proof. Straightforward. □

[9] A *transformation* $t : H \to K$ is simply a family of maps $t_X : HX \to KX$, not necessarily natural.

Just like for Hilbert modules [20, p. 8], if H is a Hilbert presheaf, then one can also apply the Banach-Steinhaus theorem to show that the components $t_X : HX \to KX$ of adjointable t are bounded. However, adjointability does not guarantee that they are uniformly bounded. For example if C is the discrete W*-category on an infinite set I (Example 2.4.1), then a Hilbert presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ is the same thing as a family of Hilbert spaces $(\mathcal{H}_i)_{i\in I}$, and an adjointable transformation $(\mathcal{H}_i)_{i\in I} \to (\mathcal{K}_i)_{i\in I}$ is an arbitrary family of bounded linear maps $(f_i : \mathcal{H}_i \to \mathcal{K}_i)_{i\in I}$. We therefore need to postulate the uniform boundedness separately:

Definition 2.5.10. *Let H and K be pre-Hilbert presheaves on a W*-category C. Then a **Hilbert transformation** from H to K is an adjointable transformation $t : H \to K$ such that*

$$\|t\| := \sup_{X\in\mathsf{C}} \|t_X\| \; < \; \infty.$$

With respect to this norm, the collection of Hilbert transformations $H \to K$ is a (possibly large) normed space. It is easy to see that if K is a Hilbert presheaf, then this normed space is complete. Hence the collection of Hilbert presheaves becomes a category enriched in (possibly large) Banach spaces.

Here is the generalization of self-duality from Hilbert modules to their many-object versions.

Definition 2.5.11. *A pre-Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Vect}$ is **self-dual** if for every $X \in \mathsf{C}$ and every uniformly bounded natural transformation*

$$t \; : \; H \longrightarrow \mathsf{C}(-, X),$$

there is $\beta \in HX$ such that $t = \langle \beta, - \rangle$.

By the Yoneda lemma, we could equivalently just require every such t to be adjointable.

Example 2.5.12. Every hom-presheaf $\mathsf{C}(-, Y)$ is self-dual by the Yoneda lemma.

The following generalizes a known result for Hilbert modules [21, Proposition 3.6].

Lemma 2.5.13. *If H and K are pre-Hilbert presheaves with H self-dual, then every bounded natural transformation $t : H \to K$ is a Hilbert transformation.*

Proof. For given $\beta \in KX$, consider the transformation

$$\langle \beta, t- \rangle \; : \; H \to \mathsf{C}(-, X).$$

As H is self-dual, there is a unique $t^*\beta \in HX$ such that

$$\langle \beta, t\alpha \rangle = \langle t^*\beta, \alpha \rangle$$

for all α. By the uniqueness, it is straightforward to see that t^* is linear in β, and this implies that t is adjointable. □

The following result generalizes the observation that a self-dual pre-Hilbert space is actually a Hilbert space.

Corollary 2.5.14. *Every self-dual pre-Hilbert presheaf is a Hilbert presheaf.*

Proof. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in HX for $X \in \mathsf{C}$. Then the $\langle \alpha_n, -\rangle$ form a Cauchy sequence of Hilbert transformations $H \to \mathsf{C}(-, X)$. Their limit is a Hilbert transformation t, which by the assumption of self-duality is also of the form $\langle \alpha, -\rangle$ for some $\alpha \in HX$. The fact that α is the limit of the sequence $(\alpha_n)_{n\in\mathbb{N}}$ now follows by

$$\|\alpha - \alpha_n\|^2 = \|\langle \alpha - \alpha_n, \alpha - \alpha_n\rangle\| = \|\langle \alpha - \alpha_n, \alpha\rangle\| + \|\langle \alpha - \alpha_n, \alpha_n\rangle\|,$$

where both terms on the right tend to zero as $n \to \infty$ thanks to the Cauchy-Schwarz inequality (14), where the second also uses the fact that the α_n are uniformly bounded. □

Our next statement is a vast generalization of the familiar correspondence between bounded linear operators on a Hilbert space and bounded sesquilinear forms on it. We will need it only further down the line.

Lemma 2.5.15. *Let H and K be pre-Hilbert presheaves with K self-dual. Let*

$$((-,-)) \,:\, KY \times HX \longrightarrow \mathsf{C}(X, Y)$$

be a sesquilinear form defined for all $X, Y \in \mathsf{C}$ such that the naturality equations

$$((\beta, \alpha f)) = ((\beta, \alpha))f \qquad \forall f : X' \to X,$$
$$((\beta g, \alpha)) = g^*((\beta, \alpha)) \qquad \forall g : Y' \to Y$$

hold, and such that there is $C > 0$ with

$$\|((\beta, \alpha))\| \leq C\|\langle \beta, \alpha\rangle\| \qquad \forall \alpha, \beta.$$

Then there is a unique Hilbert transformation $t : H \to K$ such that

$$((\beta, \alpha)) = \langle \beta, t\alpha\rangle \qquad \forall \alpha, \beta. \tag{15}$$

Conversely, it is obvious that every Hilbert transformation t defines a family of sesquilinear forms as in the statement. The proof is quite analogous to the corresponding arguments we had made in Example 2.4.5.

Proof. For the uniqueness, it is enough to note that if t is such that (15) vanishes identically, then already $t = 0$, and this is a consequence of the positive definiteness of $\langle -, -\rangle$.

For existence, fix $\alpha \in HX$ for now. Letting β vary, we obtain a transformation

$$K \longrightarrow \mathsf{C}(-, X)$$
$$\beta \longmapsto (\beta, \alpha)^*$$

which is natural by the second naturality assumption and uniformly bounded by the boundedness assumption and the Cauchy-Schwarz inequality (14). The self-duality of K therefore provides a unique element $t\alpha \in KX$ such that

$$(\beta, \alpha)^* = \langle t\alpha, \beta \rangle,$$

for all β, or equivalently such that the desired (15) holds. The adjointability of t follows by constructing t^* in the same manner from the sesquilinear form $(-,-)^*$, and the uniform boundedness holds by assumption. $\square$

Our next definition is the analogue of the notion of *small* presheaf in ordinary category theory, a property which plays an important role in the context of the representable functor theorem [27, Theorem 4.84].[10], of which we will provide a W*-categorical version in Theorem 3.8.16.

Definition 2.5.16. *A self-dual*[11] *Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ is* ***small*** *if there is a family of elements $(\alpha_i \in HX_i)_{i \in I}$ such that for every $\beta \in HY$ there are $i \in I$ and $f : Y \to X_i$ such that $\beta = \alpha_i f$.*

Clearly if C is small, then every Hilbert presheaf on C is small since we can simply consider the family consisting of all elements of all sets HX. Even if C is not small, every representable Hilbert presheaf $\mathsf{C}(-, X)$ is trivially small (with a single generator given by id_X). In fact, naturally occurring Hilbert presheaves are typically small, and finding an example of a non-small Hilbert presheaf is not obvious. Here is one.

Example 2.5.17. Let C be the discrete W*-category on a large set I (Example 2.4.1). Then there is a Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ given by $HX := \mathbb{C}$ for every object $X \in I$, and with the unique action on morphisms which makes this assignment into a Ban-enriched functor. This is a Hilbert presheaf with respect to the inner products given by necessarily $\langle \beta, \alpha \rangle = 0$ for $\alpha \in HX$ and $\beta \in HY$ with $X \neq Y$, and $\langle \beta, \alpha \rangle := \overline{\beta} \alpha$ otherwise. It is easy to see that this H is self-dual by noting that a Hilbert transformation $t : H \to \mathsf{C}(-, X)$ has only zero components except possibly for t_X. However, H is clearly not small.

Proposition 2.5.18. *Let H and K be pre-Hilbert presheaves with H small and K self-dual. Then the space of Hilbert transformations $H \to K$ is a (small) Banach space with a predual.*

Proof. The smallness is clear since a Hilbert transformation $t : H \to K$ is uniquely determined by its values on a family of generating elements as in Definition 2.5.16.

[10] In [27], Kelly uses the term *accessible* instead of *small*. It is also worth noting that the *solution set condition* appearing in Freyd's adjoint functor theorem is an instance of this property.

[11] While the definition still makes sense without self-duality, the smallness criteria we develop in Corollary 3.2.9 use self-duality in the proof. So without self-duality, there is some ambiguity as to what the "right" definition is, and it seems likely to us that the alternative condition given in Corollary 3.2.9(iii) would be more useful than the definition given here.

The existence of a predual can be proven in a way that directly generalizes the corresponding argument in the Hilbert module case (Example 2.4.5). Let us write $\hat{\mathsf{C}}(H,K)$ for the Banach space under consideration, and let

$$\hat{\mathsf{C}}(H,K)_* \subseteq \hat{\mathsf{C}}(H,K)^*$$

be the closed subspace spanned by all functionals of the form

$$\mathrm{ev}^{\eta}_{\alpha,\beta} \;:\; \begin{aligned} \hat{\mathsf{C}}(H,K) &\longrightarrow \mathbb{C} \\ t &\longmapsto \eta(\langle \beta, t\alpha\rangle), \end{aligned} \tag{16}$$

where $\alpha \in HX$ for some $X \in \mathsf{C}$ and $\beta \in KY$ for some $Y \in \mathsf{C}$ as well as $\eta \in \mathsf{C}(X,Y)_*$. Then the canonical map

$$\hat{\mathsf{C}}(H,K) \longrightarrow \big(\hat{\mathsf{C}}(H,K)_*\big)^*$$

is an isometry by the same argument as in the Hilbert module case. To see that it is surjective, we again consider a given element of $\big(\hat{\mathsf{C}}(H,K)_*\big)^*$ as defining maps

$$\mathcal{T}_{X,Y} \;:\; HX \times KY \times \mathsf{C}(X,Y)_* \longrightarrow \mathbb{C}$$

by restriction to the $\mathrm{ev}^{\eta}_{\alpha,\beta}$. Varying η turns this into a map $HX \times KY \to \mathsf{C}(X,Y)$, which we also denote by $\mathcal{T}_{X,Y}$ by abuse of notation. The fact that the ev maps satisfy the naturality equations

$$\mathrm{ev}^{\eta}_{\alpha f,\beta} = \mathrm{ev}^{\eta(-f)}_{\alpha,\beta}, \qquad \mathrm{ev}^{\eta}_{\alpha,\beta g} = \mathrm{ev}^{\eta(g^*-)}_{\alpha,\beta}$$

implies the naturality equations for $\mathcal{T}$ that are needed for the application of Lemma 2.5.15. This lets us finish the proof by the same argument as in Example 2.4.5. □

From now on, let us write

$$\hat{\mathsf{C}}(H,K)$$

to denote the Banach space of Hilbert transformation $H \to K$, whenever both H and K are small self-dual. This notation will be motivated by Corollary 2.5.22 and Theorem 3.8.20.

Theorem 2.5.19. *Let $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ be a small self-dual Hilbert presheaf. Then:*

(i) Every HX for $X \in \mathsf{C}$ has a unique Banach space predual.
(ii) For every $X, Y \in \mathsf{C}$, the inner product (12) is ultraweakly continuous in each argument,
(iii) For every $X, Y \in \mathsf{C}$, the functoriality

$$\begin{aligned} HY \times \mathsf{C}(X,Y) &\longrightarrow HX \\ (\alpha, f) &\longmapsto \alpha f \end{aligned}$$

is ultraweakly continuous in each argument.

(iv) For every $X \in \mathsf{C}$ and any other small self-dual Hilbert presheaf K, the evaluation map

$$\begin{aligned} \hat{\mathsf{C}}(H,K) \times HX &\longrightarrow KX \\ (t,\alpha) &\longmapsto t(\alpha) \end{aligned}$$

is ultraweakly continuous in each argument.

Proof. (i) This is clear by Corollary 2.5.22 and the Yoneda isometric isomorphism (18). More explicitly, applying (18) to the construction of the predual in the previous proof shows that the predual of HX is the closed subspace of $(HX)^*$ spanned by all functionals of the form

$$\alpha \longmapsto \eta(\langle \gamma, \alpha f \rangle)$$

where $\gamma \in HZ$ and $f : Y \to X$ for arbitrary Y and Z as well as $\eta \in \mathsf{C}(Y,Z)_*$, or with $\beta := \gamma f^*$ equivalently the subspace spanned by the functionals of the form

$$\alpha \longmapsto \eta(\langle \beta, \alpha \rangle) \tag{17}$$

for $\beta \in HY$ and $\eta \in \mathsf{C}(X,Y)_*$.

(ii) Based on this characterization of $(HX)_*$, the ultraweak continuity of $\alpha \mapsto \langle \beta, \alpha \rangle$ now follows upon specializing this class of functionals to $f = \mathrm{id}_X$, and ultraweak continuity in the first argument follows by symmetry.

(iii) Also straightforward by the above characterization of $(HX)_*$ and $(HY)_*$.

(iv) Consider ultraweak continuity in t first. Using the concrete form of $(KX)_*$ as spanned by functionals of the type (17), it is enough to show that

$$\begin{aligned} \hat{\mathsf{C}}(H,K)_* &\longrightarrow (KX)_* \\ t &\longmapsto \eta(\langle \beta, t\alpha \rangle) \end{aligned}$$

is ultraweakly continuous for every $\beta \in HY$ and $\eta \in \mathsf{C}(X,Y)_*$, but this holds by construction of $\hat{\mathsf{C}}(H,K)_*$. The proof of ultraweak continuity in α is similar. $\square$

A useful consequence is a version of Lemma 2.1.18 for infinite sums of elements in Hilbert presheaves, where the proof is the same (based on HX being a dual space).[12]

Lemma 2.5.20. *Let H be a small self-dual Hilbert presheaf, X an object in C, and $(\alpha_i)_{i \in I}$ a family of elements of HX. If the partial sums $\sum_{i \in F} \alpha_i$ for finite $F \subseteq I$ are uniformly bounded in norm, then $\sum_{i \in I} \alpha_i$ converges absolutely ultraweakly.*

We can now generalize Example 2.4.5, where we had constructed W*-categories of Hilbert modules, to the many-object case.

We also have a W*-categorical strengthening of the Yoneda lemma.

[12] It would also be possible to derive this as an instance of Lemma 2.1.18 based on the Yoneda isometric isomorphism $HX \cong \hat{\mathsf{C}}(\mathsf{C}(-,X), HX)$.

Theorem 2.5.21 (Yoneda lemma)**.** *Let $H\colon \mathsf{C} \to \mathsf{Vect}$ be a pre-Hilbert presheaf on a W*-category C and $X \in \mathsf{C}$. Then the Yoneda bijection*

$$\begin{aligned} HX &\longrightarrow \hat{\mathsf{C}}(\mathsf{C}(-,X),H) \\ \alpha &\longmapsto (f \mapsto \alpha f) \\ t(\mathrm{id}_X) &\longleftarrow t \end{aligned} \tag{18}$$

is an isometric isomorphism.

Proof. Let us first ensure that for every $\alpha \in HX$, assigning $f \mapsto \alpha f$ indeed defines a Hilbert transformation. The adjointability holds because for all $f : A \to X$ and $\beta \in HB$,

$$\langle \beta, \alpha f \rangle = \langle \alpha, \beta \rangle^* f,$$

which shows that the adjoint is given by the transformation[13]

$$\begin{aligned} H &\longrightarrow \mathsf{C}(-,X) \\ \beta &\longmapsto \langle \alpha, \beta \rangle. \end{aligned}$$

The uniform boundedness is obvious by $\|\alpha f\| \leq \|\alpha\| \, \|f\|$. Thanks to the proof of the Yoneda lemma in ordinary category theory, we only need to note two additional properties.

Second, we need to show that (18) is actually norm-preserving. But this is also clear by taking $f = \mathrm{id}_X$. □

Corollary 2.5.22. *Let C be a W*-category. Then the small self-dual Hilbert presheaves on C form a W*-category with Hilbert transformations as morphisms, and we denote it by $\hat{\mathsf{C}}$. Furthermore, the Yoneda embedding*

$$\begin{aligned} よ_{\mathsf{C}} \; : \; \mathsf{C} &\longrightarrow \hat{\mathsf{C}} \\ X &\longmapsto \mathsf{C}(-,X) \end{aligned}$$

is a W-functor that is an isometric isomorphism on hom-spaces.*

Proof. All basic properties are straightforward; for example, the C*-identity holds since it holds componentwise. What is less clear is local smallness and the existence of preduals for the hom-spaces, which were both established in Definition 2.5.16.

The statement about the Yoneda embedding is a straightforward consequence of Theorem 2.5.21. □

Remark 2.5.23. We will often use the Yoneda embedding in order to translate results back and forth between statements about W*-categories and statements on Hilbert presheaves. Indeed given a statement on W*-categories, we can apply it to the W*-category $\hat{\mathsf{C}}$ in order to obtain a statement about Hilbert presheaves. Conversely, given a statement about Hilbert presheaves, we can instantiate it on those of the form $\mathsf{C}(-,X)$ in order to obtain a statement about W*-categories. For an example of this, the upcoming Theorem 3.2.6 and Corollary 3.2.8 correspond in this way.

[13] This seems like an observation worth noting in itself, and we are tempted to write it as $(\alpha -)^* = \langle \alpha, - \rangle$.

Remark 2.5.24. We will obtain further results on self-dual Hilbert presheaves in Section 3, including the existence of orthonormal bases (Theorem 3.5.12).

It is now natural to move on to universal properties of objects in W*-categories.

Definition 2.5.25. *A Hilbert presheaf $H := \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ is* ***representable*** *if there is a unitary isomorphism $H \cong \mathsf{C}(-, X)$ for some object $X \in \mathsf{C}$.*

We turn to a more high-level perspective on an observation made in [28, Section 3]: universal properties in W*-categories can be formulated purely in terms of Banach space enrichment.

Lemma 2.5.26. *Let H be a Hilbert presheaf. Then H is representable as a Hilbert presheaf if and only if it is representable as a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$.*

In fact, the proof will show something a bit stronger, namely a bijection between the representations of H as a Hilbert presheaf and those as a functor. It follows that the representing object is unique up to unique *unitary* isomorphism, a known result [28, Corollary 3.5].

Proof. It is enough to show that every functor representation can be lifted to a Hilbert presheaf representation. A representation as a functor is equivalent to a choice of element $\alpha \in HX$ for some X (the universal object) such that for every A, the map

$$\begin{aligned} \mathsf{C}(A, X) &\longrightarrow HA \\ f &\longmapsto \alpha f \end{aligned}$$

is an isometric isomorphism. But this is obviously a Hilbert presheaf representation as well. $\square$

Example 2.5.27. Evaluation at any $X \in \mathsf{C}$ defines a functor

$$\begin{aligned} \hat{\mathsf{C}}^{\mathrm{op}} &\longrightarrow \mathsf{Ban} \\ H &\longmapsto \overline{H}X \\ t &\longmapsto \overline{f}^{*}, \end{aligned}$$

where we take conjugates since Hilbert presheaves are required to be contravariant, and we therefore need to act on morphisms by taking the involution, which is conjugate linear. It may not be clear a priori whether this a Hilbert presheaf, since there is no immediate candidate for an inner product. However, since the Yoneda lemma in the form Theorem 2.5.21 amounts to representing it by the Hilbert presheaf $\mathsf{C}(-, X)$, we can in particular conclude that it is a Hilbert presheaf. And indeed an inner product can be constructed by assigning to any $\beta \in \overline{K}X$ and $\alpha \in \overline{H}X$ the Hilbert transformation $H \to K$ given by $\beta\langle\alpha, -\rangle$.

Proposition 2.5.28. *If $H, K : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ are Hilbert presheaves and $t : H \to K$ is a natural transformation with isometric isomorphism components, then t is a unitary isomorphism.*

In other words, the inner products on a Hilbert presheaf are uniquely determined by its structure as a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$, and being a Hilbert presheaf is a property rather than a structure of such a functor. For Hilbert modules, this result is due to Lance [29].

Proof. This is because $\|t\| = \|t^{-1}\| = 1$ implies that $t^{-1} = t^*$, and therefore t preserves the inner products. □

Our next goal is to develop a less trivial example of representability, which provides a universal property for the **GNS construction**. In order to state it, it is convenient to also introduce **Hilbert copresheaves**, which are the covariant analogs $\mathsf{C} \to \mathsf{Ban}$ of Hilbert presheaves. Since $\mathsf{C}^{\mathrm{op}} \cong \overline{\mathsf{C}}$ by Example 2.4.14, we can equivalently consider these as Hilbert presheaves on the complex conjugate W*-category $\overline{\mathsf{C}}$. In particular, this implies that a Hilbert copresheaf $H : \mathsf{C} \to \mathsf{Ban}$ has inner products of the form

$$\langle -, - \rangle \;:\; HY \times HX \longrightarrow \mathsf{C}(X, Y)$$

but now these are linear in the first and conjugate linear in the second argument. With the covariant functoriality written as an action of morphisms from the left, the naturality (13) now turns into

$$\langle f\beta, \alpha \rangle = f \langle \beta, \alpha \rangle \qquad \forall f : Y \to Y'.$$

For example, every hom-functor $\mathsf{C}(X, -)$ is a Hilbert copresheaf with inner product $\langle g, f \rangle := g f^*$.

The GNS construction in its most basic form starts with a unital C*-algebra $\mathcal{A}$ and state $\phi : \mathcal{A} \to \mathbb{C}$. It is a representation of $\mathcal{A}$ on a certain Hilbert space $\mathcal{H}_\phi$, which we henceforth write as a left action. So naturally the universal property that we are looking for amounts to representability of a Hilbert (co)presheaf on the category of representations $\mathsf{Rep}(A)$. So for any representation $\mathcal{H}$, let us write

$$L^2(\mathcal{H}, \phi) := \overline{\{\xi \in \mathcal{H} \mid \exists C \geq 0 \text{ s.t. } \langle \xi, a^* a \xi \rangle \leq C\, \phi(a^* a)\ \forall a \in \mathcal{A}\}}^{\|-\|_\phi},$$

where the norm $\|\xi\|_\phi$ is defined as the smallest C for which the given inequalities hold and the overline denotes completion.[14] Furthermore, if $f : \mathcal{H} \to \mathcal{K}$ is any intertwiner, then a short calculation shows that we obtain an induced map

$$L^2(f, \phi) \;:\; L^2(\mathcal{H}, \phi) \longrightarrow L^2(\mathcal{K}, \phi).$$

In this way, we are dealing with a functor $L^2(-, \phi) : \mathsf{Rep}(\mathcal{A}) \to \mathsf{Ban}$. Note that the existence of inner products which would turn this functor into a Hilbert copresheaf is not obvious. We will come back to this point in Remark 2.5.30.

Proposition 2.5.29 (Universal property of the GNS construction)**.** *Let $\mathcal{A}$ be a unital C*-algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a state. Then $L^2(-, \phi) : \mathsf{Rep}(\mathcal{A}) \to \mathsf{Ban}$ is a representable Hilbert copresheaf with the GNS representation of ϕ as representing object.*

[14] The fact that it is a subspace of $\mathcal{H}$ follows by the Cauchy-Schwarz inequality, while the completeness with respect to $\|-\|_\phi$ is straightforward to see. Note that $L^2(\mathcal{H}, \phi)$ is generally not closed as a subspace of $\mathcal{H}$.

This universal property is similar to—but not the same as—the more involved universal property of the GNS construction due to Parzygnat [30, Section 5].

Proof. Let $\mathcal{G}_\phi$ be the GNS representation of ϕ. Recall that the C*-algebra $\mathcal{A}$ is dense in the Hilbert space $\mathcal{G}_\phi$, and on this dense subspace the inner product is $\langle b, a\rangle := \phi(b^* a)$. We then prove that the functor $L^2(-,\phi)$ is isometrically isomorphic to

$$\mathsf{Rep}(\mathcal{A})(\mathcal{G}_\phi, -) : \mathsf{Rep}(\mathcal{A}) \to \mathsf{Ban}.$$

This is enough to prove the claim thanks to Proposition 2.5.28.

More precisely, we show that for every Hilbert space $\mathcal{H}$ with a left action of $\mathcal{A}$, evaluating an intertwiner on $1 \in \mathcal{G}_\phi$ gives an isometric isomorphism

$$\mathsf{Rep}(\mathcal{A})(\mathcal{G}_\phi, \mathcal{H}) \cong L^2(\mathcal{H}, \phi).$$

Indeed for an intertwiner $f : \mathcal{G}_\phi \to \mathcal{H}$, we have

$$\langle f(1), a^* a f(1)\rangle = \langle a f(1), a f(1)\rangle = \langle f(a), f(a)\rangle \leq \|f\|^2 \langle a, a\rangle = \|f\|^2 \phi(a^* a),$$

and hence the evaluation map indeed lands in $L^2(\mathcal{H}, \phi)$. To see that it is well-defined and an isometry, note that

$$\|f\|^2 = \sup_{a\in\mathcal{A}\,:\,\|a\|\leq 1 \text{ in } \mathcal{G}_\phi} \langle f(a), f(a)\rangle = \sup_{a\in\mathcal{A}\,:\,\phi(a^*a)\leq 1} \langle f(1), a^* a f(1)\rangle = \|f(1)\|_\phi^2,$$

as was to be shown.

It remains to prove the surjectivity. It is enough to prove surjectivity on those ξ which satisfy the indicated inequalities. On such an element, the map $f : \mathcal{G}_\phi \to \mathcal{H}$ given by $f(a) := a\xi$ is easily seen to be well-defined and is an intertwiner with $f(1) = \xi$ by construction. □

Remark 2.5.30. To determine the inner product on $L^2(-,\phi)$, we simply transfer the inner product on $\mathsf{Rep}(\mathcal{A})(\mathcal{G}_\phi, -)$ along the isomorphism of Proposition 2.5.29. For given $\xi \in L^2(\mathcal{H}, \phi)$ and $\eta \in L^2(\mathcal{K}, \phi)$, the associated intertwiners are represented by $a \mapsto a\xi$ and $a \mapsto a\eta$, respectively, and the inner product $\langle \eta, \xi\rangle$ coincides with the composition of the adjoint of the first by the second. Since the required adjoint of $a \mapsto a\xi$ does not seem to have a simple explicit description, let us denote its action on any $\mu \in \mathcal{H}$ by

$$\frac{\mathrm{d}\langle \xi, -\mu\rangle}{\mathrm{d}\phi} \;:\; \mathcal{H} \longrightarrow \mathcal{G}_\phi,$$

to be thought of as a Radon-Nikodym derivative. This interpretation is justified by the adjointness relation, which reads[15]

$$\phi\left(a\,\frac{\mathrm{d}\langle \xi, -\mu\rangle}{\mathrm{d}\phi}\right) = \langle \xi, a\mu\rangle \qquad \forall a \in \mathcal{A},\, \mu \in \mathcal{H}.$$

[15] For $\mu = \xi$, this recovers

By the above discussion, we can now write the inner product on $L^2(-,\phi)$ as

$$\langle \eta, \xi \rangle \ : \ \begin{array}{rcl} \mathcal{H} & \longrightarrow & \mathcal{K} \\ \mu & \longmapsto & \frac{\mathrm{d}\langle \xi, -\mu \rangle}{\mathrm{d}\phi} \eta. \end{array}$$

Question 2.5.31. *Does this universal property generalize to a universal property for Kasparov's GNS construction?*

2.6 Hilbert profunctors

In ordinary category theory, profunctors generalize functors similarly to how relations generalize functions and to how bimodules between rings generalize ring homomorphisms. Transferring the notion of profunctor to W*-category theory is not immediate, since one needs to decide how to deal with the inner products: do they take values in one of the W*-categories involved, or should there be two families of inner products with respective values in both categories? In line with the standard definition of Hilbert bimodule, we opt for the former.

Throughout our consideration of Hilbert profunctors, we restrict ourselves to small W*-categories; avoiding this restriction would require us to introduce a notion of smallness for Hilbert profunctors, and it is not clear to us which notion would be the "right" one. With this in mind, a Hilbert profunctor is just a W*-functor $\mathsf{C} \to \hat{\mathsf{D}}$. Let us spell this out in some detail.

Definition 2.6.1. *For small W*-categories* C *and* D*, a* ***Hilbert profunctor***[16] $P : \mathsf{C} \nrightarrow \mathsf{D}$ *is a* Ban*-enriched functor*

$$P \ : \ \mathsf{D}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{Ban} \tag{19}$$

such that $P(-,X)$ *is a Hilbert presheaf on* D *for every* $X \in \mathsf{C}$ *and the action of every* $f : A \to B$ *in* C *is by adjointable morphisms,*

$$\langle \beta, f\alpha \rangle = \langle f^*\beta, \alpha \rangle \qquad \forall \alpha \in P(X,A), \quad \beta \in P(Y,B).$$

Similarly to a Hilbert presheaf, we write the action of morphisms from C on P by juxtaposition from the left and the contravariant action of morphisms on P by juxtaposition from the right. Note that a Hilbert profunctor is equivalently just a W*-functor $\mathsf{C} \to \hat{\mathsf{D}}$, and our definition merely explicates this further.[17]

Example 2.6.2. For every small W*-category C, we have a Hilbert hom-profunctor $\mathsf{C} \nrightarrow \mathsf{C}$ given by $(X,Y) \mapsto \mathsf{C}(X,Y)$, with the action on morphisms given by composition. The inner product is once again of the form

$$\begin{array}{rcl} \mathsf{C}(B,X) \times \mathsf{C}(A,X) & \longrightarrow & \mathsf{C}(A,B) \\ (g,f) & \longmapsto & g^* f \end{array}$$

[16] Our choice of notation, and in particular on which argument comes first, follows the standard conventions in ordinary category theory.

[17] Likewise in ordinary category theory, and this is why one writes $P : \mathsf{C} \nrightarrow \mathsf{D}$ rather than $P : \mathsf{D} \nrightarrow \mathsf{C}$.

This example explains why we write the contravariant argument first in (19). As a W*-functor $\mathsf{C} \to \hat{\mathsf{C}}$, this Hilbert profunctor is simply the Yoneda embedding $よ_{\mathsf{C}} : \mathsf{C} \to \hat{\mathsf{C}}$.

Example 2.6.3. More generally, for a W*-functor $F : \mathsf{C} \to \mathsf{D}$ between small W*-categories, we have a Hilbert profunctor $\mathsf{D}(-, F-) : \mathsf{C} \nrightarrow \mathsf{D}$ defined in the obvious way. We call a Hilbert profunctor isomorphic to one of this type **representable**. Considered as a W*-functor $\mathsf{C} \to \hat{\mathsf{D}}$, this Hilbert profunctor is given by the composition

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{よ_{\mathsf{D}}} \hat{\mathsf{D}},$$

which gives a more explicit description.

While the choice of which category the inner products take values in is conventional, it is customary that the inner product on a Hilbert module is associated with the C*-algebra acting on the right. Acting on the right corresponds to contravariance for us, and this is part of why our Hilbert presheaves have been developed as contravariant. Nevertheless, there are two opposing conventions which we cannot satisfy both:

Example 2.6.4. For W*-algebras M and N, a Hilbert profunctor $\mathfrak{B}M \to \mathfrak{B}N$ is by definition the same thing as a N-Hilbert M-module.

Definition 2.6.5. *A Hilbert profunctor $P : \mathsf{C} \to \mathsf{D}$ is* ***self-dual*** *if $P(-, X) : \mathsf{D}^{\mathrm{op}} \to \mathsf{Ban}$ is self-dual as a Hilbert presheaf for every $X \in \mathsf{C}$.*

The notion of **Hilbert transformation** extends from Hilbert presheaves to Hilbert profunctors, meaning that for $P, Q : \mathsf{C} \nrightarrow \mathsf{D}$, a Hilbert transformation $t : P \to Q$ is a family of Hilbert transformations $P(-, X) \to Q(-, X)$ for every $X \in \mathsf{C}$ which is natural with respect to the action of C and uniformly bounded in X. In this way, we obtain a W*-category of Hilbert profunctors $\mathsf{C} \nrightarrow \mathsf{D}$, which by definition is just

$$\mathsf{HProf}(\mathsf{C}, \mathsf{D}) \coloneqq \mathsf{Fun}(\mathsf{C}, \hat{\mathsf{D}}).$$

We will return to Hilbert profunctors and investigate their composition in Section 4.2, once we are better equipped with a number of auxiliary results.

3 W*-categories: structure theory and rigidity phenomena

In this section, we develop further properties of W*-categories. The general theme will be that W*-categories are quite rigid objects, as exemplified by strong classification results like Theorem 3.8.16 and Corollary 3.9.5. In particular, the examples of W*-categories from the previous section will turn out be quite exhaustive.

3.1 Square summable families of morphisms

We will often consider families of objects $(X_i)_{i \in I}$ in a W*-category C indexed by a set I. Unless otherwise noted, this I will be a *small* set; on a few occasions we will also consider families one Grothendieck universe level up, but in these cases we will note this explicitly. If I is clear from context, we also often use the placeholder notation

$$X_{\_} \coloneqq (X_i)_{i \in I}$$

to denote such a family. Further, we write

$$X_{\exists}$$

as shorthand for "X_i for some $i \in I$". For example, $f : X_{\exists} \to Y$ indicates that f is a morphism of type $X_i \to Y$ for some i, and similarly for $g : Y \to X_{\exists}$. We also write $\mathsf{C}(X_{\exists}, Y)$, respectively $\mathsf{C}(Y, X_{\exists})$, for the set of all morphisms of this kind, meaning the disjoint union of the individual hom-sets.

Given a family $X_{\_}$ as above, we similarly write $f_{\_}$ as shorthand for a family $(f_i : A \to X_i)_{i \in I}$ of morphisms, and we think of such a family as a column vector. Dually, the family $f_{\_}^* = (f_i^* : X_i \to B)_{i \in I}$ can be considered a row vector. We say that the family is **square summable** if

$$f_{\_}^* f_{\_} = \sum_{i \in I} f_i^* f_i < \infty, \tag{20}$$

where by the above $f_{\_}^* f_{\_}$ is a formal matrix multiplication. This serves to define the norm of a square summable family as

$$\|f_{\_}\| \coloneqq \sqrt{\|f_{\_}^* f_{\_}\|}, \tag{21}$$

which is finite by assumption. If $f_{\_}$ and $g_{\_}$ are square summable families, then the formal matrix multiplication

$$g_{\_}^* f_{\_} \coloneqq \sum_{i \in I} g_i^* f_i$$

converges ultraweakly, as per the following result.

Lemma 3.1.1. *For square summable families $f_{\_} = (f_i : A \to X_i)_{i \in I}$ and $g_{\_} = (g_i : B \to X_i)_{i \in I}$ in any W*-category C, the product*

$$g_{\_}^* f_{\_} \coloneqq \sum_{i \in I} g_i^* f_i$$

converges absolutely ultraweakly, and

$$\|g_{\_}^* f_{\_}\| \leq \|g_{\_}\| \, \|f_{\_}\|.$$

Although we feel that the use of linking W*-algebras in the following proof is not in the spirit of W*-categories, we have not been able to find any alternative proof that would avoid their use.

Proof. Suppose first that I is finite, say $I = \{1, \ldots, n\}$. Then we can consider all morphisms involved as elements of the linking W*-algebra $L(\mathsf{C}|_{A,B,X_1,\ldots,X_n})$, and the $*$-algebra of $n \times n$-matrices with entries in it is again a W*-algebra. Then the claimed inequality follows by submultiplicativity of the norm on there and the equation

$$\begin{pmatrix} g_-^* f_- & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} g_1^* & \cdots & g_n^* \\ 0 & \cdots & 0 \\ \vdots & & \vdots \end{pmatrix} \begin{pmatrix} f_1 & 0 & \cdots \\ \vdots & \vdots & \\ f_n & 0 & \cdots \end{pmatrix}.$$

For infinite I, it is enough to prove the claim about absolute ultraweak convergence, since then the inequality follows by the finite case. But this is a straightforward application of Lemma 2.1.18 based on the inequality in the finite case. □

Occasionally, we will want to consider families of morphisms "with multiplicity", in the sense that we will want to generalize from families of morphisms to $(f_i : A \to X_i)_{i \in I}$ to those where the same object X_i may appear multiple times.

Definition 3.1.2. *Let $A_- = (A_i)_{i \in I}$ be a family of objects. Then a **family with multiplicity** is a family of morphisms $(f_j : A \to X_{i(j)})_{j \in J}$ for some set J and function $i : J \to I$.*

We abbreviate such a family with multiplicity as $f_{(-)} : A \to X_{(-)}$.

Of course, the notion of square summability and Lemma 3.1.1 apply to families with multiplicity just as well.

3.2 Source, range and factorization theorems

We proceed by stating some basic results on source and range projections before developing a deeper criterion for one morphism in a W*-category to factor across another. Throughout this subsection, we work in an arbitrary W*-category C.

Definition 3.2.1. *Every morphism $f : X \to Y$ has:*

- ▷ *A **source projection** $s(f) : X \to X$ defined as the support projection of $f^* f$ in $\mathsf{C}(X, X)$.*
- ▷ *A **range projection** $r(f) : Y \to Y$ defined as the support projection of $f f^*$ in $\mathsf{C}(Y, Y)$.*

For concrete constructions, one can obtain $s(f)$ as the ultraweak limit of a suitable sequence of polynomials with vanishing constant coefficient applied to $f^* f$, and likewise for $r(f)$. Alternatively, the more explicit ultraweak limits

$$s(f) = \lim_{\varepsilon \to 0} f^* f \, (f^* f + \varepsilon)^{-1},$$
$$r(f) = \lim_{\varepsilon \to 0} f f^* \, (f f^* + \varepsilon)^{-1}$$

may be used. The source and range projections have the following elementary property.

Lemma 3.2.2. *(i) $s(f)$ is the smallest projection on X such that*

$$f\, s(f) = f.$$

(ii) $r(f)$ is the smallest projection on Y such that

$$r(f)\, f = f.$$

For a projection p, we write $p^\perp := \mathrm{id} - p$ for its complement.

Proof. For (i), standard W*-algebra theory applied to $\mathsf{C}(X, X)$ tells us that $s(f)^\perp$ is the largest projection with $f^* f\, s(f)^\perp = 0$. Then the claim follows since this equation is equivalent to $f\, s(f)^\perp = 0$ by

$$\left\| f\, s(f)^\perp \right\| = \left\| s(f)^\perp\, f^* f s(f)^\perp \right\| = 0.$$

Property (ii) follows by duality. □

Lemma 3.2.3. *For a composable pair of morphisms*

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z \tag{22}$$

the following are equivalent:

(i) $gf = 0$.
(ii) $g\, r(f) = 0$.
(iii) $s(g)\, f = 0$.
(iv) $s(g)\, r(f) = 0$.

Proof. All three upwards implications are obvious by the previous lemma. The downwards implications all work the same way, so let us just consider the first one explicitly. For this, it is enough to use the fact that $gf = 0$ implies $g(ff^*)^n$ for all $n \geq 1$, and $r(f)$ can be ultraweakly approximated by polynomials in ff^* with vanishing constant coefficient. □

By Lemma 3.2.3, the complementary source projection $s(g)^\perp$ can be thought of as the kernel of g, while $r(f)^\perp$ plays the role of the cokernel of f; we will make this precise in Lemma 3.7.11.

Further relevant properties to keep in mind are the equations

$$s(f) = s(f^* f) = s(\sqrt{f^* f}) = r(f^*),$$
$$r(f) = r(ff^*) = r(\sqrt{ff^*}) = s(f^*),$$

and these are straightforward to prove.

Lemma 3.2.4. *For parallel morphisms $f, g : X \to Y$,*

$$s(f+g) \leq s(f) \vee s(g),$$
$$r(f+g) \leq r(f) \vee r(g).$$

Proof. Lemma 3.2.2(i) implies the claim by

$$(f+g)(s(f)\vee s(g)) = f(s(f)\vee s(g)) + g(s(f)\vee s(g)) = 0,$$

and the second inequality works dually. □

Lemma 3.2.5. *For a composable pair of morphisms (22),*

$$s(gf)\le s(f), \qquad r(gf)\le r(g).$$

Proof. Straightforward again by Lemma 3.2.2. □

We can now already prove a surprising factorization result. As far as we know, it goes back for the case of $\mathcal{B}(\mathcal{H})$ to Douglas [31, Theorem 1] and to Westerbaan for the case of general W*-algebras [32, §81$_{\mathrm{V}}$]. It seems to be new for W*-categories.

Theorem 3.2.6. *For morphisms $f : A \to X$ and $g : A \to Y$, the following are equivalent:*

(i) There is $h : X \to Y$ such that $g = hf$.
(ii) There is a number $C \ge 0$ such that

$$g^*g \le C f^* f.$$

If these equivalent conditions hold, then there is a unique $h : X \to Y$ satisfying $g = hf$ and $s(h) \le r(f)$. It also satisfies $r(h) = r(g)$ and

$$\|h\| = \inf\{C \ge 0 \mid g^*g \le C f^*f\}.$$

Proof. Assuming (i), the inequality in (ii) follows with $C := \|h^*h\|$, since then $C\,\mathrm{id}_X - h^*h \ge 0$ in $\mathsf{C}(X,X)$, and hence

$$g^*g = f^*h^*hf \le C f^*f,$$

where we use the fact that compression by f preserves positivity.

So then assume that (ii) holds, and also assume $\|f\| \le 1$ without loss of generality. Then proving (i) is a bit more difficult, and we adapt an argument due to Westerbaan [32, §80$_{\mathrm{V}}$] to the W*-category setting. For $n \in \mathbb{N}$, let $|f|_n^{-2} : X \to X$ be the morphism obtained by applying the function

$$x \longmapsto \begin{cases} x^{-2} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n}, \\ 0 & \text{otherwise} \end{cases}$$

to $|f|$. Since these functions have disjoint support and sum to $x \mapsto x^{-2}$ for $x > 0$, functional calculus in $\mathsf{C}(X,X)$ shows that we have

$$\sum_{n=1}^{\infty} |f|_n^{-2} f^* f = s(f), \qquad \sum_{n=1}^{\infty} f|f|_n^{-2} f^* = r(f),$$

where the sums converge ultraweakly and their terms are pairwise orthogonal subprojections. Our goal is now to define

$$h := \sum_{n=1}^{\infty} g|f|_n^{-2} f^*,$$

and to use Lemma 2.1.18 in order to argue that this converges ultraweakly. The partial sum up to the k-th term satisfies

$$\begin{aligned}
\left\| \sum_{n=1}^{k} g|f|_n^{-2} f^* \right\|^2 &= \left\| \sum_{n,m=1}^{k} f|f|_n^{-2} g^* g |f|_n^{-2} f^* \right\| \\
&\leq C \left\| \sum_{n,m=1}^{k} f|f|_n^{-2} f^* f |f|_m^{-2} f^* \right\| \\
&= C \left\| \sum_{n=1}^{k} f|f|_n^{-2} f^* \right\| \\
&\leq C,
\end{aligned}$$

where the third step uses the pairwise orthogonality mentioned above. Therefore the defining series of h converges ultraweakly, and the ultraweak continuity of composition gives

$$hf = \sum_{n=1}^{\infty} g|f|_n^{-2} f^* f = g \sum_{n=1}^{\infty} |f|_n^{-2} f^* f = g s(g) = g,$$

as desired.

We now turn to the additional claims. Since $h^* h \leq C\, r(f)$ by the same calculation as in the norm estimate above, we indeed obtain $s(h) \leq r(f)$, and we have $\|h\| \leq C$ by the partial sum norm estimate above. For the uniqueness, suppose that $\tilde{h} : X \to Y$ is another morphism with $g = \tilde{h} f$ and $s(\tilde{h}) \leq r(f)$. Then the equation

$$(h - \tilde{h}) f = 0$$

implies $s(h - \tilde{h})\, r(f) = 0$ by Lemma 3.2.3. But since we also have $s(h - \tilde{h}) \leq r(f)$ by Lemma 3.2.4, we can conclude $s(h - \tilde{h}) = 0$, or equivalently $h - \tilde{h} = 0$.

Finally, we have $r(g) \leq r(h)$ by Lemma 3.2.5, and the other direction of inequality follows by $r(g) h = h$, which is itself a consequence of the uniqueness: since $\tilde{h} := r(g) h$ also satisfies the required properties, we obtain the claim. □

A good way to understand this result is in terms of a certain Hilbert copresheaf, which is of the Hilbert copresheaf of the GNS construction from Proposition 2.5.29. Indeed for given $f : A \to X$ and any object Y, consider

$$L^2(f, Y) := \{ g : A \to Y \mid \exists C \geq 0 \text{ s.t. } g^* g \leq C\, f^* f \},$$

which is a Banach space with respect to the norm $\|g\|$ defined as the smallest constant C which satisfies the given inequality. Furthermore, this construction is clearly functorial in Y, resulting in a functor $L^2(f,-) : \mathsf{C} \to \mathsf{Ban}$. As in the case of the GNS construction from Proposition 2.5.29, the existence of inner products turning this functor into a Hilbert copresheaf is not a priori obvious, but rather a consequence of the following.

Corollary 3.2.7. *There is an isometric isomorphism of functors*

$$L^2(f,-) \cong \mathsf{C}(X,-)\, r(f),$$

and in particular $L^2(f,-)$ is a Hilbert copresheaf.

Here, $\mathsf{C}(X,-)\, r(f)$ denotes the subfunctor of $\mathsf{C}(X,-)$ consisting of all morphisms h out of X which satisfy $h\, r(f) = h$, or equivalently $s(h) \leq r(f)$, and this subfunctor is a Hilbert copresheaf with respect to the induced inner product.[18]

Proof. For every Y, composition with f induces a map

$$\mathsf{C}(X,Y) \to L^2(f,Y),$$

and this is an isometric isomorphism by Theorem 3.2.6. □

From Theorem 3.2.6 we also automatically obtain a corresponding factorization result for Hilbert presheaves.

Corollary 3.2.8. *Let C be a W^*-category and $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ a small self-dual Hilbert presheaf. Then for $\alpha \in HX$ and $\beta \in HY$, the following are equivalent:*

(i) There is $f : Y \to X$ such that $\beta = \alpha f$.

(ii) There is $C \geq 0$ such that[19]

$$\beta\langle\beta,-\rangle \leq C\alpha\langle\alpha,-\rangle.$$

in $\hat{\mathsf{C}}(H,H)$.

Proof. Using Corollary 2.5.22, we take f and g to be given by

$$\langle\alpha,-\rangle : H \longrightarrow \mathsf{C}(-,X),$$
$$\langle\beta,-\rangle : H \longrightarrow \mathsf{C}(-,Y).$$

Then f^*f is the Hilbert transformation $H \to H$ given by $\alpha\langle\alpha,-\rangle$, and similarly for g^*g. This reduces the claim to Theorem 3.2.6. □

We obtain two useful criteria for witnessing smallness.

Corollary 3.2.9. *For a self-dual Hilbert presheaf H, the following are equivalent:*

(i) H is small.

(ii) There is a family of elements $\alpha_{\_} \in HX_{\_}$ such that the pre-Hilbert presheaf they generate is locally[20] *ultraweakly dense.*

[18] We will encounter the contravariant version of such functors in Section 3.7, and in particular in (53) and (54).

[19] Here, $\beta\langle\beta,-\rangle$ denotes the composite of the Hilbert transformation $\langle\beta,-\rangle : H \to \mathsf{C}(-,Y)$ with the Hilbert transformation $\beta- : \mathsf{C}(-,Y) \to H$, which is its adjoint. Likewise for $\alpha\langle\alpha,-\rangle$.

[20] That is, ultraweakly dense in every space HY.

(iii) The collection of Hilbert transformations $H \to H$ is a small set.

This equivalence seems surprising, as (ii) looks a priori much weaker than our definition of smallness.

Proof. Assuming (i), we trivially have (ii). Given (ii), the claimed (iii) follows since a Hilbert transformation $t : H \to H$ is uniquely determined by its values on the generating family α_{-}, and therefore $\hat{\mathsf{C}}(X, X)$ is small.

Finally, assume (iii). Then for every Hilbert transformation $H \to H$ which can be realized in the form $\beta\langle\beta, -\rangle$, choose some $\beta \in HX$ which witnesses this. Then this family of elements generates H by Corollary 3.2.8.[21] □

The following result is a slight generalization of [28, Lemma 3.2], and we adapt the proof correspondingly.

Lemma 3.2.10. *Let $u : X \to Y$ and $v : Y \to X$ be morphisms such that $\|u\|, \|v\| \leq 1$ and such that*

$$vu = s(u) = r(v), \qquad s(v) = r(u).$$

Then u and v are partial isometries with $v = u^*$.

Proof. Using the first assumed equation and the norm inequalities, we have

$$s(u) = u^* v^* v u \leq u^* u \leq s(u),$$

and hence $s(u) = u^* u$, making u a partial isometry. Similarly,

$$r(v) = vuu^* v^* \leq vv^* \leq r(v),$$

and hence v is a partial isometry. But then the second assumed equation gives $v^* v = uu^*$, and hence

$$v = vv^* v = vuu^* = s(u)\, u^* = u^*,$$

as was to be shown. □

There is a polar decomposition for morphisms going back to Ghez, Lima and Roberts [3, Corollary 2.7], who had proved it by reduction to polar decomposition in the linking W*-algebra $L(\mathsf{C}|_{X,Y})$. More in the spirit of W*-categories, we derive it as a consequence of our new factorization criterion based on the following variant of Theorem 3.2.6.

Theorem 3.2.11. *Let $f : A \to X$ and $g : A \to Y$ be such that $f^* f = g^* g$. Then there is a unique partial isometry $u : X \to Y$ such that:*

(i) $f = ug$,
(ii) $u^* u = r(f)$,
(iii) $uu^* = r(g)$.

[21] Technically, this application of Corollary 3.2.8 requires used it within the next Grothendieck universe in case that H is not small.

Proof. By Theorem 3.2.6, the equation $f^*f = g^*g$ gives us a unique $u : X \to Y$ such that $f = ug$ and $s(u) \leq r(g)$. This already establishes the uniqueness claim (under notably weaker hypotheses). Moreover, we know that this u satisfies $\|u\| \leq 1$ and $r(u) = r(f)$. Applying Theorem 3.2.6 the other way around likewise gives us a unique $v : Y \to X$ such that $g = vf$ and $s(v) \leq r(f)$, which also satisfies $\|v\| \leq 1$ and $r(v) = r(g)$.

Our goal is now to finish the proof by an application of Lemma 3.2.10. By $g = vug$ and $s(vu) \leq s(u) \leq r(g)$, the uniqueness part of Theorem 3.2.6 implies $vu = r(g)$. Therefore $r(g) \leq s(u)$ by Lemma 3.2.5, and hence $s(u) = r(g)$ since we already know $s(u) \leq r(g)$. Hence we have $vu = s(u) = r(v)$ as needed for Lemma 3.2.10. We likewise obtain $uv = s(v) = r(u)$, and hence the claim follows by Lemma 3.2.10 indeed. □

The polar decomposition of Ghez, Lima and Roberts follows now as an immediate special case.

Corollary 3.2.12 (Polar decomposition)**.** *For any morphism $f : X \to Y$, there is a unique partial isometry $u : X \to Y$ such that:*

(i) $f = u|f|$,
(ii) $u^*u = s(f)$,
(iii) $uu^* = r(f)$.

Proof. Apply Theorem 3.2.11 with $g = |f|$. □

Remark 3.2.13. We saw in Remark 2.1.5 that the involution on a C*-algebra is uniquely determined by its Banach algebra structure, and similarly in Proposition 2.5.28 that the inner products on a Hilbert presheaf are uniquely determined by its structure as a Ban-enriched presheaf. We can now show a similar statement: the involution on a W*-category is uniquely determined by its structure as a Ban-enriched category. Indeed on the endomorphism W*-algebras $\mathsf{C}(A, A)$, we can reconstruct the involution thanks to Remark 2.1.5. But then we in particular we know which morphisms are projections, and by Lemma 3.2.2 we therefore can reconstruct sources and ranges of all morphisms. Hence from Lemma 3.2.10 we can learn which morphisms are partial isometries as well as their adjoints. But this implies that we can recognize the polar decomposition of any given morphism and therefore compute its adjoint.

For future use, we record some observations on when a morphism intertwines two projections, or more generally representations of any W*-algebra.

Lemma 3.2.14. *Let $p : X \to X$ and $q : Y \to Y$ be projections, and let $f : X \to Y$ be a morphism with $\|f\| \leq 1$. Then the following are equivalent:*

(i) The inequalities

$$f^*qf \leq p, \qquad f^*q^\perp f \leq p^\perp.$$

(ii) $fp = qf$.

Proof. Assuming (i), we have $s(qf) = s(f^*qf) \leq p$, and hence $qfp = qf$. Likewise the other inequality gives $q^\perp fp^\perp = q^\perp f$, or equivalently $qfp = fp$. Hence (ii) holds. The converse is straightforward by $f^*qf = pf^*fp \leq p$, and similarly for the other inequality. □

This provides us with a way of characterizing intertwiners between representations of a W*-algebra in any W*-category, which will come in very handy later.

Proposition 3.2.15. *Let* C *be a W*-category and* N *a W*-algebra. Then for any two normal representations*

$$\pi : N \to \mathsf{C}(X, X), \qquad \rho : N \to \mathsf{C}(Y, Y),$$

we have

$$\mathsf{NRep}(N, \mathsf{C})((X, \pi), (Y, \rho)) = \{f : X \to Y \mid f^*\rho(a^*a)f \leq \|f\|^2 \pi(a^*a)\ \forall a \in N\}.$$

Proof. Recall that being an intertwiner means that

$$f\pi(a) = \rho(a)f \qquad \forall a \in N. \tag{23}$$

Clearly if this condition holds, then we have $f^*\rho(a^*a)f = \pi(a)^*f^*f\pi(a) \leq \|f\|^2\pi(a^*a)$, as needed. For the converse, we prove the intertwiner equation only in case that $a \in N$ is a projection, which is enough since N is generated by projections. But in this case, taking $p := \pi(a)$ and $q := \rho(a)$ gives the claim as an instance of Lemma 3.2.14. □

Question 3.2.16. *How does Proposition 3.2.15 generalize to a characterization of bounded natural transformations between W*-functors* $\mathsf{D} \to \mathsf{C}$*?*

3.3 Central supports

Ghez, Lima and Roberts had also considered the following different notion of support [3, Definition 5.1].

Definition 3.3.1. *For a family of objects* $A_\_ = (A_i)_{i \in I}$*, the* ***central support*** *at any object* X *is the smallest projection* $\mathrm{c}(A_\_)_X : X \to X$ *such that*

$$\mathrm{c}(A_\_)_X f = f \qquad \forall f : A_\_ \to X.$$

Equivalently, $\mathrm{c}(A_\_)_X$ is the supremum of the range projections of all morphisms $A_\exists \to X$. Taking adjoints shows that the defining equation is equivalent to $g\mathrm{c}(A_\_)_X = g$ for all $g : X \to A_\_$, and hence $\mathrm{c}(A_\_)_X$ is also the supremum of the source projections of all morphisms $X \to A_\exists$.

Lemma 3.3.2. *Let* $p : X \to X$ *be a projection with* $0 < p \leq \mathrm{c}(A_\_)_X$*. Then there is a partial isometry* $u : A_\exists \to X$ *with* $0 < uu^* \leq p$*.*

Proof. By the minimality property of $c(A_{\_})_X$, there must be a morphism $f : A_\exists \to X$ such that $pf \neq 0$. Using polar decomposition in the form $pf = u|pf|$ with a partial isometry $u : A_\exists \to X$, we obtain

$$uu^* = r(pf) \leq p,$$

and hence $pu = u$. Since $u \neq 0$ by $pf \neq 0$, we conclude that $0 < uu^* \leq p$. □

The following sequence of propositions exhibits the main properties of central supports, through which we will make use of them later. For the first one, recall the notion of *family with multiplicity* from Definition 3.1.2.

Proposition 3.3.3. *For every family of objects $A_{\_}$ and every object X, there is a family with multiplicity of partial isometries $u_{(\_)} : X \to A_{(\_)}$ such that*

$$c(A_{\_})_X = u_{(\_)} u^*_{(\_)}.$$

Note that such partial isometries are necessarily pairwise orthogonal.

Proof. By Zorn's lemma, there is a largest set of partial isometries $U \subseteq \mathsf{C}(A_\exists, X)$ with the property that

$$\sum_{u \in U} uu^* \leq c(A_{\_})_X.$$

Since $c(A_{\_})_X$ is a projection itself, the summands uu^* are pairwise orthogonal and the left-hand side is a projection too, so let us denote it by q. Now suppose that $q \neq c(A_{\_})_X$, so that $p := c(A_{\_})_X - q$ is a nonzero projection. Then an application of Lemma 3.3.2 shows that the set U was not maximal to begin with, a contradiction. □

Proposition 3.3.4. *For a morphism $f : X \to Y$, the following are equivalent:*

(i) $f\, c(A_{\_})_X = f$.

(ii) $c(A_{\_})_Y f = f$.

(iii) *There are square summable families with multiplicity $g_{(\_)} : X \to A_{(\_)}$ and $h_{(\_)} : Y \to A_{(\_)}$ such that*

$$f = h_{(\_)} g^*_{(\_)}.$$

Proof. As (iii) is self-dual, it is enough to prove the equivalence of (i) and (iii). Assuming (i), by Proposition 3.3.3 we have

$$f = f u_{(\_)} u^*_{(\_)},$$

where the bracketing does not matter thanks to ultraweak continuity of composition. Therefore taking $g_{(\_)} := u_{(\_)}$ and $h_{(\_)} := f u_{(\_)}$ works, where the square summability of both families with multiplicity follows by the square summability $u_{(\_)} u^*_{(\_)} < \infty$.

Assuming (iii), suppose that the families with multiplicity are indexed by $j \in J$. Then we have $s(h_j g_j^*) \leq s(g_j^*)$ for each j, and therefore

$$s(f) \leq \bigvee_{j \in J} s(g_j^*) \leq c(A_{\_})_X,$$

which is enough to get (i). □

We can now explain the term "central support": the **centre** of a category C is the set of natural transformations from id_{C} to itself, and we can now conclude that $\mathrm{c}(A_{\_})$ belong to this centre: it is a **central projection**.

Proposition 3.3.5. $\mathrm{c}(A_{\_})$ *is natural: for all* $f : X \to Y$*, the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{c}(A_{\_})_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\mathrm{c}(A_{\_})_Y} & Y \end{array}$$

commutes.

Proof. Proposition 3.3.4, and in particular the equivalence of (i) and (ii), shows that both composites equal $c(A)_Y f\, c(A)_X$. ☐

Proposition 3.3.6. *For any morphism* $f : X \to Y$*,*

$$\|f\,\mathrm{c}(A_{\_})_X\| = \|\mathrm{c}(A_{\_})_Y f\| = \sup_{u:A_\exists \to X} \|fu\| = \sup_{v:A_\exists \to Y} \|v^* f\| = \sup_{u:A_\exists \to X,\, v:A_\exists \to Y} \|v^* f u\|.$$

where the suprema range over all partial isometries of the specified type.

Proof. We only prove the equation $\|f\,\mathrm{c}(A_{\_})_X\| = \sup_{u:A_\exists \to X}\|fu\|$; the others follow by duality and naturality of central supports.

For the inequality direction $\geq$, it is enough to use the trivial $c(A_{\_})_X\, u = u$ for any partial isometry $u : A_\exists \to X$, since then

$$\|fu\| = \|f\,c(A_{\_})_X\, u\| \leq \|f\,c(A_{\_})_X\|.$$

The direction $\leq$ is more difficult. Again because of $c(A_{\_})_X\, u = u$, it is enough to prove the desired inequality with $f\,\mathrm{c}(A_{\_})_X$ in place of f. Therefore we can assume $f\,\mathrm{c}(A_{\_})_X = f$ without loss of generality, or equivalently $s(f) \leq \mathrm{c}(A_{\_})_X$. Under this assumption we will prove $\|f\| \leq \sup_{u:A_\exists \to X}\|fu\|$. Restricting further to $\|f\| = 1$ without loss of generality, it is then enough to prove that $\sup_{u:A_\exists \to X}\|fu\| \geq 1$.

For given $\varepsilon > 0$, let $p : X \to X$ be the spectral projection of $|f|$ with respect to the interval $[1-\varepsilon, 1]$. This projection satisfies $0 \neq p \leq s(f) \leq \mathrm{c}(A_{\_})_X$ and $f^*f \geq p - \varepsilon\,\mathrm{id}_X$, which implies

$$\sup_{u:A_\exists \to X}\|fu\| \geq \sup_{u:A_\exists \to X}\|f^*fu\| \geq \sup_{u:A_\exists \to X}\|(p-\varepsilon)u\| \geq \sup_{u:A_\exists \to X}\|pu\| - \varepsilon.$$

Since ε was arbitrary, the claim follows if we can show that $\sup_{u:A_\exists \to X}\|pu\| = 1$ for any nonzero projection $p : X \to X$ with $p \leq \mathrm{c}(A_{\_})_X$. Indeed an application of Lemma 3.3.2 provides us with a nonzero partial isometry $u : A_\exists \to X$ such that $pu = u$. Then $\|pu\| = \|u\| = 1$ as desired. ☐

For self-adjoint morphisms, a simpler formula can be given.

Proposition 3.3.7. *For any morphism $f : X \to X$ with $f^* = f$, we have*

$$\|f\,\mathrm{c}(A_\_)_X\| = \sup_{u:A_\exists \to X} \|u^* f u\|$$

and

$$f\,\mathrm{c}(A_\_)_X \geq 0 \qquad \Longleftrightarrow \qquad u^* f u \geq 0 \quad \forall u : A_\exists \to X.$$

where u ranges over partial isometries.

Proof. The norm formula follows by the C*-identity and an application of Proposition 3.3.6 to $|f|$. The characterization of positive elements is implied by the norm formula via the standard C*-algebra fact that $f\,\mathrm{c}(A_\_)_X \geq 0$ if and only if $\|\lambda \cdot \mathrm{id} - f\,\mathrm{c}(A_\_)_X\| \leq \lambda$ with $\lambda = \|f\,\mathrm{c}(A_\_)_X\|$, since the assumption implies that

$$u^*(\lambda \cdot \mathrm{id} - f\,\mathrm{c}(A_\_)_X)u \leq \lambda u^* u,$$

which gives $\|\lambda \cdot \mathrm{id} - f\,\mathrm{c}(A_\_)_X\| \leq \lambda$. □

Here is a useful consequence which already does not refer to central supports anymore.

Proposition 3.3.8. *Let $A_\_$ be a family of objects, and let $f_\_ : X \to A_\_$ and $g_\_ : Y \to A_\_$ be square summable. Then*

$$\|g_\_^* f_\_\| = \sup_{u:A_\exists \to X,\, v:A_\exists \to Y} \|v^* g_\_^* f_\_ u\|,$$

where the supremum is over all partial isometries u and v of the specified type.

Proof. This follows by Proposition 3.3.6 and the fact that

$$\mathrm{c}(A_\_)_Y g_\_^* f_\_ \mathrm{c}(A_\_)_X = g_\_^* f_\_,$$

which in turn holds because it holds for each summand $g_i^* f_i$ in place of $g_\_^* f_\_$ and ultraweak continuity of composition. □

3.4 Fullness theorems

Roberts proved in [33, Lemma 2.1] that if C is a W*-category and $A, X, Y \in \mathsf{C}$, then any bounded linear map

$$t \;:\; \mathsf{C}(A, X) \longrightarrow \mathsf{C}(A, Y) \tag{24}$$

which commutes with the canonical action of $\mathsf{C}(A, A)$ on both sides arises from a morphism $X \to Y$ (though generally not uniquely so). Note that this is once again a statement that is far from true in ordinary category theory. Our goal is to generalize Roberts's statement and proof to the setting of full W*-subcategories and Hilbert transformations.

Lemma 3.4.1. *Let C be a W*-category, $\mathsf{D} \subseteq \mathsf{C}$ a full W*-subcategory and $H \in \hat{\mathsf{C}}$. Then there is a (small) family of objects $B_\_ = (B_i)_{i \in I}$ in D and a family of elements $\beta_\_ : HB_\_$ such that:*

(i) With $\mathrm{c}(\mathsf{D})$ *denoting the central support in* $\hat{\mathsf{C}}$ *of the (potentially large) family of Hilbert presheaves represented by objects in* D*, we have*

$$\mathrm{c}(\mathsf{D})_H = \beta_{\text{-}}\langle \beta_{\text{-}}, - \rangle,$$

(ii) $\langle \beta_{i'}, \beta_i \rangle$ *is a nonzero projection whenever* $i' = i$ *and vanishes otherwise.*

Proof. Applying Corollary 3.5.9(i) to the potentially large family consisting of all objects in D gives a potentially large family of objects $B_{\text{-}} = (B_i)_{i \in I}$ in D and a family of partial isometries $u_{\text{-}} : H \to \mathsf{C}(-, B_{\text{-}})$ such that

$$\mathrm{c}(\mathsf{D})_H = u_{\text{-}}^* u_{\text{-}}. \tag{25}$$

By the Yoneda lemma, there is a family of elements $\beta_{\text{-}} \in HB_{\text{-}}$ such that $u_{\text{-}}^*(f) = \beta_{\text{-}} f$ and such that $\langle \beta_i, \beta_i \rangle$ is a projection for all i. By simply dropping all $i \in I$ with $\beta_i = 0$, we can assume all of these projections to be nonzero. Moreover, Since the adjoint of such a u_i is the Hilbert transformation $H \to \mathsf{C}(-, B_i)$ given by $\alpha \mapsto \langle \beta_i, \alpha \rangle$, we obtain the desired form

$$\mathrm{c}(\mathsf{D})_H = \beta_{\text{-}}\langle \beta_{\text{-}}, - \rangle.$$

Since each term in this sum is a projection and the sum is as well, also the orthogonality relation $\langle \beta_{i'}, \beta_i \rangle = 0$ for $i' \neq i$ follows.

While the index set I is a priori large, in every equation[22] $\alpha = \beta_{\text{-}}\langle \beta_{\text{-}}, \alpha \rangle$ only a small set of β_i's can make a nonzero contribution. Therefore by the smallness of H, we can assume without loss of generality that the index set I is small. □

This results in our generalization which we present now.

Theorem 3.4.2. *Let* C *be a W*-category and* $\mathsf{D} \subseteq \mathsf{C}$ *a full W*-subcategory. Then:*

(i) For every $H \in \hat{\mathsf{C}}$*, the restricted Hilbert presheaf* $H|_{\mathsf{D}}$ *is small self-dual.*

(ii) Writing

$$\hat{\mathsf{C}}(H, K)_{\mathsf{D}} := \{ t \in \hat{\mathsf{C}}(H, K) \mid t\, \mathrm{c}(\mathsf{D})_H = t \},$$

for $H, K \in \hat{\mathsf{C}}$*, the restriction of Hilbert presheaves from* C *to* D *defines a natural isometric isomorphism*

$$\begin{array}{ccc} \hat{\mathsf{C}}(H, K)_{\mathsf{D}} & \xrightarrow{\cong} & \hat{\mathsf{D}}(H|_{\mathsf{D}}, K|_{\mathsf{D}}) \\ t & \longmapsto & t|_{\mathsf{D}} \end{array} \tag{26}$$

that is compatible with composition in the sense that for all $H, K, L \in \hat{\mathsf{C}}$*, the diagram*

$$\begin{array}{ccc} \hat{\mathsf{C}}(H, K)_{\mathsf{D}} \times \hat{\mathsf{C}}(K, L)_{\mathsf{D}} & \longrightarrow & \hat{\mathsf{C}}(H, L)_{\mathsf{D}} \\ \downarrow \cong & & \downarrow \cong \\ \hat{\mathsf{D}}(H|_{\mathsf{D}}, K|_{\mathsf{D}}) \times \hat{\mathsf{D}}(K|_{\mathsf{D}}, L|_{\mathsf{D}}) & \longrightarrow & \hat{\mathsf{D}}(H|_{\mathsf{D}}, L|_{\mathsf{D}}) \end{array} \tag{27}$$

commutes, where the horizontal arrows are the (bilinear) composition maps.

[22]Note that we still have ultraweak convergence in this equation by Theorem 2.5.19(iv).

Since the condition $t\,\mathsf{c}(\mathsf{D})_H = t$ is equivalent to $\mathsf{c}(\mathsf{D})_K t = t$ by the naturality of central supports (Proposition 3.3.5), there is no asymmetry between H and K in (26), and this also makes clear that the right-hand side of the isomorphism is functorial in the obvious way in both H and K.

A good way to think about this result is that $\hat{\mathsf{D}}(H|_\mathsf{D}, K|_\mathsf{D})$ can be realized in a canonical way as a 1-complemented subspace of $\hat{\mathsf{C}}(H, K)$.

Proof. It is clear that if $H \in \hat{\mathsf{C}}$, then the restriction of H to a presheaf $H|_\mathsf{D} : \mathsf{D}^{\mathrm{op}} \to \mathsf{Ban}$ is still a Hilbert presheaf, where the relevant inner products are still the same as those of H. With this in mind, (ii) still makes sense even without (i) if one interprets $\hat{\mathsf{D}}(H|_\mathsf{D}, K|_\mathsf{D})$ as the space of Hilbert transformations, regardless of smallness and self-duality of the restricted Hilbert presheaves. We will prove this version of statement (ii) below, while we now indicate how to derive (i) from it.

For self-duality of $H|_\mathsf{D} : \mathsf{D}^{\mathrm{op}} \to \mathsf{Ban}$, consider a Hilbert transformation $t : H|_\mathsf{D} \to \mathsf{D}(-, X)$. Then (ii) shows that t is itself the restriction of a Hilbert transformation $H \to \mathsf{C}(-, X)$. This transformation has an adjoint by the self-duality assumption on H, and its restriction to D provides the required adjoint for t. For the smallness of $H|_\mathsf{D}$, we can apply (ii) together with the smallness criterion of Corollary 3.2.9.

The main statement to be proven is (ii). We first show that if a Hilbert transformation $t : H \to K$ satisfies $t\,\mathsf{c}(\mathsf{D})_H = t$, then $\|t\| = \|t|_\mathsf{D}\|$, which means that the restriction map is isometric. Indeed Proposition 3.3.6 yields

$$
\begin{aligned}
\|t\| &= \sup_{X,Y \in \mathsf{D}} \ \sup_{u:\mathsf{C}(-,X)\to H,\, v:\mathsf{C}(-,Y)\to H} \|v^* t u\| \\
&= \sup_{X,Y \in \mathsf{D}} \ \sup_{\alpha \in HX,\, \beta \in HY} \|\langle \beta, t(\alpha)\rangle\|,
\end{aligned}
$$

where the supremum over u and v is over partial isometries of the specified type, the second equality holds by the Yoneda lemma, and the supremum over α and β is correspondingly over elements with $\langle \alpha, \alpha\rangle, \langle \beta, \beta\rangle \in \{0, 1\}$. This implies the desired $\|t\| \leq \|t|_\mathsf{D}\|$ by the Cauchy-Schwarz inequality, while the other inequality direction is trivial.

It remains to be shown that every Hilbert transformation

$$
t \;:\; H|_\mathsf{D} \longrightarrow K|_\mathsf{D} \tag{28}
$$

can be extended to a Hilbert transformation $\tilde{t} : H \to K$ with $\tilde{t}\,\mathsf{c}(\mathsf{D})_H = \tilde{t}$. Using a family $\beta_\_$ as in Lemma 3.4.1, we put

$$
\tilde{t}(\alpha) \coloneqq t(\beta_\_)\,\langle \beta_\_, \alpha\rangle. \tag{29}
$$

To conclude the ultraweak convergence of this expression from Lemma 2.1.18, we must show that the partial sums are uniformly bounded in norm. Indeed the restriction to any finite subfamily satisfies

$$\begin{aligned}\|t(\beta_\text{-})\,\langle\beta_\text{-},\alpha\rangle\| &= \sup_{A\in\mathsf{D},\, v:A\to X} \|t(\beta_\text{-})\,\langle\beta_\text{-},\alpha\rangle\, v\| \\ &= \sup_{A\in\mathsf{D},\, v:A\to X} \|t(\beta_\text{-}\langle\beta_\text{-},\alpha v\rangle)\| \\ &\le \|t\| \sup_{A\in\mathsf{D},\, v:A\to X} \|\beta_\text{-}\langle\beta_\text{-},\alpha v\rangle\| \\ &\le \|t\|\,\|\alpha\|\end{aligned}$$

where the first equation, with v ranging over partial isometries $v : A \to X$, holds again by Proposition 3.3.8 and a Yoneda argument, and the second step crucially uses the assumed naturality of t on D. Hence the defining sum (29) converges ultraweakly and we obtain a Hilbert transformation $\tilde{t} : H \to K$. The relevant equation $\tilde{t}\,\mathrm{c}(\mathsf{D})_H = \tilde{t}$ follows from $\mathrm{c}(\mathsf{D})_H = \beta_\text{-}\langle\beta_\text{-},-\rangle$ and the pairwise orthogonality of the $\beta_\text{-}$.

To finish the proof of this claim, $\tilde{t}$ indeed recovers t on D since for all $\alpha \in HA$ with $A \in \mathsf{D}$, we have

$$\tilde{t}(\alpha) = t(u_\text{-})\langle u_\text{-},\alpha\rangle = t(u_\text{-}\langle u_\text{-},\alpha\rangle) = t(\alpha), \tag{30}$$

where the second step is again by the assumed naturality on D.

The final claim on compatibility with composition is clear by naturality of the central support $\mathrm{c}(\mathsf{D})$. □

Corollary 3.4.3. *Let C be a W*-category and $\mathsf{D} \subseteq \mathsf{C}$ a full W*-subcategory. Then restriction defines a full W*-functor $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$.*

Proof. The fact that restriction defines a $*$-functor is clear, while the ultraweak continuity holds as a consequence of the characterization of preduals of spaces of Hilbert transformations in the proof of Corollary 2.5.22. The fullness is part of Theorem 3.4.2(ii). □

Remark 3.4.4. If D contains a generating family, then the restriction W*-functor $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$ is even fully faithful. This can be thought of as saying that a generating family is automatically dense, although we will not given a formal definition of density in W*-category theory.[23]

The following immediate specialization of Corollary 3.4.3 recovers Roberts' fullness result mentioned at (24) upon specializing further to the case where D consists of a single object.

Corollary 3.4.5. *If $\mathsf{D} \subseteq \mathsf{C}$ is a full W*-subcategory, then the restricted Yoneda embedding*

$$\begin{aligned}\mathsf{C} &\longrightarrow \hat{\mathsf{D}} \\ X &\longmapsto \mathsf{C}(-,X)|_{\mathsf{D}}\end{aligned}$$

is a full W-functor.*

[23] Recall that in ordinary category theory, a full subcategory $\mathsf{D} \subseteq \mathsf{C}$ is dense if $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$ is fully faithful [9, Proposition 4.5.14]. In ordinary category theory, not every generating family is dense.

As a trivial space case with $\mathsf{C} = \mathsf{Hilb}$ and D consisting of $\mathbb{C}$ only, one recovers the fact that the morphisms $\mathcal{H} \to \mathcal{K}$ in Hilb are exactly the bounded linear maps.[24] Note also that f is not unique in general, as one can see already in Hilb with D consisting of the zero Hilbert space only.

We now turn to similar fullness results for W*-functors instead of Hilbert presheaves.

Theorem 3.4.6. *Let C be a W*-category and $\mathsf{D} \subseteq \mathsf{C}$ a full W*-subcategory. Then for every W*-category E and W*-functors $F, G : \mathsf{C} \to \mathsf{E}$, restriction from C to D defines an isometric isomorphism*

$$\{\alpha \in \mathsf{Fun}(\mathsf{C},\mathsf{E})(F,G) \mid \alpha_X\, F(\mathrm{c}(\mathsf{D})_X) = \alpha_X \;\; \forall X \in \mathsf{C}\} \xrightarrow{\;\cong\;} \mathsf{Fun}(\mathsf{D},\mathsf{E})(F|_{\mathsf{D}}, G|_{\mathsf{D}}).$$

Without any smallness assumption on the W*-categories involved, this will in general be an isometric isomorphism between large Banach spaces. Also, note that the equation $\alpha_X\, F(\mathrm{c}(\mathsf{D})_X) = \alpha_X$ for all $X \in \mathsf{C}$ can also be read as saying that the natural transformation α is invariant under horizontal post-composition with $\mathrm{c}(\mathsf{D})$ in the 2-category $\mathbb{W}^*\mathsf{CAT}$. By the naturality of $\mathrm{c}(\mathsf{D})$ from Proposition 3.3.5, we can also equivalently require $G(\mathrm{c}(\mathsf{D})_X)\, \alpha_X = \alpha_X$, which amounts to α being preserved by horizontal pre-composition with $\mathrm{c}(\mathsf{D})$.

Similar as for Theorem 3.4.2, the result says that $\mathsf{Fun}(\mathsf{D},\mathsf{E})(F|_{\mathsf{D}}, G|_{\mathsf{D}})$ can be realized in a canonical way as a 1-complemented subspace of $\mathsf{Fun}(\mathsf{C},\mathsf{E})(F,G)$.

Proof. As in the proof of Theorem 3.4.2(ii), we first show that this map is isometric. So for W*-functors $F, G : \mathsf{C} \to \mathsf{E}$, let $\alpha : F \to G$ be a bounded natural transformation such that $\alpha_X\, F(\mathrm{c}(\mathsf{D})_X) = \alpha_X$ for all $X \in \mathsf{C}$. We then need to prove that

$$\|\alpha\| = \sup_{Y \in \mathsf{D}} \|\alpha_Y\|. \tag{31}$$

Applying the C*-identity on both sides lets us reduce to the case where $G = F$ and $\alpha : F \to F$ is positive, and we assume this from now on. Then for any $X \in \mathsf{C}$, Proposition 3.3.3 gives us a family of objects $B_{\text{-}}$ in D and a family of partial isometries $u_{\text{-}} : X \to B_{\text{-}}$ such that

$$\mathrm{c}(\mathsf{D})_X = u_{\text{-}}^* u_{\text{-}}, \tag{32}$$

and hence

$$\begin{aligned} 0 \;\le\; \alpha_X \;&=\; \alpha_X F(u_{\text{-}}^*) F(u_{\text{-}}) \;=\; F(u_{\text{-}}) \alpha_{B_{\text{-}}} F(u_{\text{-}}^*) \\ &\le\; \sup_{Y \in \mathsf{D}} \|\alpha_Y\|\, F(u_{\text{-}}) F(u_{\text{-}}^*) \;=\; \sup_{Y \in \mathsf{D}} \|\alpha_Y\|\, F(\mathrm{c}(\mathsf{D})_X), \end{aligned}$$

where the second equation holds by naturality of α. Since $F(\mathrm{c}(\mathsf{D})_X)$ is a projection, this implies that $\|\alpha_X\| \le \sup_{Y \in \mathsf{D}} \|\alpha_Y\|$, and hence we get the desired (31).

[24] Basic examples like this already illustrate that Theorem 3.4.2 is far from true in ordinary category theory. For example if C is any category and D the full subcategory on a terminal object I, then the statement would be that every map $\mathsf{C}(I, X) \to \mathsf{C}(I, Y)$ is induced by a morphism $X \to Y$, which is clearly not true in general.

It remains to be shown that every $\alpha : F|_{\mathsf{D}} \to G_{\mathsf{D}}$ can be extended to a bounded natural transformation $\tilde{\alpha} : F \to G$ satisfying the invariance under $\mathrm{c}(\mathsf{D})$. For a given $X \in \mathsf{C}$, consider again (32) and let us define

$$\tilde{\alpha}_X \coloneqq G(u_{\_})\alpha_{B_{\_}} F(u_{\_}^*).$$

This converges ultraweakly by Lemma 2.1.18, where the uniform boundedness of the finite partial sums is a consequence of the C*-identity and the fact that the $u_{\_}^*$ are pairwise orthogonal partial isometries. This in fact shows that $\|\alpha_X\| \leq \sup_{Y \in \mathsf{D}} \|\alpha_Y\|$, which already establishes the boundedness. Moreover, $\tilde{\alpha}_X$ is clearly invariant under pre-composition by $\mathrm{c}(\mathsf{D})_X$ and post-composition by $\mathrm{c}(\mathsf{D})_Y$.

For naturality with respect to a morphism $f : X \to Y$, consider another family of partial isometries $v_{\_} : Y \to C_{\_}$ in D such that $\mathrm{c}(\mathsf{D})_Y = v_{\_}^* v_{\_}$. Then we need to prove that

$$\tilde{\alpha}_Y \, F(f) = G(f) \, \tilde{\alpha}_X,$$

or equivalently that

$$G(v_{\_}) \, \alpha_{C_{\_}} \, F(v_{\_}^* f) = G(f u_{\_}) \, \alpha_{B_{\_}} \, F(u_{\_}^*).$$

By the invariance under $\mathrm{c}(\mathsf{D})$, it is enough to prove this equation upon pre-composition by $F(w)$ for any partial isometry $w : D \to X$ in C, which reduces the problem to showing that

$$G(v_{\_}) \, \alpha_{C_{\_}} \, F(v_{\_}^* f w) = G(f u_{\_}) \, \alpha_{B_{\_}} \, F(u_{\_}^* w)$$

holds. But now naturality of α on D applies and further reduces the problem to showing that

$$G(v_{\_} v_{\_}^* f w) \, \alpha_D = G(f u_{\_} u_{\_}^* w) \, \alpha_D,$$

which is of course true by $v_{\_} v_{\_}^* f w = \mathrm{c}(\mathsf{D})_Y \, f = f \, \mathrm{c}(\mathsf{D})_X = u_{\_} u_{\_}^* w$. In particular, applying this naturality proof with $f = \mathrm{id}_X$ shows that $\tilde{\alpha}_X$ is independent of the choice of the family $u_{\_}$.

Finally, we need to prove that $\tilde{\alpha}_X = \alpha_X$ for all $X \in \mathsf{D}$. But this holds trivially, since then we have $\mathrm{c}(\mathsf{D})_X = \mathrm{id}_X$ and we can take $u_{\_}$ to be the singleton family consisting of id_X only. □

Corollary 3.4.7. *Let C be a W*-category and $\mathsf{D} \subseteq \mathsf{C}$ a full W*-subcategory. Then for every W*-category E, the restriction functor*

$$\mathsf{Fun}(\mathsf{D}, \mathsf{E}) \longrightarrow \mathsf{Fun}(\mathsf{C}, \mathsf{E}) \tag{33}$$

is full.

Proof. Clear from Theorem 3.4.6. □

If D and C are small, then the categories appearing in (33) are W*-categories, and in this case the restriction functor is of course a W*-functor.

Remark 3.4.8. Theorem 3.4.2(ii) and Theorem 3.4.6 are quite similar in both the statements and the proofs. In particular, both establish a fullness result by providing a *canonical* extension of a given transformation between restricted functors.[25] Therefore, it is tantalizing to think that there may be a common generalization of both theorems. However, we have not been able to find such a generalization.

Analyzing similar problems for non-full W*-functors $F : \mathsf{D} \to \mathsf{C}$ appears much more difficult. However, we do have some results in the special case $\mathsf{D} = \mathfrak{B}\mathbb{C}$ and $\mathsf{C} = \mathfrak{B}N$ for a W*-algebra N, in which case the restriction functor (33) becomes the forgetful W*-functor

$$\mathsf{NRep}(N, \mathsf{E}) \longrightarrow \mathsf{E}.$$

In the following, we write C again in place of E. We then have the following result about this kind of situation, which amounts to an equivariant version of Theorem 3.2.6.

Theorem 3.4.9. *Let C be a W*-category, N any W*-algebra and*

$$\pi \ : \ N \to \mathsf{C}(X, X), \qquad \rho \ : \ N \to \mathsf{C}(Y, Y)$$

two normal representations. Then for arbitrary morphisms $f : X \to Y$ and $g : Y \to X$, the following are equivalent:

(i) There is an intertwiner $h : X \to Y$ such that $g = hf$ and $\|h\| \leq 1$.
(ii) We have

$$g^* \rho(-)\, g \ \leq_{\mathrm{CP}} \ f^* \pi(-)\, f,$$

where $\leq_{\mathrm{CP}}$ means that the difference of the two sides is completely positive as a map $N \to \mathsf{C}(S, S)$.

Proof. Given (i), we get (ii) via

$$g^* \rho(-)\, g = f^* h^* \rho(-)\, hf = f^* h^* h\, \pi(-)\, f \leq_{\mathrm{CP}} f^* \pi(-)\, f,$$

where we have used the assumptions that h is an intertwiner, and the inequality step holds because h is an intertwiner and

$$(1 - h^* h)\, \pi(-) = (1 - h^* h)^{1/2}\, \pi(-)\, (1 - h^* h)^{1/2} \geq_{\mathrm{CP}} 0.$$

Assuming (ii), the idea is to try and construct a map $\mathsf{C}(S, X) \to \mathsf{C}(S, Y)$ which takes

$$\pi(a) f r \longmapsto \rho(a) g r \qquad \forall a \in N,\, r \in \mathsf{C}(S, S). \tag{34}$$

To this end, consider the space

$$W := \overline{\pi(N) f \mathsf{C}(S, S)} \subseteq \mathsf{C}(S, X),$$

by which we mean the norm closed linear span of all morphisms of the form $\pi(a) f r$ as above. We then claim that there is a unique contraction $t : W \to \mathsf{C}(S, Y)$ which acts

[25] It reminds us of an orthogonal projection to a subspace of a Hilbert space.

like (34). Indeed for any finite family of elements $(a_i)_{i\in I}$ in N and $(r_i)_{i\in I}$ in $\mathsf{C}(S,S)$, we have in $\mathsf{C}(S,S)$

$$\begin{aligned}(\rho(a_\_)gr_\_)^*(\rho(a_\_)gr_\_) &= r_\_^* g^* \rho(a_\_^* a_\_) g r_\_ \\ &\le r_\_^* f^* \pi(a_\_^* a_\_) f r_\_ \\ &= (\pi(a_\_)fr_\_)^*(\pi(a_\_)fr_\_),\end{aligned}$$

where the inequality holds by the assumed $\le_{\mathrm{CP}}$. This establishes that the desired map is well-defined and contractive on a dense subspace of W, and therefore extends uniquely to a contraction $t : W \to \mathsf{C}(S,Y)$.

Moreover, the left action of N and right action of $\mathsf{C}(S,S)$ make $\mathsf{C}(S,X)$ into a Hilbert bimodule, and it is clear that W is a Hilbert subbimodule thereof. With $v : W \hookrightarrow \mathsf{C}(S,X)$ denoting the isometric inclusion, we can consider the composite

$$\mathsf{C}(S,X) \xrightarrow{v^*} W \xrightarrow{t} \mathsf{C}(S,Y) \tag{35}$$

in the category $\mathsf{HilbBiMod}(N,\mathsf{C}(S,S))$. By construction, this composite implements the desired mapping (34).

Finally, consider the commutative diagram of W*-categories and W*-functors given by

$$\begin{array}{ccc} \mathsf{NRep}(N,\mathsf{C}) & \longrightarrow & \mathsf{C} \\ \downarrow & & \downarrow \\ \mathsf{HilbBiMod}(N,\mathsf{C}(S,S)) & \longrightarrow & \mathsf{HilbMod}(\mathsf{C}(S,S)) \end{array}$$

where all arrows are the obvious forgetful W*-functors. By Theorem 3.4.2 and the Yoneda embedding, the right vertical arrow has a canonical section whose image consists of those morphisms in C which are invariant under the central support $c(S)$. By the naturality of $c(S)$, the same follows for the left vertical arrow, and in particular the left vertical arrow is full as well, and every morphism can be lifted to a morphism of the same norm. Applying this to the composite (35) produces the desired intertwiner h. ☐

3.5 Generating families

Generating objects and generating families of objects are a classical concept in ordinary category theory, and there are several different notions with subtle technical distinctions [34]. As it happens likewise with other categorical concepts in W*-category theory, these subtle distinctions no longer exist: as we will see, the formal theory of generators in W*-categories is quite simple and rigid.

In the following definition, we denote the generating family by "$S_\_$" in order to indicate that they *separate* the category.[26]

Corollary 3.5.1. *For a family of objects $S_\_ = (S_i)_{i\in I}$ in a W*-category C, the following are equivalent:*

[26] Note that we reserve the letter G for functors.

(i) $c(S_\_) = \mathrm{id}_{\mathrm{id}_{\mathsf{C}}}$.

(ii) The hom-functors $\mathsf{C}(S_\_, -) : \mathsf{C} \longrightarrow \mathsf{Ban}$ *are jointly faithful.*

(iii) $fg = 0$ *for all* $g : S_\exists \to X$ *implies* $f = 0$.

(iv) $fu = 0$ *for all partial isometries* $u : S_\exists \to X$ *implies* $f = 0$.

The proof of the equivalence of these properties is immediate from the results on central supports of the previous subsection.

Definition 3.5.2. *A* ***generating family*** *is a family of objects* $S_\_$ *satisfying the equivalent conditions of Corollary 3.5.1. If a generating family consists of a single object, then that object is called a* ***generator****.*

Probably the earliest reference on generators in W*-categories is Rieffel [5, p. 54], who had investigated them in categories of Hilbert modules. The general definition of generator then appears in [3, Proposition 7.3]. More recently, generating families were considered by Yamagami [35, Lemma 1.5].

Example 3.5.3. In Hilb, every nonzero object is a generator.

Example 3.5.4. If C is small, then the family of all objects C is trivially a generating family.

Example 3.5.5. If C is small, then the family of representable Hilbert presheaves $\mathsf{C}(-, X)$ is a generating family for the W*-category of Hilbert presheaves $\hat{\mathsf{C}}$ by the Yoneda lemma.

Example 3.5.6. If A is merely a C*-algebra, then a faithful representation of A is *not* necessarily a generator in its W*-category of representations $\mathsf{Rep}(A)$ (see Example 2.4.2 for background on this W*-category). Counterexamples exist even for a commutative C*-algebra like $A = C([0, 1])$. Indeed if we consider $(\mathcal{H}, \rho)$ to be the GNS representation associated to the state induced by any atomic probability measure on $[0, 1]$ with dense support, then ρ is faithful because continuous functions are determined by their values on a dense subset. But there is also the representation of $C([0, 1])$ by multiplication on $L^2([0, 1])$, which is the GNS representation associated to the Lebesgue measure considered as a state. Since the only intertwiner $\mathcal{H} \to L^2([0, 1])$ is the zero map, we conclude that $(\mathcal{H}, \rho)$ is not a generator despite being a faithful representation.

Nevertheless, generators in arbitrary W*-categories of the form $\mathsf{NRep}(N)$ exist, as we will see in Lemma 3.9.9.

Example 3.5.7. Let C_1 and C_2 be W*-categories with generators $S_1 \in \mathsf{C}_1$ and $S_2 \in \mathsf{C}_2$. Then the coproduct W*-category $\mathsf{C}_1 + \mathsf{C}_2$ (see Example 2.4.10) has a generating family given by $\{S_1, S_2\}$. However, $\mathsf{C}_1 + \mathsf{C}_2$ does not have a single generator unless C_1 or C_2 is trivial (in the sense that all morphisms are zero).

Let us consider examples of W*-categories in which no generating family exists.

Example 3.5.8. Let N be a large W*-algebra, such as $N = \ell^\infty(S)$ for a large set S. Then $\mathsf{NRep}(N)$ is a locally small W*-category that does not have a generating family.

Indeed by the upcoming Lemma 3.9.9—which still applies with N large—such a generator would have to be a faithful representation. But since N clearly does not have a faithful representation on a (small) Hilbert space, we conclude that no generator exists. By our our upcoming results on direct sums (Example 3.6.12 and Remark 3.6.14), we can hence conclude further that no generating family exists either.

We return to the general theory, noting some direct consequences of Propositions 3.3.3 to 3.3.7.

Corollary 3.5.9. *Let $S_\_$ be a generating family in a W*-category.*

(i) For any object X, there is a family with multiplicity of partial isometries $u_{(\_)} : X \to S_{(\_)}$ such that

$$\mathrm{id}_X = u_{(\_)}^* u_{(\_)}.$$

(ii) Every morphism $f : X \to Y$ can be written as an ultraweakly convergent series

$$f = h_{(\_)}^* g_{(\_)}$$

for suitable square summable families with multiplicity $g_{(\_)} : X \to S_{(\_)}$ and $h_{(\_)} : Y \to S_{(\_)}$.

(iii) For every morphism $f : X \to Y$,

$$\|f\| = \sup_{u : S_\exists \to X,\, v : S_\exists \to Y} \|v^* f u\|,$$

where u and v range over partial isometries of the specified type.

(iv) For every morphism $f : X \to X$ with $f = f^*$,

$$\|f\| = \sup_{u : S_\exists \to X} \|u^* f u\|,$$

and

$$f \geq 0 \qquad \Longleftrightarrow \qquad u^* f u \geq 0 \quad \forall u : S_\exists \to X,$$

where u ranges over partial isometries of the specified type.

Example 3.5.10. In Hilb, applying (iii) to $\mathbb{C}$ as a generator recovers a standard formula for the operator norm, namely that for any bounded linear map between Hilbert spaces $f : \mathcal{H} \to \mathcal{K}$,

$$\|f\| = \sup_{\phi, \psi} |\langle \psi, h\phi \rangle|,$$

where ϕ and ψ ranges over unit vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively.

We now record an astonishing consequence, namely that every small self-dual Hilbert presheaf has an orthonormal basis.

Definition 3.5.11. *An* ***orthonormal basis*** *of a self-dual Hilbert presheaf* $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ *is a family of objects* $B_{\_}$ *in* C *together with a family of elements* $\beta_{\_} \in HB_{\_}$ *such that:*

(i) $\langle \beta_{i'}, \beta_i \rangle = 0$ *whenever* $i' \neq i$.

(ii) $\langle \beta_i, \beta_i \rangle$ *is a nonzero projection for all* i.

(iii) For every $X \in \mathsf{C}$ *and* $\alpha \in HX$,

$$\alpha = \beta_{\_}\langle \beta_{\_}, \alpha \rangle \tag{36}$$

where the right-hand side converges ultraweakly.

This generalizes the definition of orthonormal basis for a self-dual Hilbert module over a W*-algebra [36, Lemma 8.5.23]. Interestingly, Paschke's original formulation of the definition [21, Theorem 3.12] considers direct sums decompositions into singly generated Hilbert modules instead. For our Hilbert presheaves, the analogous statement is that the data of an orthonormal basis for H is equivalent to a direct sum decomposition of H in $\hat{C}$ into subobjects of representable Hilbert presheaves; this clearly matches with Definition 3.5.11 with the transformations

$$\beta_i \;:\; \mathsf{C}(-, B_i) \longrightarrow H$$

as direct sum inclusions.

For self-dual Hilbert modules, Paschke proved that orthonormal bases always exist [21, Theorem 3.12]. Our W*-categorical methods have pretty much already established this in our more general setting.

Theorem 3.5.12. *Let* C *be a W*-category. Then every small self-dual Hilbert presheaf* $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ *has an orthonormal basis.*

Proof. This is Lemma 3.4.1 applied with $\mathsf{D} = \mathsf{C}$, since then $\mathrm{c}(\mathsf{C})_H = \mathrm{id}_H$ gives us $\mathrm{id}_H = \beta_{\_}\langle \beta_{\_}, - \rangle$. By the ultraweak continuity of evaluation from Theorem 2.5.19(iv), the claimed (36) holds as well. □

Remark 3.5.13. For self-dual Hilbert presheaves that are not necessarily small, the same arguments still go through, but now the orthonormal basis itself may have to be indexed by a large set.

Finally, let us record special cases of our fullness theorems in the case where the subcategory is generating.

Corollary 3.5.14. *Let* C *be a W*-category and* $\mathsf{D} \subseteq \mathsf{C}$ *a generating full W*-subcategory. Then:*

(i) The restriction W-functor* $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$ *is fully faithful.*[27]

[27] We will prove in Corollary 3.8.28 that it is actually a W*-equivalence.

(ii) For every W-category* E*, the restriction functor*

$$\mathsf{Fun}(\mathsf{C},\mathsf{E}) \longrightarrow \mathsf{Fun}(\mathsf{D},\mathsf{E})$$

is fully faithful.

Proof. Upon using that $\mathrm{c}(\mathsf{D}) = \mathrm{id}$ in both cases, these claims follow directly by Theorems 3.4.2 and 3.4.6. ☐

In particular, every W*-functor $F : \mathsf{C} \to \mathsf{E}$ is the (two-sided) Kan extension of its restriction $\mathsf{D} \to \mathsf{E}$ along the inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ in the 2-category $\mathbb{W}^*\mathsf{CAT}$. For some purposes, it is useful to have a more explicit description of what this Kan extension looks like. By the Yoneda lemma, this amounts to describing morphisms out of FX for $X \in \mathsf{C}$, which we now turn to.

Lemma 3.5.15. *Let* $\mathsf{D} \subseteq \mathsf{C}$ *be a generating full W*-subcategory and* $F : \mathsf{C} \to \mathsf{E}$ *any W*-functor. Then for any* $X \in \mathsf{C}$ *and morphism* $f : Y \to FX$ *in* E*, there are square summable families of morphisms* $g_{\_} : S_{\_} \to X$ *with* $S_{\_}$ *in* D *and* $h_{\_} : Y \to FS_{\_}$ *such that*

$$f = F(g_{\_})h_{\_}.$$

Proof. Using a decomposition $\mathrm{id}_X = u_{\_}^* u_{\_}$, where $u_{\_} : X \to S_{\_}$ is a family of partial isometries with $S_{\_}$ in D as in Corollary 3.5.9(i), we obtain

$$f = F(u_{\_}^*)F(u_{\_})f,$$

so that taking $g_{\_} := u_{\_}^*$ and $h_{\_} := F(u_{\_})f$ works. ☐

Assuming that D is small, let us use this to characterize the hom-space $\mathsf{E}(Y, FX)$ more precisely. To this end, we consider formal linear combinations of the form[28]

$$\sum_{i\in I} g_i \otimes h_i,$$

for square summable families of morphisms $g_{\_} : S_{\_} \to X$ and $h_{\_} : Y \to FS_{\_}$ as in Lemma 3.5.15. The space of these formal linear combinations carries a $\mathsf{E}(Y,Y)$-valued sesquilinear form given by

$$\left\langle \sum_{j\in J} g'_j \otimes h'_j, \sum_{i\in I} g_i \otimes h_i \right\rangle := \sum_{i\in I, j\in J} h'^*_j F(g'^*_j g_i) h_i.$$

[28]Formally, such a formal linear combination is formally given by a function

$$c : \left(\coprod_{A\in\mathsf{D}} \mathsf{C}(X,A) \times \mathsf{E}(Y,FA) \right) \longrightarrow \mathbb{C},$$

given by assigning to very pair of morphisms its coefficient in the formal linear combination and such that the relevant square summability conditions

$$\sum_{(g,h)} |c(g,h)|^2 gg^* < \infty, \qquad \sum_{(g,h)} |c(g,h)|^2 h^*h < \infty$$

hold. But we will not use this more cumbersome notation.

This sesquilinear form is positive semidefinite, as is straightforward to see e.g. from the formalism of bounded matrices of Definition 3.6.17 and after. Of course it typically has quite a large null space, for example containing formal linear combinations like[29]

$$gk \otimes h - g \otimes F(k)h$$

for any $g : B \to X$ and $h : Y \to FA$ and $k : A \to B$. We can now get to the point of considering all this.

Proposition 3.5.16. *The map*

$$\sum_{i \in I} g_i \otimes h_i \longmapsto \sum_{i \in I} F(g_i) h_i$$

defines an isometric isomorphism between $\mathsf{E}(Y, FX)$ *and these linear combinations modulo the null space, which thereby also form an* $\mathsf{E}(Y, Y)$*-Hilbert module.*

Proof. The inner product of these formal linear combinations was defined precisely in such a way that the isometry holds. The surjectivity of this map is a consequence of Lemma 3.5.15. □

The fullness and faithfulness statements in Corollary 3.5.14 are not to be confused with the following.

Lemma 3.5.17. *Let* C *and* E *be W*-categories and* $\mathsf{D} \subseteq \mathsf{C}$ *a generating full W*-subcategory. Then for a W*-functor* $F : \mathsf{C} \to \mathsf{E}$*, we have:*

(i) F *is faithful if and only if* $F|_{\mathsf{D}} : \mathsf{D} \to \mathsf{E}$ *is.*
(ii) F *is fully faithful if and only if* $F|_{\mathsf{D}} : \mathsf{D} \to \mathsf{E}$ *is.*

Proof. The "only if" directions are trivial, so we focus on the converses.

(i) We assume that $F|_{\mathsf{D}} : \mathsf{D} \to \mathsf{E}$ is faithful and show the same for F. To see that also F is faithful, suppose that $f : X \to Y$ in C is such that $F(f) = 0$. Then by functoriality, we also have $F(v^* f u) = 0$ for all $u : A \to X$ and $v : B \to Y$ with $A, B \in \mathsf{D}$. By the assumed faithfulness of $F|_{\mathsf{D}}$, this gives $v^* f u = 0$. Since u and v were arbitrary and D is generating, we get $f = 0$.

(ii) We assume that $F|_{\mathsf{D}} : \mathsf{D} \to \mathsf{E}$ is fully faithful and show that F is full too. So let $f : F(X) \to F(Y)$ in E for $X, Y \in \mathsf{C}$. By Corollary 3.5.9(ii), we may assume $Y \in \mathsf{D}$ without loss of generality. Upon writing $\mathrm{id}_X = u_-^* u_-$ for a suitable family of partial isometries $u_- : X \to S_-$ with the S_- in D, let $g_-^* : S_- \to Y$ be the unique family of morphisms in D such that

$$F(g_-^*) = f\, F(u_-^*).$$

By the assumed faithfulness of F, this family is still square summable since the family $F(g_-^*)$ is. Now $h := g_-^* u_- : X \to Y$ is a morphism in C such that

$$F(h) = F(g_-^*)\, F(u_-) = f\, F(u_-^*)\, F(u_-) = f,$$

and hence F is full. □

[29] The similarity to tensor products is not at all accidental, and we will return to this point in Section 4.2.

3.6 Direct sums and the direct sum completion

In Hilb, there are *direct sums*. Given a family $\mathcal{H}_\_ = (\mathcal{H}_i)_{i\in I}$ of Hilbert spaces, their direct sum is given by

$$\bigoplus_{i\in I} \mathcal{H}_i \coloneqq \left\{ (\phi_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}_i \;\middle|\; \sum_{i\in I} \|\phi_i\|^2 < \infty \right\}. \tag{37}$$

It is a Hilbert space with respect to the pointwise vector space structure and inner product given by the sum of the componentwise inner products,

$$\langle \phi, \psi \rangle \coloneqq \sum_{i\in I} \langle \phi_i, \psi_i \rangle.$$

It is worth noting explicitly that I may be uncountable, in which case each element of the direct sum has at most countable support by the square summability condition. The inner product converges absolutely by a first application of the Cauchy-Schwarz inequality in $\mathcal{H}_i$ and a second application of Cauchy-Schwarz in $\ell^2(I)$. The completeness is a simple consequence of the completeness of the summands $\mathcal{H}_i$.

We will next state the general definition of direct sums in W*-categories. We follow [28] in using the universal property as the definition, and we will subsequently state the equivalence with the original definition given in [3]. To begin, for any finite or infinite family of objects $(X_i)_{i\in I}$ and any other object A in a W*-category C, we define the Banach space of square summable families of morphisms,

$$\bigoplus_{i\in I} \mathsf{C}(A, X_i) \; \coloneqq \; \left\{ (f_i : A \to X_i)_{i\in I} \;\middle|\; \sum_{i\in I} f_i^* f_i < \infty \right\}.$$

With $A = \mathbb{C}$ in Hilb, this specializes to (37). In general, $\bigoplus_{i\in I} \mathsf{C}(A, X_i)$ is a vector space with respect to the componentwise operations, and in addition a Banach space with respect to the norm $(f_i) \mapsto \|\sum_i f_i^* f_i\|^{1/2}$ [28, Lemma 4.1]. This generalizes the norm on the Hilbert space $\ell^2(I)$, since with $A = X_i = \mathbb{C}$ in Hilb, our $f_\_$ is exactly an element of $\ell^2(I)$. By our underscore notation, this norm can also be formally written as

$$\|f_\_\| \coloneqq \|f_\_^* f_\_\|^{\frac{1}{2}}.$$

Our underscore notation also applies to the direct sums themselves, for which we write

$$\bigoplus \mathsf{C}(A, X_\_) = \bigoplus_{i\in I} \mathsf{C}(A, X_i).$$

Precomposing a square summable family $f_\_ : A \to X_\_$ with any $g : B \to A$ results in another square summable family $f_\_ g \coloneqq (f_i g)_{i\in I}$, now in $\mathsf{C}(A, X_\_)$, which satisfies

$$\|f_\_ g\| \leq \|f_\_\| \, \|g\|.$$

It follows that mapping A to the set of square summable families with values in $X_\_$ is a functor

$$\bigoplus \mathsf{C}(-, X_\_) \; : \; \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}.$$

We aim at showing that this functor can be made into a Hilbert presheaf on C. By Proposition 2.5.28, there is at most one way to choose inner products in order for this to be the case.

Proposition 3.6.1. $\bigoplus \mathsf{C}(-, X_{\_}) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ *is a Hilbert presheaf with inner product given by*

$$\langle g_{\_}, f_{\_} \rangle := g_{\_}^* f_{\_}.$$

For $|I| = 1$, this of course recovers the representable Hilbert presheaves from Example 2.5.2.

Proof. Straightforward. □

Applying the Cauchy-Schwarz inequality for Hilbert presheaves (14) then sharpens Lemma 3.1.1 to the stronger inequality

$$f_{\_}^* g_{\_} g_{\_}^* f_{\_} \leq \|g\|^2 f_{\_}^* f_{\_}.$$

Definition 3.6.2 ([28])**.** *Given a W*-category* C *and an arbitrary family* $X_{\_} = (X_i)_{i \in I}$ *of objects in* C*, a* ***direct sum*** *is any object* $X_{\oplus}$ *which represents the* Ban*-enriched functor*

$$\bigoplus \mathsf{C}(-, X_{\_}) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}.$$

By Lemma 2.5.26, such a representation is automatically also a representation of $\bigoplus \mathsf{C}(-, X_{\_})$ as a Hilbert presheaf, and this implies uniqueness up to unique unitary isomorphism.

Let us spell out the definition. The defining property of the representing object $X_{\oplus}$ is that there must be isometric isomorphisms

$$\mathsf{C}(A, X_{\oplus}) \cong \bigoplus \mathsf{C}(A, X_{\_}) \tag{38}$$

that are natural in A. Thus, having a morphism into a direct sum is the same thing as having a square summable family of morphisms into its summands.

Notation 3.6.3. *If* $X_{\_}$ *is any family of objects as above and* $X_{\oplus}$ *is their direct sum, then we also denote the defining isometric isomorphism* (38) *by* $f_{\_} \mapsto f_{\oplus}$*.*

Thus every square summable family of morphisms $f_{\_}$ maps to the single morphism $f_{\oplus}$ and conversely, and this correspondence must satisfy

$$\|f_{\oplus}\| = \|f_{\_}\|.$$

Following [28], we now relate Definition 3.6.2 to the original definition of direct sum given by Ghez, Lima and Roberts [3, p. 100]. To this end, note that for every family of objects $X_{\_} = (X_i)_{i \in I}$, there is a canonical matrix of morphisms

$$\kappa_{\_\_} = (\kappa_{i'i} : X_i \to X_{i'})_{i, i' \in I}$$

with entries given by

$$\kappa_{ii'} \coloneqq \begin{cases} \mathrm{id}_{X_i} & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $i \in I$, applying the isomorphism (38) with $A = X_i$ to the trivially square-summable family of morphisms $\kappa_{\_i}$ on the right produces a morphism $\kappa_{\oplus i} : X_i \to X_\oplus$. We think of the resulting family $\kappa_{\oplus\_} : X_\_ \to X_\oplus$ as a row vector.

Theorem 3.6.4 ([28, Theorem 5.1])**.** *Let $X_\_ = (X_i)_{i\in I}$ be a family of objects in a W*-category. Then the following pieces of structure on an object $X_\oplus$ are equivalent:*

(i) Families of isometries $\kappa_{\oplus\_} : X_\_ \to X_\oplus$ such that

$$\kappa_{\oplus\_}\kappa^*_{\oplus\_} = \mathrm{id}_{X_\oplus}. \tag{39}$$

(ii) A natural isometric isomorphism

$$\mathsf{C}(-, X_\oplus) \;\cong\; \bigoplus \mathsf{C}(-, X_\_). \tag{40}$$

By (39), the isometries $\kappa_{\oplus i}$ in this definition are necessarily pairwise orthogonal,

$$\kappa^*_{\oplus i}\kappa_{\oplus i'} = \delta_{ii'}\mathrm{id}_{X_i}.$$

We now give a concise proof based on the methods developed so far. These facilitate significantly shorter arguments than the ones used in [28].

Proof. Assuming (i), the isomorphism (40) is given explicitly by

$$f_\_ = \kappa^*_{\oplus\_} f_\oplus, \qquad f_\oplus = \kappa_{\oplus\_} f_\_$$

from left to right and right to left, respectively. The fact that these maps are isometric and inverse of each other is straightforward to see from the assumptions and the ultraweak continuity of composition, and the naturality is obvious. Conversely, if (ii) holds, then we obtain the family $\kappa^*_{\oplus\_}$ by applying (40) with $\mathrm{id}_{X_\oplus}$ on the left. The isomorphism from left to right is then given by $f_\oplus \mapsto \kappa^*_{\oplus\_} f_\oplus$ by naturality, and we obtain (39) by the fact that the isomorphism must be an isomorphism of Hilbert presheaf and therefore preserve inner products.

We leave it to the reader to check that the two constructions are inverse to each other. □

Example 3.6.5. In Hilb, the W*-categorical direct sum of a family of Hilbert spaces $\mathcal{H}_\_ = (\mathcal{H}_i)_{i\in I}$ coincides with the usual direct sums (37), with $\kappa_{\oplus i} : \mathcal{H}_i \to \mathcal{H}_\oplus$ given by the canonical inclusion operators. Their adjoints $\kappa^*_{\oplus i} : \mathcal{H}_\oplus \to \mathcal{H}_i$ are the canonical projections $\phi \mapsto \phi_i$.

We now turn to the construction of direct sums of self-dual Hilbert presheaves and prove their universal property. These generalize direct sums for self-dual Hilbert modules [21, p. 457/458][30].

For a W*-category C and a family $H_{\_} = (H_i)_{i\in I}$ of small self-dual Hilbert presheaves on it, we put

$$H_{\oplus}X \coloneqq \bigoplus_{i\in I} H_i X,$$

where the direct sum on the right-hand side by definition consists of all families $\alpha_{\_} \in H_{\_}X$ that are square summable in the sense that

$$\langle \alpha_{\_}, \alpha_{\_} \rangle = \sum_{i\in I} \langle \alpha_i, \alpha_i \rangle < \infty.$$

$H_{\oplus}$ is functorial in the obvious way: if $\alpha_{\_} \in H_{\_}X$ is square summable and $f : Y \to X$, then also $\alpha_{\_}f \in H_{\_}Y$ is square summable, since

$$\langle \alpha_{\_}f, \alpha_{\_}f \rangle = f^* \langle \alpha, \alpha \rangle f < \infty.$$

If I is finite, then we can declare the inner product of $\alpha_{\_} \in H_{\_}X$ and $\beta_{\_} \in H_{\_}Y$ to be given by

$$\langle \beta_{\_}, \alpha_{\_} \rangle \coloneqq \sum_{i\in I} \langle \beta_i, \alpha_i \rangle.$$

This converges ultraweakly by the assumed square summability and Lemma 3.1.1, where we also use the Yoneda lemma again in order to view the $\alpha_{\_}$ and $\beta_{\_}$ as Hilbert transformations.

Lemma 3.6.6. *These definitions make $H_{\oplus}$ into a small self-dual Hilbert presheaf, and the finitely supported families $\alpha_{\_} \in H_{\oplus}X$ are ultraweakly dense.*

Proof. It is straightforward to see that $H_{\oplus}$ is a pre-Hilbert presheaf. We thus start by proving self-duality of $H_{\oplus}$ by Corollary 2.5.14, which takes some work.[31] So let

$$t_{\oplus}^* \;:\; H_{\oplus} \longrightarrow \mathsf{C}(-, X)$$

be a Hilbert transformation, where our notation already indicates that there will be an adjoint $t_{\oplus}$, which amounts to the desired self-duality. By assumption, for every $i \in I$ the restricted transformation

$$t_i^* \;:\; H_i \longrightarrow \mathsf{C}(-, X)$$

can be represented by some $\beta_i \in H_i X$,

$$t_i^*(\alpha_i) = \langle \beta_i, \alpha_i \rangle \qquad \forall \alpha_i \in HY_i.$$

[30] Note that the notion of direct sum that is relevant for us is what Paschke calls the *ultraweak direct sum*, which is in general different from the direct sum of Hilbert modules considered e.g. at [20, p. 6], since the latter contains fewer elements and does not necessarily preserve self-duality.

[31] In the Hilbert module case, Paschke claims that verifying the self-duality is "routine" [21, p. 458]. We do not find this to be the case, since the application of Egoroff's theorem that we use does not seem obvious, and specializing to the single-object case of Hilbert modules does not simplify the problem.

We then argue first that $\beta_{\text{-}} = (\beta_i)_{i\in I}$ is a square summable family. For a finite subset $F \subseteq I$, let us write $\beta_F \in H_\oplus X$ for the family that is given by β_i for $i \in F$ and zero otherwise. Then

$$\|\beta_F\|^2 = \|\langle \beta_F, \beta_F\rangle\| = \left\|t_\oplus^*(\beta_F)\right\| \leq \left\|t_\oplus^*\right\| \|\beta_F\|.$$

Since the resulting bound $\|\beta_F\| \leq \left\|t_\oplus^*\right\|$ is independent of F, the square summability follows by the fact that the implied bound $\langle \beta_F, \beta_F\rangle \leq \left\|t_\oplus^*\right\|^2$ is independent of F. We now know that

$$t_\oplus^*(\alpha_{\text{-}}) = \langle \beta_{\text{-}}, \alpha_{\text{-}}\rangle \tag{41}$$

for all finitely supported $\alpha_{\text{-}} \in H_\oplus Y$, and we still need to prove this equation for general $\alpha_{\text{-}} \in H_\oplus Y$. A straightforward approximation argument shows that the equation also holds if $\alpha_{\text{-}}^* \alpha_{\text{-}} = \sum_{i\in I} \langle \alpha_i, \alpha_i\rangle$ converges in norm. For the fully general case, we argue that id_X can be ultraweakly approximated by projections $p \in \mathsf{C}(X, X)$ for which the sum

$$\sum_{i\in I} p\langle \alpha_i, \alpha_i\rangle p = \sum_{i\in I} \langle \alpha_i p, \alpha_i p\rangle$$

converges in norm. Indeed for given normal states $\eta_1, \ldots, \eta_k \in \mathsf{C}(X, X)_*$ and any $\varepsilon > 0$, Saito's noncommutative Egoroff theorem [15, Theorem 4.13][32] provides us with a projection $p : X \to X$ such that

$$\eta_1(p), \ldots, \eta_k(p) > 1 - \varepsilon$$

and such that $\sum_i \langle \alpha_i, \alpha_i\rangle p$ converges in norm. But then also

$$t_\oplus^*(\alpha_{\text{-}})p = t_\oplus^*(\alpha_{\text{-}}p) = \langle \beta_{\text{-}}, \alpha_{\text{-}}p\rangle = \langle \beta_{\text{-}}, \alpha_{\text{-}}\rangle p,$$

and the claimed (41) follows upon taking $p \nearrow \mathrm{id}_X$. Therefore $H_\oplus$ is indeed self-dual. Incidentally, these approximations also establish the second claim on the ultraweak density of finitely supported families, since $\alpha_{\text{-}}p$ converges ultraweakly to $\alpha_{\text{-}}$ by Theorem 2.5.19(iii).

Finally, the smallness of $H_\oplus$ now follows straightforwardly by the smallness criterion given in Corollary 3.2.9(ii): choosing generating families of elements for each H_i and including them into $H_\oplus$ generates the pre-Hilbert presheaf of all finitely supported families, which we already proved to be ultraweakly dense. □

Proposition 3.6.7. *$H_\oplus$ is indeed a direct sum of the family $H_{\text{-}}$ in $\hat{\mathsf{C}}$.*

Proof. To verify the universal property, let $K \in \hat{\mathsf{C}}$ be another small self-dual Hilbert presheaf and let $t_{\text{-}} : K \to H_{\text{-}}$ be a family of Hilbert transformations satisfying the square summability condition $t_{\text{-}}^* t_{\text{-}} < \infty$ in $\hat{\mathsf{C}}(K, K)$. Then for every $\alpha \in KX$ with $X \in \mathsf{C}$, we also have

$$\langle t_{\text{-}}(\alpha), t_{\text{-}}(\alpha)\rangle = \langle \alpha, (t_{\text{-}}^* t_{\text{-}})(\alpha)\rangle \leq \|t_{\text{-}}^* t_{\text{-}}\| \langle \alpha, \alpha\rangle,$$

[32] Note that we are dealing with a bounded directed set of positive elements, so that ultraweak convergence implies strong convergence.

and therefore also $t_{\_}(\alpha) \in H_{\_}X$ is a square summable family. But then if we define $t_\oplus : K \to H_\oplus$ via

$$t_\oplus(\alpha) \coloneqq (t_i(\alpha))_{i\in I}, \tag{42}$$

then the above inequality also shows that the resulting transformation $t_\oplus$ is bounded uniformly in X. Since the naturality is obvious by naturality of each t_i, Lemma 2.5.13 implies that $t_\oplus$ is a Hilbert transformation.

It is clear that $t_\oplus$ recovers t_i upon composing with the inclusion $H_i \to H$. The uniqueness is obvious as the required (42) determines $t_\oplus$ uniquely. □

Example 3.6.8. Let $X_{\_}$ be any family of objects in a W*-category C. Then the Hilbert presheaf $\bigoplus \mathsf{C}(-, X_{\_})$ of Proposition 3.6.1 is the direct sum of the family of representable Hilbert presheaves $\mathsf{C}(-, X_{\_})$.

The following concrete example adds an important caveat.

Example 3.6.9 ([37, Example 2.5.6])**.** Let $\mathcal{H} \coloneqq \ell^2(\mathbb{N})$ be a Hilbert space with orthonormal basis given by $\{e_i\}_{i\in\mathbb{N}}$. Then $\bigoplus_{i\in\mathbb{N}} \mathcal{H}$, the $\mathbb{N}$-fold direct sum of $\mathcal{H}$ with itself in Hilb, is canonically isomorphic to $\ell^2(\mathbb{N} \times \mathbb{N})$. In particular, the sequence of basis vectors $(e_i)_{i\in\mathbb{N}}$ does *not* define an element of $\bigoplus_{i\in\mathbb{N}} \mathcal{H}$, since it obviously fails the relevant square summability condition by $\sum_{i\in\mathbb{N}} \langle e_i, e_i\rangle = \infty$.

However, it is instructive to note that $\mathcal{H}$ can also be considered as a $\mathcal{B}(\mathcal{H})$-module with $\mathcal{B}(\mathcal{H})$-valued inner product $\langle\!\langle \beta, \alpha \rangle\!\rangle \coloneqq \alpha\langle \beta, \cdot\rangle$. In this setting, the direct sum $\bigoplus_{i\in\mathbb{N}} \mathcal{H}$ is a $\mathcal{B}(\mathcal{H})$-Hilbert module whose underlying set *is different* from the one of the Hilbert space direct sum of the previous paragraph: now the sequence $(e_i)_{i\in\mathbb{N}}$ *does* define a valid element, since the square summability condition

$$\sum_{i\in\mathbb{N}} \langle\!\langle e_i, e_i \rangle\!\rangle = \sum_{i\in\mathbb{N}} e_i\langle e_i, \cdot\rangle = \mathrm{id}_{\mathcal{H}} < \infty$$

is satisfied.

Remark 3.6.10. More generally for any *finite* set I, a direct sum $\bigoplus_{i\in I} X_i$ is in particular a biproduct in C.[33] However, the universal property of a biproduct characterizes it only up to unique isomorphism, while the universal property of a W*-categorical direct sum characterizes it up to unique *unitary* isomorphism.

When I is infinite, then $\bigoplus_{i\in I} X_i$ is typically neither a product nor a coproduct. For example in Hilb, this happens for a family of Hilbert spaces $(\mathcal{H}_i)_{i\in I}$ as soon as infinitely many of them are nonzero. What is still true is that every morphism between direct sums

$$f \;:\; X_\oplus \longrightarrow Y_\oplus \tag{43}$$

of families $X_{\_} = (X_i)_{i\in I}$ and $Y_{\_} = (Y_j)_{j\in J}$ can be expressed as a matrix of morphisms

$$f_{\_\_} = (f_{ji} : X_i \to Y_j)_{i\in I, j\in J}$$

[33] Which for $I = \emptyset$ specializes to a zero object.

whose components are obtained by composing f with the direct sum inclusions and projections. The universal properties of the two direct sums involved imply that f is uniquely specified by this matrix. However, just as for morphisms between Hilbert spaces, not every such matrix defines a morphism, since an additional boundedness condition is needed. We will investigate this in detail below.

Definition 3.6.11. *We say that a W*-category* ***has direct sums*** *if a direct sum exists for every (small) family of objects.*

Example 3.6.12. As we just showed, every W*-category of small self-dual Hilbert presheaves $\hat{\mathsf{C}}$ has direct sums. Let us see which of our examples of W*-categories from Section 2.4 have direct sums.

- ▷ The discrete W*-category on a set I from Example 2.4.1 clearly does not have direct sums unless $I = \emptyset$.
- ▷ For a W*-algebra N, the W*-category of normal representations $\mathsf{NRep}(N)$ from Example 2.4.2 has direct sums, constructed as in Hilb. Similarly for the W*-categories of representations of topological groups (Example 2.4.3) and operator systems (Example 2.4.4).
- ▷ A W*-category of self-dual Hilbert modules $\mathsf{HilbMod}(N)$ has direct sums because of $\mathsf{HilbMod}(N) = \widehat{\mathfrak{B}N}$.
- ▷ A product W*-category $\prod_{i\in I} \mathsf{C}_i$ as in Example 2.4.9 has direct sums if and only if the individual C_i do, and they can be constructed componentwise.[34]
- ▷ As with the first item, a coproduct W*-category $\coprod_{i\in I} \mathsf{C}_i$ as in Example 2.4.10 does not have direct sums unless it is highly degenerate: any family of nonzero objects which does not belong to one component C_i only fails to have a direct sum.
- ▷ If C and D are W*-categories with C having direct sums and D small, then also the functor W*-category $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ has direct sums. This is straightforward to see upon constructing them objectwise on D and proving the universal property in the obvious manner.
  As special cases of this, we obtain that a W*-category of the form $\mathsf{NRep}(N, \mathsf{C})$ has direct sums if C does, and that all W*-categories of the form $\mathsf{HilbBiMod}(M, N)$ and $\mathsf{Connes}(M, N)$ do.
- ▷ Provided that a W*-category C has direct sums, its arrow W*-category $\mathsf{C}^{\to}$ as in Example 2.4.13 has a direct sum for a family $f_{\_} : X_{\_} \to Y_{\_}$ if and only if $\|f_{\_}\|$ is bounded. Indeed if this is the case, then their upcoming sum is the induced morphism $f_{\oplus} : X_{\oplus} \to Y_{\oplus}$ as in the upcoming (44). Conversely if a direct sum exists, then using the fact that the domain and codomain W*-functors $\mathsf{C}^{\to} \to \mathsf{C}$ must preserve direct sums by the upcoming Proposition 3.6.16, we conclude that

[34] For the "only if" direction, note that the empty W*-category does not have direct sums as it is missing a zero object. Hence if $\prod_{i\in I} \mathsf{C}_i$ has direct sums, then neither C_i is empty. The fact that the product preservation W*-functors must preserve direct sums by the upcoming Proposition 3.6.16 implies that the direct sum of a family of objects in some C_i can be constructed by lifting arbitrarily to the product W*-category, which is possible by none of the W*-categories involved being empty, forming the direct sum there, and then projecting back.

the direct sum must be of the claimed form. It follows that the contractive arrow W*-category $\mathsf{C}^{\to,1}$ has direct sums if and only if C does.

- ▷ Given a W*-category C with direct sums, clearly also the conjugate W*-category $\overline{\mathsf{C}}$ as in Example 2.4.14 has direct sums.

Remark 3.6.13. The only way for a nontrivial W*-category to have direct sums is if the W*-category is large. Indeed if $X \in \mathsf{C}$ is such that $\mathsf{C}(X, X) \neq 0$, then the hom-spaces between direct sums of X with itself will be of arbitrary large cardinality, and therefore C cannot be small.

In the original paper on W*-categories [3], this size issue has not been dealt with consistently.[35] A case in point is given by their Propositions 1.14 and 2.13, which purport to show that *every* C*-category—regardless of size—has a faithful (normal) functor into Hilb. Granted that we assume local smallness, the given proof requires direct sums of cardinality $|\mathsf{C}|$ in Hilb. So either C needs to be a small W*-category in order for the argument to go through, or one needs to use a W*-category of large Hilbert spaces (which would itself be very large).

Remark 3.6.14. If a W*-category C has direct sums and a generating family $S_{\_}$, then it also has a single generator given by $S_\oplus$. This is a straightforward consequence of the universal property of direct sums.

The universal property of direct sums implies that their formation is functorial: for families of objects $X_{\_} = (X_i)_{i \in I}$ and $Y_{\_} = (Y_i)_{i \in I}$, any uniformly bounded family of morphisms $f_{\_} = (f_i : X_i \to Y_i)_{i \in I}$ induces a morphism

$$f_\oplus \; : \; X_\oplus \longrightarrow Y_\oplus. \tag{44}$$

In terms of the W*-category from (7), the formation of direct sums with index set I thus becomes a functor

$$\bigoplus \; : \; \ell^\infty(I, \mathsf{C}) \longrightarrow \mathsf{C}. \tag{45}$$

In fact, this is a W*-functor: it is clearly faithful, and thus by Lemma 2.2.3 we only need to argue that its action on each hom-set has ultraweakly closed image. But this is because a generic morphism $f : X_\oplus \to Y_\oplus$ is of the form (44) if and only if it is block diagonal, by which we mean that the composites

$$X_{i'} \xrightarrow{\kappa_{\oplus i'}} X_\oplus \xrightarrow{f} Y_\oplus \xrightarrow{\kappa^*_{\oplus i}} Y_i$$

vanish for $i' \neq i$, and each one of these conditions is ultraweakly closed.

Remark 3.6.15. Naively, one might hope that the direct sum functor (45) would be adjoint to the diagonal functor $\Delta : \mathsf{C} \to \ell^\infty(I, \mathsf{C})$ similar to what happens with products in ordinary category theory. This is of course the case for finite I, in which case direct

[35] However, it is worth noting that their Footnote 2 indicates awareness of size issues.

sums are just finite biproducts, but not for infinite I. Indeed if this held for infinite I, then we would in particular have a bijection

$$\mathsf{C}(A, X_\oplus) \cong \ell^\infty(I, \mathsf{C})(\Delta A, X_\_)$$

for every object A and family of objects $X_\oplus$ implemented by composing with the direct sum projections. With $\mathsf{C} = \mathsf{Hilb}$ and all objects A and $X_\_$ given by $\mathbb{C}$, we obtain $\ell^2(I)$ on the left but $\ell^\infty(I)$ on the right, and the canonical map $\ell^2(I) \to \ell^\infty(I)$ which corresponds to composition with the projections is of course not surjective.

Every additive functor between additive categories preserves biproducts (whenever these exist). The following abstract version of Rieffel's [5, Proposition 4.9] extends this to infinite direct sums in W*-categories.

Proposition 3.6.16. *A W*-functor $F : \mathsf{C} \to \mathsf{D}$ preserves all direct sums that exist in C.*

Proof. In full, the statement to be proved is that if a family of morphisms $(\kappa_{\oplus i} : X_i \to X_\oplus)_{i \in I}$ makes $X_\oplus$ into a direct sum of a family of objects $(X_i)_{i\in I}$ in C, then also the family

$$(F(\kappa_{\oplus i}) : F(X_i) \to F(X_\oplus))_{i\in I}$$

makes $F(X_\oplus)$ into a direct sum of the family $(F(X_i))_{i\in I}$. This is a straightforward consequence of the equational characterization of direct sums in Theorem 3.6.4 and the ultraweak continuity of F. □

Our next goal is to construct, for every W*-category C, a **direct sum completion** $\mathsf{C}^\oplus$, which is another W*-category obtained by formally adjoining direct sums to C. In doing so, the main difficulty consists in finding the right definition of the hom-spaces and in proving that they have preduals.

To this end, we first characterize which matrices of morphisms correspond to morphisms between direct sums $X_\oplus \to Y_\oplus$ of arbitrary families $X_\_ = (X_i)_{i\in I}$ and $Y_\_ = (Y_j)_{j\in J}$. Using the norm of square summable families defined in (21), we can introduce the relevant notion of boundedness of matrices of morphisms $f_{\_\_} : X_\_ \to Y_\_$.

Definition 3.6.17. *A matrix of morphisms $f_{\_\_} : X_\_ \to Y_\_$ is **bounded** if for every finitely supported $\phi_\_ : A \to X_\_$, the product $f_{\_\_}\phi_\_$ is square summable, and in addition*

$$\|f_{\_\_}\| \coloneqq \sup_{A\in\mathsf{C}} \sup_{\|\phi_\_\|\leq 1} \|f_{\_\_}\phi_\_\| < \infty. \tag{46}$$

Example 3.6.18. In $\mathsf{C} = \mathsf{Hilb}$ and with $X_i = Y_j = \mathbb{C}$, a matrix as above is bounded if and only if it is bounded as an operator $\ell^2(I) \to \ell^2(J)$.

The following observation lets us apply our existing machinery in order to obtain lots of results on bounded matrices for free.

Lemma 3.6.19. *There is an isometric isomorphism between bounded matrices $f_{--} : X_- \to Y_-$ and Hilbert transformations*

$$\bigoplus \mathsf{C}(-, X_-) \longrightarrow \bigoplus \mathsf{C}(-, Y_-), \tag{47}$$

given by $\phi_- \longmapsto f_{--}\phi_-$.

Proof. We first show that a bounded matrix defines such a transformation of the same norm. This holds by definition if the index set I is finite. In general, for given $\phi : A \to X_-$ and finite $F \subseteq I$, we have

$$\|f_{-F}\phi_F\|^2 = \|\phi_F^* f_{F-}^* f_{-F}\phi_F\| \leq \|\phi_F^* f_{-F}^*\| \left\| f_{-F} \|\phi_F\|^2 \right\| \leq \|f_{--}\|^2 \|\phi_-\|^2,$$

and therefore Lemma 2.5.20 applies to show that $f_{--}\phi_- : A \to Y_-$ is square summable as well. The same inequality also shows that the norm of this transformation is exactly $\|f_{--}\|$ while the naturality is obvious. Hence we indeed get a Hilbert transformation of the same norm.

Conversely, given such a given Hilbert transformation we can plug in vectors α_- and β_- with singleton support and apply the Yoneda lemma produces a candidate matrix of morphisms f_{--}, which is the unique matrix such that

$$\langle \beta_-, t(\alpha_-)\rangle = \langle \beta_-, f_{--}\alpha_-\rangle = \langle f_{--}^*\beta_-, \alpha_-\rangle = \langle t^*(\beta_-), \alpha_-\rangle$$

holds for all finitely supported α_- and β_-. Linearity and naturality together with boundedness of the Hilbert transformation then imply that the resulting matrix is bounded as well. The last equation implies $t^*(\beta_-) = f_{--}^*\beta_-$ for all finitely supported β_-; since both adjunctions apply generally, this implies that the first equation holds for all α_- and all finitely supported β_-. But this is enough to conclude $t(\alpha_-) = f_{--}\alpha_-$ in general.[36] □

As one immediate consequence of Lemma 3.6.19 and the Yoneda lemma, we obtain that if the direct sums $X_\oplus$ and $Y_\oplus$ exist, then the morphisms $f_{\oplus\oplus} : X_\oplus \to Y_\oplus$ are exactly the bounded matrices $f_{--} : X_- \to Y_-$, where the matrix entries are constructed as the composites

$$X_i \xrightarrow{\kappa_{\oplus i}} X_\oplus \xrightarrow{f_{\oplus\oplus}} Y_\oplus \xrightarrow{\kappa_{\oplus j}^*} Y_j.$$

By (39), the composition of Hilbert transformations of the form (47) corresponds to matrix multiplication of the corresponding bounded matrices. Similarly, forming the adjoint of a Hilbert transformation like this corresponds to forming the entrywise adjoint of the transposed matrix.

Also by Lemma 3.6.19 or by Lemma 3.1.1, we can derive alternative formulas for the operator norm,

$$\|f_{--}\| = \sup_{A,B\in\mathsf{C}} \sup_{\|\phi_-\|,\|\psi_-\|\leq 1} \|\psi_-^* f_{--}\phi_-\| = \sup_{B\in\mathsf{C}} \sup_{\|\psi_-\|\leq 1} \|f_{--}^*\psi_-\|.$$

[36] We can also conclude this from the fact that it holds on finitely supported α_- by construction, that these are ultraweakly dense by Lemma 3.6.6, and the application of a Hilbert transformation is ultraweakly continuous by Theorem 2.5.19(iv).

where $\phi_{\_} : A \to X_{\_}$ and $\psi_{\_} : B \to Y_{\_}$ can be assumed finitely supported, and $f^*_{\_\_}$ denotes the transposed matrix with the involution applied entrywise.

Definition 3.6.20. *For any W*-category* C*, its* ***direct sum completion*** $\mathsf{C}^{\oplus}$ *is either of the following two equivalent W*-categories:*

(i) The full subcategory of $\hat{\mathsf{C}}$ *on Hilbert presheaves of the form* $\bigoplus \mathsf{C}(-, X_{\_})$*.*
(ii) The W-category with*

- *families* $X_{\_} = (X_i)_{i \in I}$ *of objects of* C *as objects,*
- *hom-spaces given by*

$$\mathsf{C}^{\oplus}(X_{\_}, Y_{\_}) \coloneqq \{\textit{bounded } f_{\_\_} : X_{\_} \to Y_{\_}\}, \tag{48}$$

with respect to matrix multiplication as composition, operator norm (46) as norm and the usual involution on matrices.

While we have not explicitly shown that the latter category is indeed a W*-category, this follows by the already established equivalence with the first category. Nevertheless, it is useful to have a more explicit description of the preduals. To obtain this, for given families $X_{\_}$ and $Y_{\_}$, consider matrices of predual elements

$$\eta_{\_\_} = (\eta_{ji} \in \mathsf{C}(X_i, Y_j)_*)_{i \in I, j \in J}.$$

If such η is finitely supported, then

$$\mathrm{tr}(\eta_{\_\_} f_{\_\_}) \coloneqq \sum_{i \in I, j \in J} \eta_{ji}(f_{ij}) \tag{49}$$

trivially converges. Since $\|f_{ij}\| \leq \|f_{\_\_}\|$ for all i and j, (49) defines a bounded linear functional on the space of bounded matrices.

Proposition 3.6.21. *For a hom-space* $\mathsf{C}^{\oplus}(X_{\_}, Y_{\_})$ *of a direct sum completion* $\mathsf{C}^{\oplus}$*, we have:*

(i) Its predual is the norm closure in $\mathsf{C}^{\oplus}(X_{\_}, Y_{\_})^*$ *of the set of functionals of the form (49) for finitely supported* $\eta_{\_\_}$*.*
(ii) The finitely supported matrices are ultraweakly dense.

Proof. (i) Using the preduals of spaces of Hilbert transformations constructed in the proof of Corollary 2.5.22 shows that the predual is the closed subspace of $\mathsf{C}^{\oplus}(X_{\_}, Y_{\_})^*$ spanned by the functionals of the form

$$f_{\_\_} \longmapsto \eta(\beta^*_{\_} f_{\_\_} \alpha_{\_})$$

for $\alpha_{\_} \in \mathsf{C}(A, X_{\_})$ and $\beta_{\_} \in \mathsf{C}(B, Y_{\_})$ and $\eta \in \mathsf{C}(A, B)_*$. Using finitely supported $\alpha_{\_}$ and $\beta_{\_}$ then produces a functional of the claimed form. These functionals are ultraweakly dense in $\mathsf{C}^{\oplus}(X_{\_}, Y_{\_})$ by Lemma 3.6.6 and the ultraweak continuity (in each argument) of matrix multiplication. But then they are also norm dense by Lemma 2.1.17.

(ii) The finitely supported η_{--} also induce the ultraweak topology on (48) by Lemma 2.1.17. Now the claim is obvious. □

Example 3.6.22. Let C be a small W*-category. Then we define the **linking W*-algebra** $L(\mathsf{C})$ to be the endomorphism W*-algebra of the Hilbert presheaf

$$\bigoplus_{X \in \mathsf{C}} \mathsf{C}(-, X). \tag{50}$$

When C has finitely many objects, then this specializes to our earlier Definition 2.1.10. A simple nontrivial example with infinitely many objects is the discrete W*-category on a set I (Example 2.4.1), whose linking W*-algebra is exactly $\mathcal{B}(\ell^2(I))$, arising as the endomorphism W*-algebra of the Hilbert presheaf that maps every object to C.

With D ranging over all full subcategories $\mathsf{D} \subseteq \mathsf{C}$ with finitely many objects ordered by inclusion, $L(\mathsf{C})$ is the filtered colimit of the $L(\mathsf{D})$ in the category of W*-algebras and not necessarily unital ultraweakly continuous $*$-homomorphisms[37]. This characterization was originally used as the definition [3, p. 90]. Proving the relevant universal property as a filtered colimit is straightforward based on the facts that every $L(\mathsf{D})$ is included in $L(\mathsf{C})$, and the union of these subalgebras $\bigcup_{\mathsf{D}} L(\mathsf{D})$ is ultraweakly dense in $L(\mathsf{C})$ by Proposition 3.6.21.

The importance of the linking W*-algebra is grounded in the fact that (50) is a generator in $\hat{\mathsf{C}}$. As we will see in Example 3.9.7, the associated fully faithful W*-functor $\mathfrak{B}L(\mathsf{C}) \to \hat{\mathsf{C}}$ induces a W*-equivalence $\hat{\mathsf{C}} \cong \mathsf{HilbMod}(L(\mathsf{C}))$.

We need two more lemmas before we can state and prove a 2-categorical universal property of $\mathsf{C}^{\oplus}$. First, it is useful to have an algebraic certificate for the boundedness of a matrix.

Lemma 3.6.23. *A matrix of morphisms $f_{--} : X_{-} \to Y_{-}$ is bounded if and only if there is $\lambda \geq 0$ and a matrix $g_{--} : X_{-} \to X_{-}$ such that*

$$f_{--}^* f_{--} + g_{--}^* g_{--} = \lambda^2 \,\mathrm{id}_{--}.$$

Furthermore, such g_{--} exists if and only if $\lambda \geq \|f_{--}\|$.

Proof. If f_{--} is bounded, then the fact that $\mathsf{C}^{\oplus}(X_{-}, X_{-})$ is a W*-algebra implies that $f_{--}^* f_{--} \leq \lambda^2 \,\mathrm{id}_{--}$ for any $\lambda \geq \|f_{--}\|$, so that we can take e.g. $g_{--} := \sqrt{\lambda^2 - f_{--}^* f_{--}}$ and the stated equation follows.

Conversely, suppose that f_{--} satisfies the stated condition. Then it implies that for finitely supported $\phi_{-} : A \to X_{-}$, we have in the W*-algebra $\mathsf{C}(A, A)$,

$$\phi_{-}^* f_{--}^* f_{--} \phi_{-} = \lambda^2 \,\phi_{-}^* \phi_{-} - \phi_{-}^* g_{--}^* g_{--} \phi_{-} \leq \lambda^2 \phi_{-}^* \phi_{-}.$$

Hence we obtain the claimed boundedness with $\|f_{--}\| \leq \lambda$. □

[37] This is exactly Guichardet's category of W*-algebras [22, §2].

Lemma 3.6.24. *Let $F : \mathsf{C} \to \mathsf{D}$ be a W*-functor and $(f_{\_\_} : X_\_ \to Y_\_)$ a bounded matrix of morphisms in C. Then the matrix of morphisms $(F(f_{\_\_}) : F(X_\_) \to F(Y_\_))$ in D is bounded as well.*

Note that this is not obvious directly from the definition of boundedness.

Proof. This follows by applying the previous Lemma 3.6.23 together with ultraweak continuity of F. ☐

Our 2-categorical universal property of $\mathsf{C}^\oplus$ is based on the canonical inclusion $\mathsf{C} \hookrightarrow \mathsf{C}^\oplus$ corresponding to including every object of C as the singleton family in $\mathsf{C}^\oplus$. It is clear that this is fully faithful, for example since the composite $\mathsf{C} \hookrightarrow \mathsf{C}^\oplus \hookrightarrow \hat{\mathsf{C}}$ is the Yoneda embedding.

Theorem 3.6.25. *Let C be a W*-category. Then:*

(i) The W-category $\mathsf{C}^\oplus$ has direct sums.*

(ii) For any W-category D with direct sums, composition with the fully faithful embedding $\mathsf{C} \hookrightarrow \mathsf{C}^\oplus$ establishes a W*-equivalence*

$$\mathsf{Fun}(\mathsf{C}^\oplus, \mathsf{D}) \cong \mathsf{Fun}(\mathsf{C}, \mathsf{D}). \tag{51}$$

Moreover, if a W-functor $\mathsf{C} \to \mathsf{D}$ is fully faithful, then its extension to $\mathsf{C}^\oplus \to \mathsf{D}$ is fully faithful as well.*

In particular, this shows that $\mathsf{Fun}(\mathsf{C}^\oplus, \mathsf{D})$ is locally small as soon as C is small.

Proof. (i) The embedding $\mathsf{C} \hookrightarrow \mathsf{C}^\oplus$ identifies every $X_\_ \in \mathsf{C}^\oplus$ with the corresponding Hilbert presheaf $\bigoplus \mathsf{C}(-, X_\_)$. A family of such objects corresponds to a doubly indexed family $(X_\_)_\_$, and Proposition 3.6.7 shows that its direct sum in $\hat{\mathsf{C}}$ is given by the Hilbert presheaf

$$\bigoplus \bigoplus \mathsf{C}(-, (X_\_)_\_).$$

The relevant square summability condition that defines the elements of this Hilbert presheaf is equivalently square summability of the total family obtained by considering the double index as a single index. Therefore the double direct sum can equivalently be written as a single direct sum, thereby showing that this Hilbert presheaf is of the required form for being an object of $\mathsf{C}^\oplus$.

(ii) For W*-functors $F, G : \mathsf{C}^\oplus \to \mathsf{D}$, we already know from Corollary 3.5.14(ii) that the space of bounded natural transformations $F \to G$ is isometrically isomorphic to the space of bounded natural transformation $F|_\mathsf{C} \to G|_\mathsf{C}$ through the restriction map, since C is a generating full subcategory of $\hat{\mathsf{C}}$.

It thus remains to be shown that every W*-functor $F : \mathsf{C} \to \mathsf{D}$ can be extended to a W*-functor $\tilde{F} : \mathsf{C}^\oplus \to \mathsf{D}$, where extended means that $\tilde{F}|_\mathsf{C}$ is naturally isomorphic to F. Indeed by Lemma 3.6.24, the formation of the direct sum completion is itself functorial, and this is ultraweakly continuous on hom-spaces by the characterization of the preduals from Proposition 3.6.21(i). Therefore F induces a W*-functor $F^{\oplus} : \mathsf{C}^{\oplus} \to \mathsf{D}^{\oplus}$ which is such that the diagram

$$\begin{array}{ccc} \mathsf{C} & \hookrightarrow & \mathsf{C}^{\oplus} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^{\oplus}} \\ \mathsf{D} & \hookrightarrow & \mathsf{D}^{\oplus} \end{array}$$

commutes up to unitary natural isomorphism. But now the claim follows since the inclusion $\mathsf{D} \hookrightarrow \mathsf{D}^{\oplus}$ is a W*-equivalence by the assumption that D has direct sums. The final statement is an instance of Lemma 3.5.17. □

This ends the main part of our treatment of the direct sum completion, and we turn to a few auxiliary results that will be useful later on in the considerations on composition of Hilbert profunctors. The first is a simplification of the definition of operator norm of a bounded matrix, showing that we only need to take the supremum over square summable families originating in the family of objects $X_{\_}$ itself.

Lemma 3.6.26. *For any bounded $f_{\_\_} : X_{\_} \to Y_{\_}$, we have*

$$\|f_{\_\_}\| = \sup_{u_{\_}:X_{\exists}\to X_{\_}} \; \sup_{v:Y_{\exists}\to Y_{\_}} \|v_{\_}^{*} f_{\_\_} u_{\_}\|,$$

where both suprema are over partial isometries of the given type.

Proof. Working in $\mathsf{C}^{\oplus}$ reduces this to $\mathrm{c}(X_{\_})_{X_{\oplus}} = \mathrm{id}_{X_{\oplus}}$ and Proposition 3.3.6. □

There is a similar statement in the self-adjoint case, which similarly follows by Proposition 3.3.7.

Lemma 3.6.27. *For a self-adjoint bounded matrix $f_{\_\_} : X_{\_} \to X_{\_}$, we similarly have*

$$\|f_{\_\_}\| = \sup_{u_{\_}:X_{\exists}\to X_{\_}} \|u_{\_}^{*} f_{\_\_} u_{\_}\|,$$

and

$$f_{\_\_} \geq 0 \qquad \Longleftrightarrow \qquad u_{\_}^{*} f_{\_\_} u \geq 0 \quad \forall u : X_{\exists} \to X_{\_}, \tag{52}$$

where u ranges over partial isometries.

The following consequence for Hilbert presheaves is what we will actually need, where we restrict to finite families in order to avoid the question of boundedness.

Proposition 3.6.28. *Let H be a Hilbert presheaf on a W*-category C and $\alpha_{\_} \in HX_{\_}$ a finite family of elements. Then the matrix*

$$\langle \alpha_{\_}, \alpha_{\_} \rangle$$

is positive in $\mathsf{C}^{\oplus}(X_{\_}, X_{\_})$, and vanishes if and only if $\alpha_{\_} = 0$.

This is a standard result in the case where C has only a single object, so that H is simply a Hilbert module [20, Lemma 4.2].

Proof. We apply (52) to $f_{--} = \langle \alpha_{-}, \alpha_{-} \rangle$, which reduces the problem to showing that

$$u_{-}^{*} \langle \alpha_{-}, \alpha_{-} \rangle u_{-} = \langle \alpha_{-} u_{-}, \alpha_{-} u_{-} \rangle \geq 0,$$

which is clearly the case. □

3.7 Projection splittings and the projection completion

In ordinary category theory, we have the concept of *idempotent splitting* and *idempotent completion*. We now develop the analogous concepts for W*-categories. After that, we will show that the existence of such splittings together with direct sums already makes a W*-category "W*-complete".

Ordinarily, a subobject of an object X is an equivalence class of monomorphisms with codomain X, where two monomorphisms are considered equivalent if they are isomorphic as objects over X. In W*-categories, isometries play the role of monomorphisms.

Definition 3.7.1. *A* ***subobject*** *of an object X in a W*-category C is an equivalence class of isometries $v : A \hookrightarrow X$, where $v : A \hookrightarrow X$ and $w : B \hookrightarrow X$ are considered equivalent if there is invertible $u : A \to B$ such that $v = wu$.*

We write $\mathsf{Sub}(X)$ for the collection of subobjects of X.

Remark 3.7.2. (i) As is commonplace in category theory, we will abuse terminology a bit by leaving the distinction between a subobject and a representing isometry implicit.

(ii) Any $u : A \to B$ satisfying $v = wu$ for subobjects v and w is not only unique (as in ordinary category theory), but it is itself is an isometry since

$$u^* u = u^* w^* w u = v^* v = \mathrm{id}_A.$$

(iii) As in ordinary category theory, $\mathsf{Sub}(X)$ becomes a partially ordered set with $v \leq w$ if and only if such u exists. The proof of antisymmetry is straightforward.

(iv) If v and w are equivalent, then (by the previous item) this equivalence is implemented by a unique unitary u.

(v) A subobject represented by $v : Y \hookrightarrow X$ is uniquely determined by the projection $vv^* : X \to X$. Indeed if $vv^* = ww^*$, then Theorem 3.2.11 gives a partial isometry u with $v = wu$, and this is necessarily an isometry by (ii), and therefore $v \leq w$. The other direction $w \leq v$ holds by symmetry. In this way, $\mathsf{Sub}(X)$ becomes a subposet of the projection lattice of X.

Definition 3.7.3. *A projection $p : X \to X$ in a W*-category C is* ***split*** *if there is an isometry $v : A \to X$ with $p = vv^*$. We say that C is* ***projection complete*** *if every projection is split.*

It is straightforward to see that if a splitting exists, then it is unique up to unique unitary isomorphism. Therefore in any case, $\mathsf{Sub}(X)$ embeds into the projection lattice of X. C is projection complete if for every $X \in \mathsf{C}$, this inclusion is surjective, so that subobjects of X can be identified with projections on X. A nonempty projection complete W*-category must have a zero object (as the splitting of any zero projection).

If projection completeness holds, then in particular every subobject $v : A \hookrightarrow X$ has a **complement** $w : B \hookrightarrow X$ in the sense that

$$vv^* + ww^* = \mathrm{id}_X,$$

and this identifies $X \cong A \oplus B$. This immediately proves the following.

Lemma 3.7.4. *Let $X \in \mathsf{C}$ for projection complete C, and let $p : X \to X$ be a projection. Then*

$$X \cong X_p \oplus X_{\mathrm{id}-p},$$

where X_p splits p and $X_{\mathrm{id}-p}$ splits $\mathrm{id}_X - p$.

Remark 3.7.5. In practice, one can often construct the splitting of a projection p as the kernel of the complementary projection $\mathrm{id}_X - p$. This applies for example to prove that Hilb is projection complete.

Example 3.7.6. As we had done for direct sums in Example 3.6.12, let us see which of the W*-categories of Section 2.4 are projection complete.

- ▷ The discrete W*-category on a set I is not projection complete (unless $I = \emptyset$) as it lacks a zero object.
- ▷ With splittings constructed as for Hilb, any W*-category of (normal) representations is projection complete.
- ▷ A W*-category of self-dual Hilbert modules $\mathsf{HilbMod}(N)$ is projection complete. This will be a special case of the projection completeness of W*-categories of small self-dual Hilbert presheaves which we consider below.
- ▷ A product W*-category $\prod_{i\in I} \mathsf{C}_i$ is projection complete if and only if the individual C_i do, since projection splittings can be constructed componentwise.[38]
- ▷ A coproduct W*-category $\coprod_{i\in I} \mathsf{C}_i$ has projection splittings as soon as each individual C_i does. The converse is not true: for example, the coproduct $\mathfrak{B}\mathbb{C} \amalg \mathfrak{B}0$ has projection splittings, although $\mathfrak{B}\mathbb{C}$ does not: taking the coproduct with $\mathfrak{B}0$ adds the missing zero object to $\mathfrak{B}\mathbb{C}$.
- ▷ If a W*-category C is projection complete and D is any small W*-category, then $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ is projection complete as well. The proof is straightforward upon using the objectwise projection splitting.
  In particular, a W*-category of the form $\mathsf{NRep}(N, \mathsf{C})$ is projection complete if C is. Even more particularly, all W*-categories of the form $\mathsf{HilbBiMod}(M, N)$ and $\mathsf{Connes}(M, N)$ are projection complete.

[38] As in the case of direct sums, the other direction follows upon noting that neither C_i can be empty if $\prod_{i\in I} \mathsf{C}_i$ is projection complete, and then using that the product projection W*-functors must preserve splittings by the upcoming Lemma 3.7.12.

- ▷ An arrow W*-category $\mathsf{C}^{\to}$ is projection complete if and only if C is, and likewise for the full W*-subcategory $\mathsf{C}^{\to,1}$. Indeed projection splittings in $\mathsf{C}^{\to}$ can be constructed by functoriality of projection splittings in C, and the converse follows from the existence of W*-functors $\mathsf{C} \to \mathsf{C}^{\to,1} \to \mathsf{C}$ which compose to the identity and the fact that W*-functors preserve projection splittings as per the upcoming Lemma 3.7.12.
- ▷ Finally, a W*-category C has projection splittings if and only if its conjugate W*-category $\overline{\mathsf{C}}$ does.

Example 3.7.7. If C is a W*-category, then the W*-category of small self-dual Hilbert presheaves $\hat{\mathsf{C}}$ is projection complete.

Indeed if $H \in \hat{\mathsf{C}}$ and $p : H \to H$ is any projection, then consider the Hilbert presheaf $pH : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ defined by

$$(pH)(X) \coloneqq \{\alpha \in HX \mid p(\alpha) = \alpha\}, \tag{53}$$

with the induced action on morphisms. The naturality of p implies that pH is a Hilbert presheaf as well with respect to the induced inner products. The smallness holds by the smallness of H, while the self-duality follows upon applying the self-duality of H to composite transformations of the form

$$H \xrightarrow{p} pH \longrightarrow \mathsf{C}(-, X).$$

Finally, the natural inclusion $pH \hookrightarrow H$ then is clearly an isometry that splits p.

Example 3.7.8. A W*-category with a single nonzero[39] object is not projection complete since the zero projection does not split.

As a first application of projection completeness, we can derive a simplified criterion for the existence of a generator.

Lemma 3.7.9. *Let C be projection complete. Then an object $S \in \mathsf{C}$ is a generator if and only if for every $X \in \mathsf{C}$,*

$$\mathsf{C}(S, X) = \{0\} \qquad \Longrightarrow \qquad \mathrm{id}_X = 0.$$

Note that $\mathrm{id}_X = 0$ is equivalent to X being a zero object, i.e. an object which is both initial and terminal.

Proof. The "only if" direction follows e.g. by the properties of generators developed in Corollary 3.5.9. For the "if" direction, suppose that S satisfies the stated condition. By Corollary 3.5.1, it is enough to show that $\mathrm{c}(S)_X = \mathrm{id}_X$ for every $X \in \mathsf{C}$. Applying Lemma 3.7.4 with $e \coloneqq \mathrm{c}(S)_X$ gives a decomposition

$$X \cong X_{\mathrm{c}(S)_X} \oplus X_{\mathrm{id}_X - \mathrm{c}(S)_X}.$$

The definition of central support shows that the only morphism $S \to X_{\mathrm{id}_X - \mathrm{c}(S)_X}$ is the zero morphism. But then we conclude $\mathrm{id}_X - \mathrm{c}(S)_X = 0$ from the assumption. □

[39] An object X is nonzero if $\mathrm{id}_X \neq 0$.

Projection completeness implies that every partial isometry can be decomposed as follows.

Lemma 3.7.10. *Let* C *be a projection complete W*-category. Then for every partial isometry* $u : X \to Y$ *in* C*, there is* $A \in \mathsf{C}$ *together with isometries* $v : Z \to X$ *and* $w : Z \to Y$ *such that* $u = wv^*$*.*

Proof. Let $v : Z \to X$ be an isometry which splits the projection $u^*u : X \to X$. Then $w := uv$ is also an isometry, since

$$w^*w \;=\; v^*u^*uv \;=\; v^*vv^*v \;=\; v^*v \;=\; \mathrm{id}_Z,$$

and we have

$$wv^* \;=\; uvv^* \;=\; uu^*u \;=\; u,$$

as was to be shown. □

In relation to kernels as in Remark 3.7.5, projection completeness is a powerful condition which implies the existence of kernels and cokernels, as per the following abstract version of Rieffel's [5, Lemma 1.2].

Lemma 3.7.11. *Let* C *have projection splittings. For a morphism* $f : X \to Y$*, we have:*

(i) The splitting of $\mathrm{id}_X - s(f)$ *is a kernel of* f*.*
(ii) The splitting of $\mathrm{id}_Y - r(f)$ *is a cokernel of* f*.*

Proof. By Lemma 3.2.3, we can assume without loss of generality that f itself is a projection. Then the statement is straightforward to check. □

We already saw in Proposition 3.6.16 that every W*-functor preserves all direct sums that exist. The same holds for projection splittings.

Lemma 3.7.12. *A W*-functor* $F : \mathsf{C} \to \mathsf{D}$ *preserves all projection splittings that exist in* C*.*

Proof. As in ordinary category theory, this is a consequence of the fact that projection splittings have an equational characterization, namely that $v^*v = \mathrm{id}_A$ and $ww^* = p$. □

Lemma 3.7.13. *A W*-category* C *is projection complete if and only if every Hilbert presheaf that is a subobject in* $\hat{C}$ *of a representable presheaf is representable as well.*

Proof. For the "only if" direction, assume that C is projection complete and that $t : H \hookrightarrow \mathsf{C}(-, X)$ is a subobject of a representable presheaf. Then $t^*t : \mathsf{C}(-, X) \to \mathsf{C}(-, X)$ corresponds to composition with a projection $p : X \to X$ by the Yoneda lemma. With $p = vv^*$ a splitting for some isometry $v : A \to X$, the subobject $t : H \hookrightarrow \mathsf{C}(-, X)$ becomes unitarily isomorphic to the subobject $\mathsf{C}(-, A) \hookrightarrow \mathsf{C}(-, X)$ given by composition with v by Remark 3.7.2(iv), and hence H is representable.

For the "if" direction, we construct a splitting for a given projection $p : X \to X$. To this end, consider the Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ given by

$$HA := p\mathsf{C}(A, X) = \{f : A \to X \mid pf = f\} \tag{54}$$

with the obvious action on morphisms and inner products induced from $\mathsf{C}(-, X)$. It is straightforward to see that this subobject splits the projection $\mathsf{C}(-, X) \to \mathsf{C}(-, X)$ given by composition with p. Since H is in the image of the Yoneda embedding by assumption, it follows that p already splits in C. □

As in the case of direct sums and the direct sum completion, we can formally adjoin splittings in order to form a "projection completion" for any given W*-category. This is essentially the Karoubi envelope from ordinary category theory turned into a W*-category.

Definition 3.7.14. *Let* C *be a W*-category. Then its* ***projection completion*** *or* ***Karoubi envelope*** $\mathsf{Kar}(\mathsf{C})$ *is the W*-category where:*

- ▷ *Objects are pairs* (X, p) *where* $X \in \mathsf{C}$ *and* $p \in \mathsf{C}(X, X)$ *is a projection.*
- ▷ *A morphism* $(X, p) \to (Y, q)$ *is a morphism* $f \in \mathsf{C}(X, Y)$ *such that* $fp = qf = f$.
- ▷ *Composition, involution and norm are induced from* C *in the obvious way.*

Of course, it needs to be shown that $\mathsf{Kar}(\mathsf{C})$ is actually a W*-category. To this end, we still need to establish the existence of preduals, which can also be done as in the direct sum case by constructing an extended Yoneda embedding

$$\mathsf{Kar}(\mathsf{C}) \hookrightarrow \hat{\mathsf{C}}$$

which maps every (X, p) to the Hilbert presheaf (54) and acts on morphisms by composition. By a Yoneda-style argument, it is straightforward to see that this is fully faithful, and hence $\mathsf{Kar}(\mathsf{C})$ is a W*-category as well.[40]

In terms of the obvious faithful embedding $\mathsf{C} \hookrightarrow \mathsf{Kar}(\mathsf{C})$ given by mapping every $X \in \mathsf{C}$ to $(X, \mathrm{id}_X) \in \mathsf{Kar}(\mathsf{C})$ and every morphism to itself, the universal property of the projection splitting is exactly what one would expect and reads as follows.

Theorem 3.7.15. *Let* C *be a W*-category. Then:*

(i) The W-category* $\mathsf{Kar}(\mathsf{C})$ *has projection splittings.*
(ii) For any W-category* D *with projection splittings, composition with the fully faithful embedding* $\mathsf{C} \hookrightarrow \mathsf{Kar}(\mathsf{C})$ *establishes a W*-equivalence*

$$\mathsf{Fun}(\mathsf{Kar}(\mathsf{C}), \mathsf{D}) \cong \mathsf{Fun}(\mathsf{C}, \mathsf{D}).$$

[40] For an alternative argument, recall the W*-subcategory criterion of Lemma 2.2.3. Although this does not technically apply since forgetting the projections is not a functor $\mathsf{Kar}(\mathsf{C}) \to \mathsf{C}$ as identities are not preserved, we can argue in the exact same way as in its proof, which reduces the problem to showing that every hom-space

$$\mathsf{Kar}(\mathsf{C})((X, p), (Y, q)) \subseteq \mathsf{C}(X, Y) \tag{55}$$

is ultraweakly closed. But this is an immediate consequence of the definition and the ultraweak continuity of $f \mapsto fp$ and $f \mapsto qf$.

Moreover, if a W-functor* $\mathsf{C} \to \mathsf{D}$ *is fully faithful, then its extension to* $\mathsf{Kar}(\mathsf{C}) \to \mathsf{D}$ *is fully faithful as well.*

Proof. (i) This is quite clear, as a projection $(X, p) \to (X, p)$ in $\mathsf{Kar}(\mathsf{C})$ is a projection $q : X \to X$ with $qp = q = pq$, or equivalently $q \le p$. Then (X, q) splits (X, q) via the isometry $(X, q) \hookrightarrow (X, p)$ represented by id_X.

(ii) While one can given a proof that is quite analogous to the one in ordinary category theory [9, Proposition 6.5.9], in light of our results so far it is more advantageous to conduct a proof similar to the one in the direct sum case (Theorem 3.6.25). Indeed it is clear that the full W*-subcategory $\mathsf{C} \hookrightarrow \mathsf{Kar}(\mathsf{C})$ is generating. Hence Corollary 3.5.14(ii) already implies that the restriction functor

$$\mathsf{Fun}(\mathsf{Kar}(\mathsf{C}), \mathsf{D}) \longrightarrow \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \tag{56}$$

is fully faithful. It remains to be shown that it is essentially surjective. To this end, note that every the projection completion is functorial, in the sense that every W*-functor $F : \mathsf{C} \to \mathsf{D}$ yields a W*-functor $\mathsf{Kar}(F) : \mathsf{Kar}(\mathsf{C}) \to \mathsf{Kar}(\mathsf{D})$ such that the diagram

$$\begin{array}{ccc} \mathsf{C} & \hookrightarrow & \mathsf{Kar}(\mathsf{C}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathsf{Kar}(F)} \\ \mathsf{D} & \hookrightarrow & \mathsf{Kar}(\mathsf{D}) \end{array}$$

commutes. Since the lower horizontal arrow is a W*-equivalence, the essential surjectivity claim follows.

The final statement is again an instance of Lemma 3.5.17. □

3.8 W*-limits and the W*-completion

The goal of this subsection will be to introduce and study notions of W*-limits and W*-completeness. This is close in spirit to Henry's work on completeness for C*-categories [26]. Before turning to that, let us quickly note a consequence of the preceding two subsections.

Corollary 3.8.1. *If* C *is a W*-category with finite direct sums and projection splittings, then* C *has finite limits and finite colimits.*

Proof. Finite direct sums are in particular biproducts (Remark 3.6.10), and these make C into an additive category. Furthermore, C has kernels and cokernels by Lemma 3.7.11. Finally, it is a standard fact that an additive category with kernels and cokernels has finite limits and colimits. □

Throughout the following, we focus on the discussion of limits, and leave it understood that the involution allows us to regard every limit also as a colimit at the same time. First, we generalize a result of Ghez, Lima and Roberts [3, Proposition 7.3(f)] from single generators to generating subcategories.[41]

[41] See also Rieffel's earlier [5, Proposition 1.1], which shows this in the special case $\mathsf{C} = \mathsf{NRep}(N)$.

Lemma 3.8.2. *Let* C *be a* W^**-category with direct sums and projection splittings and* $\mathsf{D} \subseteq \mathsf{C}$ *a generating full* W^**-subcategory. Then for every* $X \in \mathsf{C}$ *there is a family of objects* Y_- *in* D *and a subfamily of subobjects* $Z_- \hookrightarrow Y_-$ *in* C *together with a unitary isomorphism* $X \cong Z_\oplus$*.*

Proof. By Corollary 3.5.9(i), we have a family of objects $Y_- = (Y_i)_{i \in I}$ in D with a family of partial isometries $u_- : X \to Y_-$ such that $\mathrm{id}_X = u_- u_-^*$. While this family is a priori large, only a small set of members can make a nonzero contribution, and hence we restrict to that.

Splitting these partial isometries as in Lemma 3.7.10 results in subobject inclusions $v_- : Z_- \hookrightarrow Y_-$ and $w_- : Z_- \hookrightarrow X$ such that $u_i = w_i v_i^*$ for all $i \in I$. Therefore

$$\mathrm{id}_X = u_- u_-^* = \sum_i w_i v_i^* v_i w_i^* = w_- w_-^*,$$

which by Theorem 3.6.4 makes X into a direct sum of the family Z_-. □

The following special case will be important for us in this subsection.

Proposition 3.8.3. *Every small self-dual Hilbert presheaf* $H \in \hat{\mathsf{C}}$ *is a direct sum of subobjects of representables.*

Proof. Apply Lemma 3.8.2 to $\hat{\mathsf{C}}$ with C as generating full subcategory. Alternatively, use the existence of an orthonormal basis (Theorem 3.5.12). □

The most general notion of limit in W*-category theory may involve weightings by Hilbert presheaves or something along these lines. Since the technical details of such a putative definition are unclear to us, we contend ourselves with a simpler and more restrictive definition that will be sufficient for our purposes; as we will see, both direct sums and projection splittings are instances of it.

Definition 3.8.4. *Let* C *be a* W^**-category,* J *a small category and* $D : \mathsf{J} \to \mathsf{C}$ *a functor. For an object* $A \in \mathsf{C}$*, a cone* $(f_X : A \to DX)_{X \in \mathsf{J}}$ *is* ***square summable*** *if*

$$f_-^* f_- = \sum_{X \in \mathsf{J}} f_X^* f_X < \infty. \tag{57}$$

We denote the set of these cones by $\mathsf{Cone}(A, D)$*.*

With the cone considered as a family of morphisms with common domain, this is really the same notion of square summability that we have been using throughout since (20). Note that J is merely an ordinary category rather than a W*-category.

Remark 3.8.5. An unpleasant feature of our definition is that the square summability is not invariant under equivalence. For example, let J be the terminal category consisting of a single object and its identity morphism. Similarly, let J' be a category with infinitely many isomorphic objects with trivial endomorphism monoid. Then J and J' are equivalent as categories. However, while a cone over a functor $\mathsf{J} \to \mathsf{C}$ is trivially square summable, a cone over the associated functor $\mathsf{J}' \to \mathsf{C}$ is square summable only if it vanishes, since all of the infinitely many terms in (57) are then equal.

We will prove easily that cones as above form a Hilbert presheaf, but proving its self-duality requires a bit more work, so let us be prepared by considering first how to "project" the two morphisms at the apex of a triangle such that the triangle commutes.

Lemma 3.8.6. *Let* C *be a* W^**-category and* $Y, Z \in \mathsf{C}$*. Then for every* $f : Y \to Z$*, the matrix*

$$\begin{pmatrix} (\mathrm{id}_Y + f^* f)^{-1} & (\mathrm{id}_Y + f^* f)^{-1} f^* \\ f(\mathrm{id}_Y + f^* f)^{-1} & f(\mathrm{id}_Y + f^* f)^{-1} f^* \end{pmatrix}$$

represents a projection in $\hat{\mathsf{C}}$ *on the Hilbert presheaf* $\mathsf{C}(-, Y) \oplus \mathsf{C}(-, Z)$*. Its image is the set of all pairs* $(g : A \to Y, h : A \to Z)$ *such that* $fg = h$*.*

Proof. Straightforward calculation. □

Proposition 3.8.7. *For any diagram* $D : \mathsf{J} \to \mathsf{C}$*, the square summable cones form a small self-dual Hilbert presheaf*

$$\mathsf{Cone}(-, D) \;:\; \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}. \tag{58}$$

Proof. By definition, the square summable cones with apex A are a subset of the square summable families of morphisms $(f_X : A \to DX)_{X \in \mathsf{J}}$, namely those that make all the relevant triangles commute. Since cones are clearly closed under precomposition by morphisms in C, the square summable cones form a subfunctor

$$\mathsf{Cone}(-, D) \;\subseteq\; \bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX), \tag{59}$$

and we can equip it with the induced inner product to see that it is a Hilbert presheaf with isometric inclusion $v : \mathsf{Cone}(-, D) \hookrightarrow \mathsf{C}(-, DX)$. It remains to prove self-duality of $\mathsf{Cone}(-, D)$, which is the actually tricky part.

We first prove the adjointability of v by constructing a projection on $\bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX)$ that projects to $\mathsf{Cone}(-, D)$. For any morphism $j : Y \to Z$ in J, let

$$q_j \;:\; \bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX) \longrightarrow \bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX)$$

be the projection represented by the matrix which coincides with (58) on the Y and Z summands and is identity elsewhere. Then our desired projection is the infimum

$$p \coloneqq \bigwedge_j q_j,$$

where j ranges over all morphisms in J, and this infimum is taken in the lattice of projections of the endomorphism W*-algebra of $\bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX)$. By construction, p projects onto all families of morphisms $(f_X : A \to DX)_{X \in \mathsf{J}}$ which make all the relevant triangles commute, or in other words it projects onto $\mathsf{Cone}(-, D)$, and this provides the desired adjoint v^*.

Since we can now project onto $\mathsf{Cone}(-, D)$, its self-duality follows as in our proof of projection completeness of $\hat{\mathsf{C}}$ given in Example 3.7.7. □

Definition 3.8.8. *The* ***W*-limit*** *of a diagram* $D : \mathsf{J} \to \mathsf{C}$ *is a representation of the Hilbert presheaf* $\mathsf{Cone}(-, D)$*, meaning an object* $L \in \mathsf{C}$ *and a natural isometric isomorphism*

$$\mathsf{C}(-, L) \cong \mathsf{Cone}(-, D). \tag{60}$$

Let us spell this out more explicitly based on the definition of $\mathsf{Cone}(-, D)$ as a subfunctor of $\bigoplus_{X \in \mathsf{J}} \mathsf{C}(-, DX)$. This shows that a W*-limit consists of an object $L \in \mathsf{C}$ and a square summable cone $(\ell_X : L \to DX)_{X \in \mathsf{J}}$ that is universal in the sense that for every square summable cone $(f_X : A \to DX)_{X \in \mathsf{J}}$, there is a unique morphism $g : A \to L$ such that $f_X = \ell_X g$ for all X and

$$\|g\|^2 = \|f_{\text{-}}\|^2 = \|f_{\text{-}}^* f_{\text{-}}\|.$$

Example 3.8.9. If J is a discrete category consisting of a set of objects and only identity morphisms, then a diagram $\mathsf{J} \to \mathsf{C}$ is simply a family of objects in C indexed by the objects of J, and a W*-limit is a direct sum of this family.

Example 3.8.10. Let J be the category consisting of a single object and a single non-identity morphism squaring to itself. Then a diagram $\mathsf{J} \to \mathsf{C}$ is given by an object $X \in \mathsf{C}$ together with a (not necessarily self-adjoint!) morphism $e : X \to X$ satisfying $e^2 = e$. The Hilbert presheaf assigns to an object A the set of all $f : A \to X$ with $ef = f$, with the usual inner product $\langle g, f \rangle = g^* f$.

A limit of this diagram thus consists of an object L together with an isometry $v : L \to X$ such that $ef = f$ is equivalent to $vv^* f = f$. By Lemma 3.7.11, this equivalently means that v splits the projection $s(\mathrm{id}_X - e)^\perp$. If e is self-adjoint, then this is equivalent to v splitting e. In general, W*-limits of this shape exist as soon as C is projection complete.

Example 3.8.11. Let G be a group and C any W*-category. Then recall that a unitary representation of G in C consists of an object $X \in \mathsf{C}$ and a group homomorphism $\pi : G \to \mathsf{C}(X, X)$ with values in unitaries. The *invariant subobject* of such a representation is the W*-limit of the diagram π itself, if it exists.

Generalizing the equational characterization of direct sums (Theorem 3.6.4) and projection splittings (by definition), we obtain the following equational characterization of W*-limits in general.

Proposition 3.8.12. *Let* $D : \mathsf{J} \to \mathsf{C}$ *be a diagram in a W*-category* C*. Then for any* $L \in \mathsf{C}$ *and cone* $(\ell_X : L \to DX)_{X \in \mathsf{J}}$*, the following are equivalent:*

(i) This cone makes L *into the W*-limit of* D*.*

(ii) We have

$$\ell_{\text{-}}^* \ell_{\text{-}} = \mathrm{id}_L, \tag{61}$$

and the matrix $\ell_{\text{-}} \ell_{\text{-}}^*$ *represents the projection* p *from the proof of Proposition 3.8.7.*

Proof. Condition (ii) states exactly that the Hilbert transformation $\mathsf{C}(-,L) \to \bigoplus_{X\in\mathsf{J}} \mathsf{C}(-,DX)$ represented by the cone $\ell_{\_}$ is a splitting of the projection p from the proof of Proposition 3.8.7. The claim therefore follows by the uniqueness of projection splittings up to unique unitary isomorphism. □

Since the condition on the matrix $\ell_{\_}\ell_{\_}^*$ in Proposition 3.8.12 is somewhat unwieldy, this result is arguably not useful when dealing with W*-limits in practice, but it has the following important theoretical consequence.

Theorem 3.8.13. *Let $F : \mathsf{C} \to \mathsf{D}$ be a W*-functor. Then F preserves all W*-limits that exist in C.*

Proof. This follows by the equational characterization of Proposition 3.8.12 together with the fact that the ultraweakly continuous $*$-homomorphism

$$\bigoplus_{X\in\mathsf{J}} \mathsf{C}(-,DX) \longrightarrow \bigoplus_{X\in\mathsf{J}} \mathsf{D}(F-,FDX)$$

given by entrywise application of F preserves meets of projections.[42] □

Example 3.8.14. Let J be the "walking isomorphism", i.e. the category consisting of two objects with a pair of inverse isomorphisms between them and no other non-identity morphisms. Then a functor $\mathsf{J} \to \mathsf{Hilb}$ is a diagram of the form

$$\mathcal{H} \underset{t^{-1}}{\overset{t}{\rightleftarrows}} \mathcal{K}$$

where t is an isomorphism of Hilbert spaces (not necessarily unitary). Of course, a cone to this diagram is determined uniquely by either of its two components; let us just use the $\mathcal{H}$ component. Then the cones at any $A \in \mathsf{Hilb}$ are in bijection with the hom-set $\mathsf{Hilb}(A,\mathcal{H})$, but the inner product is different: using the prescription above gives

$$\langle g, f\rangle = g^*f + g^*t^*tf = g^*(\mathrm{id}_{\mathcal{H}} + t^*t)f. \tag{62}$$

Hence the W*-limit of the above diagram exists and is given by $\mathcal{H}$ with the inner product weighted by the kernel $\mathrm{id}_{\mathcal{H}} + t^*t$ (see also Example 2.5.5).

Remark 3.8.15. Although it is unclear to us if and how our W*-limits can be generalized further by introducing (something like) an additional Hilbert presheaf $\mathsf{J}^{\mathrm{op}} \to \mathsf{Ban}$ as weight, a different way of introducing weights seems to be more straightforward, as anticipated for the case of direct sums at [28, Remark 4.5]. The idea is to introduce a positive semidefinite *kernel matrix* $K_{\_\_} = (K_{X,Y})_{X,Y\in\mathsf{J}} \in \mathbb{R}^{|\mathsf{J}|\times|\mathsf{J}|}$ as weight by modifying

[42] Let us argue instead that joins of projections are preserved, so that the preservation of meets follows upon taking complements. The preservation of a binary join $p \vee q$ follows by $p \vee q = s(p+q)$. The preservation of an arbitrary join follows as this is the ultraweak limit of the finite joins below it.

the definition of $\bigoplus_{X\in\mathsf{J}} \mathsf{C}(-, DX)$ accordingly, namely by introducing K_{--} in both the definition of square summability and the definition of the inner product as[43]

$$\langle g_-, f_- \rangle := g_-^* K_{--} f_-.$$

In this way, we can hope to address the main shortcoming of our present definition, namely the lack of invariance under equivalence from Remark 3.8.5: it is conceivable that replacing J by an equivalent category preserves the limit provided that one also adjusts the kernel K_{--} accordingly; see (62) for an example where this seems to happen.

The following theorem is our main result on W*-completeness. Its part (iii) amounts to a W*-categorical version of Freyd's representable functor theorem; no preservation of limits appears as such preservation is automatic (Theorem 3.8.13).

Theorem 3.8.16. *Let C be a W*-category. Then the following are equivalent:*

(i) *C has all W*-limits.*
(ii) *C has direct sums and projection splittings.*
(iii) *Every small self-dual Hilbert presheaf $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ is representable.*
(iv) *The Yoneda embedding $\mathsf{C} \hookrightarrow \hat{\mathsf{C}}$ is a W*-equivalence.*
(v) *There is a W*-category D and a W*-equivalence $\mathsf{C} \cong \hat{\mathsf{D}}$.*

Proof. Given (i), we obtain (ii) since direct sums and projection splittings are W*-limits of a particular shape by Examples 3.8.9 and 3.8.10. Assuming (ii), let $H : \mathsf{C}^{\mathrm{op}} \to \mathsf{Ban}$ be a small self-dual Hilbert presheaf. By Proposition 3.8.3, we can write it as a direct sum of subobjects of representables. These subobjects are representable by the existence of projection splittings and Lemma 3.7.13. Their direct sums are therefore representable by the existence of direct sums in C and we obtain (iii). Conversely, (iii) clearly implies (i), so that the first three items are all equivalent.

If (iii) holds, then the Yoneda embedding is indeed a W*-equivalence since it is fully faithful anyway and essentially surjective by assumption. Given (iv), we obtain (v) by taking $\mathsf{D} := \mathsf{C}$. Finally assuming (v), we get (ii) by the existence of direct sums (Proposition 3.6.7) and projection splittings (Example 3.7.7). □

Definition 3.8.17. *A W*-category C is **W*-complete** if it satisfies the equivalent conditions of Theorem 3.8.16.*

So by (v) above, the *only* W*-complete W*-categories are those equivalent to $\hat{\mathsf{C}}$ for some W*-category C. Here is a family of examples which are not manifestly of this form.

Example 3.8.18. Let us see which of our examples of W*-categories from Section 2.4 are W*-complete. Thanks to Theorem 3.8.16, we can combine Examples 3.6.12 and 3.7.6 to obtain the following:

▷ The discrete W*-category on a set I is W*-complete if and only if $I = \emptyset$.

[43] In general, one will also have to quotient by the resulting null space.

- ▷ A W*-category of normal representations $\mathsf{NRep}(N)$ is W*-complete, as is $\mathsf{HilbMod}(N)$.
- ▷ A product W*-category $\prod_{i\in I}\mathsf{C}_i$ is W*-complete if and only if the individual C_i are.
- ▷ A coproduct W*-category $\coprod_{i\in I}\mathsf{C}_i$ is not W*-complete, provided that at least two of the components C_i contain a nonzero object.
- ▷ For C a W*-complete W*-category and D any small W*-category, the functor W*-category $\mathsf{Fun}(\mathsf{D},\mathsf{C})$ is also W*-complete. In particular, all W*-categories of the form $\mathsf{HilbBiMod}(M,N)$ and $\mathsf{Connes}(M,N)$ are W*-complete.
- ▷ For a W*-complete W*-category C, also its contractive arrow W*-category $\mathsf{C}^{\to,1}$ is W*-complete.
- ▷ A W*-category C is W*-complete if and only if its conjugate $\overline{\mathsf{C}}$ is.

An important step in the proof of Theorem 3.8.16 was that every small self-dual Hilbert presheaf can be written as a direct sum of subobjects of representables. Let us reformulate this observation a bit more abstractly.

Proposition 3.8.19. *For every W*-category* C*, there is a W*-equivalence* $\mathsf{Kar}(\mathsf{C})^{\oplus} \xrightarrow{\cong} \hat{\mathsf{C}}$ *such that the diagram*

$$\begin{array}{ccc} & \mathsf{C} & \\ \swarrow & & \searrow \\ \mathsf{Kar}(\mathsf{C})^{\oplus} & \xrightarrow{\cong} & \hat{\mathsf{C}} \end{array}$$

commutes up to natural unitary isomorphism, where the upper arrows are the inclusions. Moreover, this W-functor is the unique one (up to unitary isomorphism) which makes the triangle commute.*

Proof. With $\mathsf{Kar}(\mathsf{C})$ in place of $\mathsf{Kar}(\mathsf{C})^{\oplus}$, the existence and essential uniqueness follow by Theorem 3.7.15. In a second step, one can then extend from $\mathsf{Kar}(\mathsf{C})$ to $\mathsf{Kar}(\mathsf{C})^{\oplus}$ through Theorem 3.6.25.

It remains to be shown that the extension is a W*-equivalence. The fact that it is fully faithful also follows by the corresponding statements in Theorems 3.6.25 and 3.7.15. Its essential surjectivity is a restatement of Proposition 3.8.3. □

As a consequence, we obtain a universal property of $\hat{\mathsf{C}}$ as the **W*-completion**. This is analogous to the fact that small presheaves on an ordinary category form its free cocompletion [38].

Theorem 3.8.20. *Let* C *be a W*-category. Then:*

(i) The W-category* $\hat{\mathsf{C}}$ *is W*-complete.*

(ii) For any W-complete W*-category* D*, composition with the Yoneda embedding* $\mathsf{C} \hookrightarrow \hat{\mathsf{C}}$ *establishes a W*-equivalence*

$$\mathsf{Fun}(\hat{\mathsf{C}},\mathsf{D}) \cong \mathsf{Fun}(\mathsf{C},\mathsf{D}).$$

Moreover, if a W-functor* $\mathsf{C}\to\mathsf{D}$ *is fully faithful, then its extension to* $\hat{\mathsf{C}}\to\mathsf{D}$ *is fully faithful as well.*

Proof. (i) See Theorem 3.8.16.

(ii) This W*-equivalence follows by Proposition 3.8.19 together with the universal properties of the projection completion (Theorem 3.7.15) and direct sum completion (Theorem 3.6.25). The final claim likewise follows by the final claims in the referenced statements. □

Let us record another immediate consequence of Theorem 3.8.20, where we abbreviate "up to unitary isomorphism" to "essentially".

Corollary 3.8.21. *If* C *and* D *are W*-categories and* $F : \mathsf{C} \to \mathsf{D}$ *is a W*-functor, then there is a essentially unique W*-functor* $\hat{F}$ *such that the diagram*

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{F} & \mathsf{D} \\ \big\downarrow & & \big\downarrow \\ \hat{\mathsf{C}} & \xrightarrow{\hat{F}} & \hat{\mathsf{D}} \end{array} \tag{63}$$

essentially commutes, where the vertical arrows are the Yoneda embeddings.

Given W*-functors $F, G : \mathsf{C} \to \mathsf{D}$ and a bounded natural transformation $\alpha : F \to G$, we can again apply the 2-categorical universal property of Theorem 3.8.20 to obtain a bounded natural transformation $\hat{\alpha} : \hat{F} \to \hat{G}$. In this way, the W*-completion becomes a weak 2-functor $\mathbb{W}^*\mathsf{CAT} \to \mathbb{W}^*\mathsf{CAT}$.

Since $\hat{F}$ extends small self-dual Hilbert presheaves from C to D, we also use the term **extension along** F to refer to the action of $\hat{F}$.

Remark 3.8.22. The extension of $H \in \mathsf{C}$ along F can be concretely understood in terms of orthonormal bases: if $\beta_{\_} \in HX_{\_}$ is an orthonormal basis for H, then $\hat{F}H$ likewise has an orthonormal basis consisting of a family of elements $\gamma_{\_} \in (\hat{F}H)(FX_{\_})$ with $\langle \gamma_i, \gamma_i \rangle = F(\langle \gamma_i, \gamma_i \rangle)$. This follows by the essential commutativity of (63) and the fact that as a W*-functor, $\hat{F}$ preserves direct sums and projection splittings.

Remark 3.8.23. The universal property Theorem 3.8.20 shows that $\hat{\mathsf{C}}$ is the **free W*-completion** of C. While free cocompletion in ordinary category theory is a *lax-idempotent 2-monad* [39], this free W*-completion is even better behaved: every W*-functors preserves W*-limits, and therefore the W*-complete W*-categories simply form a reflective subbicategory of $\mathbb{W}^*\mathsf{CAT}$.

In particular, using the identification $\hat{\hat{D}} \cong \hat{D}$, we write $\hat{F} : \hat{\mathsf{C}} \to \hat{\mathsf{D}}$ for the extension of a W*-functor $F : \mathsf{C} \to \hat{\mathsf{D}}$ along the Yoneda embedding $\mathsf{C} \hookrightarrow \hat{\mathsf{C}}$.

We add an important caveat to the notion of extension.

Remark 3.8.24. One might hope that extension might be adjoint to some sort of "restriction" W*-functor $\hat{\mathsf{D}} \to \hat{\mathsf{C}}$. It is not clear how one might define such a restriction, given that $\hat{\mathsf{D}}$-valued inner products cannot be turned into C-valued inner products in general. However, in general there is no W*-functor $F^* : \hat{\mathsf{D}} \to \hat{\mathsf{C}}$ which would be adjoint to $\hat{F}$ in the sense of an isometric isomorphism

$$\hat{\mathsf{C}}(H, F^*K) \cong \hat{\mathsf{D}}(\hat{F}H, K)$$

natural in $H \in \hat{\mathsf{C}}$ and $K \in \hat{\mathsf{D}}$. Indeed this already fails with $\mathsf{C} = \mathfrak{B}\mathbb{C}$ and $H = \mathbb{C}$, in which case F simply picks out an object $F(*) \in \mathsf{D}$, we have $\hat{F}H = \mathsf{D}(-, F(*))$, and therefore by Yoneda a putative isomorphism as above would instantiate to an isometric isomorphism

$$F^*K \cong K(F(*)).$$

Since $\hat{\mathsf{C}} = \mathcal{H}$, this implies that $K(F(*))$ would have to be isometrically isomorphic to a Hilbert space, which of course happens very rarely.

On the positive side, if $F : \mathsf{C} \to \mathsf{D}$ is *fully faithful*, then of course we do have the restriction W*-functor already considered in Theorem 3.4.2 and Corollary 3.4.7, though with C and D used the other way around. Let us denote it by $F^* : \hat{\mathsf{D}} \to \hat{\mathsf{C}}$.

Theorem 3.8.25. *Let $F : \mathsf{D} \to \mathsf{C}$ a fully faithful W*-functor. Then there is an isometric adjunction*

$$\hat{\mathsf{D}} \underset{F^*}{\overset{\hat{F}}{\rightleftarrows}} \hat{\mathsf{C}} \quad (\hat{F} \dashv F^*)$$

which is such that:

(i) The unit $\eta : \mathrm{id}_{\hat{\mathsf{D}}} \to F^*\hat{F}$ is a unitary isomorphism.
(ii) The counit $\varepsilon : \hat{F}F^* \to \mathrm{id}_{\hat{\mathsf{C}}}$ is an isometry.

In particular, $\hat{\mathsf{D}}$ is a coreflective W-subcategory of $\hat{\mathsf{C}}$.*

Item (ii) in particular means that the counit has split monomorphism components. By general abstract nonsense [16, Theorem IV.3.1], this is equivalent to fullness of the right adjoint F^*, which we already saw in Corollary 3.4.3. In this sense, (ii) strengthens that earlier result.

By the existence of the involution, we can equivalently regard F^* as the left and $\hat{F}$ as the right adjoint, so that $\hat{\mathsf{D}}$ is also a reflective W*-subcategory of $\hat{\mathsf{C}}$. With this in mind, the isometry property of the counit means equivalently that the composition of the counit above and the unit of the reversed adjunction

$$\hat{F}F^* \xrightarrow{\ \varepsilon\ } \mathrm{id}_{\hat{\mathsf{C}}} \xrightarrow{\ \varepsilon^*\ } \hat{F}F^* \tag{64}$$

is the identity. This additional coherence property shows that $\hat{\mathsf{D}}$ is actually a *bireflective subcategory* of $\hat{\mathsf{C}}$ in the sense of [40, Definition 8].

Proof. Throughout, we assume without loss of generality that $\mathsf{D} \subseteq \mathsf{C}$ with F being the inclusion in order to simplify notation. This gives $F^*K = K|_{\mathsf{D}}$ for any $K \in \hat{\mathsf{C}}$. By Theorem 3.8.20, we already know that $\hat{F}$ is fully faithful, since it is constructed through the universal property apply to the W*-functor

$$\mathsf{D} \hookrightarrow \mathsf{C} \longrightarrow \hat{\mathsf{C}}$$

which is fully faithful as a composite of fully faithful functors.

To construct the adjunction, let us fix $K \in \hat{\mathsf{C}}$ and consider two Hilbert presheaves $\hat{\mathsf{D}}^{\mathrm{op}} \to \mathsf{Ban}$, namely

$$\hat{\mathsf{C}}(\hat{F}-, K) \qquad \text{and} \qquad \hat{\mathsf{D}}(-, K|_{\mathsf{D}}).$$

The first of these is a Hilbert presheaf as in Example 2.5.3, while the second is manifestly representable. Proving the adjunction amounts to showing that they are isometrically isomorphic. To this end, we apply Corollary 3.5.14(i) to the Yoneda embedding $\mathsf{D} \to \hat{\mathsf{D}}$, which reduces the problem to constructing a natural isometric isomorphism on representable Hilbert presheaves. Indeed for $X \in \mathsf{D}$, using $\hat{F}\mathsf{D}(-, X) = \mathsf{C}(-, X)$ gives

$$\hat{\mathsf{C}}(\hat{F}\mathsf{D}(-, X), K) \;=\; \hat{\mathsf{C}}(\mathsf{C}(-, X), K) \;\cong\; KX \;\cong\; \hat{\mathsf{D}}(\mathsf{D}(-, X), K|_{\mathsf{D}}),$$

as was to be shown; it is clear that all steps are isometric isomorphisms and natural in both X and K.

We show that every unit component $\eta_H : H \to (\hat{F}H)_{\mathsf{D}}$ is a unitary isomorphism. By the same reasoning as above, it is enough to show this on representable presheaves $H = \mathsf{D}(-, X)$. But by definition of the unit, the diagram

$$\begin{array}{ccc} \hat{\mathsf{D}}(H', \mathsf{D}(-, X)) & \xrightarrow{\eta_{\mathsf{D}(-,X)} \circ -} & \hat{\mathsf{D}}(H', (\hat{F}\mathsf{D}(-, X))_{\mathsf{D}}) \\ & \searrow^{\text{apply } \hat{F}} & \big\downarrow \cong \\ & & \hat{\mathsf{C}}(\hat{F}H', \hat{F}\mathsf{D}(-, X)) \end{array}$$

commutes for every $H' \in \hat{\mathsf{D}}$. This implies that composition by $\eta_{\mathsf{D}(-,X)}$ is an isometric isomorphism since the other two arrows are, and hence $\eta_{\mathsf{D}(-,X)}$ is unitary.

Next, we prove that every counit component

$$\varepsilon_K : \hat{F}(K|_{\mathsf{D}}) \longrightarrow K$$

is an isometry. To this end, consider the commuting diagram[44]

$$\begin{array}{ccc} \hat{\mathsf{C}}(\hat{F}(K|_{\mathsf{D}}), K) \times \hat{\mathsf{C}}(K, \hat{F}(K|_{\mathsf{D}})) & \longrightarrow & \hat{\mathsf{C}}(\hat{F}(K|_{\mathsf{D}}), \hat{F}(K|_{\mathsf{D}})) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \hat{\mathsf{D}}(K|_{\mathsf{D}}, K|_{\mathsf{D}}) \times \hat{\mathsf{D}}(K|_{\mathsf{D}}, K|_{\mathsf{D}}) & \longrightarrow & \hat{\mathsf{D}}(K|_{\mathsf{D}}, K|_{\mathsf{D}}) \end{array}$$

[44] This diagram can also be seen as an instance of (27), since in this case the invariance under the central support $\mathrm{c}(\mathsf{D})$ is obviously automatic since $\hat{F}(K|_{\mathsf{D}})$ lies in the subcategory.

Here, the vertical arrows are restriction to D, where we have identified $\hat{F}(K|_{\mathsf{D}})|_{\mathsf{D}}$ with $K|_{\mathsf{D}}$ to simplify the bottom row, and the horizontal arrows are composition. The vertical arrows are isometric isomorphisms by the fact that restriction is a (co)reflector, as was already shown. Now the counit ε_K is the counterpart of $\mathrm{id}_{K|_{\mathsf{D}}}$ at the very bottom left. Thus if we start with $(\varepsilon_K, \varepsilon_K^*)$ top left, then its composite (top right) is the counterpart of $\mathrm{id}_{K|_{\mathsf{D}}}$, which is $\mathrm{id}_{\hat{F}(K|_{\mathsf{D}})}$, as was to be shown. □

A similar result holds for W*-functors landing in a W*-complete W*-category. In this setting, we can phrase this as a (left or right) Kan extension, which is why we denote the W*-functor adjoint to restriction as Kan.

Theorem 3.8.26. *Let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful W*-functor and E any W*-complete W*-category. Then there is an isometric adjunction*

$$\mathsf{Fun}(\mathsf{C},\mathsf{E}) \underset{-\circ F}{\overset{\mathrm{Kan}_F}{\rightleftarrows}} \mathsf{Fun}(\mathsf{D},\mathsf{E}) \qquad (\bot)$$

which is such that:

(i) The unit with components $\eta_G : G \to \mathrm{Kan}_F(G) \circ F$ is a unitary isomorphism.
(ii) The counit with components $\varepsilon_G : \mathrm{Kan}_F(GF) \to G$ is an isometry.

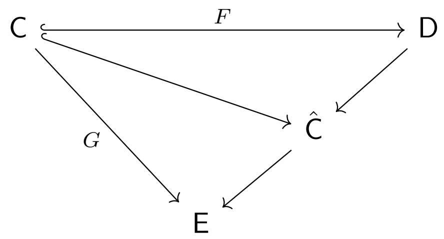
In particular, every W*-functor $\mathsf{C} \to \mathsf{E}$ can extended to a W*-functor $\mathsf{D} \to \mathsf{E}$. More specifically, $\mathsf{Fun}(\mathsf{C},\mathsf{E})$ becomes a bireflective W*-subcategory of $\mathsf{Fun}(\mathsf{D},\mathsf{E})$.

Proof. In the diagram (63), we know that also $\hat{F}$ is fully faithful by Theorem 3.8.25. Since furthermore restricting along the Yoneda embedding defines a W*-equivalence $\mathsf{Fun}(\mathsf{C},\mathsf{E}) \cong \mathsf{Fun}(\hat{\mathsf{C}},\mathsf{E})$, and similarly for D, it is enough to prove the statement with "hats everywhere". But in this case we get an isometric adjunction

$$\mathsf{Fun}(\hat{\mathsf{C}},\mathsf{E}) \underset{-\circ F_*}{\overset{-\circ \hat{F}}{\rightleftarrows}} \mathsf{Fun}(\hat{\mathsf{D}},\mathsf{E}) \qquad (\bot)$$

from Theorem 3.8.25 on purely formal grounds. Since composing with a W*-functor $\mathsf{C} \to \mathsf{E}$ (resp. $\mathsf{D} \to \mathsf{E}$) preserves unitaries (resp. isometries), the claim (i) (resp. (ii)) follows from the corresponding claim in Theorem 3.8.25. □

Remark 3.8.27. The previous proof can be turned into the following explicit construction of $\mathrm{Kan}_F(G)$ for any $G : \mathsf{C} \to \mathsf{E}$. It is given by the right composite in the diagram



where the upper right arrow is the restricted Yoneda embedding and the lower right arrow is the essentially unique extension of G to $\hat{\mathsf{C}}$.

The previous two theorems imply another variation on the universal property of Theorem 3.8.20(ii), a result which will be our workhorse in Section 4.

Corollary 3.8.28. *Let $\mathsf{D} \subseteq \mathsf{C}$ be a generating full W*-subcategory. Then restriction along the inclusion $\mathsf{D} \hookrightarrow \mathsf{C}$ implements:*

(i) A W-equivalence $\hat{\mathsf{D}} \cong \hat{\mathsf{C}}$, where extension along the inclusion is its essential inverse.*

(ii) For every W-complete W*-category E, a W*-equivalence*[45]

$$\mathsf{Fun}(\mathsf{C}, \mathsf{E}) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{E}).$$

Moreover, this W-equivalence respects faithfulness and full faithfulness.*

With D and E of the form $\mathsf{NRep}(N)$ and $\mathsf{C} \subseteq \mathsf{D}$ a single object, (ii) is due to Rieffel [5, Proposition 5.4]. For the general case with a single generator, see also [3, Corollary 7.4].

Proof. Claim (i) holds by Theorem 3.8.25 and Corollary 3.5.14(i). Similarly, the W*-equivalence of (ii) holds by Theorem 3.8.26 and Corollary 3.5.14(ii). The final statement is Lemma 3.5.17. □

Example 3.8.29. Let C be a W*-category with a generator S having endomorphism W*-algebra $N := \mathsf{C}(S, S)$, and let D a be a W*-complete W*-category. If there is a faithful representation of N on an object in D, then there also is a faithful W*-functor $\mathsf{C} \to \mathsf{D}$.

3.9 Grothendieck W*-categories

In ordinary category theory, a *Grothendieck category* is a suitably well-behaved abelian category with a generator [41]. The goal of this subsection is to develop the analogous concept for W*-categories. Once again the situation is simpler than in ordinary category theory, and in fact there is a complete classification of all Grothendieck W*-categories.

[45] Note that these functor W*-categories may be large.

Definition 3.9.1. *A **Grothendieck W*-category** is a W*-complete W*-category with a generating family.*

As noted already at Remark 3.6.14, one can equivalently postulate the existence of a single generator rather than a generating family. The simplest nontrivial example of a Grothendieck W*-category is Hilb. Given the results that we have already developed, it is an easy matter to give one version of the complete classification of Grothendieck W*-categories already.

Theorem 3.9.2. *Let C be a Grothendieck W*-category with small generating full W*-subcategory $\mathsf{D} \subseteq \mathsf{C}$. Then the restricted Yoneda embedding*

$$\begin{aligned} \mathsf{C} &\longrightarrow \hat{\mathsf{D}} \\ X &\longmapsto \mathsf{C}(-, X)|_{\mathsf{D}} \end{aligned}$$

is a W-equivalence.*

Proof. This is the composite of the Yoneda embedding $\mathsf{C} \to \hat{\mathsf{C}}$, which is a W*-equivalence by Theorem 3.8.16, and the restriction $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$, which is a W*-equivalence by Corollary 3.8.28(i). □

Corollary 3.9.3. *A W*-category C is a Grothendieck W*-category if and only if there is a W*-equivalence $\mathsf{C} \cong \hat{\mathsf{D}}$ for a small W*-category D.*

Proof. If D is a small W*-category, then $\hat{\mathsf{D}}$ is a Grothendieck W*-category since it is W*-complete (Theorem 3.8.20) and has a generating family given by the representable Hilbert presheaves (Example 3.5.5). The converse is by Theorem 3.9.2. □

In particular, if N is a W*-algebra, then $\mathsf{HilbMod}(N) = \widehat{\mathfrak{B}N}$ is a Grothendieck W*-category. Once again this is the general case, as per the following versions of Theorem 3.9.2 and Corollary 3.9.3 which have appeared as [33, Theorem 2.3] and [3, Proposition 7.6].

Theorem 3.9.4. *Let C be a Grothendieck W*-category with generator S. Then the hom-functor*

$$\mathsf{C}(-, S) \ : \ \mathsf{C} \longrightarrow \mathsf{HilbMod}(\mathsf{C}(S, S)^{\mathrm{op}})$$

is a W-equivalence.*

Proof. This is simply the special case of Theorem 3.9.2 where the generating family is a single generator. We need to take the opposite W*-algebra because the hom-functor is contravariant, which corresponds to a left action of $\mathsf{C}(S, S)$, and hence a right action (as required) of $\mathsf{C}(S, S)^{\mathrm{op}}$. □

The following characterization of Grothendieck W*-categories is a W*-analogue of the Gabriel-Popescu theorem characterizing categories of modules over rings [41, Theorem 3.7.9].

Corollary 3.9.5. *A W*-category* C *is a Grothendieck W*-category if and only if there is a W*-equivalence* $\mathsf{C} \cong \mathsf{HilbMod}(N)$ *for some W*-algebra* N*.*

Let us turn to some examples which are not manifestly of this form.

Example 3.9.6. Given a family of Grothendieck W*-categories $(\mathsf{C}_i)_{i \in I}$, also the product W*-category $\prod_{i \in I} \mathsf{C}_i$ is a Grothendieck W*-category: it is W*-complete by Example 3.8.18 and the product of a family of generators gives a generator. Together with Theorem 3.9.4, this gives us a W*-equivalence

$$\prod_{i \in I} \mathsf{HilbMod}(N_i) \cong \mathsf{HilbMod}\left(\prod_{i \in I} N_i \right)$$

for every family of W*-algebras $(N_i)_{i \in I}$.

Example 3.9.7. If C is a small W*-category, then we have a W*-equivalence

$$\hat{\mathsf{C}} \cong \mathsf{HilbMod}(L(\mathsf{C})^{\mathrm{op}}),$$

where $L(\mathsf{C})$ is the linking W*-algebra as defined in Example 3.6.22. Indeed as was noted there, $L(\mathsf{C})$ is the endomorphism W*-algebra of a generator in $\hat{\mathsf{C}}$, and therefore this follows as an instance of Theorem 3.9.4.

Proving the existence of a generator in the case of a functor W*-category relies on much of the power of our stronger results thus far.

Theorem 3.9.8. *If* C *is a Grothendieck W*-category and* D *a small W*-category, then also* $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ *is a Grothendieck W*-category.*

Proof. The W*-completeness of $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ is Example 3.8.18. Showing the existence of a generating family is significantly more difficult.

Let us start with the case that D has a single object, or equivalently $\mathsf{D} = \mathfrak{B}N$ for some W*-algebra N. Then we have $\mathsf{Fun}(\mathsf{D}, \mathsf{C}) = \mathsf{NRep}(\mathfrak{B}N, \mathsf{C})$. For a fixed generator $S \in \mathsf{C}$ and morphism $f : S \to X$, every representation $\pi : N \to \mathsf{C}(X, X)$ on any $X \in \mathsf{C}$ has a *profile*, by which we mean the linear map

$$\wp(\pi, f) \ : \ \begin{array}{ccc} N & \longrightarrow & \mathsf{C}(S, S) \\ a & \longmapsto & f^* \pi(a) f. \end{array}$$

Profiles are ordered with respect to the complete positivity order as in Theorem 3.4.9. This result also tells us that given another representation $\rho : N \to \mathsf{C}(Y, Y)$ and $g : S \to Y$, we have $\wp(\rho, g) \leq \wp(\pi, f)$ if and only if there is an intertwiner $h : X \to Y$ such that $g = hf$.

Now let S be a generator in C, and choose for every profile $N \to \mathsf{C}(S, S)$ a particular pair (π, f) which realizes this profile. Then we claim that this family of objects in $\mathsf{NRep}(N, \mathsf{C})$ is generating. Indeed for every other representation $\rho : N \to \mathsf{C}(Y, Y)$ on nonzero Y, we can find some nonzero $g : S \to Y$ since S is a generator, and hence the discussion of the previous paragraph guarantees that there is a nonzero intertwiner from one of our generating representation to this one. This proves the generating property by Lemma 3.7.9, and therefore $\mathsf{NRep}(N, \mathsf{C})$ is indeed a Grothendieck W*-category.

Now on to the general case of a functor W*-category $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$. Again by Lemma 3.7.9, it suffices to find a W*-functor $F_A : \mathsf{D} \to \mathsf{C}$ for every $A \in \mathsf{D}$ such that if $G : \mathsf{D} \to \mathsf{C}$ is a W*-functor with GA nonzero, then there is a nonzero bounded natural transformation $F_A \to G$. Considering the single-object full W*-subcategory on the object A shows that $\mathsf{NRep}(\mathsf{D}(A, A), \mathsf{C})$ is a full bireflective W*-subcategory of $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$, with the reflector given by restriction. Since GA is nonzero, also the corresponding object in $\mathsf{NRep}(\mathsf{D}(A, A), \mathsf{C})$ is nonzero. In this way, we have reduced the problem to the previously treated single-object case.[46] □

The simplest instances of this are arguably the W*-categories $\mathsf{NRep}(N)$, for which we can say more.

Lemma 3.9.9 (Rieffel [5, Proposition 1.3])**.** *The generators in* $\mathsf{NRep}(N)$ *are exactly the faithful normal representations.*

Proof. In one direction, every representation has to arise as a subobject of a direct sum of the generating one by Lemma 3.8.2, and hence has a kernel at least as large as the kernel of the generating one; but since there are faithful representations, also every generating one needs to be faithful.

Conversely, if $\pi : N \to \mathcal{B}(\mathcal{H})$ is faithful, then it is a standard fact that any normal state on N can be written as a countable convex combination of vector states from π [42, Theorem 7.1.8]. Hence the profile of any vector from any representation dominates the profile of some nonzero vector from π. Hence Theorem 3.4.9 with $S = \mathbb{C}$ gives us the desired nonzero intertwiner. □

As a simple consequence, we obtain a result of Rieffel and Roberts on reconstructing a W*-algebra from its W*-category of normal representations together with the associated forgetful functor to Hilb.

Corollary 3.9.10 (Rieffel-Roberts)**.** *Let N be a W*-algebra and* $U : \mathsf{NRep}(N) \to \mathsf{Hilb}$ *the canonical forgetful functor. Then the canonical map* $N \to \mathsf{Nat}(U, U)$ *is a $*$-isomorphism.*

Proof. Fix some generator $(\mathcal{H}, \rho)$ in $\mathsf{NRep}(N)$, such as $L^2(N)$. Then Corollary 3.8.28 identifies $\mathsf{Nat}(U, U)$ with the double commutant $\rho(N)''$, which is exactly $\rho(N)$. Hence the composite

$$N \xrightarrow{\ i\ } \mathsf{Nat}(U, U) \xrightarrow{\ \cong\ } \rho(N)'' = \rho(N)$$

is a $*$-isomorphism, and therefore so is i. □

[46] For an alternative such reduction, note that $\mathsf{Fun}(\mathsf{D}, \mathsf{C}) \cong \mathsf{Fun}(\hat{\mathsf{D}}, \mathsf{C}) \cong \mathsf{NRep}(L(\mathsf{C}), \mathsf{C})$, where the first W*-equivalence is by Theorem 3.8.20(ii) and the second by Example 3.9.7 together with the fact that $\mathsf{Fun}(-, \mathsf{C})$ is a 2-functor.

Remark 3.9.11. In particular, if A is a C*-algebra and $U : \mathsf{Rep}(A) \to \mathsf{Hilb}$ is the forgetful functor, then $\mathsf{Nat}(U, U) \cong A^{**}$. This follows upon combining Corollary 3.9.10 with the isomorphism of categories $\mathsf{Rep}(A) \cong \mathsf{NRep}(A^{**})$ from Example 2.4.2.

More generally, if C is a Grothendieck W*-category, then so is $\mathsf{NRep}(N, \mathsf{C})$ for every W*-algebra N, although in this case an explicit description of the generators seems to be more difficult to obtain. Let us note the usual further special cases of this.

Example 3.9.12. The W*-categories of Hilbert bimodules $\mathsf{HilbBiMod}(M, N)$ and Connes correspondences $\mathsf{Connes}(M, N)$ are Grothendieck W*-categories for every two W*-algebras M and N.

Let us now turn to studying W*-functors between Grothendieck W*-categories, starting with some canonical W*-equivalences.

A canonical choice of generator in $\mathsf{NRep}(N)$ is given by the **standard representation** or *standard form* of a W*-algebra N, which is a particularly well-behaved faithful representation $L^2(N)$ [43, § IX.1]. In fact $L^2(N)$ also has a right action of N which is exactly the commutant of the left action, and this is the property behind the following result, which seems to have been folklore for a long time.[47]

Theorem 3.9.13. *For every W*-algebra N, there is a W*-equivalence*

$$\boxed{\mathsf{NRep}(N^{\mathrm{op}}) \cong \mathsf{HilbMod}(N)} \tag{65}$$

implemented by

$$\mathsf{NRep}(N^{\mathrm{op}}) \underset{X \mapsto X \otimes_N L^2(N)}{\overset{\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), -)}{\rightleftarrows}} \mathsf{HilbMod}(N) \quad \cong \tag{66}$$

Here, we view $\mathsf{NRep}(N^{\mathrm{op}})$ as the W*-category of Hilbert spaces equipped with *right* actions of N. A particular such Hilbert space is $L^2(N)$. The hom-functor $\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), -)$ lands in the W*-category of Hilbert modules over the endomorphism W*-algebra of $L^2(N)$, but as mentioned above this commutant is exactly N acting on the left, so that we indeed end up in $\mathsf{HilbMod}(N)$.

In the other direction, the W*-functor $X \mapsto X \otimes_N L^2(N)$ operates by inducing, considering $L^2(N)$ now as a Hilbert module over N. The right action of N is then what turns it into a normal representation of N^{op}.

Proof. The fact that $\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), -)$ is a W*-equivalence is an instance of Theorem 3.9.4 thanks to Lemma 3.9.9.

To see that $- \otimes_N L^2(N)$ is its essential inverse, it is enough to show by Corollary 3.5.14(ii) that this is the case on a generator and its endomorphisms. Starting in

[47] See also Corollary 3.9.14 below.

$\mathsf{NRep}(N^{\mathrm{op}})$ with the generator $L^2(N)$ and going full circle gives us $N \otimes_N L^2(N) \cong L^2(N)$, and it is clear that this isomorphism respects the left action of N. Starting in $\mathsf{HilbMod}(N)$ with the generator N, going full circle gives us $\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), L^2(N))$, and it is similarly clear that this is compatible with the left action. □

In combination with Corollary 3.9.5, we can thus conclude that the Grothendieck W*-categories are precisely those W*-categories equivalent to one of the form $\mathsf{NRep}(N)$ for some W*-algebra N.

The following more general W*-equivalence seems to have been made explicit first by Baillet, Denizeau and Havet [44, Théorème 2.2].[48]

Corollary 3.9.14. *For every two W*-algebras M and N, there is a W*-equivalence*

$$\boxed{\mathsf{Connes}(M, N) \cong \mathsf{HilbBiMod}(M, N)} \tag{67}$$

implemented by

$$\mathsf{Connes}(M,N) \;\underset{X \mapsto X \otimes_N L^2(N)}{\overset{\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), -)}{\rightleftarrows}}\; \mathsf{HilbBiMod}(M,N) \qquad \cong$$

Here, the top W*-functor simply takes bounded linear maps out of $L^2(N)$ that respect the right action by N, and the resulting object still carries an obvious left action by M.

Proof. Apply the 2-functor $\mathsf{NRep}(M, -)$ to (66). □

We obtain another known result now as a simple consequence. The following was proven as [3, Proposition 2.13] via a GNS-like construction.

Corollary 3.9.15. *For any small W*-category C there is a faithful W*-functor $\mathsf{C} \to \mathsf{Hilb}$.*

Proof. Combining Example 3.9.7 and Theorem 3.9.13 shows $\hat{\mathsf{C}} \to \mathsf{NRep}(L(\mathsf{C}))$, and the latter has a faithful W*-functor to Hilb by definition. We can now compose with the Yoneda embedding $\mathsf{C} \hookrightarrow \hat{\mathsf{C}}$ to get the desired result. □

Remark 3.9.16. Since a Connes correspondence with a left action of M and a right action of N is equivalently a Connes correspondence with a left action of N^{op} and a right action of M^{op}, we also see that for Hilbert bimodules, we can reverse the direction of the inner product: there is a canonical W*-equivalence

$$\boxed{\mathsf{HilbBiMod}(M, N) \cong \mathsf{HilbMod}(N^{\mathrm{op}}, M^{\mathrm{op}})}$$

and we can clearly think of $\mathsf{HilbMod}(N^{\mathrm{op}}, M^{\mathrm{op}})$ as the W*-category of M-N-bimodules with an M-valued inner product. However, it is important to keep in mind that this equivalence does not preserve the underlying sets.

[48] See also [3, Corollary 7.11] for a closely related statement.

We can generalize Theorem 3.9.8 to bifunctor W*-categories.

Theorem 3.9.17. *If* D *and* E *are small W*-categories and* C *is a Grothendieck W*-category, then also*

$$\mathsf{BiFun}(\mathsf{D} \times \mathsf{E}, \mathsf{C})$$

is a Grothendieck W-category.*

Proof. There is a canonical W*-equivalence

$$\mathsf{BiFun}(\mathsf{D} \times \mathsf{E}, \mathsf{C}) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{Fun}(\mathsf{E}, \mathsf{C})), \tag{68}$$

so that this is an immediate consequence of Theorem 3.9.8. □

We also obtain a W*-categorical version of the **Deligne tensor product** for abelian categories [45, Section 5].

Theorem 3.9.18. *For Grothendieck W*-categories* C *and* D*, there is a Grothendieck W*-category* $\mathsf{C} \otimes \mathsf{D}$ *together with a W*-bifunctor*

$$\mathsf{C} \times \mathsf{D} \to \mathsf{C} \otimes \mathsf{D}$$

such that composition establishes a W-equivalence*

$$\mathsf{Fun}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \cong \mathsf{BiFun}(\mathsf{C} \times \mathsf{D}, \mathsf{E}).$$

for every W-complete W*-category* E.

Proof. We can assume $\mathsf{C} = \mathsf{NRep}(M)$ and $\mathsf{D} = \mathsf{NRep}(N)$ for W*-algebras M and N by Corollary 3.9.5. Then we have W*-equivalences

$$\begin{aligned} \mathsf{BiFun}(\mathsf{NRep}(M) \times \mathsf{NRep}(N), \mathsf{E}) &\cong \mathsf{Fun}(\mathsf{NRep}(M), \mathsf{Fun}(\mathsf{NRep}(N), \mathsf{E})) \\ &\cong \mathsf{NRep}(M, \mathsf{NRep}(N, \mathsf{E})) \\ &\cong \mathsf{NRep}(M \underline{\otimes} N, \mathsf{E}) \\ &\cong \mathsf{Fun}(\mathsf{NRep}(M \underline{\otimes} N), \mathsf{E}), \end{aligned}$$

where the second and fourth equivalences are by Corollary 3.8.28(ii). We thus take $\mathsf{C} \otimes \mathsf{D} := \mathsf{NRep}(M \underline{\otimes} N)$, and take the universal W*-bifunctor to be given by

$$\begin{aligned} \mathsf{NRep}(M) \times \mathsf{NRep}(N) &\longrightarrow \mathsf{NRep}(M \underline{\otimes} N) \\ (\mathcal{H}, \mathcal{K}) &\longmapsto \mathcal{H} \otimes \mathcal{K}, \end{aligned}$$

where the tensor product of Hilbert spaces carries the obvious tensor representation and the action on morphisms is also given by tensoring. Another look at the above sequence of isomorphisms above shows that the composite isomorphism is indeed given by restriction along this W*-bifunctor, since this is the case on the pair $(L^2(M), L^2(N))$ and we can apply Corollary 3.8.28(ii) again. □

4 Bicategories of W*-algebras and W*-categories

It should now be clear that W*-category theory is a rich and powerful subject. We now turn to a more detailed study of bicategories of W*-algebras and W*-categories, focusing on one particular such bicategory that we will present in four different versions. It is the W*-categorical analogue of the bicategory of small categories and profunctors. In fact, as it a bit more structure than just a bicategory, as in the following notion considered somewhat informally by Yamagami [46, 35].

Definition 4.0.1. *A **W*-bicategory** is a bicategory $\mathbb{D}$ in which the hom-categories $\mathbb{D}(A, B)$ are W*-categories such that:*

(i) The horizontal composition $\circ$ is a W-bifunctor.*
(ii) The associators and unitors are unitary.

In other words, a W*-bicategory is a bicategory enriched in the bicategory $\mathbb{W}^*\mathsf{CAT}$ [47] and such that the associators and unitors are unitary. There is also an obvious notion of equivalence of W*-bicategories: two W*-bicategories are equivalent if they are equivalent as bicategories [48] in such a way that the equivalence preserves the involution on 2-cells or the norm (in which case it necessarily preserves both).

4.1 The W*-bicategory of Grothendieck W*-categories

Perhaps the most paradigmatic example of a bicategory is the bicategory of (small) categories, which has categories as objects, functors as morphisms and natural transformations as 2-cells. Its W*-analogue was introduced already in Definition 2.3.2, and we can now consider it even as a W*-bicategory.

Definition 4.1.1. *We write $\mathbb{W}^*\mathsf{cat}$ for the strict W*-bicategory with*

- *Small W*-categories as objects,*
- *W*-functors as morphisms,*
- *Bounded natural transformations as 2-cells,*

and the obvious composition operations.

Here, we leave it understood that the W*-structure is the obvious one, i.e. the one corresponding to the functor W*-categories from Example 2.4.11. Also $\mathbb{W}^*$ is a strict W*-bicategories, so we do not need to specify any coherence isomorphisms.

We now restrict to the full subbicategory of $\mathbb{W}^*\mathsf{cat}$ consisting of the Grothendieck W*-categories.

Definition 4.1.2. *We write $\mathbb{W}^*{}_{\mathsf{Fun}}$ for the W*-bicategory with*

- *Grothendieck W*-categories as objects,*
- *W*-functors as morphisms,*
- *Bounded natural transformations as 2-cells,*

and the obvious composition operations.

The idea behind this restriction and our choice of notation for it will become clear in the following subsections.

4.2 The bicategory of small W*-categories and self-dual Hilbert profunctors

Our aim is now to define another W*-bicategory equivalent to $\mathbb{W}^*_{\mathsf{Fun}}$, where the equivalence will follow quite straightforwardly from the results of the previous section.

The formalism of Hilbert profunctors from Section 2.6 has not yet added anything other than terminology, as we simply are dealing with W*-functors $\mathsf{C} \to \hat{\mathsf{D}}$. A less trivial treatment should also develop the composition of Hilbert profunctors by analogy with profunctor composition in ordinary category theory. It is only after having developed Section 3 that we are sufficiently equipped to do so in an elegant way. Let us start by giving an abstract treatment of this composition; although this is perfectly sufficient and no concrete description is necessary for working with it, we do give such an explicit description later on.

Theorem 4.2.1. *For any small W*-categories* C *and* D*, there is a W*-equivalence*

$$\boxed{\mathsf{Fun}(\hat{\mathsf{C}}, \hat{\mathsf{D}}) \cong \mathsf{HProf}(\mathsf{C}, \mathsf{D})}$$

implemented by the adjoint equivalence

$$\mathsf{Fun}(\hat{\mathsf{C}}, \hat{\mathsf{D}}) \underset{\mathrm{Kan}_{\text{よ}_{\mathsf{C}}}}{\overset{-\circ \text{よ}_{\mathsf{C}}}{\rightleftarrows}} \mathsf{HProf}(\mathsf{C}, \mathsf{D}) \tag{69}$$

Proof. Since $\mathsf{HProf}(\mathsf{C}, \mathsf{D}) = \mathsf{Fun}(\mathsf{C}, \hat{\mathsf{D}})$ by definition, the restriction W*-functor $- \circ \text{よ}_{\mathsf{C}}$, or equivalently $F \mapsto F|_{\mathsf{C}}$, is a W*-equivalence by Theorem 3.8.20(ii). Its adjoint is given by the Kan extension W*-functor $\mathrm{Kan}_{\text{よ}_{\mathsf{C}}}$ by Theorem 3.8.26, and it is a W*-equivalence as $- \circ \text{よ}_{\mathsf{C}}$ is. □

In other words, for a Hilbert profunctor $Q : \mathsf{C} \nrightarrow \mathsf{D}$ there is an essentially unique W*-functor

$$Q \odot - \ : \hat{\mathsf{C}} \longrightarrow \hat{\mathsf{D}}$$

with the property that $\hat{Q}|_{\mathsf{C}} \cong Q$, and we call it **inducing** along Q. Its action on objects $P \in \hat{\mathsf{C}}$ can be described in terms of Kan extension along the Yoneda embedding,

$$Q \odot P \ = \ \mathrm{Kan}_{\text{よ}_{\mathsf{C}}}(Q) \circ P.$$

Before we proceed with the abstract theory, it may be helpful to understand the Hilbert presheaf $Q \odot P$ more concretely.

As noted before, we can view an element $\alpha \in PX$ as a Hilbert transformation $\mathsf{C}(-, X) \to P$. Applying $\mathrm{Kan}_{\text{よ}_{\mathsf{C}}}(Q)$ to this transformation produces a Hilbert transformation

$$Q(-, X) \longrightarrow Q \odot P.$$

Evaluating this new transformation on some $\beta \in Q(Y, X)$ thus gives us an element

$$\alpha \odot \beta \in (Q \odot P)(Y).$$

This observation can used in order to obtain a more explicit description of the Hilbert presheaf $Q \odot P$ which generalizes the tensor product of Hilbert bimodules (Section 4.3) to the many-object case. This will be explicit even in the sense that it does not require the formation of any sort of completion, as the resulting Hilbert presheaf will automatically be self-dual. Perhaps surprisingly, proving that this explicit description indeed gives $Q \odot P$ is once again a simple application of our general results on W*-categories.[49] To see what it looks like, define first $(Q \odot P)_0(Z)$ to be the space of all formal linear combinations of the form[50]

$$\sum_{i \in I} \alpha_i \otimes \beta_i \tag{70}$$

for families $\alpha_{\_} \in PY_{\_}$ and $\beta_{\_} \in Q(Z, Y_{\_})$, for a family of objects $Y_{\_}$ in C, which are square summable in the sense that

$$\alpha_{\_}\langle \alpha, - \rangle < \infty, \qquad \langle \beta_{\_}, \beta_{\_} \rangle < \infty,$$

and indexed by any set I.[51] The set of these formal linear combinations is equipped with the D-valued inner product given by

$$\left\langle \sum_{j \in J} \alpha'_j \otimes \beta'_j, \sum_{i \in I} \alpha_i \otimes \beta_i \right\rangle := \sum_{i \in I, j \in J} \langle \beta'_j, \langle \alpha'_j, \alpha_i \rangle \beta_i \rangle. \tag{71}$$

This sesquilinear form typically has null spaces: as is standard for tensor products of bimodules, elements corresponding to formal linear combinations like

$$\alpha f \otimes \beta - \alpha \otimes f\beta \tag{72}$$

for $\alpha \in PY'$ and $\beta \in Q(Z, Y)$ and $f : Y \to Y'$ are null. We suspect that the null space consists precisely of the (infinite) linear combinations of elements of this form, but this has not been proven at this point.[52] In any case, using $(Q \odot P)_0$ we indeed obtain a concrete description of the induced Hilbert presheaf $Q \odot P$ in the following sense.

[49] This follows Blecher's identification of the tensor product of Hilbert modules with the *extended Haagerup tensor product* of operator spaces [4], which is going to be the single-object special case of our construction.

[50] The reader may be put off by the order of P and Q having switched now, as the α_i belong to P and the β_i to Q. *Some* mismatch like this seems to be impossible to avoid completely in any choice of conventions, and we follow the existing conventions for profunctors in ordinary category theory as used at ncatlab.org/nlab/show/profunctor.

[51] Formally, these formal linear combinations should be understood as maps

$$c : \left(\coprod_{Y \in \mathsf{C}} PY \times Q(Z, Y) \right) \longrightarrow \mathbb{C}$$

assigning to every pair (α, β) its coefficient, subject to the square summability requirements $\sum_{(\alpha,\beta)} |c(\alpha, \beta)|^2 \alpha\langle \alpha, - \rangle < \infty$ and $\sum_{(\alpha,\beta)} |c(\alpha, \beta)|^2 \langle \beta, \beta \rangle < \infty$.

[52] The known characterization in the single-object case [49, Remark 3.7] may be helpful here.

Proposition 4.2.2. *For $Q : \mathsf{C} \nrightarrow \mathsf{D}$ and $P \in \hat{\mathsf{C}}$ and $Z \in \mathsf{D}$, the map*

$$(Q \odot P)_0(Z) \longrightarrow (Q \odot P)(Z)$$
$$\sum_{i \in I} \alpha_i \otimes \beta_i \longmapsto \sum_{i \in I} \alpha_i \odot \beta_i$$

is a surjective isometry for every $Z \in \mathsf{D}$.

Proof. This is an instance of Proposition 3.5.16 applied to the generating full W*-subcategory $よ_{\mathsf{C}} : \mathsf{C} \hookrightarrow \hat{\mathsf{C}}$ and the W*-functor $\mathrm{Kan}_{よ_{\mathsf{C}}} : \hat{\mathsf{C}} \to \hat{\mathsf{D}}$. □

So to summarize, we can describe $(Q \odot P)(Z)$ as consisting of the formal linear combinations (70) modulo the null space of the positive semidefinite sesquilinear form (71), and this correspondence also preserves the D-valued inner product. Moreover, the W*-functoriality of $Q \odot P$ in P is reproduced: if $s : P \to P'$ is any Hilbert transformation, then is can be described in terms of the concrete construction as mapping

$$\sum_{i \in I} \alpha_i \otimes \beta_i \longmapsto \sum_{i \in I} s(\alpha_i) \otimes \beta_i.$$

This can be seen perhaps most easily in the general categorical setting of Proposition 3.5.16.

Let us return to the abstract theory and consider functoriality in the other argument. For every Hilbert transformation $t : Q \to R$ between Hilbert profunctors in $\mathsf{HProf}(\mathsf{C}, \mathsf{D})$, we get an induced bounded natural transformation

$$t \odot - \; : \; Q \odot - \longrightarrow R \odot -$$

via the universal property of the Kan extensions as $\mathrm{Kan}_{よ_{\mathsf{C}}}(t) : \mathrm{Kan}_{よ_{\mathsf{C}}}(Q) \longrightarrow \mathrm{Kan}_{よ_{\mathsf{C}}}(R)$. In the concrete construction, a brief look back at Proposition 3.5.16 will show that this is given by

$$\sum_{i \in I} \alpha_i \otimes \beta_i \longmapsto \sum_{i \in I} \alpha_i \otimes t(\beta_i).$$

Let us finish this discussion with another abstract perspective on the same operation.

Proposition 4.2.3. *There is an essentially unique functor*

$$\hat{\mathsf{C}} \times \mathsf{HProf}(\mathsf{C}, \mathsf{D}) \longrightarrow \hat{\mathsf{D}}, \tag{73}$$

which is a W-bifunctor and such that its restriction to C in the first argument is the application functor*

$$\mathsf{C} \times \mathsf{HProf}(\mathsf{C}, \mathsf{D}) \longrightarrow \hat{\mathsf{D}}.$$

Proof. Note first that the application functor is indeed a W*-bifunctor as an instance of (8). Now the statement is clear as for fixed $Q \in \mathsf{HProf}(\mathsf{C}, \mathsf{D})$, we had defined $Q \odot -$ as the essentially unique extension of $Q : \mathsf{C} \to \hat{\mathsf{D}}$. □

Suppose that we now consider a third small W*-category B. Then applying (73) objectwise in B results in a W*-bifunctor

$$\mathsf{HProf}(\mathsf{B},\mathsf{C}) \times \mathsf{HProf}(\mathsf{C},\mathsf{D}) \longrightarrow \mathsf{HProf}(\mathsf{B},\mathsf{D}).$$

This is now the composition of Hilbert profunctors in general. More explicitly, for $P : \mathsf{B} \nrightarrow \mathsf{C}$ and $Q : \mathsf{C} \nrightarrow \mathsf{D}$, their composition is

$$Q \odot P \coloneqq \mathrm{Kan}_{よ_\mathsf{C}}(Q) \circ P. \tag{74}$$

For $\mathsf{B} = \mathfrak{B}\mathbb{C}$, this specializes back to (73) by construction. In general, Proposition 4.2.2 also provides us with an explicit construction: $(Q \odot P)(Z, X)$ can be obtained concretely as the space of formal linear combinations

$$\sum_{i \in I} \alpha_i \otimes \beta_i$$

for square summable families $\alpha_- \in P(Y_-, X)$ and $\beta_- \in Q(Z, Y_-)$, modulo the null space of the positive semidefinite D-valued sesquilinear form given by

$$\left\langle \sum_{j \in J} \alpha'_j \otimes \beta'_j, \sum_{i \in I} \alpha_i \otimes \beta_i \right\rangle \coloneqq \sum_{i \in I} \sum_{j \in J} \langle \beta'_j, \langle \alpha'_j, \alpha_i \rangle \beta_i \rangle$$

which at the same time defines the D-valued inner product on $Q \odot P$.

Remark 4.2.4. It is plausible that the composition of Hilbert profunctors has a universal property generalizing Blecher's universal property for the tensor product of Hilbert bimodules [4]. However this is not easily expressible in our setting as it requires operator space structure, and finding suitable extensions of this to the many-object setting remains open.

Our next goal is to show that (74) can be used as composition in a bicategory, we need some auxiliary results.

Lemma 4.2.5. *There is a unitary isomorphism of W*-functors $\hat{\mathsf{B}} \to \hat{\mathsf{D}}$ of the form*

$$\mathrm{Kan}_{よ_\mathsf{B}}(Q \odot P) \cong \mathrm{Kan}_{よ_\mathsf{C}}(Q) \circ \mathrm{Kan}_{よ_\mathsf{B}}(P)$$

natural in $P \in \mathsf{HProf}(\mathsf{B},\mathsf{C})$ and $Q \in \mathsf{HProf}(\mathsf{C},\mathsf{D})$.

Proof. By Theorem 3.8.20(ii) again since both sides restrict to $Q \odot P$ on B. □

It follows that our composition of Hilbert profunctors is naturally associative.

Lemma 4.2.6. *For given self-dual Hilbert profunctors*

$$P : \mathsf{B} \nrightarrow \mathsf{C}, \qquad Q : \mathsf{C} \nrightarrow \mathsf{D}, \qquad R : \mathsf{D} \nrightarrow \mathsf{E},$$

there is a unitary isomorphism

$$R \odot (Q \odot P) \xrightarrow{\cong} (R \odot Q) \odot P. \tag{75}$$

natural in all three Hilbert profunctors. Moreover, this associator satisfies the pentagon equation.

Proof. We can construct such an isomorphism via

$$\begin{aligned} R \odot (Q \odot P) &= \mathrm{Kan}_{よ_{\mathsf{D}}}(R) \circ \left(\mathrm{Kan}_{よ_{\mathsf{C}}}(Q) \circ P\right) \\ &= \left(\mathrm{Kan}_{よ_{\mathsf{D}}}(R) \circ \mathrm{Kan}_{よ_{\mathsf{C}}}(Q)\right) \circ P \\ &\cong \mathrm{Kan}_{よ_{\mathsf{C}}}(R \odot Q) \circ P \\ &= (R \odot Q) \odot P, \end{aligned}$$

where we have applied Lemma 4.2.5 in the third step. For the pentagon equation, suppose that we are given another Hilbert profunctor $S : \mathsf{E} \to \mathsf{F}$. Suppressing all composition symbols and indices for ease of notation, what we need to prove is that the outer diagram in

$$\begin{tikzcd} S(R(QP)) \arrow[rr, "\cong"] \arrow[dddd, "\cong"] \arrow[dr, equal] & & (SR)(QP) \arrow[rr, "\cong"] \arrow[d, equal] & & ((SR)Q)P \\ & \mathrm{Kan}(S)\mathrm{Kan}(R)\mathrm{Kan}(Q)P \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Kan}(SR)\mathrm{Kan}(Q)P \arrow[r, "\cong"] & \mathrm{Kan}((SR)Q)P \arrow[ur, equal] \arrow[d, "\cong"] & \\ & \mathrm{Kan}(S)\mathrm{Kan}(RQ)P \arrow[rr, "\cong"] & & \mathrm{Kan}(S(RQ))P & \\ (S(RQ))P \arrow[ur, equal] \arrow[rrrr, "\cong"] & & & & ((SR)Q)P \arrow[uuuu, "\cong"] \arrow[ul, equal] \end{tikzcd}$$

commutes, where all arrows are (induced from) the already constructed isomorphisms. Since the outer rectangles commute by definition, it is enough to show that the inner pentagon commutes. But this can be done without the P on the right. Moreover, by Theorem 3.8.20(ii) again, we can prove it after restriction along $よ_{\mathsf{C}}$, which reduces the problem to showing commutativity of

$$\begin{tikzcd} \mathrm{Kan}(S) \circ \mathrm{Kan}(R) \circ Q \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Kan}(S \odot R) \circ Q \arrow[r, "\cong"] & (S \odot R) \odot Q \\ \mathrm{Kan}(S) \circ (R \odot Q) \arrow[rr, "\cong"] & & S \odot (R \odot Q) \arrow[u, "\cong"] \end{tikzcd}$$

where we have reinstated the composition symbols for clarity. Upon now plugging in the definition of $\odot$ via (74), all arrows turn into identities except for the upper left horizontal one and the right vertical one, which coincide. Therefore the pentagon diagram commutes. □

For every W*-category C we have its Hilbert hom-profunctor $\mathsf{C}(-,-) : \mathsf{C} \nrightarrow \mathsf{C}$ from Example 2.6.2, which as a W*-functor is precisely the Yoneda embedding $よ_{\mathsf{C}} : \mathsf{C} \to \hat{\mathsf{C}}$, which is why we also use that notation. It plays the role of an **identity Hilbert profunctor** in the following sense.

Lemma 4.2.7. *For every Hilbert profunctor* $P : \mathsf{B} \nrightarrow \mathsf{C}$*, there are unitary isomorphisms*

$$P \odot \text{よ}_{\mathsf{B}} \cong P, \qquad \text{よ}_{\mathsf{C}} \odot P \cong P, \tag{76}$$

natural in P*, and these satisfy the triangle equation.*

Proof. The first isomorphism must be explicitly given by

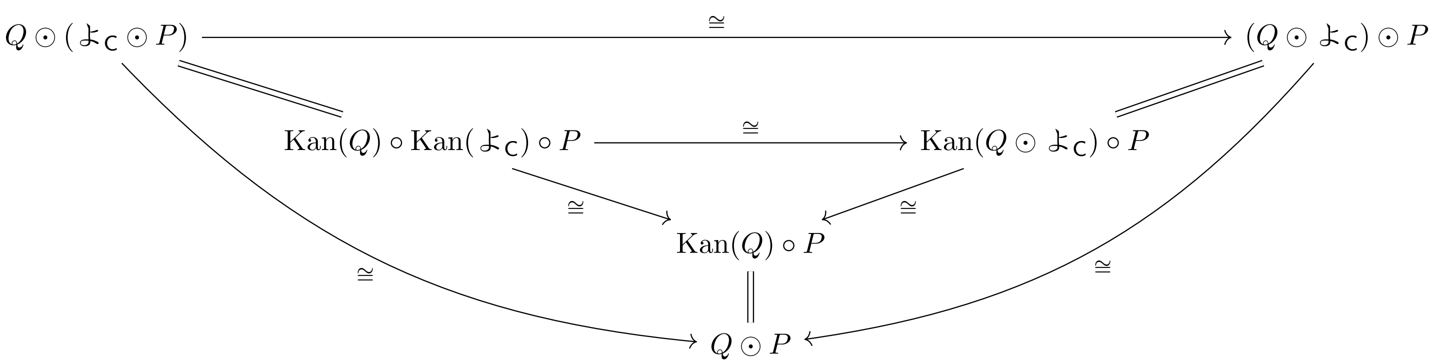
$$\mathrm{Kan}_{\text{よ}_{\mathsf{B}}}(P) \circ \text{よ}_{\mathsf{B}} \cong P,$$

and we take this to be given by the isomorphism associated to the equivalence (69), which is the unit of the adjunction from Theorem 3.8.26. The second isomorphism is of the type

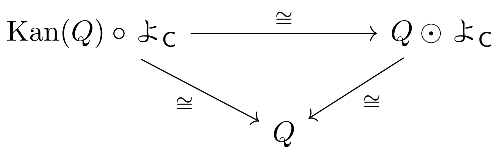
$$\mathrm{Kan}_{\text{よ}_{\mathsf{C}}}(\text{よ}_{\mathsf{C}}) \circ P \cong P,$$

and we take it to correspond to the unitary isomorphism $\mathrm{Kan}_{\text{よ}_{\mathsf{C}}}(\text{よ}_{\mathsf{C}}) \cong \mathrm{id}_{\hat{\mathsf{C}}}$ which arises from the fact that both sides restrict to よ_{C} on C.

The triangle equation takes the form, with any other $Q : \mathsf{C} \nrightarrow \mathsf{D}$,



where as for the pentagon equation, the commutativity of the outer triangle is reduced to that of the inner. To show its commutativity in turn, it is enough to show the commutativity without P and upon restriction to C, where we get



At this point, the horizontal arrow is an identity, while the other two coincide. Hence the triangle equation holds. □

We have thus established that the following indeed is a W*-bicategory.

Definition 4.2.8. *We write* $\mathbb{W}^*{}_{\mathsf{HProf}}$ *for the W*-bicategory with*

- ▷ *Small W*-categories as objects,*

▷ *Self-dual Hilbert profunctors $M \nrightarrow N$ as morphisms $M \to N$,*
▷ *Bounded natural transformations maps as 2-cells,*

and the obvious composition operations.

This W*-bicategory is no longer strict, and we leave it understood that the associators and unitors are the isomorphisms constructed in the proofs of Lemmas 4.2.6 and 4.2.7, respectively.

Proposition 4.2.9. *The W*-bicategories $\mathbb{W}^*{}_{\mathsf{Fun}}$ and $\mathbb{W}^*{}_{\mathsf{HProf}}$ are equivalent, where the equivalence is implemented by*

$$\mathbb{W}^*{}_{\mathsf{HProf}}(\mathfrak{B}\mathbb{C}, -) \;:\; \mathbb{W}^*{}_{\mathsf{HProf}} \longrightarrow \mathbb{W}^*{}_{\mathsf{Fun}}.$$

For the detailed definition of such a hom-2-functor, we refer to [48, Proposition 4.5.2].

Proof. For any W*-category C, a self-dual Hilbert profunctor $\mathfrak{B}\mathbb{C} \nrightarrow \mathsf{C}$ is by definition a W*-functor $\mathfrak{B}\mathbb{C} \to \hat{\mathsf{C}}$, or equivalently an object of $\hat{\mathsf{C}}$. Therefore the 2-functor under consideration can also be described as acting on objects by $\mathsf{C} \mapsto \hat{\mathsf{C}}$. On a morphism $P : \mathsf{C} \nrightarrow \mathsf{D}$, it induces the W*-functor $P \odot -$, which is $\mathrm{Kan}_{よ_{\mathsf{C}}}(P) : \hat{\mathsf{C}} \to \hat{\mathsf{D}}$, and the action on 2-cells is obvious.

The fact that the 2-functor under consideration is a local equivalence is now exactly the statement of Theorem 4.2.1. Its essential surjectivity is Corollary 3.9.3. □

Remark 4.2.10. The preservation of composition of 1-morphisms is exactly the statement of Lemma 4.2.5.

Recall also representable Hilbert profunctors from Example 2.6.3. In terms of these, we can generalize the right unitality from Lemma 4.2.7.

Lemma 4.2.11. *Let $\mathsf{C}(-, F-) : \mathsf{B} \nrightarrow \mathsf{C}$ be the representable Hilbert profunctor associated to a W*-functor $F : \mathsf{B} \to \mathsf{C}$. Then for every $Q : \mathsf{C} \nrightarrow \mathsf{D}$, there is a unitary isomorphism*

$$Q \odot \mathsf{C}(-, F-) \;\cong\; Q(-, F-)$$

natural in F and Q.

Proof. We have

$$Q \odot \mathsf{C}(-, F-) \;=\; \mathrm{Kan}_{よ_{\mathsf{C}}}(Q) \circ よ_{\mathsf{C}} \circ F \;\cong\; Q \circ F,$$

where the isomorphism is again because $\mathrm{Kan}_{よ_{\mathsf{C}}}(Q)$ reproduces Q on C. □

4.3 The bicategory of W*-algebras and self-dual Hilbert bimodules

Specializing the composition of Hilbert profunctors to the single-object case produces the standard **W*-tensor product** of Hilbert bimodules [49, 4]. For any three W*-algebras M, N, and O, this is given by a W*-bifunctor of the form

$$\begin{aligned} \mathsf{HilbBiMod}(M,N) \times \mathsf{HilbBiMod}(N,O) &\longrightarrow \mathsf{HilbBiMod}(M,O) \\ (X,Y) &\longmapsto X \otimes_N Y, \end{aligned}$$

where the only difference relative to the previous subsection is that we now write $\otimes$ and swap the order of the factors for consistency with standard notation. The concrete construction of Proposition 4.2.2 now takes the following form.

Proposition 4.3.1. *Let $X \in \mathsf{HilbBiMod}(M,N)$ and $Y \in \mathsf{HilbBiMod}(N,O)$. Then $X \otimes_N Y$ is modelled by the space of formal linear combinations*

$$\sum_{i \in I} x_i \otimes y_i$$

of families x_- in X and y_- in Y which are square summable in the sense that

$$x_-\langle x_-, -\rangle < \infty, \qquad \langle y_-, y_-\rangle < \infty,$$

and modulo the null space corresponding to the O-valued inner product

$$\left\langle \sum_{j \in J} x'_j \otimes y'_j, \sum_{i \in I} x_i \otimes y_i \right\rangle = \sum_{i \in I, j \in J} \langle y'_j, \langle x'_j, x_i\rangle y_i\rangle.$$

For fixed X, the operation of forming this tensor product is often called **inducing** by X. Considering the special case $M = \mathbb{C}$, we get for fixed $Y \in \mathsf{HilbBiMod}(N,O)$ a W*-functor

$$\begin{aligned} \mathsf{HilbMod}(N) &\longrightarrow \mathsf{HilbMod}(O) \\ X &\longmapsto X \otimes_N Y. \end{aligned}$$

Similarly for $O = \mathbb{C}$ and fixed $X \in \mathsf{HilbBiMod}(M,N)$, we get a W*-functor

$$\begin{aligned} \mathsf{NRep}(N) &\longrightarrow \mathsf{NRep}(M) \\ Y &\longmapsto X \otimes_N Y. \end{aligned}$$

Now as a W*-categorical analogue of the **Eilenberg–Watts theorem** for rings, we note as a special case of our earlier results that these constructions implement a W*-equivalence between the relevant W*-category of Hilbert bimodules and the respective functor W*-category.

Theorem 4.3.2. *For any W*-algebras M and N, there is a W*-equivalence*

$$\boxed{\mathsf{Fun}(\mathsf{HilbMod}(M), \mathsf{HilbMod}(N)) \cong \mathsf{HilbBiMod}(M, N)}$$

implemented by the adjoint equivalence

$$\mathsf{Fun}(\mathsf{HilbMod}(M), \mathsf{HilbMod}(N)) \underset{P\mapsto(-\otimes_M P)}{\overset{F\mapsto F(M)}{\rightleftarrows}} \cong \quad \mathsf{HilbBiMod}(M, N) \tag{77}$$

Here, we regard M itself as a generator in $\mathsf{HilbMod}(M)$, and note that its endomorphism W*-algebra is also M acting on itself by left multiplication. For the proof, note that this statement is simply the single-object special case of Theorem 4.2.1. It is due to Rieffel [5, Theorems 5.5 and 6.9].[53]

Let us now return to bicategorical considerations. We can restrict $\mathbb{W}^*{}_{\mathsf{HProf}}$ even further to the full subbicategory consisting of the single-object W*-categories, or equivalently the W*-algebras, which results in the following.

Definition 4.3.3. *We write $\mathbb{W}^*{}_{\mathsf{HilbMod}}$ for the W*-bicategory with*

- ▷ *W*-algebras as objects,*
- ▷ *Self-dual Hilbert bimodules as morphisms $M \to N$,*
- ▷ *Bounded bimodule maps as 2-cells,*

and the obvious composition operations.

We can now show that this restriction does not lose anything, which is arguably part of why W*-categories have not received all that much attention so far: as far as bimodules are concerned, moving to the many-object case is not necessary.

Proposition 4.3.4. *The inclusion of W*-bicategories $\mathbb{W}^*{}_{\mathsf{HilbMod}} \hookrightarrow \mathbb{W}^*{}_{\mathsf{HProf}}$ is an equivalence.*

Proof. It is enough to prove essential surjectivity. But upon using the equivalence with $\mathbb{W}^*{}_{\mathsf{Fun}}$ from Proposition 4.2.9, this is an application of the W*-equivalences $\hat{\mathsf{C}} \cong \mathsf{HilbMod}(L(\mathsf{C})^{\mathrm{op}})$ from Example 3.9.7. □

[53] Rieffel's formulation uses W*-categories of normal representations $\mathsf{NRep}(N)$ rather than W*-categories of Hilbert modules $\mathsf{HilbMod}(M)$, but by Theorem 3.9.13 these results can be directly translated.

4.4 The bicategory of W*-algebras and Connes correspondences

For W*-algebras M and N, we had introduced the W*-category of Connes correspondences $\mathsf{Connes}(M,N)$ in Example 2.4.8, whose objects are simply Hilbert spaces equipped with commuting actions of M and N^{op}, and we also saw in Corollary 3.9.14 that it is equivalent to $\mathsf{HilbBiMod}(M,N)$. Our goal is now to extend this equivalence to a W*-bicategorical equivalence, and in particular a description of the tensor product of Hilbert bimodules in terms of the associated Connes correspondences. So for a third W*-algebra O, **Connes fusion** is supposed to be a W*-bifunctor

$$\mathsf{Connes}(M,N) \times \mathsf{Connes}(N,O) \longrightarrow \mathsf{Connes}(M,O), \tag{78}$$

analogous to the tensor product of Hilbert bimodules. To see what this might look like, we use a minor variation on an idea of Sauvageot [50] and ignore the actions of M and O for the moment.

Proposition 4.4.1. *For every W*-algebra N, there is an essentially unique W*-bifunctor*

$$-\boxtimes_N - \;:\; \mathsf{NRep}(N^{\mathrm{op}}) \times \mathsf{NRep}(N) \longrightarrow \mathsf{Hilb} \tag{79}$$

that satisfies

$$L^2(N) \boxtimes_N L^2(N) \cong L^2(N).$$

Proof. Recall that $L^2(N)$, when considered as an object in either categories, has endomorphism W*-algebra given by

$$\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), L^2(N)) \cong N, \qquad \mathsf{NRep}(N)(L^2(N), L^2(N)) \cong N^{\mathrm{op}}, \tag{80}$$

where the isomorphisms are given by N acting on the left and right, respectively. Moreover, we have a functor

$$\mathfrak{B}N \times \mathfrak{B}N^{\mathrm{op}} \longrightarrow \mathsf{Hilb}$$

given by sending the unique object to $L^2(N)$ as a Hilbert space and letting N act on each side, and this is a W*-functor in each argument by construction. Now since $\mathsf{NRep}(N)$ is W*-complete and $L^2(N)$ is a generator in it, Corollary 3.8.28(ii) lets us extend this functor in an essentially unique way to[54]

$$\mathfrak{B}N \times \mathsf{NRep}(N) \longrightarrow \mathsf{Hilb}$$

thanks to (80) and the W*-completeness of Hilb. Now for every $\mathcal{H}$ in $\mathsf{NRep}(N)$, we can apply the same idea for the first argument and get a W*-functor

$$-\boxtimes \mathcal{H} : \mathsf{NRep}(N^{\mathrm{op}}) \longrightarrow \mathsf{Hilb}.$$

Another application of Corollary 3.8.28(ii) extends this to (79) as desired. □

[54] It may be worth noting that the restriction of this functor to $\mathrm{id}_{\mathfrak{B}N}$ in the first argument is essentially the forgetful functor $\mathsf{NRep}(N) \to \mathsf{Hilb}$.

Similarly to how we had defined the composition of Hilbert profunctors in (74), we can now use Proposition 4.4.1 to define Connes fusion in general: if we have $\mathcal{H} \in \mathsf{Connes}(M, N)$ and $\mathcal{K} \in \mathsf{Connes}(N, O)$, then forgetting the left action of M on $\mathcal{H}$ and the right action of O on $\mathcal{K}$ makes (79) applicable, and its functoriality shows that we obtain the desired W*-bifunctor (78).

Before we show that Connes fusion can be used as the horizontal composition in a bicategory, let us consider an alternative formulation. The following reduction of Connes fusion to the tensor product of Hilbert bimodules has also been used as the definition [51, Section 3].

Proposition 4.4.2. *Given $\mathcal{H} \in \mathsf{NRep}(N^{\mathrm{op}})$ and $\mathcal{K} \in \mathsf{NRep}(N)$, there are unitary isomorphisms*

$$\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), \mathcal{H}) \otimes_N \mathcal{K} \cong \mathcal{H} \boxtimes_N \mathcal{K} \cong \mathcal{H} \, {}_N\!\otimes \mathsf{NRep}(N)(L^2(N), \mathcal{K})$$

natural in $\mathcal{H}$ and $\mathcal{K}$, where $\otimes_N$ denotes the usual tensor of Hilbert bimodules while ${}_N\otimes$ is the "reverse" one where the N-valued inner product on comes from the second factor.

Proof. In this description, what happens is that the functors $\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), -)$ and $\mathsf{NRep}(N)(-, L^2(N))$ first convert the given correspondences into Hilbert modules per Theorem 3.9.13, where the tensor product of Hilbert modules can then be used.

It is clear that the left-hand side and right-hand side are indeed W*-bifunctors with arguments $\mathcal{H}$ and $\mathcal{K}$. For the first isomorphism, by Proposition 4.4.1 it is enough to evaluate

$$\mathsf{NRep}(N^{\mathrm{op}})(L^2(N), L^2(N)) \otimes_N L^2(N) \cong N \otimes_N L^2(N) \cong L^2(N),$$

where we used the fact that the left and right actions of N are each others commutants. The other isomorphism works similarly. The claimed naturality in $\mathcal{H}$ and $\mathcal{K}$ is a consequence of Corollary 3.8.28(ii), again applied with respect to the Yoneda embedding. ☐

To get some idea of what Connes fusion looks like more concretely, we can apply our earlier construction of Hilbert profunctor composition from Proposition 4.3.1 to Proposition 4.4.2. This shows that $\mathcal{H} \boxtimes_N \mathcal{K}$ can be defined as the space of formal linear combinations

$$\sum_{i \in I} f_i \otimes \xi_i$$

where the families of intertwiners $f_{\_} : L^2(N) \to \mathcal{H}$ and elements $\xi_{\_} \in \mathcal{K}$ are square summable[55], modulo the null space of the sesquilinear form

$$\left\langle \sum_{j \in J} f'_j \otimes \xi'_j, \sum_{i \in I} f_i \otimes \xi_i \right\rangle := \sum_{i \in I, j \in J} \langle \xi'_j, f_i f'^*_j \xi_i \rangle.$$

[55] In this situation, this means $\sum_{i \in I} f_i f_i^* < \infty$ and $\sum_{i \in I} \langle \xi_i, \xi_i \rangle < \infty$.

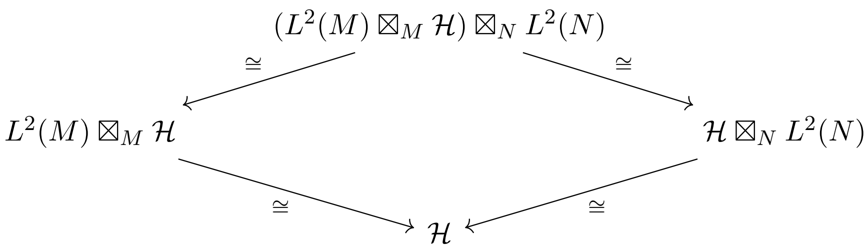
However, we will not need any explicit description in the rest of this work.[56] We also encourage the reader to work with the abstract characterization of Proposition 4.4.1 whenever possible, as this can help to avoid some of the technicalities.

In order for Connes fusion to serve as horizontal composition in a bicategory, we need to construct coherent associators and unitors. We show how to do this and prove the relevant coherences based on the abstract approach. Starting with the unitors, for every $\mathcal{H} \in \mathsf{Connes}(M, N)$ we get natural unitary isomorphisms

$$L^2(M) \boxtimes_M \mathcal{H} \cong \mathcal{H} \cong \mathcal{H} \boxtimes_N L^2(N)$$

again by ignoring the action of N on the right (resp. of M on the left) and reducing to the case $\mathcal{H} = L^2(M)$ (resp. $\mathcal{H} = L^2(N)$) by applying Corollary 3.8.28(ii). We will take these isomorphisms to be the unitors, and note that the two unitors $L^2(N) \boxtimes_N L^2(N) \cong L^2(N)$ coincide by construction.

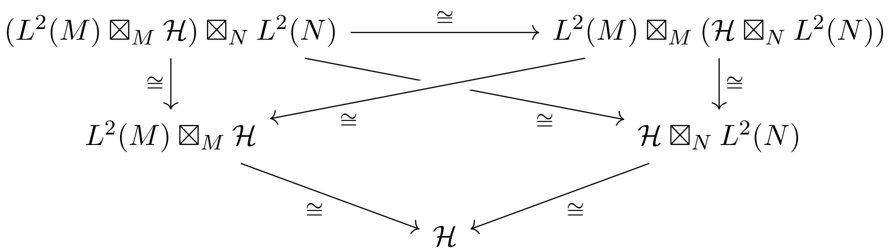
Lemma 4.4.3. *With these unitors, the diagram*



commutes.

Proof. This is a naturality square for the unitor $L^2(M) \boxtimes_M \mathcal{H} \cong \mathcal{H}$. □

For the associators, we use the unitors to obtain a composite unitary isomorphism



(81)

where the diagram without the upper horizontal arrow commutes by Lemma 4.4.3. We use the same idea as before to extend this to an associator for every composable triple of correspondences.

Showing the triangle equation is now a simple matter.

[56]For Connes' own more explicit description of Connes fusion, see also [52, Section V.B.δ].

Lemma 4.4.4. *For all* $\mathcal{H} \in \mathsf{Connes}(M,N)$ *and* $\mathcal{K} \in \mathsf{Connes}(N,O)$*, the diagram*

$$\begin{array}{ccc} (\mathcal{H} \boxtimes_N L^2(N)) \boxtimes_N \mathcal{K} & \xrightarrow{\cong} & \mathcal{H} \boxtimes_N (L^2(N) \boxtimes_N \mathcal{K}) \\ & \searrow^{\cong} \quad \swarrow^{\cong} & \\ & \mathcal{H} \boxtimes_N \mathcal{K} & \end{array}$$

commutes.

Proof. Taking $\mathcal{H} = \mathcal{K} = L^2(N)$ without loss of generality reduces this to either of the two triangles in (81), which both commute by definition of the associator. □

Before we can prove the pentagon equation for the associators, it is helpful to show another coherence involving the unitors first.

Lemma 4.4.5. *With unitors and associators as above, the diagram*

$$\begin{array}{ccc} (\mathcal{H} \boxtimes_N \mathcal{K}) \boxtimes_O L^2(O) & \xrightarrow{\cong} & \mathcal{H} \boxtimes_N (\mathcal{K} \boxtimes_O L^2(O)) \\ & \searrow^{\cong} \quad \swarrow^{\cong} & \\ & \mathcal{H} \boxtimes_N \mathcal{K} & \\ & \nearrow^{\cong} \quad \nwarrow^{\cong} & \\ L^2(M) \boxtimes_M (\mathcal{H} \boxtimes_N \mathcal{K}) & \xrightarrow[\cong]{} & (L^2(M) \boxtimes_M \mathcal{H}) \boxtimes_N \mathcal{K} \end{array}$$

commutes for all correspondences $\mathcal{H} \in \mathsf{Connes}(M,N)$ *and* $\mathcal{K} \in \mathsf{Connes}(N,O)$.

Proof. We only consider the upper triangle as the lower one is analogous. Then it is enough to consider $\mathcal{H} = L^2(M)$ by another application of Corollary 3.8.28(ii) □

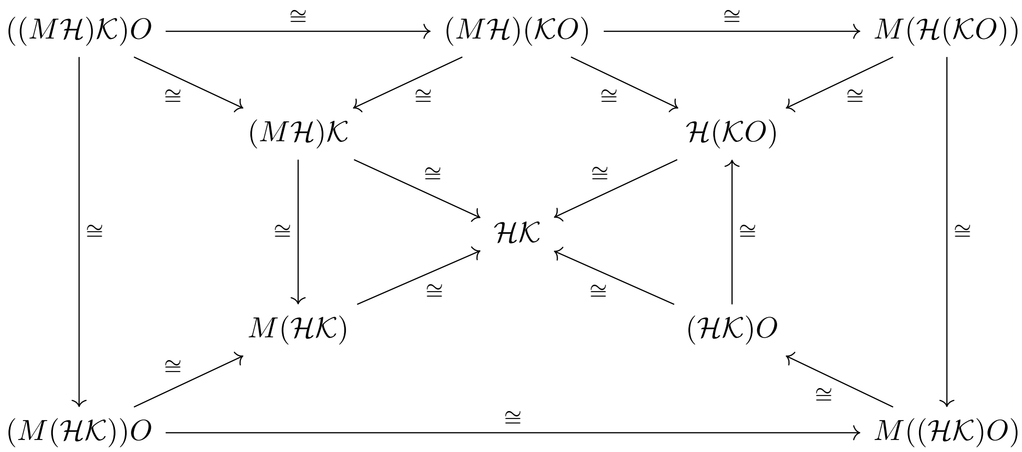
We can now turn to the pentagon equation, the proof of which reduces by the usual argument to the following statement, where we omit the subscript on $\otimes$ for simplicity.

Lemma 4.4.6. *For every* $\mathcal{H} \in \mathsf{Connes}(M,N)$ *and* $\mathcal{K} \in \mathsf{Connes}(N,O)$*, the diagram*

$$\begin{array}{ccccc} ((L^2(M) \boxtimes \mathcal{H}) \boxtimes \mathcal{K}) \boxtimes L^2(O) & \xrightarrow{\cong} & (L^2(M) \boxtimes \mathcal{H}) \boxtimes (\mathcal{K} \boxtimes L^2(O)) & \xrightarrow{\cong} & L^2(M) \boxtimes (\mathcal{H} \boxtimes (\mathcal{K} \boxtimes L^2(O)) \\ \downarrow \cong & & & & \uparrow \cong \\ (L^2(M) \boxtimes (\mathcal{H} \boxtimes \mathcal{K})) \boxtimes L^2(O) & & \xrightarrow[\cong]{} & & L^2(M) \boxtimes ((\mathcal{H} \boxtimes \mathcal{K}) \boxtimes L^2(O)) \end{array}$$

commutes.

Proof. To simplify the notation further, we also drop the ⊠ symbol and abbreviate $L^2(M)$ to M and $L^2(O)$ to O for this proof. Then the diagram can be filled in as



where all square-shaped subdiagrams commute as naturality diagrams of the unitors, the triangles commute by Lemma 4.4.4 and the pentagon commutes by definition of the associator (81). □

Remark 4.4.7. The way in which Corollary 3.8.28(ii) has allowed us to reduce everything to identity correspondences $L^2(N)$ is strongly reminiscent of path induction in homotopy type theory [53]. Especially in light of the role of W*-algebras and correspondences in topology, this connection may merit further exploration.

It is now clear that we indeed obtain a bicategory, and in fact a W*-bicategory.

Definition 4.4.8 (Brouwer [54])**.** *We write* $\mathbb{W}^*{}_{\mathsf{Connes}}$ *for the W*-bicategory with*

- ▷ *W*-algebras as objects,*
- ▷ *Connes correspondences as morphisms* $M \to N$*,*
- ▷ *Bounded intertwiners as* 2*-cells,*

and the composition operations described above.

Before stating the equivalence with the W*-bicategories considered in the previous subsections, let us mention Rieffel's Eilenberg–Watts theorem, which is the Connes correspondence analog of Theorem 4.3.2.

Theorem 4.4.9 ([5])**.** *For any W*-algebras* M *and* N*, there is a W*-equivalence*

$$\boxed{\mathsf{Fun}(\mathsf{NRep}(M^{\mathrm{op}}), \mathsf{NRep}(N^{\mathrm{op}})) \cong \mathsf{Connes}(M, N)}$$

implemented by the adjoint equivalence

$$\mathsf{Fun}(\mathsf{NRep}(M^{\mathrm{op}}),\mathsf{NRep}(N^{\mathrm{op}})) \;\underset{\mathcal{H}\mapsto(-\otimes_M\mathcal{H})}{\overset{F\mapsto F(L^2(M))}{\overset{\cong}{\rightleftarrows}}}\; \mathsf{Connes}(M,N) \tag{82}$$

Proof. By Corollary 3.8.28(ii), we have a W*-equivalence

$$\mathsf{Fun}(\mathsf{NRep}(M^{\mathrm{op}}),\mathsf{NRep}(N^{\mathrm{op}})) \;\cong\; \mathsf{Fun}(\mathfrak{B}M,\mathsf{NRep}(N^{\mathrm{op}}))$$

given by restriction to $L^2(M)$. But the W*-category on the right-hand side is $\mathsf{Connes}(M,N)$ by definition, and this establishes the equivalence given by the upper arrow.

It is now enough to show that the lower arrow is its adjoint. Indeed for given $\mathcal{H}\in\mathsf{Connes}(M,N)$ and W*-functor $F:\mathsf{NRep}(M^{\mathrm{op}})\to\mathsf{NRep}(N^{\mathrm{op}})$, we have a natural isometric isomorphism between bounded natural transformations

$$-\otimes_M\mathcal{H}\longrightarrow F$$

and intertwiners $\mathcal{H}\longrightarrow F(L^2(M))$ given by restriction and another application of Corollary 3.8.28(ii). □

We have now a statement analogous to Proposition 4.2.9.

Proposition 4.4.10. *The W*-bicategories* $\mathbb{W}^*{}_{\mathsf{Fun}}$ *and* $\mathbb{W}^*{}_{\mathsf{Connes}}$ *are equivalent, where the equivalence is implemented by*

$$\mathbb{W}^*{}_{\mathsf{Connes}}(\mathbb{C},-)\;:\;\mathbb{W}^*{}_{\mathsf{Connes}}\longrightarrow\mathbb{W}^*{}_{\mathsf{Fun}}.$$

Proof. To a W*-algebra N, this construction assigns the W*-category $\mathsf{Connes}(\mathbb{C},N)=\mathsf{NRep}(N^{\mathrm{op}})$. A Connes correspondence $\mathcal{H}\in\mathsf{Connes}(M,N)$ induces the W*-functor

$$-\otimes_M\mathcal{H}\;:\;\mathsf{NRep}(M^{\mathrm{op}})\longrightarrow\mathsf{NRep}(N^{\mathrm{op}})$$

and this is a W*-equivalence by Theorem 4.4.9. Finally, essential surjectivity holds as every Grothendieck W*-category is W*-equivalent to some $\mathsf{NRep}(N^{\mathrm{op}})$ by Corollary 3.9.5 and Theorem 3.9.13. □

Remark 4.4.11. Also

$$\mathbb{W}^*{}_{\mathsf{Connes}}(-,\mathbb{C})\;:\;\mathbb{W}^*{}_{\mathsf{Connes}}\longrightarrow\mathbb{W}^{*\mathrm{op}}_{\mathsf{Fun}}$$

is an equivalence of W*-bicategories, assigning to every W*-algebra N the W*-category $\mathsf{Connes}(\mathbb{C},N)=\mathsf{NRep}(N^{\mathrm{op}})$ and to every $\mathcal{H}\in\mathsf{Connes}(M,N)$ the W*-functor

$$-\otimes_M\mathcal{H}\;:\;\mathsf{NRep}(N)\longrightarrow\mathsf{NRep}(M)$$

going in the other direction. This arises from $\mathbb{W}^*{}_{\mathsf{Connes}}(-,\mathbb{C})$ by composing with the obvious equivalence

$$-^{\mathrm{op}} : \mathbb{W}^{*\mathrm{op}}_{\mathsf{Fun}} \longrightarrow \mathbb{W}^*{}_{\mathsf{Fun}}$$

which acts as $N \mapsto N^{\mathrm{op}}$ on objects and applies the canonical equivalences $\mathsf{Connes}(M,N) \cong \mathsf{Connes}(N^{\mathrm{op}}, M^{\mathrm{op}})$ on morphisms and 2-cells.

Let us briefly summarize the main results so far.

Theorem 4.4.12. *There are canonical equivalences of W*-bicategories*

$$\boxed{\mathbb{W}^*{}_{\mathsf{Fun}} \cong \mathbb{W}^*{}_{\mathsf{HProf}} \cong \mathbb{W}^*{}_{\mathsf{HilbMod}} \cong \mathbb{W}^*{}_{\mathsf{Connes}} \cong \mathbb{W}^{*\mathrm{op}}_{\mathsf{Fun}}.}$$

We thus simply write $\mathbb{W}^*$ for any of these W*-bicategories from now on, and we will work with whichever one of them is most convenient in a given situation.

4.5 Consequences for Morita equivalence of W*-algebras

Rieffel's Eilenberg–Watts theorem gives one module-theoretic definition of Morita equivalence for W*-algebras, which can be defined categorically as equivalence of the categories of normal representations. One interprets the abstract definition of equivalence in the 2-category $\mathbb{W}^*$. One has an N-Hilbert M-module ${}_N X_M$ and a M-Hilbert N-module ${}_M Y_N$, such that ${}_N X_M \otimes_M {}_M Y_N \cong {}_N N_N$, the right hand side meaning A interpreted as a bimodule over itself, and that ${}_M Y_N \otimes_N {}_N X_M \cong {}_M M_M$.

There is also Rieffel's *Morita theorem*. For this theorem, one has instead only a single bimodule showing the equivalence of $\mathsf{NRep}(N)$ and $\mathsf{NRep}(M)$. The bimodule is an equivalence bimodule [5, Definition 7.5].

Definition 4.5.1. *An N-M-equivalence bimodule X is a module that is both a normal right M-Hilbert left N-module and a normal left N-Hilbert right M-module,* i.e. *it is a triple $(X, \langle -,- \rangle_N, \langle -,- \rangle_M)$, such that*

(i) $\langle x,y \rangle_N z = x \langle y,z \rangle_M$
(ii) $\langle X,X \rangle_N$ *is weak-* dense in N, and* $\langle X,X \rangle_M$ *is weak-* dense in M.*

An equivalence bimodule is called self-dual *if it is self-dual in both inner products, although [5, Proposition 7.7] shows that it is equivalent to ask that it be self-dual in either inner product alone.*

For an N-M-equivalence bimodule X, one can define $\widetilde{X}$, which is a M-N-equivlence bimodule. The underlying $\mathbb{C}$-vector space is $\overline{X}$, and one defines $a \cdot x = x \cdot a^*$, for $a \in N$ and $x \in X$. The action of M is defined similarly. The inner products are the same. One then has

Theorem 4.5.2 (Rieffel-Morita, [5, Theorem 7.9])**.** *Let N, M be W*-algebras. If there exists an N-M-equivalence bimodule X, then $X \otimes_N -$ and $\widetilde{X} \otimes_M -$ define an equivalence between $\mathsf{NRep}(N)$ and $\mathsf{NRep}(M)$. From any equivalence $F : \mathsf{NRep}(N) \to \mathsf{NRep}(M)$ one may define a self-dual equivalence bimodule defining an equivalence naturally isomorphic to F. These constructions define a bijection between isomorphism classes of equivalences $\mathsf{NRep}(N) \to \mathsf{NRep}(M)$ and isomorphism classes of self-dual N-M-equivalence bimodules.*

4.6 $\mathbb{W}^*$ as a (weak) proarrow equipment

Since the pioneering work of Wood [55, 56], it has become more and more clear that every flavour of category theory is formalized most adequately not as a bicategory, but as a special kind of *double category*, namely a **proarrow equipment**. The basic idea is that there are two relevant notions of morphism in any flavour of category theory, namely functors and profunctors. Moreover, every functor $F : \mathsf{C} \to \mathsf{D}$ induces profunctors

$$\mathsf{D}(-, F-) \; : \; \mathsf{C} \nrightarrow \mathsf{D}, \qquad \mathsf{D}(F-, -) \; : \; \mathsf{D} \nrightarrow \mathsf{C},$$

respectively. For W*-categories, the relevant notions of morphism are W*-functors and Hilbert profunctors. Since a W*-functor $F : \mathsf{C} \to \mathsf{D}$ induces a Hilbert profunctor $\mathsf{C} \nrightarrow \mathsf{D}$, but often not the other way around (Example 4.6.3), we need to consider a weaker notion.

Definition 4.6.1 (Verity [57, Definition 1.2.1])**.** *A* ***weak proarrow equipment*** *consists of:*

- *A strict bicategory* $\mathbb{D}_t$*.*
- *A bicategory* $\mathbb{D}_\ell$*.*
- *A* 2*-functor*

$$(-)_* \; : \; \mathbb{D}_t \longrightarrow \mathbb{D}_\ell$$

  which is identity-on-objects and locally fully faithful.

Note that the strictness condition is not part of Verity's definition, but is usually assumed in more recent work. The reason is that it strongly simplifies the treatment of many coherence issues, and this will be crucial below in the construction of the symmetric monoidal structure.

We follow [58] in calling the morphisms in $\mathbb{D}_t$ **tight** and those $\mathbb{D}_\ell$ **loose**, which also explains our choice of subscripts. This terminology rests on the fact that $\mathbb{D}_t$ is often a strict bicategory while $\mathbb{D}_\ell$ is weak, and this is also what happens in our case.

Proposition 4.6.2. *There is a weak proarrow equipment with:*

- $\mathbb{W}^*\mathsf{cat}$ *as its tight bicategory,*
- $\mathbb{W}^*$ *as its loose bicategory,*
- *The* 2*-functor* $(-)_* : \mathbb{W}^*\mathsf{cat} \to \mathbb{W}^*$ *sending a W*-functor* $F : \mathsf{C} \to \mathsf{D}$ *to the representable Hilbert profunctor* $\mathsf{C} \nrightarrow \mathsf{D}$ *given by*

$$F_* \coloneqq \mathsf{D}(-, F-) = \text{よ}_{\mathsf{D}} F \tag{83}$$

  as in Example 2.6.3, and the induced action on bounded natural transformations.

By abuse of notation, we write $\mathbb{W}^*$ for this weak proarrow equipment, and leave the distinction between this and $\mathbb{W}^*$ as its loose bicategory to context.

Proof. As the proof of Proposition 4.2.9 shows, $\mathbb{W}^*$ is equivalent to the strict bicategory with:

- ▷ Small W*-categories as objects,
- ▷ A morphism $\mathsf{C} \to \mathsf{D}$ being a W*-functor $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$,
- ▷ Bounded natural transformations as 2-cells.

We can thus work with this picture of $\mathbb{W}^*$, where we use $(-)_* : F \mapsto \hat{F}$ with $\hat{F} : \hat{\mathsf{C}} \to \hat{\mathsf{D}}$ the essentially unique extension from Corollary 3.8.21. This is a equivalent to (83) by the commutativity of (63). Moreover, we can assume without loss of generality that $\hat{F}(\mathsf{C}(-, X)) = \mathsf{D}(-, FX)$ strictly for all objects $X \in \mathsf{C}$, and in addition that $\widehat{\mathrm{id}_{\mathsf{C}}} = \mathrm{id}_{\hat{\mathsf{C}}}$ for all identity W*-functors.

Constructing the coherence isomorphisms of a 2-functor and proving the relevant equations is now routine. Concerning the coherence isomorphisms for composition, for W*-categories B, C, and D with W*-functors $F : \mathsf{B} \to \mathsf{C}$ and $G : \mathsf{C} \to \mathsf{D}$, we need a natural isomorphism

$$\widehat{GF} \cong \hat{G}\hat{F},$$

and this can be obtained by Corollary 3.5.14(ii), as both sides are canonically isomorphic to $よ_{\mathsf{D}} GF$ on C.

Since both bicategories under consideration are strict, the hexagon identity involving a third W*-functor $H : \mathsf{D} \to \mathsf{E}$ amounts to showing that the diagram

$$\begin{array}{ccccc} & & \hat{H}\hat{G}\hat{F} & & \\ & \overset{\cong}{\swarrow} & & \overset{\cong}{\searrow} & \\ \hat{H}\widehat{GF} & & & & \widehat{HG}\hat{F} \\ & \underset{\cong}{\searrow} & & \underset{\cong}{\swarrow} & \\ & & \widehat{HGF} & & \end{array}$$

commutes. Once again by Corollary 3.5.14(ii), it is enough to show this commutativity on representable Hilbert presheaves $\mathsf{B}(-, X)$. On this, the diagram commutes trivially, as all objects become $HGFX$ and all morphisms become the identity.

Finally, we also need to show that the coherence isomorphisms

$$\widehat{F\mathrm{id}_{\mathsf{B}}} \cong \hat{F}\mathrm{id}_{\hat{\mathsf{B}}}, \qquad \widehat{\mathrm{id}_{\mathsf{C}}F} \cong \mathrm{id}_{\hat{\mathsf{C}}}\hat{F}$$

are identities. These both follow in the same way by evaluation on a representable Hilbert presheaf. □

A proarrow equipment distinguishes itself from a weak proarrow equipment in that for every morphism F in $\mathbb{D}_t$, the associated loose morphism F_* has a right adjoint F^* in $\mathbb{D}_\ell$. Unfortunately these adjoints do generally not exist in $\mathbb{W}^*$, not even in the single-object case.

Example 4.6.3. Let N be a W*-algebra and $F : \mathfrak{B}\mathbb{C} \to \mathfrak{B}N$ the unique W*-functor associated to the unique $*$-homomorphism $\mathbb{C} \to N$. Then we have

$$\begin{aligned} F_* \; : \; \mathsf{Hilb} &\longrightarrow \mathsf{HilbMod}(N) \\ \mathcal{H} &\longmapsto \mathsf{Hilb} \otimes_{\mathbb{C}} N. \end{aligned}$$

Therefore a putative right adjoint F^* will in particular implement a bijection

$$N \cong \mathsf{HilbMod}(N)(F_*\mathbb{C}, N) \cong \mathsf{Hilb}(\mathbb{C}, F^*N) \cong F^*N,$$

which has to be a (not necessarily isometric) isomorphism of Banach spaces since the unit and counit of the adjunction are bounded linear maps. In particular, N should be isomorphic to a Hilbert space, which is not the case if N is infinite-dimensional.[57]

It therefore may be appropriate to restrict the tight morphisms to be those W*-functors for which this adjoint exists. In the single-object case, these are the *finite* $*$-homomorphisms in the terminology of Bartels, Douglas and Henriques [51]. These are most commonly encountered in subfactor theory, where the inclusion of a subfactor $M \subseteq N$ is finite if and only if the Jones index $[N : M]$ is finite.

Definition 4.6.4. *A W*-functor $F : \mathsf{C} \to \mathsf{D}$ is* ***finite*** *if $F_* : \mathsf{C} \nrightarrow \mathsf{D}$ has a (necessarily two-sided) adjoint F^* in $\mathbb{W}^*$.*

So if one takes the finite W*-functors as the tight morphisms, then $\mathbb{W}^*$ becomes a proarrow equipment.

Nowadays, proarrow equipments are more commonly defined as certain **double categories** [59][58] These involve 2-cells of the square form

$$\begin{array}{ccc} \mathsf{C} & \overset{P}{\nrightarrow} & \mathsf{D} \\ {\scriptstyle F}\downarrow & s & \downarrow{\scriptstyle G} \\ \mathsf{C}' & \underset{Q}{\nrightarrow} & \mathsf{D}' \end{array} \tag{84}$$

and it is known that every weak proarrow equipment gives rise to a double category by defining such 2-cells s to be given by 2-cells

$$G_*P \longrightarrow QF_*$$

in $\mathbb{D}_\ell$ [57, Definition 1.2.4]. In particular, we can also consider $\mathbb{W}^*$ as a double category. Let us spell this out in a bit more detail.

Proposition 4.6.5. *There is a double category with:*

- ▷ *Small W*-categories as objects,*
- ▷ *Tight morphisms $\mathsf{C} \to \mathsf{D}$ being W*-functors $\mathsf{C} \to \mathsf{D}$,*
- ▷ *Loose morphisms $\mathsf{C} \to \mathsf{D}$ being W*-functors $\hat{\mathsf{C}} \to \hat{\mathsf{D}}$,*
- ▷ *2-cells (84) being bounded natural transformations $\hat{G}P \to Q\hat{F}$,*
- ▷ *all composition operations being the obvious ones.*

[57] For example because a Hilbert space is reflexive while an infinite-dimensional W*-algebra is not.
[58] Note that this reference uses the term "framed bicategory" instead of "proarrow equipment".

In particular, the horizontal composition of 2-cells

$$\begin{array}{ccccc} \mathsf{C} & \xrightarrow{P} & \mathsf{D} & \xrightarrow{R} & \mathsf{E} \\ {\scriptstyle F}\big\downarrow & s & \big\downarrow{\scriptstyle G} & t & \big\downarrow{\scriptstyle H} \\ \mathsf{C}' & \xrightarrow[Q]{} & \mathsf{D}' & \xrightarrow[S]{} & \mathsf{E}' \end{array}$$

is given by the composite bounded natural transformation

$$\hat{H}RP \xrightarrow{tP} S\hat{G}P \xrightarrow{Ss} SQ\hat{F},$$

and the axioms of a double category are now straightforward to verify directly.

Proarrow equipments can equivalently be seen as **fibrant double categories** [60], which are double categories satisfying additional lifting conditions.

4.7 $\mathbb{W}^*$ as a symmetric monoidal bicategory

Our next goal is to turn $\mathbb{W}^*$ into a symmetric monoidal bicategory. This is most easily done by using a construction due to Shulman [60], who gave a general recipe for constructing symmetric monoidal bicategories which avoids explicit verifications of many of the coherences. Shulman's construction applies to fibrant double categories equipped with a monoidal structure. We do not repeat the full definition here but refer to [60, Section 2] for the details. Nevertheless, the upcoming proof of Theorem 4.7.1 will go through the relevant conditions, so that the general definition can also be extracted from that.

Although the most commonly considered tensor product of W*-algebras is the spatial tensor product, in our context the Guichardet–Dauns tensor product seems more appropriate, due to its universal property which results in isomorphisms like (5). In Section 4.8, we will see that this is what makes $\mathbb{W}^*$ into a compact closed bicategory.

Throughout this subsection, we consider $\mathbb{W}^*$ as a proarrow equipment with W*-algebras as objects and finite $*$-homomorphisms as tight morphisms, and with $\mathbb{W}^*{}_{\mathsf{Connes}}$ as the bicategory of loose morphisms.

Theorem 4.7.1. *$\mathbb{W}^*$ is a symmetric monoidal double category with respect to the Guichardet–Dauns tensor product of W*-algebras.*

Proof. It is standard to see that this makes $\mathbb{W}^*\mathsf{cat}$ into a symmetric monoidal category. The loose morphisms are themselves the objects of a category, with 2-cells

$$\begin{array}{ccc} L & \xrightarrow{P} & M \\ {\scriptstyle f}\big\downarrow & s & \big\downarrow{\scriptstyle g} \\ L' & \xrightarrow[Q]{} & M' \end{array}$$

as morphisms $P \to Q$ which compose vertically. The tensor product of objects is given by

$$(L \overset{P}{\longmapsto} M) \quad \otimes \quad (N \overset{R}{\longmapsto} O) \quad := \quad L \otimes N \overset{P \otimes R}{\longmapsto} M \otimes O,$$

where the W*-functor

$$P \otimes R \,:\, \mathsf{NRep}((L \otimes N)^{\mathrm{op}}) \longrightarrow \mathsf{NRep}((M \otimes O)^{\mathrm{op}})$$

□

4.8 $\mathbb{W}^*$ as a compact closed bicategory

Theorem 4.8.1. *The symmetric monoidal bicategory* $\mathbb{W}^*$ *is compact closed.*

Proof. □

5 Outlook

2-Hilbert spaces, universal property of direct integrals

A Hilbert modules

A.1 Definition and basic properties

The following definition is standard; a good textbook account can be found in [20].

Definition A.1.1. *Let A be a C^*-algebra. Then a **right A-Hilbert module** is a right A-module X together with a map*

$$\langle -, - \rangle : X \times X \to A$$

that is additive in both arguments, satisfies the conditions

- $\langle x, ya \rangle = \langle x, y \rangle a$.
- $\langle x, y \rangle^* = \langle y, x \rangle$.
- $\langle x, x \rangle \geq 0$ *in* A.
- $\langle x, x \rangle = 0 \iff x = 0$.

and is such that X is complete with respect to the norm $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$.

For example, a $\mathbb{C}$-Hilbert module is the same thing as a Hilbert space. Rieffel also calls A-Hilbert modules *A-rigged spaces* [5, Definition 3.1 and 3.3]. Among the most important basic observations of their theory is the Cauchy-Schwarz inequality in the form

$$\langle x, y \rangle^* \langle x, y \rangle \;\leq\; \|\langle x, x \rangle\| \langle y, y \rangle. \tag{85}$$

Every Hilbert space is self-dual by virtue of the Riesz representation theorem. The analogous statement is generally not true for Hilbert modules, where self-duality becomes an additional property that a particular Hilbert module may or may not enjoy:

Definition A.1.2. *A A-Hilbert module X is **self-dual** if for each bounded A-linear map $\phi : X \to A$ there exists an $x \in X$ such that*

$$\phi(y) = \langle x, y \rangle$$

for all $y \in Y$.

Note that such x is necessarily unique.

For example, if A is unital, then A itself is a self-dual A-Hilbert module with $\langle x, y \rangle = x^* y$. For $A = \mathbb{C}$, it is a standard textbook fact that a pre-Hilbert space X is self-dual if and only if it is norm complete (a Hilbert space) if and only if it has an orthonormal basis. Westerbaan [61, §149_{V}] provides an elegant generalization of this fact to general W*-algebras A, extending earlier results of Paschke and Frank. Paschke also proved that every A-Hilbert module can be completed to a self-dual one [21, Theorem 3.2], and this completion has the expected universal property [61, §151_{I}].

Definition A.1.3. *Let X and Y be A-Hilbert modules. A map $T : X \to Y$ is **adjointable** if there exists $T^* : Y \to X$ such that*

$$\langle y, Tx \rangle = \langle T^* y, x \rangle$$

for all $x \in X$ and $y \in Y$.

As is easy to see, an adjointable map T is automatically A-linear and satisfies $T^{**} = T$. Furthermore, it is necessarily bounded, as follows for example by an application of the Banach-Steinhaus theorem [20, p. 8].

Lemma A.1.4. *If X and Y are A-Hilbert modules with X self-dual, then every bounded A-linear map $X \to Y$ is adjointable.*

Proof. For $y \in Y$, the element $T^* y \in X$ must be the unique element representing the bounded A-linear map $x \mapsto \langle y, Tx \rangle$. □

Remark A.1.5. Also (by definition), X is self-dual if every bounded A-linear map $X \to A$ is adjointable.

The space of adjointable operators from X to Y is a Banach space with respect to the operator norm, and we denote it by $L(X, Y)$. For example, $L(A, A)$ is the multiplier algebra of A. In the special case $A = \mathbb{C}$, we have $L(X, Y) = \mathsf{Hilb}(X, Y)$.

It is often of interest to consider an additional left action by another C*-algebra on a Hilbert module, as in the following standard definition.

Definition A.1.6. *Let A and B be C*-algebras. An A-**Hilbert** B-**module** is a right A-Hilbert module X which is also a left B-module by virtue of a $*$-homomorphism $B \to L(X, X)$ such that BX is norm dense in X (non-degeneracy).*

If B is unital, then the non-degeneracy condition says exactly that $1x = x$ for all $x \in X$. In particular, the A-Hilbert $\mathbb{C}$-modules are precisely the A-Hilbert modules from Definition A.1.1.

Example A.1.7. Considering B itself as a B-Hilbert module over itself, a $*$-homomorphism $f : A \to B$ makes B into a B-Hilbert A-module if and only if it is non-degenerate, i.e. if the elements of the form $f(a)b$ are dense in B.

In the W*-algebra case, it is natural to expect a further condition on A-Hilbert B-modules. This reads as follows.

Definition A.1.8. *Let M and N be W*-algebras. Then a* ***normal N-Hilbert M-module*** *is an N-Hilbert M-module X such that for all $x, y \in X$, the map $M \to N$ defined by*

$$a \longmapsto \langle x, ay \rangle$$

is normal.

Rieffel calls such X a *normal N-rigged M-module* [5, Definition 5.1].

For example, a $\mathbb{C}$-Hilbert M-module is the same thing as a normal representation of M on a Hilbert space. As the opposite special case, a N-Hilbert $\mathbb{C}$-module is the same thing as an N-Hilbert module since the normality condition is trivial.

Example A.1.9. Continuing Example A.1.7, the Hilbert module induced by a $*$-homomorphism $f : M \to N$ is normal if and only if the map $a \mapsto x^* f(a) y$ is normal for all $x, y \in N$, which holds if and only if f itself is normal.

Sometimes a normal self-dual N-Hilbert M-module is also called a Hilbert W*-module. This combination of conditions is the one that we use in the main text.

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