Characterizing symmetric powers bialgebraically

A first step into the logic of usual linear mathematics

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Introduction

Linear mathematics and Graded Bialgebraic Linear Logics

The specialty property of symmetric powers

Binomial theorem, ideas of polynomial linear logic and vectorial categories

Characterization of symmetric powers: statement ~~and overview of the proof~~

Towards further characterizations: Schur functors, $(A^{\otimes n})^{G_n} = (A^{\otimes n})_{G_n}$, Cyclic homology, Positive characteristic...

Introduction

W. Lawvere (p.213 in **Foundations and applications: axiomatization and education**, Bulletin of Symbolic Logic 213-224, 2003):

> *In my own education I was fortunate to have two teachers who used the term "foundations" in a common-sense way (rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition). This way is exemplified by their work in Foundations of Algebraic Topology, published in 1952 by Eilenberg (with Steenrod), and The Mechanical Foundations of Elasticity and Fluid Mechanics, published in the same year by Truesdell. The orientation of these works seemed to be "concentrate the essence of practice and in turn use the result to guide practice".*

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That's the opposite of an "internal" approach where we think about what's going on inside the person, or "inside" the word ie. thinking of the word as an abstract independent concept, which can be different from its concrete use.

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These two definitions are equivalent. But to do proof-theoretic logic, we need the second type of definitions. We'll see this with homogenous polynomials/symmetric powers. We're going to give an external characterization which is equivalent to the classical definition but is better to build a logic of them.

Finite exponentials vs Infinite exponentials

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We will use graded/bounded exponentials. They are more concrete, but most important, we can characterize symmetric powers as a particular graded exponential but I don't know how to characterize symmetric algebras as a non-graded exponential.

Linear mathematics and Graded Bialgebraic Linear Logics

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Morover, in Linear Logic $!A$ is a coalgebra
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But that's not enough.

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Let's see how it works in vector spaces.

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Alternatively, in characteristic 0, it can be seen as the subspace of $A^{\otimes n}$ constituted by the vectors which are invariant under this action.

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The comultiplication is like this:

$$\begin{aligned}\mathbb{K}_5[X, Y, Z] &\to \mathbb{K}_2[X, Y, Z] \otimes \mathbb{K}_3[X, Y, Z] \\ X^2Y^2Z &\mapsto X^2 \otimes Y^2Z + 4XY \otimes XYZ \\ &\qquad + 2XZ \otimes XY^2 + Y^2 \otimes X^2Z + 2YZ \otimes X^2Y\end{aligned}$$

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How does it work exactly?

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$$\begin{aligned}
\Delta_{n,p}(P = X_1^{n_1}...X_q^{n_q}) &= \sum_{\substack{M \in \mathcal{M}_p(X_1,...,X_q) \\ M|P}} D_M(P) \otimes M \\
&= \sum_{\substack{0 \le m_1 \le n_1 \\ ... \\ 0 \le m_q \le n_q}} D_{X_1^{m_1}...X_q^{m_q}}(P) \otimes X_1^{(n_1-m_1)}...X_q^{(n_q-m_q)} \\
&= \sum_{\substack{0 \le m_1 \le n_1 \\ ... \\ 0 \le m_q \le n_q}} \binom{n_1}{m_1}...\binom{n_q}{m_q}X_1^{n_1-m_1}...X_q^{n_q-m_q} \otimes X_1^{(n_1-m_1)}...X_q^{(n_q-m_q)}
\end{aligned}$$

It is much simpler to write it with symmetric tensors:

$$\Delta_{n,p}(y_1 \otimes_s \dots \otimes_s y_{n+p}) = \sum_{X \in \mathcal{P}_p([1,n+p])} y_{[1,n+p] \setminus X} \otimes y_X$$

It shows directly that it is a natural transformation:

$$\begin{tikzcd}
S_{n+p}E \arrow[r, "\Delta_{n,p}E"] \arrow[d, "S_{n+p}\phi"'] & S_nE \otimes S_pE \arrow[d, "S_n\phi \otimes S_p\phi"] \\
S_{n+p}F \arrow[r, "\Delta_{n,p}E"'] & S_nF \otimes S_pF
\end{tikzcd}$$

It would be more difficult to show directly that for any linear map

$$u : \mathbb{K}_1[X_1, ..., X_q] \to \mathbb{K}_1[Y_1, ..., Y_r]$$

this diagram commute:

$$\begin{tikzcd}
\mathbb{K}_{n+p}[X_1, ..., X_q] \arrow[r, "\Delta_{n,p}"] \arrow[d, "\mathbb{K}_{n+p}(u)"'] & \mathbb{K}_n[X_1, ..., X_q] \otimes \mathbb{K}_p[X_1, ..., X_q] \arrow[d, "\mathbb{K}_n(u) \otimes \mathbb{K}_p(u)"] \\
\mathbb{K}_{n+p}[Y_1, ..., Y_r] \arrow[r, "\Delta_{n,p}"'] & \mathbb{K}_n[Y_1, ..., Y_r] \otimes S_p[Y_1, ..., Y_r]
\end{tikzcd}$$

by using the first definition of $\Delta_{n,p}$ and the matrix of u...

With polynomials: $\mathbb{K}[X_i, i \in I]$

$$\Delta_{n,1} : \mathbb{K}_{n+1}[X_i, i \in I] \to \mathbb{K}_n[X_i, i \in I] \otimes \mathbb{K}_1(X_i, i \in I)$$

is given by

$$\Delta_{n,1}(P) = \sum_{i \in I} \frac{\partial P}{\partial X_i} \otimes X_i$$

We then have $\nabla_{n,1}(\Delta_{n,1}(P)) = \sum_{i \in I} \frac{\partial P}{\partial X_i} X_i = (n+1).P$ by a theorem of Euler which says that this identity is a characterization of homogeneous polynomials of degree n among smooth functions!

It is much easier to view the identity without coordinates:

$$\Delta_{n,1} : S_{n+1}E \to S_nE \otimes E$$

is given by

$$\Delta_{n,1}(x_1 \otimes_s ... \otimes_s x_n) = \sum_{1 \leq i \leq n} (x_1 \otimes_s ... \otimes_s \widehat{x_i} \otimes_s ... \otimes_s x_n) \otimes x_i$$

and thus

$$\begin{aligned} \nabla_{n,1}(\Delta_{n,1}(x_1 \otimes_s ... \otimes_s x_n)) &= \sum_{1 \leq i \leq n} x_1 \otimes_s ... \otimes_s x_i \otimes_s ... \otimes_s x_n \\ &= n.x_1 \otimes_s ... \otimes_s x_n \end{aligned}$$

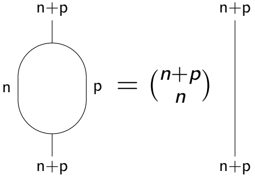
More generally, we have:

$$\begin{aligned}\nabla_{n,p}(\Delta_{n,p}(x_1 \otimes_s \dots \otimes_s x_{n+p})) &= \sum_{X \in \mathcal{P}_p([1,n+p])} y_1 \otimes_s \dots \otimes_s y_{n+p} \\ &= |\mathcal{P}_p([1,n+p])|.y_1 \otimes_s \dots \otimes_s y_{n+p} \\ &= \binom{n+p}{p}.y_1 \otimes_s \dots \otimes_s y_{n+p}\end{aligned}$$

We thus have:

$$\begin{tikzcd}
S_{n+p}A \arrow[r, "\Delta_{n,p}"] \arrow[rr, bend left, "\binom{n+p}{n} Id"] & S_nA \otimes S_pA \arrow[r, "\nabla_{n,p}"] & S_{n+p}A
\end{tikzcd}$$

In string diagrams, it looks:



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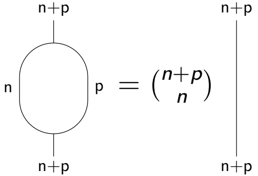

We'll see in a minute how to characterize symmetric powers combining this with the graded bialgebraic structure.

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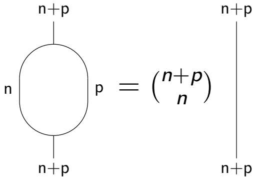

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But before, let's see something else...

Binomial theorem, ideas of polynomial linear logic and vectorial categories

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We'll see in a minute which categories are the models.

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It should be equivalent (by cut elimination/rewriting) to the other binomial proof:

$$(x, y) \mapsto \sum_{0 \leq k \leq n} \binom{n}{k} x^{\otimes_s k} \otimes_s y^{\otimes_s (n-k)}$$

Cartesian left additive categories and biadditve maps

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- $(u \times (v_1 + v_2)); f = (u \times v_1); f + (u \times v_2); f$

Proposition:

- If $u : A \to C$ and $v : B \to D$ are additive and $f : C \times D \to E$ is biadditive, then $(u \times v); f$ is biadditive.
- If $f : A \times B \to C$ is biadditive and $u : C \to D$ is additive, then $f; u$ is biadditive.

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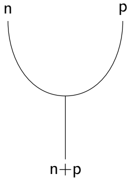
- probably some additional boring conditions.

In a vectorial category $\mathcal{C}$ such that $\mathcal{C}_+$ has the symmetric powers, we should have the binomial theorem verified...

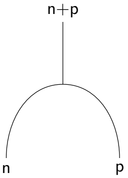
Characterization of symmetric powers: statement ~~and overview of the proof~~

Definition: Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal $\mathbb{Q}^+$-linear category. A **symmetric bialgebra** is a family $(A_n)_{n \geq 0}$ of objects with:

$(\nabla_{n,p} \colon A_n \otimes A_p \to A_{n+p})_{n,p \geq 0}$:



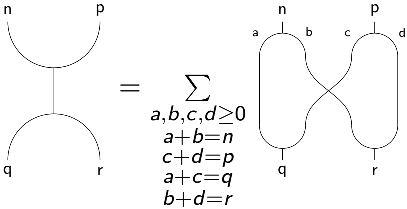
$(\Delta_{n,p} \colon A_{n+p} \to A_n \otimes A_p)_{n,p \geq 0}$:



$\eta \colon I \to A_0$:

$\epsilon \colon A_0 \to I$:

such that: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r
=
$\sum_{\substack{a,b,c,d\geq 0\\ a+b=n\\ c+d=p\\ a+c=q\\ b+d=r}}$
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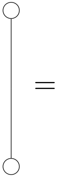
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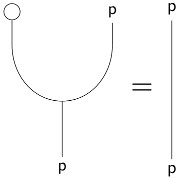
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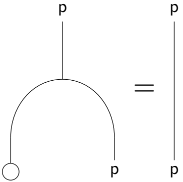
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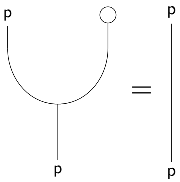
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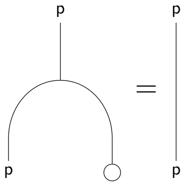
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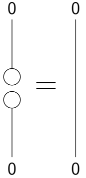
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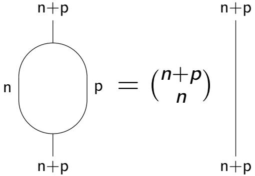
n+p
n
p
$= \binom{n+p}{n}$
n+p
n+p
n+p

Definition: Let $\mathcal{C}$ be a symmetric monoidal $\mathbb{Q}^+$-linear category. Define a **family of symmetric powers** as a family $(A_n)_{n\geq 0} \in \mathcal{C}$ together with morphisms

$$\left(\begin{tikzcd} A_1^{\otimes n} \arrow[r, shift left, "r_n"] & A_n \arrow[l, shift left, "s_n"] \end{tikzcd} \right)_{n\in\mathbb{N}\backslash\{1\}}$$

such that:

$$\forall n \in \mathbb{N}\backslash\{1\} \quad \begin{cases} r_n ; s_n & = \frac{1}{n!} \sum\limits_{\sigma\in\mathfrak{S}_n} \sigma \\ s_n ; r_n & = Id \end{cases}$$

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$$\left(\begin{tikzcd} A_1^{\otimes n} \arrow[r, shift left, "\nabla^n"] & A_n \arrow[l, shift left, "\Delta^n"] \end{tikzcd} \right)_{n \in \mathbb{N} \setminus \{1\}}$$

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Then $(A_n)_{n\geq 0}$ together with $\eta^*, \epsilon^*, (\nabla^*_{n,p})_{n,p\geq 0}$, $(\Delta^*_{n,p})_{n,p\geq 0}$ is a symmetric bialgebra.

3) Given a family $(A_n)_{n\geq 0}$ of objects, the two preceding transformations give a bijection between the sets of morphisms which define a structure of symmetric bialgebra and the sets of morphisms which define a structure of family of symmetric powers.

Proof:

The proof is about showing that the combinatorics of "paths with fixed flow" is equivalent to the combinatorics of symmetrization.

It is really interesting but quite long. And I'm still trying to really finish it and to polish it.

Maybe I could talk of the proof another day :) because it seems to be a technique applicable to a lot of situations (I talk of that in a minute), so it's useful to make it crystal clear.

Towards further characterizations: Schur functors, $(A^{\otimes n})^{G_n} = (A^{\otimes n})_{G_n}$, Cyclic homology, Positive characteristic...

Theorem:

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- a limit $(A^{\otimes n})^{G_n} \xrightarrow{s_n} A^{\otimes n}$ of this diagram:

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- a splitting $A^{\otimes n} \xrightarrow{r_n} B \xrightarrow{s_n} A^{\otimes n}$ of this idempotent:

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And the bjjection preserves the objects: $(A^{\otimes n})^{G_n} = B = (A^{\otimes_n})_{G_n}$.

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We can then resume the useful equations by this diagram:

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(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \arrow[d, "\nabla_{n,p}"'] \\
(A^{\otimes(n+p)})_{G_{n+p}}
\end{tikzcd}
= \frac{|G_{n+p}|}{|G_n|^2.|G_p|^2} \sum_{\substack{g\in G_n \\ h\in G_p}}
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(A^{\otimes(n+p)})_{G_{n+p}} \arrow[d, "s_{n+p}"'] \\
A^{\otimes(n+p)} \arrow[d, "g\otimes h"'] \\
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\end{tikzcd}$$

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(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \arrow[d, "\nabla_{n,p}"] \\
(A^{\otimes (n+p)})_{G_{n+p}} \arrow[d, "\Delta_{q,r}"] \\
(A^{\otimes q})_{G_q} \otimes (A^{\otimes r})_{G_r}
\end{tikzcd}
= \frac{1}{|G_q|.|G_r|} \sum_{g \in G_{n+p}}
\begin{tikzcd}
(A^{\otimes n})_{G_n} \otimes (A^{\otimes p})_{G_p} \arrow[d, "s_n \otimes s_p"] \\
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- Exterior powers are different because we need a symmetric monoidal $\mathbb{Q}$-linear category and we must put signs. We look at splitting of the idempotents:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma)\sigma : A^{\otimes n} \to A^{\otimes n}$$

- Symmetric powers and exterior powers are example of Schur functors which can be defined in any symmetric monoidal $\mathbb{Q}$-linear category as a functor $S_\lambda : \mathcal{C} \to \mathcal{C}$ such that $S_\lambda A$ is the intermediate object in the splitting of some idempotent

$$e_\lambda : A^{\otimes n} \to A^{\otimes n}$$

for every partition $\lambda \vdash n$

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- In every symmetric monoidal $\mathbb{Q}$-linear category, we can look at the n^{th} object of the cyclic homology complex of an object $A^{\otimes n}$. It is the set spanned by vectors of the form:

$$\begin{aligned} x_1 \otimes^a_{\mathbb{Z}_n} \ldots \otimes^a_{\mathbb{Z}_n} x_n &= (-1)^{n-1} x_n \otimes^a_{\mathbb{Z}_n} x_1 \otimes^a_{\mathbb{Z}_n} \ldots \otimes^a_{\mathbb{Z}_n} x_{n-1} \\ &= (-1)^{n-1} x_{n-1} \otimes^a_{\mathbb{Z}_n} x_n \otimes^a_{\mathbb{Z}_n} x_1 \otimes^a_{\mathbb{Z}_n} \ldots \otimes^a_{\mathbb{Z}_n} x_{n-2} \\ &\ldots \\ &= (-1)^{n-1} x_2 \otimes^a_{\mathbb{Z}_n} \ldots \otimes^a_{\mathbb{Z}_n} x_n \otimes^a_{\mathbb{Z}_n} x_1 \end{aligned}$$

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- In the same way, in a symmetric monoidal **Ab**-category $\mathcal{C}$, Schur functors would divide into Schur functors $S_\lambda : \mathcal{C} \to \mathcal{C}$ (eg. symmetric powers) and Weyl or co-Schur functors $S^\lambda : \mathcal{C} \to \mathcal{C}$ (eg. divided powers).
- I've also seen things like skew Schur functors $S_{\lambda,\mu}$...
- And we can maybe look at more complicated groups than $\mathfrak{S}_n$ or $\mathbb{Z}_n$ acting on $A^{\otimes n}$...

There is (a lot of) work to do!