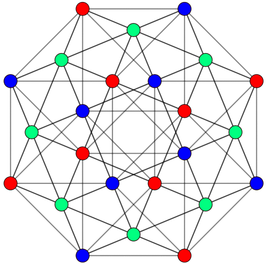
# TWO OF MY FAVORITE NUMBERS:

# 8 AND 24



John Baez

Quantum Matter Seminar

2023/8/24

The **Clifford algebra** $\mathrm{Cliff}_n$ is the algebra over $\mathbb{R}$ freely generated by $n$ anticommuting square roots of $-1$:

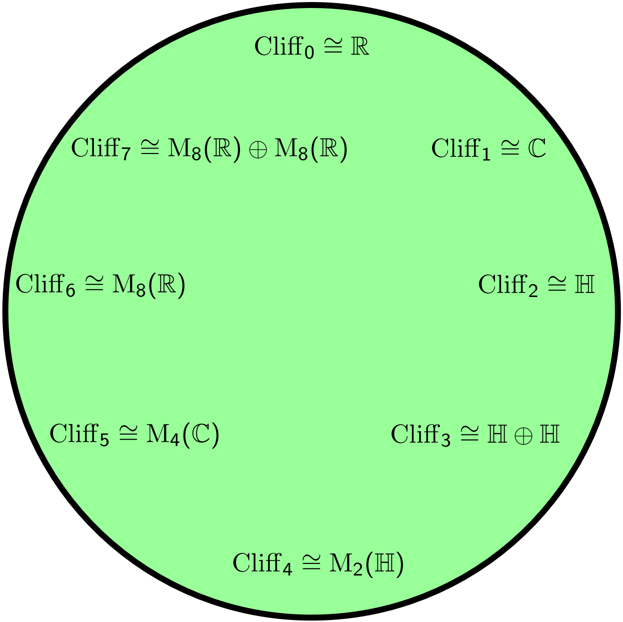
$$e_i e_j + e_j e_i = -2\delta_{ij}$$

The **Clifford algebra** $\mathrm{Cliff}_n$ is the algebra over $\mathbb{R}$ freely generated by $n$ anticommuting square roots of $-1$:

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

In 1908, Cartan showed that $\mathrm{Cliff}_{n+8}$ consists of $16 \times 16$ matrices with entries in $\mathrm{Cliff}_n$:

$$\mathrm{Cliff}_{n+8} \cong \mathrm{M}_{16}(\mathrm{Cliff}_n)$$


$\mathrm{Cliff}_0 \cong \mathbb{R}$
$\mathrm{Cliff}_7 \cong \mathrm{M}_8(\mathbb{R}) \oplus \mathrm{M}_8(\mathbb{R})$
$\mathrm{Cliff}_1 \cong \mathbb{C}$
$\mathrm{Cliff}_6 \cong \mathrm{M}_8(\mathbb{R})$
$\mathrm{Cliff}_2 \cong \mathbb{H}$
$\mathrm{Cliff}_5 \cong \mathrm{M}_4(\mathbb{C})$
$\mathrm{Cliff}_3 \cong \mathbb{H} \oplus \mathbb{H}$
$\mathrm{Cliff}_4 \cong \mathrm{M}_2(\mathbb{H})$

As a consequence, we get **Bott periodicity**:

$$\pi_{n+8}(\mathrm{O}(\infty)) \cong \pi_n(\mathrm{O}(\infty))$$

As a consequence, we get **Bott periodicity**:

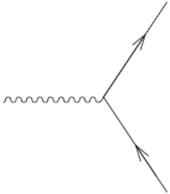
$$\pi_{n+8}(\mathrm{O}(\infty)) \cong \pi_n(\mathrm{O}(\infty))$$

$$
\begin{array}{rclrl}
\pi_0(\mathrm{O}(\infty)) & \cong & \mathbb{Z}_2 & \text{real numbers:} & \mathbb{R} \\
\pi_1(\mathrm{O}(\infty)) & \cong & \mathbb{Z}_2 & \text{complex numbers:} & \mathbb{C} \\
\pi_2(\mathrm{O}(\infty)) & \cong & 0 & & \\
\pi_3(\mathrm{O}(\infty)) & \cong & \mathbb{Z} & \text{quaternions:} & \mathbb{H} \\
\pi_4(\mathrm{O}(\infty)) & \cong & 0 & & \\
\pi_5(\mathrm{O}(\infty)) & \cong & 0 & & \\
\pi_6(\mathrm{O}(\infty)) & \cong & 0 & & \\
\pi_7(\mathrm{O}(\infty)) & \cong & \mathbb{Z} & \text{octonions:} & \mathbb{O}
\end{array}
$$

The rotation group $\mathrm{SO}(n)$ acts on *vectors*,
but its double cover also acts on *spinors*,
which are defined using Clifford algebras.

The rotation group $\mathrm{SO}(n)$ acts on *vectors*,
but its double cover also acts on *spinors*,
which are defined using Clifford algebras.

There's a way to 'multiply' a spinor and a vector and get a spinor:



When the space of spinors and the space of vectors have the same dimension, this gives a normed division algebra!

| $n$ | vectors | spinors | normed division algebra? |
|---|---|---|---|
| 1 | $\mathbb{R}$ | $\mathbb{R}$ | YES: REAL NUMBERS |
| 2 | $\mathbb{R}^2$ | $\mathbb{R}^2$ | YES: COMPLEX NUMBERS |
| 3 | $\mathbb{R}^3$ | $\mathbb{R}^4$ | NO |
| 4 | $\mathbb{R}^4$ | $\mathbb{R}^4$ | YES: QUATERNIONS |
| 5 | $\mathbb{R}^5$ | $\mathbb{R}^4$ | NO |
| 6 | $\mathbb{R}^6$ | $\mathbb{R}^4$ | NO |
| 7 | $\mathbb{R}^7$ | $\mathbb{R}^8$ | NO |
| 8 | $\mathbb{R}^8$ | $\mathbb{R}^8$ | YES: OCTONIONS |

Bott periodicity $\implies$ spinors in dimension 8 more have dimension 16 times as big.

So, we only get 4 normed division algebras.

The normed division algebras are connected to lattices!

A lattice $L \subseteq \mathbb{R}^n$ is **integral** if $v \cdot w$ is an integer for all $v, w \in L$.

A lattice $L \subseteq \mathbb{R}^n$ is **even** if $v \cdot v$ is an even number for all $v \in L$.

Any even lattice is integral.

A lattice $L \subseteq \mathbb{R}^n$ is **unimodular** if the volume of its unit cell is 1.

The normed division algebras are connected to lattices!

A lattice $L \subseteq \mathbb{R}^n$ is **integral** if $v \cdot w$ is an integer for all $v, w \in L$.

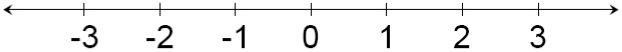
A lattice $L \subseteq \mathbb{R}^n$ is **even** if $v \cdot v$ is an even number for all $v \in L$.

Any even lattice is integral.

A lattice $L \subseteq \mathbb{R}^n$ is **unimodular** if the volume of its unit cell is 1.

**Witt's Theorem** There exists an even unimodular lattice in $\mathbb{R}^n$ if and only if $n$ is a multiple of 8.

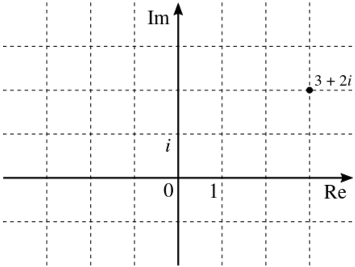
The integers $\mathbb{Z} \subset \mathbb{R}$ are an integral unimodular lattice, but not an even lattice:



The Gaussian integers

$$\{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C} \cong \mathbb{R}^2$$

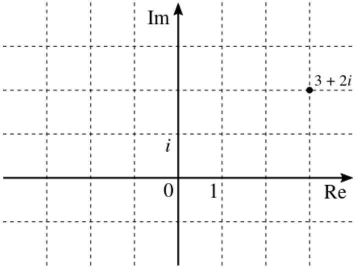
are an integral unimodular lattice, but not an even lattice:



The Gaussian integers

$$\{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C} \cong \mathbb{R}^2$$

are an integral unimodular lattice, but not an even lattice:

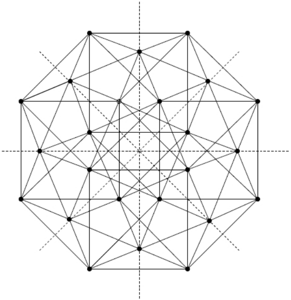



They're closed under multiplication.

## The **Hurwitz integral quaternions**

$$\{a + bi + cj + dk \mid a, b, c, d \text{ all in } \mathbb{Z} \text{ or all in } \mathbb{Z} + \tfrac{1}{2}\} \subset \mathbb{H} \cong \mathbb{R}^4$$

are closed under multiplication.

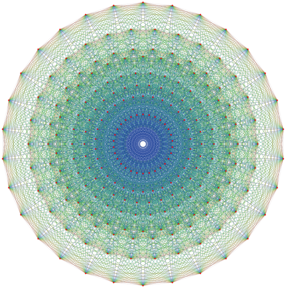


They give an integral unimodular lattice when rescaled by $\sqrt{2}$, but not an even lattice.

The ‘Cayley integral octonions’

$$\mathbb{K} \subset \mathbb{O} \cong \mathbb{R}^8$$

are closed under multiplication.

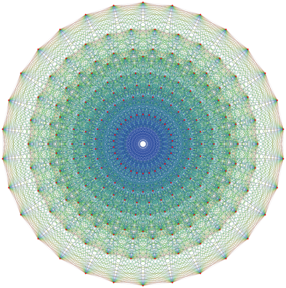


They give an integral unimodular lattice when rescaled by $\sqrt{2}$,
*and this is an even lattice!*

The 'Cayley integral octonions'

$$\mathbb{K} \subset \mathbb{O} \cong \mathbb{R}^8$$

are closed under multiplication.



They give an integral unimodular lattice when rescaled by $\sqrt{2}$, *and this is an even lattice!*

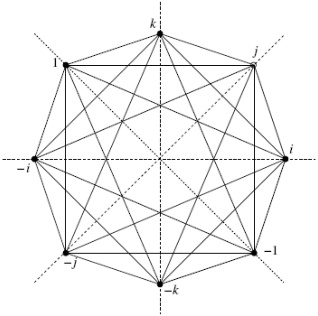
To get this lattice, just pack equal-sized balls in 8 dimensions so that each touches 240 others. It's called the **$\mathbf{E_8}$ lattice**.

Of the Hurwitz integral quaternions

$$\{a + bi + cj + dk \mid a, b, c, d \text{ all in } \mathbb{Z} \text{ or all in } \mathbb{Z} + \tfrac{1}{2}\} \subset \mathbb{H} \cong \mathbb{R}^4$$

exactly 24 lie on the unit sphere!

8 are the vertices of a 'hyperoctahedron':



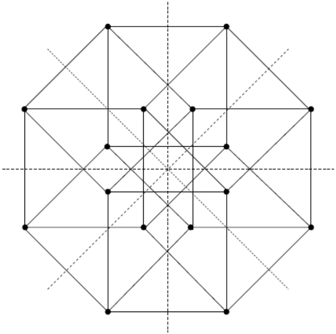

$\pm 1,\ \pm i,\ \pm j,\ \pm k$

Of the Hurwitz integral quaternions

$$\{a + bi + cj + dk \mid a, b, c, d \text{ all in } \mathbb{Z} \text{ or all in } \mathbb{Z} + \tfrac{1}{2}\} \subset \mathbb{H} \cong \mathbb{R}^4$$

exactly 24 lie on the unit sphere!

16 are the vertices of a hypercube:



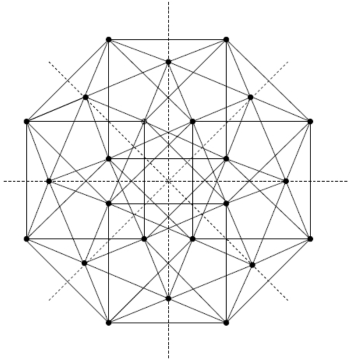
$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

Of the Hurwitz integral quaternions

$$\{a + bi + cj + dk \mid a, b, c, d \text{ all in } \mathbb{Z} \text{ or all in } \mathbb{Z} + \tfrac{1}{2}\} \subset \mathbb{H} \cong \mathbb{R}^4$$

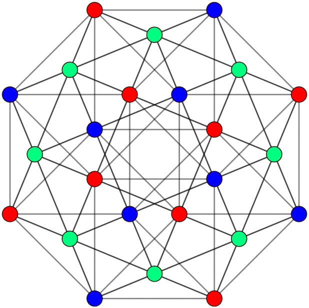
exactly 24 lie on the unit sphere!

Together they are the vertices of the **24-cell**:



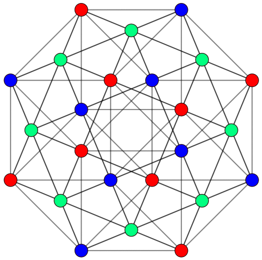
They form a group called the **binary tetrahedral group**.

Even better, the 16 vertices of a hypercube form the vertices of two hyperoctahedra! So the vertices of the 24-cell can be partitioned into the vertices of 3 hyperoctahedra:



$$24 = 8 + 8 + 8$$

Rescaling the Hurwitz integral quaternions by $\sqrt{2}$, we get an integral unimodular lattice called the **$D_4$ lattice**. This controls the representation theory of $\mathrm{Spin}(8)$, the double cover of $\mathrm{SO}(8)$.



The vertices of the 24-cell break up into 3 sets of 8. These give bases for the vector, left-handed spinor, and right-handed spinor representations of $\mathrm{Spin}(8)$.

Each of these representations can be seen as the octonions $\mathbb{O}$.

A superstring in 10 dimensions can be described by an $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$-valued field on the 2-dimensional string worldsheet.

This field transforms under rotations in 8 spatial dimensions transverse to the worldsheet via this representation of $\mathrm{Spin}(8)$:

**vector $\oplus$ left-handed spinor $\oplus$ right-handed spinor**

A superstring in 10 dimensions can be described by an $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$-valued field on the 2-dimensional string worldsheet.

This field transforms under rotations in 8 spatial dimensions transverse to the worldsheet via this representation of $\mathrm{Spin}(8)$:

**vector $\oplus$ left-handed spinor $\oplus$ right-handed spinor**

The 'vector' $\mathbb{O}$ describes the motion of the string in the 8 directions transverse to the worldsheet:
its bosonic degrees of freedom.

The left- and right-handed spinors, $\mathbb{O} \oplus \mathbb{O}$ describe the string's fermionic degrees of freedom.

So, we have seen the numbers 8 and 24 interacting in superstring theory. But the number 24 also shows up starting from the simplest field theory of all!

First consider the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

on the cylinder of radius 1:

$$(t, x) \in \mathbb{R} \times S^1 \qquad \phi\colon \mathbb{R} \times S^1 \to \mathbb{C}$$

Since

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)$$

any solution of the wave equation
is the sum of a left-moving and right-moving waves:

$$\phi(t, x) = f(t - x) + g(t + x)$$

Keep just the left-moving waves using this equation:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x}$$

This is arguably the simplest field theory of all!

Keep just the left-moving waves using this equation:

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}$$

This is arguably the simplest field theory of all!

If we quantize this field theory on the cylinder of radius 1, its vacuum energy is

$$-\frac{1}{24}$$

Why???

Discarding the constant function,
every left-moving solution is a linear combination of waves

$$\phi_k(t,x) = \exp(ik(t-x))$$

where $k = 1, 2, 3, \ldots$. The frequency of the wave $\phi_k$ is just $k$.

Thus, the left-moving wave equation is isomorphic to a collection of classical harmonic oscillators, one of frequency $k$ for each $k = 1, 2, 3, \ldots$.

Let's use units where $\hbar = 1$. Then the ground state energy of a quantum harmonic oscillator of frequency $\omega$ is $\frac{1}{2}\omega$.

Let's use units where $\hbar = 1$. Then the ground state energy of a quantum harmonic oscillator of frequency $\omega$ is $\frac{1}{2}\omega$.

When we have a bunch of oscillators, their ground state energies add. Since the left-moving wave equation is isomorphic to a collection of oscillators of frequencies $1, 2, 3, \ldots,$ its ground state energy is apparently

$$\frac{1}{2}(1 + 2 + 3 + \cdots) \ = \ \infty$$

We could set the ground state energy to zero.

But around 1735, Leonhard Euler gave a bizarre ‘proof’ that

$$1 + 2 + 3 + 4 + \cdots \; = \; -\frac{1}{12}$$

This would mean the ground state energy of the quantized left-moving wave equation is

$$\frac{1}{2}(1 + 2 + 3 + \cdots) \; = \; -\frac{1}{24}$$

Euler started with this:

$$1 + x + x^2 + x^3 + \cdots \; = \; \frac{1}{1-x}$$

He differentiated both sides:

$$1 + 2x + 3x^2 + \cdots \; = \; \frac{1}{(1-x)^2}$$

He set $x = -1$ and got this:

$$1 - 2 + 3 - 4 + \cdots \; = \; \frac{1}{4}$$

Then Euler considered this function:

$$\zeta(s) \;=\; 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

He multiplied by $2^{-s}$:

$$2^{-s}\zeta(s) \;=\; 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} + \cdots$$

Then he subtracted twice the second equation from the first:

$$(1 - 2 \cdot 2^{-s})\zeta(s) \;=\; 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

Taking this result:

$$(1 - 2 \cdot 2^{-s})\zeta(s) \ = \ 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

and setting $s = -1$, he got:

$$-3(1 + 2 + 3 + 4 + \cdots) = 1 - 2 + 3 - 4 + \cdots$$

Since he already knew the right-hand side equals $1/4$, he concluded:

$$1 + 2 + 3 + 4 + \cdots \ = \ -\frac{1}{12}$$

Euler's calculation looks crazy, but now we understand it better!
The sum

$$1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$$

converges for $\mathrm{Re}(s) > 1$ to an analytic function:
the Riemann zeta function, $\zeta(s)$.

This function can be analytically continued to $s = -1$,
and one can prove

$$\zeta(-1) = -\frac{1}{12}$$

Assuming Euler's calculation is right, what is the partition function of the left-moving scalar field?

Assuming Euler's calculation is right, what is the partition function of the left-moving scalar field?

For any system with energy eigenvalues $E_j$, define its **partition function** to be

$$Z(\beta) = \sum_j e^{-\beta E_j}$$

To calculate it quickly, we'll use this fact:

When we combine several sysems, we can multiply their partition functions to get the partition function of the combined system.

First: what's the partition function of
a quantum harmonic oscillator?

An oscillator with frequency $\omega$ can have energies

$$\frac{1}{2}\omega,\ (1+\frac{1}{2})\omega,\ (2+\frac{1}{2})\omega,\ (3+\frac{1}{2})\omega, \ldots$$

So, its partition function is:

$$\sum_{n=0}^{\infty} e^{-i(n+\frac{1}{2})\beta\omega} = e^{-\frac{1}{2}\beta\omega} \sum_{k=0}^{\infty} e^{-n\beta\omega} = \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta\omega}}$$

Since the left-moving scalar field is isomorphic to a collection of oscillators with frequencies $1, 2, 3, \ldots,$ its partition function is a product:

$$Z(\beta) \;=\; \prod_{k=1}^{\infty} \frac{e^{-\frac{1}{2}k\beta}}{1 - e^{-k\beta}} \;=\; e^{-\frac{1}{2}(1+2+3+\cdots)\beta} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-k\beta}}$$

According to Euler's crazy calculation, we get

$$Z(\beta) \;=\; e^{\frac{1}{24}\beta} \; \prod_{k=1}^{\infty} \frac{1}{1 - e^{-k\beta}}$$

This partition function

$$Z(\beta) = e^{\frac{1}{24}\beta} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-k\beta}}$$

is essentially the reciprocal of the **Dedekind eta function** — introduced in 1877, long before quantum field theory!

This partition function

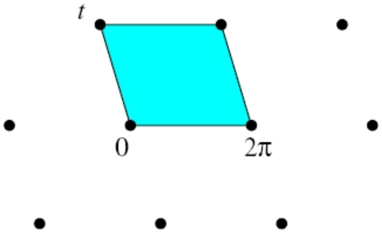
$$Z(\beta) \;=\; e^{\frac{1}{24}\beta} \;\prod_{k=1}^{\infty} \frac{1}{1-e^{-k\beta}}$$

is essentially the reciprocal of the **Dedekind eta function** — introduced in 1877, long before quantum field theory!

Next, let $\beta = it$. Inverse temperature is like imaginary time!

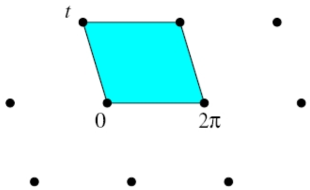
$$Z \;=\; e^{\frac{1}{24}it} \;\prod_{k=1}^{\infty} \frac{1}{1-e^{-ikt}}$$

This converges when $t$ is in the complex upper half-plane.

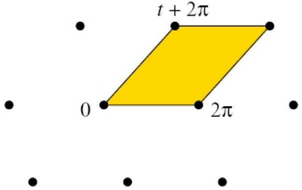



Now $Z$ is the partition function for the torus-shaped spacetime $\mathbb{C}/L$ where $L$ is a lattice in $\mathbb{C}$.

But the torus coming from this parallelogram:




is the same as the torus coming from this one:



So: our calculation only gives a well-defined partition function for the torus $\mathbb{C}/L$ if

$$Z = e^{\frac{1}{24}it} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-ikt}}$$

is unchanged when we add $2\pi$ to $t$.

Alas, $Z$ *does* change: it gets multiplied by

$$e^{\frac{2\pi i}{24}}$$

So: our calculation only gives a well-defined partition function for the torus $\mathbb{C}/L$ if

$$Z = e^{\frac{1}{24}it} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-ikt}}$$

is unchanged when we add $2\pi$ to $t$.

Alas, $Z$ *does* change: it gets multiplied by

$$e^{\frac{2\pi i}{24}}$$

But $Z^{24}$ does *not* change!

*So, the left-moving wave equation has a well-defined partition function on $\mathbb{C}/L$ when the field has 24 components!*

In bosonic string theory, we use such a field to describe the motion of the string in the 24 directions to the worldsheet.

*So, the left-moving wave equation has a well-defined partition function on $\mathbb{C}/L$ when the field has 24 components!*

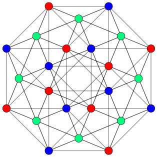
In bosonic string theory, we use such a field to describe the motion of the string in the 24 directions to the worldsheet.

But this partition function, $Z^{24}$, was famous long before string theory. Its reciprocal is called the **modular discriminant** $\Delta$.

$\Delta$ is the simplest 'modular form' that vanishes in the limit where the torus $\mathbb{C}/L$ becomes infinitely skinny.

$$\Delta = e^{-it} \left( \prod_{k=1}^{\infty} 1 - e^{-ikt} \right)^{24}$$

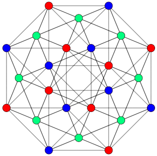
We've seen both superstrings and bosonic strings involve a 24-component field on the string worldsheet. For superstrings the 24 components take values in $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, and are thus connected to the 24-cell:



For bosonic strings the 24 components are connected to the modular discriminant

$$\Delta = e^{-it} \left( \prod_{k=1}^{\infty} 1 - e^{-ikt} \right)^{24}$$

We've seen both superstrings and bosonic strings involve a 24-component field on the string worldsheet. For superstrings the 24 components take values in $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, and are thus connected to the 24-cell:

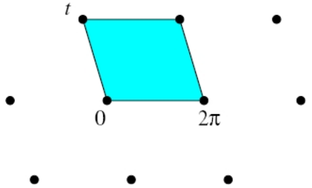


For bosonic strings the 24 components are connected to the modular discriminant

$$\Delta = e^{-it} \left( \prod_{k=1}^{\infty} 1 - e^{-ikt} \right)^{24}$$

Is this function related to the 24-cell? Yes!

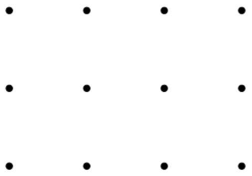
Each point $t$ in the complex upper half-plane $H$ gives a flat Riemannian torus:

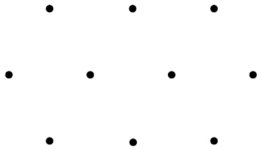



But many different choices of $t \in H$ give conformally equivalent tori! If we only care about the conformal structure on the torus, we call it an **elliptic curve**.

Thus, the 'moduli space' of elliptic curves is a quotient of $H$. In fact it is $H/\mathrm{SL}(2,\mathbb{Z})$.

But $\mathrm{SL}(2, \mathbb{Z})$ doesn't act freely on $H$, because there are elliptic curves with extra symmetries corresponding to the square and hexagonal lattices.





However, the subgroup $\Gamma(3) \subset \mathrm{SL}(2, \mathbb{Z})$ *does* act freely. This subgroup consists of integer matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1, such that each entry is congruent to the corresponding entry of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

modulo 3.

The quotient $H/\Gamma(3)$ has no 'points of greater symmetry'.

The group

$$\mathrm{SL}(2, \mathbb{Z})/\Gamma(3) \cong \mathrm{SL}(2, \mathbb{Z}/3)$$

acts on $H/\Gamma(3)$. To get the moduli space of elliptic curves from $H/\Gamma(3)$, we just need to mod out by the action of this group.

But this group $\mathrm{SL}(2, \mathbb{Z}/3)$ has 24 elements.

In fact, it's isomorphic to our friend the binary tetrahedral group!

$$\pm 1, \ \pm i, \ \pm j, \ \pm k$$
$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$