Definition 1 (Complete lattice)**.** A **complete lattice** is a partially ordered set $(\mathcal{L}, \leq)$[1] where all subsets $S \subseteq \mathcal{L}$ have a infimum and a supremum denoted by $\inf S$ and $\sup S$ respectively. We use $\mathcal{L}$ to refer to the lattice and its underlying set.

[1] i.e. $\mathcal{L}$ is a set and $\leq \subseteq \mathcal{L} \times \mathcal{L}$ is a binary relation on $\mathcal{L}$ that is reflexive, transitive and anti-symmetric.

Definition 2 ($\mathcal{L}$-relation)**.** Given a complete lattice $\mathcal{L}$ and a set X, an **$\mathcal{L}$-relation** on X is a function $d : X \times X \to \mathcal{L}$. We often refer to the pair (X, d) as a **$\mathcal{L}$-space**, and we will also use a single bold-face symbol $\mathbf{X}$ to refer to a $\mathcal{L}$-space with underlying set X and $\mathcal{L}$-relation $d_{\mathbf{X}}$.[2]

[2] We will try to match the symbol for the space and the one for its underlying set only modifying the former with `mathbf`.

A **nonexpansive** (or short) map from $\mathbf{X}$ to $\mathbf{Y}$ is a function $f : X \to Y$ between the underlying sets of $\mathbf{X}$ and $\mathbf{Y}$ that does not increase the distance between points:

$$\forall x, x' \in X, \quad d_{\mathbf{Y}}(f(x), f(x')) \leq d_{\mathbf{X}}(x, x'). \tag{1}$$

The identity maps $\mathrm{id}_X : X \to X$ and the composition of two nonexpansive maps are always nonexpansive[3], therefore we have a category whose objects are $\mathcal{L}$-spaces and morphisms are nonexpansive maps. We denote it by $\mathcal{L}\mathbf{Rel}$.

[3] Fix three $\mathcal{L}$-spaces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ with two nonexpansive maps $f : X \to Y$ and $g : Y \to Z$, we have by nonexpansiveness of g then f:

$$\begin{aligned} d_{\mathbf{Z}}(g \circ f(x), g \circ f(x')) &\leq d_{\mathbf{Y}}(f(x), f(x')) \\ &\leq d_{\mathbf{X}}(x, x'). \end{aligned}$$

Definition 3 ($\mathcal{L}$-structure)**.** Given a complete lattice $\mathcal{L}$, an **$\mathcal{L}$-structure**[4] is a set X equipped with a family of binary relations $R_\varepsilon \subseteq X \times X$ indexed by $\varepsilon \in \mathcal{L}$ satisfying

[4] We borrow the name "structure" from the very abstract notion of relational structure used in [**?**, **?**, **?**].

- **monotonicity** in the sense that if $\varepsilon \leq \varepsilon'$, then $R_\varepsilon \subseteq R_{\varepsilon'}$, and
- **continuity** in the sense that for an I-indexed family of elements $\varepsilon_i \in \mathcal{L}$,

$$\bigcap_{i \in I} R_{\varepsilon_i} = R_\delta, \text{ where } \delta = \inf_{i \in I} \varepsilon_i.$$

Intuitively[5] $(x, y) \in R_\varepsilon$ should be interpreted as bounding the distance from x to y above by ε. Then, monotonicity means the points that are at a distance below ε are also at a distance below ε' when $\varepsilon \leq \varepsilon'$. Continuity means the points that are at a distance below a bunch of bounds ε_i are also at distance below the infimum of those bounds $\inf_{i \in I} \varepsilon_i$.

[5] The proof of Proposition 4 will shed more light on these objects by equating them with $\mathcal{L}$-spaces.

The names for these conditions come from yet another equivalent definition (this time more directly equivalent). Organising the data of an $\mathcal{L}$-structure into a function $R : \mathcal{L} \to \mathcal{P}(X \times X)$ sending ε to R_ε, we can recover monotonicity and continuity by seeing $\mathcal{P}(X \times X)$ as a complete lattice like in **??**. Indeed, monotonicity is equivalent to R being a monotone function between the posets $(\mathcal{L}, \leq)$ and $(\mathcal{P}(X \times X), \subseteq)$, and continuity is equivalent to R preserving infimums, which in turn is equivalent to R being a continuous functor between these posets viewed as posetal categories.[6]

[6] Limits in a posetal category are always computed by taking the infimum of all the points in the diagram.

A morphism between two $\mathcal{L}$-structures $(X, \{R_\varepsilon\})$ and $(Y, \{S_\varepsilon\})$ is a function $f : X \to Y$ satisfying

$$\forall \varepsilon \in \mathcal{L}, \forall x, x' \in X, (x, x') \in R_\varepsilon \implies (f(x), f(x')) \in S_\varepsilon. \tag{2}$$

This should feel similar to nonexpansive maps. Let us call $\mathcal{L}\mathbf{Str}$ the category of $\mathcal{L}$-structures.

Proposition 4. *Given a complete lattice $\mathcal{L}$, the categories $\mathcal{L}\mathbf{Rel}$ and $\mathcal{L}\mathbf{Str}$ are isomorphic.*

Proof. Given an $\mathcal{L}$-relation (X, d), we define the binary relations $R^d_\varepsilon \subseteq X \times X$ by

$$(x, x') \in R^d_\varepsilon \iff d(x, x') \leq \varepsilon. \tag{3}$$

This family satisfies monotonicity because for any $\varepsilon \leq \varepsilon'$ we have

$$(x, x') \in R^d_\varepsilon \overset{(3)}{\iff} d(x, x') \leq \varepsilon \implies d(x, x') \leq \varepsilon' \overset{(3)}{\iff} (x, x') \in R^d_{\varepsilon'}.$$

It also satisfies continuity because if $(x, x') \in R_{\varepsilon_i}$ for all $i \in I$, then $d(x, x') \leq \varepsilon_i$ for all $i \in I$. By defintion of infimum, we must have $d(x, x') \leq \inf_{i \in I} \varepsilon_i$, hence $(x, x') \in R_{\inf_{i \in I} \varepsilon_i}$. We conclude the forward inclusion ($\subseteq$) of continuity holds, the converse ($\supseteq$) follows from monotonicity.

Any nonexpansive map $f : (X, d) \to (Y, \Delta)$ in $\mathcal{L}\mathbf{Rel}$ is also a morphism between the $\mathcal{L}$-structures $(X, \{R^d_\varepsilon\})$ and $(Y, \{R^\Delta_\varepsilon\})$ because for all $\varepsilon \in \mathcal{L}$ and $x, x' \in X$, we have

$$(x, x') \in R^d_\varepsilon \overset{(3)}{\iff} d(x, x') \leq \varepsilon \overset{(1)}{\implies} \Delta(f(x), f(x')) \leq \varepsilon \overset{(3)}{\iff} (f(x), f(x')) \in R^\Delta_\varepsilon.$$

It follows that the assignment $(X, d) \mapsto (X, \{R^d_\varepsilon\})$ is a functor $F : \mathcal{L}\mathbf{Rel} \to \mathcal{L}\mathbf{Str}$ acting trivially on morphisms.

Given an $\mathcal{L}$-structure $(X, \{R_\varepsilon\})$, we define the function $d_R : X \times X \to \mathcal{L}$ by

$$d_R(x, x') = \inf \{\varepsilon \in \mathcal{L} \mid (x, x') \in R_\varepsilon\}.$$

Note that monotonicity and continuity of the family $\{R_\varepsilon\}$ imply[7]

$$d_R(x, x') \leq \varepsilon \iff (x, x') \in R_\varepsilon. \tag{4}$$

This allows us to prove that a morphism $f : (X, \{R_\varepsilon\}) \to (Y, \{S_\varepsilon\})$ is nonexpansive from (X, d_R) to (Y, d_S) because for all $\varepsilon \in \mathcal{L}$ and $x, x' \in X$, we have

$$d_R(x, x') \leq \varepsilon \overset{(4)}{\iff} (x, x') \in R_\varepsilon \overset{(2)}{\implies} (f(x), f(x')) \in S_\varepsilon \overset{(4)}{\iff} d_S(f(x), f(x')) \leq \varepsilon,$$

hence putting $\varepsilon = d(x, x')$, we obtain $d_S(f(x), f(x')) \leq d_R(x, x')$. It follows that the assignment $(X, \{R_\varepsilon\}) \mapsto (X, d_R)$ is a functor $G : \mathcal{L}\mathbf{Str} \to \mathcal{L}\mathbf{Rel}$ acting trivially on morphisms.

Observe that (3) and (4) together say that $R^{d_R}_\varepsilon = R_\varepsilon$ and $d_{R^d} = d$, so F and G are inverse to each other on objects. Since both functors do nothing to morphisms, we conclude that F and G are inverse to each other, and that $\mathcal{L}\mathbf{Rel} \cong \mathcal{L}\mathbf{Str}$. □

[7] The converse implication ($\Leftarrow$) is by definition of infimum. For ($\Rightarrow$), continuity says that

$$R_{d_R(x,x')} = \bigcap_{\varepsilon \in \mathcal{L}, (x,x') \in R_\varepsilon} R_\varepsilon,$$

so $R_{d_R(x,x')}$ contains (x, x'), then by monotonicity, $d_R(x, x') \leq \varepsilon$ implies R_ε also contains (x, x').