The double category of lenses

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A thesis presented for the degree of
Doctor of Philosophy

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Macquarie University
28 February 2022

Abstract

A lens is a functor equipped with a suitable choice of lifts, generalising the notion of a split opfibration. Lenses were first introduced in computer science to model bidirectional transformations between systems, and this thesis contributes to an ongoing initiative to understand lenses using category theory. Specifically, we study the mathematical structure of lenses using double categories, and use this to generalise the theory of lenses to new and useful settings.

Arguably the most fundamental characterisation of lenses, due to Ahman and Uustalu, is as a compatible functor and cofunctor pair. We introduce the flat double category of cofunctors — consisting of categories, functors, and cofunctors — and examine several of its basic properties. Right-connected double categories were first introduced in the study of algebraic weak factorisation systems, and we establish an explicit construction which completes a double category under this property. The double category of lenses is characterised as the right-connected completion of the double category of cofunctors, and we demonstrate how many properties of lenses are inherited from functors and cofunctors via this construction. In particular, the double category of cofunctors is strongly span representable, and this leads to a diagrammatic calculus for lenses using the right-connected completion.

The Grothendieck construction, which yields split opfibrations, is among the most useful notions in category theory, and we introduce a generalised Grothendieck construction for lenses. Given a double category equipped with a functorial choice of companions, its left-connected completion is shown to admit a universal property with respect to lax double functors. The double category of split multi-valued functions is introduced as the left-connected completion of the double category of spans, and we prove that lax double functors into this double category correspond to lenses using the aforementioned universal property.

It is well known that functors between categories naturally arise as monad morphisms in the double category of spans, providing the basis for useful generalisations such as internal categories. We introduce the notion of monad retromorphism, and show that in the double category of spans, monad retromorphisms are precisely cofunctors.

Using the right-connected completion, lenses between monads are defined, and they are used to develop the theory of lenses and split opfibrations in internal category theory.

It is natural to ask if any functor or cofunctor may be equipped with the structure of a lens. We show that lenses arise as both algebras for a monad, and coalgebras for a comonad. It follows that a lens is a functor with additional algebraic structure, and also a cofunctor with additional coalgebraic structure. In particular, the monad for lenses generates an algebraic weak factorisation system, for which every functor factorises through its corresponding free lens. The link between lenses and algebraic weak factorisations systems provides a new setting in which many of the properties of lenses can be understood.

While lenses are a generalisation of the notion of split opfibration, the results in this thesis also have implications for split opfibrations themselves, including new characterisations using décalage, strict factorisation systems, and lax double functors.

Altogether, the thesis demonstrates that double categories provide a valuable unified framework for the theory of lenses.

Statement of originality

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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28 February 2022

Acknowledgements

First and foremost, I would like to thank my supervisor Michael “Mike” Johnson. You have helped me grow to become the mathematician that I am today, and everything I have achieved throughout my PhD was made possible through your support and encouragement.

Thank you to my associate supervisor, Dominic Verity, for all of the advice and wisdom that you have shared with me. I would also like to thank Richard Garner, Steve Lack, John Power, and Ross Street for making Macquarie University such a wonderful place to learn category theory, and for always being happy to answer my questions, no matter how big or small. Thank you to Alexander Campbell, Paul Lessard, and Sophie Raynor for being my academic role models; I am a better person for knowing each of you.

To my friends and category theory companions: Adrian, Branko, David, Eli, Geoff, Giacomo, Jason, Nicola, Paula, Ramon, Ruben, Yao, and Yuki. Thank you for all of the fun times we have shared together and the memories we have made. A special thanks to Matthew Di Meglio for all of the helpful feedback you have offered on my work, and sharing in the excitement of new ideas.

Thank you to Daniel Cicala and Jules Hedges for organising the Applied Category Theory School in 2019 and for allowing me the opportunity to participate. Thank you to everyone in the “Traversal optics and profunctors” group: Bartosz Milewski, Derek Elkins, Fosco Lorgian, Emily Pillmore, and Mario Román.

Thank you also to Carmen Constantin, Eliana Lorch, and Paolo Perrone for organising the Applied Category Theory School in 2020, especially throughout the most difficult time of the pandemic. Thank you to Emma Chollet, Maurine Songa, Vincent Wang, and Gioele Zardini for making the “Categories of maintainable relations” project a success; I am very grateful to have worked with you all.

Thank you to the many others in the category theory community who have inspired and helped me along the way, especially Robin Cockett, Zinovy Diskin, Brendan Fong, Tim Hosgood, Bob Paré, Dorette Pronk, Fara Renaud, Marcy Robertson, David Roberts, Bob Rosebrugh, David Spivak, and Tarmo Uustalu.

Thank you to Chakaveh for your warm friendship and our many lunchtime chats. Thank you to Eline for being my best friend, and keeping me grounded throughout the last three years; I could not have done it without you. Thank you to my family: Lochie, Zoe, Emilee, Amber, Kiernan, Noah, and Paisley. Finally, I would like to thank my parents, Daniel and Michelle, for their continued love and support as I take on new challenges.

Contents

Chapter 1

Introduction

Thus, each lens must include not one but two functions, one for extracting an abstract view from a concrete one and another for putting an updated abstract view back into the original concrete view to yield an updated concrete view.

Foster et al. [Fos+07]

Category theory is the study of morphisms, and this thesis is the study of a certain kind of morphism called a *lens*. Lenses were originally defined in computer science as morphisms between sets consisting of a pair of functions [Fos+07]. This thesis focuses on a variant called a *delta lens* (or d-lens), which is a morphism between categories consisting of a functor with certain additional structure [DXC11]. The aim of this work is to demonstrate how double categories play a fundamental role in developing the theory of delta lenses.

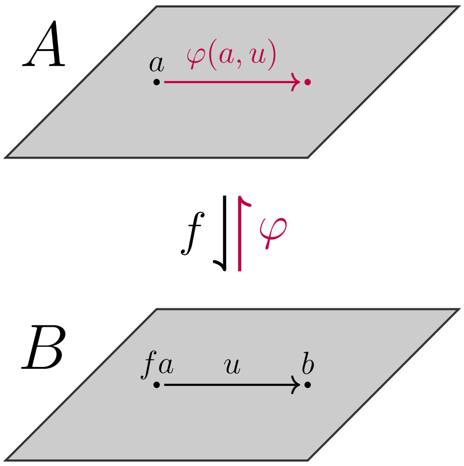


Figure 1.1: An illustration of a delta lens.

Delta lenses are functors equipped with a suitable choice of lifts. Ahman and Uustalu first recognised that such a choice of lifts is equivalent to a kind of morphism between categories called a *cofunctor* [AU17]. Thus, a succinct characterisation of a delta lens is as a compatible functor and cofunctor pair. The first main theme of this work is interplay between functors and cofunctors as two kinds of morphisms in a double category.

Cofunctors were introduced by Aguiar [Agu97] in the setting of internal category theory, and were based upon the notion of *comorphism* between Lie groupoids due to Higgins and Mackenzie [HM93]. Every cofunctor is equivalent to a span of functors, whose left leg is bijective-on-objects and whose right leg is a discrete opfibration. This leads to another simple characterisation of a delta lens as a certain commutative diagram of functors. The second main theme of this thesis is the development of an abstract diagrammatic approach to delta lenses, by generalising cofunctors via a restriction of the double category of spans in a category with pullbacks.

Fibrations and opfibrations are among the most important concepts in category theory. Johnson and Rosebrugh proved that every *split opfibration* is a delta lens [JR13]. In particular, a split opfibration is a delta lens whose chosen lifts are *opcartesian*. Thus, in a sense, the study of delta lenses is the study of the underlying structure of split opfibrations. The third main theme of this work is exploring the deep relationship between delta lenses and split opfibrations, and, in particular, establishing a double-categorical Grothendieck construction for delta lenses.

A detailed discussion of the content and motivation behind each of the main chapters is presented in the following sections. As for the other chapters, Chapter 2 introduces the background material on cofunctors, delta lenses, and split opfibrations. Appendix A recalls standard definitions and results about double categories. Chapter 7 presents concluding remarks and outlines future work.

In the remainder of this thesis delta lenses are referred to simply as *lenses*.

The double category of lenses

Whenever there are two kinds of morphisms between objects, it is natural to ask if they can be assembled into a double category. Chapter 3 begins with the definition of the flat double category $\mathbb{C}\mathrm{of}$, the *double category of cofunctors*, whose objects are categories, whose horizontal morphisms are functors, and whose vertical morphisms are cofunctors. A cell in $\mathbb{C}\mathrm{of}$ exists when the functors and cofunctors in its boundary are suitably *compatible* with each other. The central question of Chapter 3 is:

Can lenses be universally constructed from the double category of cofunctors?

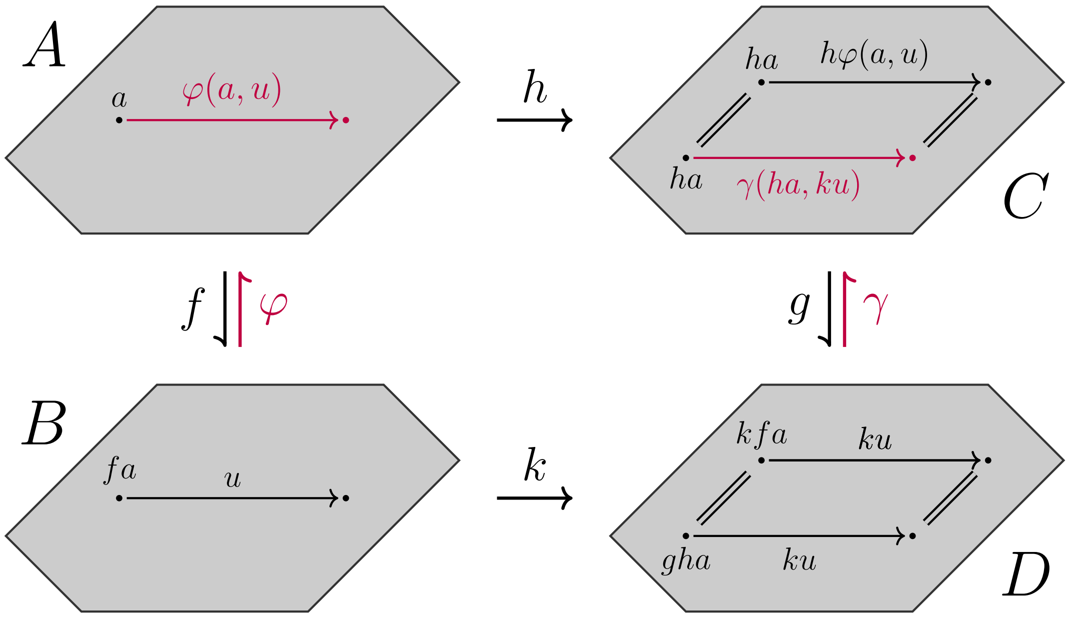


Figure 1.2: A cell in the flat double category $\mathbb{C}\mathrm{of}$, where a vertical morphism is a cofunctor given by a lifting operation relative to a function on objects.

A double category is *right-connected* if its identity map is right adjoint to its codomain map [BG16a]. A key contribution of this thesis is an explicit construction of the *right-connected completion* of a double category. The right-connected completion of $\mathbb{C}\mathrm{of}$ yields a double category whose vertical morphisms are exactly lenses, answering the central question of Chapter 3. In other words, the flat double category $\mathbb{L}\mathrm{ens}$, the *double category of lenses*, whose objects are categories, whose horizontal morphisms are functors, and whose vertical morphisms are lenses, is the right-connected completion of the double category of cofunctors.

The second main result of Chapter 3 is that the double category $\mathbb{C}\mathrm{of}$ is *span representable* [GP17], thus reinterpreting the classical representation of cofunctors as spans [HM93]. This amounts to showing that $\mathbb{C}\mathrm{of}$ has *tabulators*, and admits a faithful lax double functor to $\mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$, the double category of spans in $\mathcal{C}\mathrm{at}$. Indeed, there is actually a fully faithful pseudo double functor from $\mathbb{C}\mathrm{of}$ to $\mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$, whose essential image yields an equivalent double category to $\mathbb{C}\mathrm{of}$. The key benefit is a diagrammatic approach to cofunctors using bijective-on-objects functors and discrete opfibrations. A simple, yet immensely important, corollary of this result is that every lens admits a representation as a commutative diagram of functors.

The third and final main result of Chapter 3 is a new characterisation of split opfibrations. The *décalage construction* is a copointed endofunctor on $\mathcal{C}\mathrm{at}$, and the class of lenses to which this extends is the split opfibrations. The proof is completely diagrammatic, and determines a canonical cell in $\mathbb{L}\text{ens}$ for every split opfibration.

Lenses as lax double functors

The classical *Grothendieck construction*, which yields an equivalence between $\mathcal{C}\text{at}$-valued functors and split opfibrations, is one the most important tools in category theory. Chapter 4 investigates a fibred approach to lenses as generalised split opfibrations. The central question that Chapter 4 seeks to answer is:

Does there exist a generalised Grothendieck construction for lenses?

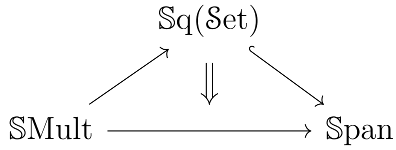
To develop an answer, it is worth reflecting upon the generalised Grothendieck construction for ordinary functors. Given the double category of squares $\mathbb{S}\text{q}(\mathcal{C}\text{at})$, the fibre over a small category B with respect to the codomain map $\text{cod}\colon \mathbb{S}\text{q}(\mathcal{C}\text{at}) \to \mathcal{C}\text{at}$ of $\mathbb{S}\text{q}(\mathcal{C}\text{at})$ is given by the slice category $\mathcal{C}\text{at}/B$. Moreover, for every category B there is a double category $\mathbb{V}(B)$, whose objects and vertical morphisms are taken from B, and whose horizontal morphisms and cells are identities. The final ingredient is to recall the double category $\mathbb{S}\text{pan} := \mathbb{S}\text{pan}(\mathcal{S}\text{et})$ of sets, functions, and spans of functions. The generalised Grothendieck construction for functors is the right-to-left direction of an equivalence of categories,

$$\mathcal{C}\text{at}/B \simeq [\mathbb{V}(B), \mathbb{S}\text{pan}]_{\text{lax}}$$

between functors into B, and lax double functors from $\mathbb{V}(B)$ into $\mathbb{S}\text{pan}$. Details of this result may be found in the work of Paré [Par11]. Note that restricting along the fully faithful double functor $\mathbb{S}\text{q}(\mathcal{S}\text{et}) \to \mathbb{S}\text{pan}$ yields a modified statement of the classical *category of elements* construction for discrete opfibrations, given by the equivalence:

$$\mathcal{D}\text{Opf}_B \simeq [\mathbb{V}(B), \mathbb{S}\text{q}(\mathcal{S}\text{et})]$$

Lenses are functors with additional structure, therefore it is natural to wonder whether modifying $\mathbb{S}\text{pan}$ could yield a generalised Grothendieck construction for lenses. Surprisingly, the answer lies in considering the left-connected completion of $\mathbb{S}\text{pan}$, given by the double category $\mathbb{S}\text{Mult}$, the *double category of split multi-valued functions*. There is a canonical horizontal transformation between strict double functors,



whose components are globular cells. Whiskering this transformation by a lax double functor $\mathbb{V}(B) \to \mathbb{S}\text{Mult}$ yields a morphism in $\mathcal{C}\text{at}/B$ which is precisely the diagrammatic representation of a lens.

The main result of Chapter 4 is an equivalence of categories,

$$\mathcal{L}\text{ens}_B \simeq [\mathbb{V}(B), \mathbb{S}\text{Mult}]_{\text{lax}}$$

where $\mathcal{L}\text{ens}_B$ is the fibre over a small category B with respect to the codomain map $\text{cod}\colon \mathcal{L}\text{ens} \to \mathcal{C}\text{at}$ of the double category $\mathbb{L}\text{ens}$ of lenses. The right-to-left direction of this equivalence is the generalised Grothendieck construction for lenses.

This conceptual shift to this fibred approach to lenses is significant and offers a new perspective on their relationship with split opfibrations. The second main contribution of Chapter 4 is a characterisation of split opfibrations as lax functors $\mathbb{V}(B) \to \mathbb{S}\text{Mult}$ with a certain property. Chapter 4 concludes with an exploration of several classes of lenses that arise through considering fully faithful double functors into $\mathbb{S}\text{Mult}$.

Lenses as monad morphisms

Many concepts in category theory can be better understood through generalising them to the setting of internal or enriched categories. Chapter 5 seeks to answer the question:

What is the definition of a (delta) lens between internal categories?

The topic of internalising lenses is not a new one. For *state-based lenses*, originally called *very well-behaved lenses* [Fos+07], an internalisation was developed both in a category with finite products [JRW10] and in a cartesian closed category [GJ12]. Since state-based lenses are equivalent to delta lenses between codiscrete categories [JR16], the correct definition of lens between internal categories should specialise at least one of these results. A straightforward answer is to simply use the seemingly ad hoc definition of internal cofunctor [Agu97] to define internal lenses, however this is unsatisfying and raises a more important question:

What is the natural setting in which cofunctors arise?

Ahman and Uustalu [AU16] characterised small categories as polynomial comonads on $\mathcal{S}\text{et}$, and found that the comonad morphisms correspond exactly to cofunctors. It is possible to internalise their definition of cofunctor to any *locally cartesian closed category*, however it not known how to define internal functors in this particular setting. An open question arising from their work was whether lenses, like cofunctors, admitted a concise characterisation as comonad morphisms.

The primary contribution of Chapter 5 comes from utilising the formal theory of monads [Str72; LS02] to characterise cofunctors. Categories are equivalent to monads in the bicategory of sets and spans, but the (lax) monad morphisms are not functors.

Instead, they correspond to *Mealy morphisms* [Par12] or *two-dimensional partial maps* [LS02]. However, restricting to the monad morphisms whose 1-cell component is a right adjoint recovers functors between categories, while those whose 1-cell component is a left adjoint are cofunctors. In this setting, the duality between functors and cofunctors becomes clear.

While initially the contributions of this chapter were originally formulated in the language of 2-categories, it makes sense to translate them to the language of double categories where they fit more naturally in the scope of this thesis. The formal theory of monads in 2-categories may be adapted to the double category setting [FGK11; FGK12]. Monads in the double category $\mathbb{S}\mathrm{pan}$ are still categories, and the monad morphisms are precisely functors. Paré introduced the notion of *retrocell* [Par19] in any double category with companions, and these may be used to define *monad retromorphisms*, which in $\mathbb{S}\mathrm{pan}$ are precisely cofunctors.

For any double category $\mathbb{D}$, there is a double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of monads in $\mathbb{D}$. If $\mathbb{D}$ is equipped with a functorial choice of companions, there is a full double subcategory $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ on the monad retromorphisms. When $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks, this construction yields the span representable double category $\mathbb{C}\mathrm{of}(\mathcal{E})$ whose objects are internal categories, whose horizontal morphisms are internal functors, and whose vertical morphisms are internal cofunctors. The right-connected completion of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ constructs a double category $\mathbb{L}\mathrm{ens}(\mathbb{D})$ whose vertical morphisms may be interpreted as *lenses between monads*. Taking $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$ produces the definition of *internal lens*, thus providing a natural answer to the main question of Chapter 5.

The final contribution of Chapter 5 is two new characterisations of *internal split opfibrations*. The first result characterises internal split opfibrations as internal lenses which satisfy a certain pullback condition, and demonstrates an equivalence to a presentation using unique structure. The second characterisation utilises strict factorisation systems, and also admits a presentation using unique structure.

Lenses as algebras and coalgebras

Many structures in mathematics may be fruitfully studied as algebras for a monad or as coalgebras for a comonad. Lenses are simultaneously functors with additional structure *and* cofunctors with additional structure, and it is natural to wonder if this structure comes from a monad or comonad. Chapter 6 addresses the question:

Are lenses algebras for a monad or coalgebras for a comonad?

Parallel to this question is the problem of finding the right level of generality for lenses to reveal an answer. The starting point is recognising that an important aspect of cofunctors is their diagrammatic representation as spans, and using this to define lenses between objects in any suitable category with pullbacks.

The double category $\mathbb{C}\mathrm{of}$ is replaced with a double category $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ of spans in $\mathcal{C}$ with left leg in a class of morphisms $\mathcal{W}$ and right leg in a class of morphisms $\mathcal{M}$. The idea is that when $\mathcal{C} = \mathcal{C}\mathrm{at}$, $\mathcal{W}$ is the class of *bijective-on-objects functors*, and $\mathcal{M}$ is the class of *discrete opfibrations*, then $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is equivalent to the double category of cofunctors. Taking the right-connected completion yields the double category $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ of generalised lenses, together with canonical double functors:

$$\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \longleftarrow \mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C})$$

Under simple conditions on the class $\mathcal{W}$, it is proved that the forgetful functor between the corresponding categories of morphisms of $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ and $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$,

$$\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \longrightarrow \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$$

is *comonadic*. The right adjoint utilises pullbacks and codiscrete objects in $\mathcal{C}$. In the case of $\mathcal{C} = \mathcal{C}\mathrm{at}$ with bijective-on-objects functors and discrete opfibrations, this establishes the first main result of the Chapter 6 which is that lenses are coalgebras for a comonad on a category of cofunctors. A basic corollary is the construction of the *cofree lens* on a cofunctor, previously introduced by Ahman and Uustalu [AU16].

Under further conditions on the triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$, it is proved that the forgetful functor between the categories of morphisms,

$$\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \longrightarrow \mathcal{S}\mathrm{q}(\mathcal{C})$$

is *monadic*. The left adjoint utilises pushouts, discrete objects in $\mathcal{C}$, and that $\mathcal{M}$ is the right class in an orthogonal factorisation system. In the case of $\mathcal{C} = \mathcal{C}\mathrm{at}$ with bijective-on-objects functors and discrete opfibrations, this establishes the second main result of the Chapter 6 which is that lenses are algebras for a monad on the arrow category on $\mathcal{C}\mathrm{at}$. A basic corollary is the construction of the *free lens* on a functor, a result which would seem to be almost impossible to obtain without the diagrammatic approach to lenses. These results are shown to share a close relationship with work of Johnson and Rosebrugh [JR13] which characterised lenses as algebras for a *semi-monad* on the same category.

The third main result of the Chapter 6 is that generalised lenses arise as the right class of an *algebraic weak factorisation system* [BG16a; BG16b]. Indeed, every functor factorises through the free lens via the unit of the monad for lenses on $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$.

Chapter 6 concludes with a notion of change of base for lenses through showing that the codomain functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is a bifibration.

In summary, the contributions of this thesis provide new perspectives on the concept of a lens using double categories. Among the many characterisations, lenses are shown to be morphisms in the right-completion of the double category of cofunctors, lax double functors into the double category of split multi-valued functions, compatible morphisms between monads in a double category, algebras for a monad, and coalgebras for a comonad. Individually, it is hoped that these characterisations of lenses provide new tools for future applications, while collectively, they present a significant advance towards understanding lenses in the setting of category theory.

Publication summary

The research conducted during my PhD studies has resulted in the publication of five papers [Cla20a; Cla20b; Cla21a; Cho+21; Cla21b]. The contributions from four of these papers to the results in this thesis are outlined below.

- My sole-authored paper *Internal lenses as functors and cofunctors* [Cla20a] contains results found in Section 5.3 and Section 5.4, including the definition of internal lens and their construction from internal functors and cofunctors.

- My sole-authored paper *Internal split opfibrations and cofunctors* [Cla20b] extends the results in [Cla20a] to include material on the characterisation of (internal) split opfibrations, and this is presented in Section 3.5 and Section 5.5.

- My sole-authored paper *A diagrammatic approach to symmetric lenses* [Cla21a] defines morphisms of lenses over a fixed codomain, an idea which led directly to the definition of cells in the double category of lenses and underlies the entire thesis. However, the main results of this paper on symmetric lenses is not included in this thesis.

- My sole-authored paper *Delta lenses as coalgebras for a comonad* [Cla21b] is covered entirely in Section 6.2.

Chapter 2

Background

This chapter introduces the background material on lenses. Several results are proven from first principles, including the characterisation of lenses as compatible functors and cofunctors. A diagrammatic formulation of lenses and cofunctors is presented, and is shown to be equivalent to their axiomatic definition. The main examples of lenses are also introduced, including discrete opfibrations, split opfibrations, and bijective-on-objects functors with a chosen section.

2.1 Lenses

Definition 2.1. A *(delta) lens* $(f, \varphi)\colon A \to B$ consists of a functor $f\colon A \to B$ together with a *lifting operation*,

$$(a \in A, u\colon fa \to b \in B) \quad \longmapsto \quad \varphi(a, u)\colon a \to p(a, u)$$

where $p(a, u) := \mathrm{cod}(\varphi(a, u))$, satisfying the following three axioms:

(L1) $f\varphi(a, u) = u$,

(L2) $\varphi(a, 1_{fa}) = 1_a$,

(L3) $\varphi(a, v \circ u) = \varphi(p(a, u), v) \circ \varphi(a, u)$.

The first axiom (L1) states that the morphisms $\varphi(a, u) \in A$ are *chosen lifts* of the morphisms $u \in B$ with respect to the functor $f\colon A \to B$. This axiom may be depicted as follows:

$$\begin{array}{ccccc} A & \qquad & a & \xrightarrow{\varphi(a,u)} & p(a,u) \\ {\scriptstyle f}\big\downarrow & & \vdots & & \vdots \\ B & & fa & \xrightarrow{\quad u \quad} & b \end{array}$$

The second axiom (L2) states that the lifting operation respects identity morphisms, and may be depicted as follows:

$$
\begin{array}{ccccc}
 & & \overset{\varphi(a,1_{fa})}{\longrightarrow} & p(a,1_{fa}) \\
 & & & \| \\
A & a & \xrightarrow{1_a} & a \\
{\scriptstyle f}\downarrow & \vdots & & \vdots \\
B & fa & \xrightarrow{1_{fa}} & fa
\end{array}
$$

Finally, the third axiom (L3) states that the lifting operation respects composition of morphisms, and may be depicted as follows:

$$
\begin{array}{cccccc}
 & & & \overset{\varphi(a,v\circ u)}{\longrightarrow} & & p(a,v\circ u) \\
 & & & & & \| \\
A & a & \xrightarrow{\varphi(a,u)} & p(a,u) & \xrightarrow{\varphi(p(a,u),v)} & p(p(a,u),v) \\
{\scriptstyle f}\downarrow & \vdots & & \vdots & & \vdots \\
B & fa & \xrightarrow{u} & b & \xrightarrow{v} & b'
\end{array}
$$

The *identity lens* on a category A consists of the identity functor $1_A \colon A \to A$ together with the trivial lifting operation:

$$(a \in A, u \colon a \to a' \in A) \quad \longmapsto \quad u$$

The *composite lens* of a pair of lenses $(f,\varphi)\colon A \to B$ and $(g,\gamma)\colon B \to C$, consists of the composite functor $gf\colon A \to C$ together with the lifting operation:

$$(a \in A, u \colon gfa \to c \in C) \quad \longmapsto \quad \varphi(a, \gamma(fa,u))$$

The lifting operation of the composite lens may be depicted as follows:

$$
\begin{array}{cccc}
A & a & \xrightarrow{\varphi(a,\gamma(fa,u))} & a' \\
{\scriptstyle f}\downarrow & \vdots & & \vdots \\
B & fa & \xrightarrow{\gamma(fa,u)} & b \\
{\scriptstyle g}\downarrow & \vdots & & \vdots \\
C & gfa & \xrightarrow{u} & c
\end{array}
\tag{2.1}
$$

Composition of lenses is unital and associative, and there is a category whose objects are (small) categories and whose morphisms are lenses. In Chapter 3, lenses are shown to be the vertical morphisms in a double category which will be denoted $\mathbb{L}\mathrm{ens}$.

2.2 Cofunctors

Closely related to lenses is the more basic notion of a cofunctor between categories, which is defined as additional structure on a function rather than a functor. The notation A_0 is used for the underlying *set of objects* of a category A, and the notation $f_0 \colon A_0 \to B_0$ is used for the underlying *object assignment* of a functor $f \colon A \to B$.

Definition 2.2. A *cofunctor* $(f, \varphi) \colon A \to B$ consists of a function $f \colon A_0 \to B_0$ together with a *lifting operation*,

$$(a \in A, u \colon fa \to b \in B) \quad \longmapsto \quad \varphi(a,u) \colon a \to p(a,u)$$

where $p(a,u) \coloneqq \mathrm{cod}(\varphi(a,u))$, satisfying the following three axioms:

(C1) $fp(a,u) = \mathrm{cod}(u)$,

(C2) $\varphi(a, 1_{fa}) = 1_a$,

(C3) $\varphi(a, v \circ u) = \varphi(p(a,u), v) \circ \varphi(a,u)$.

Remark. The notation $(f, \varphi) \colon A \to B$ is used for both lenses and cofunctors; it will always be clear form the context which kind of morphism is intended.

While it is immediate from the definition that each lens has an underlying functor, each lens has an underlying cofunctor as well.

Lemma 2.3. *Every lens $(f, \varphi) \colon A \to B$ has an underlying cofunctor $(f_0, \varphi) \colon A \to B$.*

Proof. Axioms (L2) and (L3) for a lens are identical to the axioms (C2) and (C3) for a cofunctor, after substituting the functor with its underlying object assignment. Axiom (L1) for a lens implies that axiom (C1) holds for the corresponding cofunctor. □

The following result due to Ahman and Uustalu [AU17] states that a *compatible* functor and cofunctor pair completely capture the notion of a lens.

Proposition 2.4. *A lens $(f, \varphi) \colon A \to B$ is equivalent to a functor $f \colon A \to B$ and a cofunctor $(f_0, \varphi) \colon A \to B$ such that $f\varphi(a,u) = u$ for all pairs $(a \in A, u \colon fa \to b \in B)$.*

Proof. By Definition 2.1 and Lemma 2.3, every lens has an underlying functor and cofunctor such that the underlying object assignments agree and axiom (L1) is satisfied. Conversely, it is immediate by definition that a functor and cofunctor pair satisfying the conditions is equivalent to a lens. □

Composition of cofunctors is defined in the same way as the composition of lenses (2.1), and there is a category whose objects are (small) categories and whose morphisms are cofunctors. In Chapter 3, cofunctors are shown to be the vertical morphisms in a double category $\mathbb{C}\mathrm{of}$, from which the double category of lenses $\mathbb{L}\mathrm{ens}$ will be constructed.

2.3 The category of chosen lifts

The axioms on the lifting operation of a lens or a cofunctor $(f, \varphi)\colon A \to B$ determine corresponding properties on the codomain of the chosen lifts which may be stated, using the notation $p(a, u) \coloneqq \mathrm{cod}(\varphi(a, u))$, as follows:

(P1) $fp(a, u) = \mathrm{cod}(u)$,

(P2) $p(a, 1_{fa}) = a$,

(P3) $p(a, v \circ u) = p(p(a, u), v)$.

Note that property (P1) is identical to axiom (C1) for a cofunctor.

Remark 2.5. In the literature, the lifting operation of a lens is referred to as the *Put*. The notation $p(a, u)$ for the codomain of the lift $\varphi(a, u)$ is chosen in recognition of this terminology.

These properties may be used to show that every lens or cofunctor canonically induces a category which captures its essential information. For clarity, the following result is stated for cofunctors, but immediately applies for lenses as well via its underlying cofunctor.

Proposition 2.6. *Given a cofunctor* $(f, \varphi)\colon A \to B$ *there is a category* $\Lambda(f, \varphi)$*, whose objects are the same as* A*, and whose morphisms are pairs* $(a \in A, u\colon fa \to b \in B)$ *such that* $\mathrm{dom}(a, u) = a$ *and* $\mathrm{cod}(a, u) = p(a, u) \coloneqq \mathrm{cod}(\varphi(a, u))$*.*

Proof. Given morphisms (a, u) and (a', v) in $\Lambda(f, \varphi)$, where $a' = p(a, u)$, their composite is the morphism $(a, v \circ u)$, which is well-defined by property (P3). The identity morphism on an object $a \in \Lambda(f, \varphi)$ is given by the pair $(a, 1_{fa})$ which is well-defined by property (P2). Composition is left unital by definition, right unital by property (P1), and associative due to the associativity of composition in B. □

In a sense, the category $\Lambda(f, \varphi)$ is the *image* of the lifting operation of a cofunctor. In Chapter 3, it is shown that this category is actually the *tabulator* of a vertical morphism in $\mathbb{C}\mathrm{of}$ or $\mathbb{L}\mathrm{ens}$. When the category $\Lambda(f, \varphi)$ occurs as a *wide subcategory* of the domain of a lens or cofunctor, it will be called *the category of chosen lifts*. The following result shows that this property holds for every lens.

Proposition 2.7. *Given a lens* $(f, \varphi)\colon A \to B$ *there is a faithful identity-on-objects functor* $\varphi\colon \Lambda(f, \varphi) \to A$ *whose assignment on morphisms is given by* $(a, u) \mapsto \varphi(a, u)$*.*

Proof. The assignment on morphisms respects identity morphisms due to axiom (L2) and respects composition due to axiom (L3), therefore φ is a well-defined functor. To show that the functor is faithful, consider a pair of morphisms (a, u) and (a', u') in $\Lambda(f, \varphi)$ such that $\varphi(a, u) = \varphi(a', u')$. By definition of φ we have that $a = a'$, and by axiom (L1) we have that $u = f\varphi(a, u) = f\varphi(a', u') = u'$. Therefore $(a, u) = (a', u')$. □

2.4 A diagrammatic approach

While lenses are defined as an axiomatic structure on a functor, there is also a very useful diagrammatic approach to lenses. This approach is based on representing cofunctors as certain spans of functors, an idea implicit in the original definition of *comorphism* due Higgins and Mackenzie [HM93], and later made explicit in the work of Aguiar [Agu97, Section 4.4]. First recall the following basic definitions.

Definition 2.8. A functor $f\colon A \to B$ is *bijective-on-objects* if for all $b \in B$ there exists a unique object $a \in A$ such that $fa = b$.

Definition 2.9. A functor $f\colon A \to B$ is a *discrete opfibration* if for all $a \in A$ and $u\colon fa \to b \in B$ there exists a unique morphism $w\colon a \to a' \in A$ such that $fw = u$.

Each of the classes of bijective-on-objects functors and discrete opfibrations contain the isomorphisms, is closed under composition, and is stable under pullback along arbitrary functors. Discrete opfibrations, in particular, are a very important class of functors as they have a unique lifting operation which satisfies the lens axioms; this point will be revisited in the next section.

Proposition 2.10. *For every cofunctor $(f, \varphi)\colon A \to B$, there is a span of functors,*

$$A \xleftarrow{\ \varphi\ } \Lambda(f,\varphi) \xrightarrow{\ \overline{f}\ } B$$

$$\begin{array}{ccccc} a & \cdots & a & \cdots & fa \\ {\scriptstyle \varphi(a,u)}\downarrow & & \downarrow{\scriptstyle (a,u)} & & \downarrow{\scriptstyle u} \\ p(a,u) & \cdots & p(a,u) & \cdots & fp(a,u) = b \end{array}$$

where φ is identity-on-objects and $\overline{f}$ is a discrete opfibration.

Proof. By Proposition 2.6, the category $\Lambda(f, \varphi)$ exists and is well-defined. The functors φ and $\overline{f}$ are both well-defined by the axioms of a cofunctor.

To show that $\overline{f}$ is a discrete opfibration, consider an object $a \in \Lambda(f, \varphi)$ and a morphism $u\colon fa \to b \in B$. Then, by construction, there exists a unique morphism $(a, u)\colon a \to p(a, u)$ in $\Lambda(f, \varphi)$ such that $\overline{f}(a, u) = u$. □

A way of interpreting this result is that bijective-on-objects cofunctors (which are exactly bijective-on-objects functors in the opposite direction) and discrete opfibrations form an *orthogonal factorisation system* on the category of small categories and cofunctors, although this perspective will not be emphasised. Proposition 2.10 together with the following result provides the basis for a diagrammatic approach to cofunctors and lenses.

Proposition 2.11. *Given a span of functors whose left leg ψ is bijective-on-objects and whose right leg g is a discrete opfibration,*

$$A \xleftarrow{\ \psi\ } X \xrightarrow{\ g\ } B \tag{2.2}$$

there is a cofunctor $(f, \varphi)\colon A \to B$ and an isomorphism $j\colon \Lambda(f, \varphi) \to X$ such that $\psi j = \varphi$ and $gj = \overline{f}$.

Proof. Since ψ is bijective-on-objects, let $f\colon A_0 \to B_0$ be the composite $g\psi^{-1}$. Given a pair $(a \in A, u\colon fa \to b \in B)$, consider the the pair $(\psi^{-1}a, u)$. Since g is a discrete opfibration, there exists a unique morphism $\gamma(\psi^{-1}a, u)\colon \psi^{-1}a \to x \in X$ such that $g\gamma(\psi^{-1}a, u) = u$. Define the lifting operation φ by:

$$(a \in A, u\colon fa \to b \in B) \quad \longmapsto \quad \psi(\gamma(\psi^{-1}a, u))$$

It satisfies (C1) by construction. By uniqueness of lifts of g and functoriality of ψ, this lifting operation also satisfies axioms (C2) and (C3), and thus defines a cofunctor $(f, \varphi)\colon A \to B$. Finally, define an isomorphism $j\colon \Lambda(f, \varphi) \to X$ with assignment on objects $a \mapsto \psi^{-1}a$ and assignment on morphisms $(a, u) \mapsto \gamma(\psi^{-1}a, u)$. By construction, $\psi j = \varphi$ and $gj = \overline{f}$. □

Altogether, Proposition 2.10 and Proposition 2.11 imply that cofunctors and the special spans (2.2) are essentially equivalent to each other; this idea will be explored closely in Chapter 3. Using Proposition 2.4, these results for cofunctors enable a approach to lenses as commutative diagrams of functors which is central for many results in this thesis.

Proposition 2.12. *For every lens $(f, \varphi)\colon A \to B$, there is a commutative diagram of functors,*

$$\begin{array}{ccc} & \Lambda(f, \varphi) & \\ {}^{\varphi}\swarrow & & \searrow^{f\varphi} \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

where φ is a faithful identity-on-objects functor and $f\varphi$ is a discrete opfibration.

Proposition 2.13. *Given a commutative diagram of functors,*

$$\begin{array}{ccc} & X & \\ {}^{\psi}\swarrow & & \searrow^{f\psi} \\ A & \xrightarrow[f]{} & B \end{array} \tag{2.3}$$

where ψ is bijective-on-objects and $f\psi$ is a discrete opfibration, there exists a lens $(f, \varphi) \colon A \to B$ together with an isomorphism $j \colon \Lambda(f, \varphi) \to X$ such that $\psi j = \varphi$.

2.5 Basic examples

Arguably the most important class of examples of lenses in category theory are split opfibrations: these are lenses with a certain *property*.

Example 2.14. A *split opfibration* is a lens $(f, \varphi) \colon A \to B$ such that each morphism $\varphi(a, u)$ is opcartesian. That is, for each morphism $w \colon a \to a'$ in A such that $fw = v \circ u$, there exists a unique morphism $\hat{v}$ in A such that $w = \hat{v} \circ \varphi(a, u)$ and $f\hat{v} = v$.

A special case of split opfibrations is discrete opfibrations (Definition 2.9), which are examples of functors with a unique lens structure. These can also be understood as lenses with a certain property.

Example 2.15. A functor $f \colon A \to B$ is a discrete opfibration if and only if there is a lens $(f, \varphi) \colon A \to B$ such that $\varphi(a, fw) = w$ for all morphisms $w \colon a \to a' \in A$. Discrete opfibrations are equivalent to diagrams (2.3) of the form:

$$\begin{array}{ccc} & A & \\ /\!\!/ & & \searrow^{f} \\ A & \xrightarrow[f]{} & B \end{array}$$

Note that every isomorphism, in particular every identity functor, is a discrete opfibration and thus has a unique lens structure. Every functor between discrete categories is also a discrete opfibration. While Example 2.15 exhibits the conventional way in which discrete opfibrations are equivalent to certain lenses, discrete opfibrations may also be considered as lenses via the bijective-on-objects, fully faithful factorisation.

Example 2.16. Every lens $(f, \varphi) \colon A \to B$ such that f is fully faithful is equivalent to a discrete opfibration, namely, the discrete opfibration $f\varphi \colon \Lambda(f, \varphi) \to B$ defined in Proposition 2.12.

Lenses between codiscrete categories have been of historical importance in computer science [Fos+07].

Example 2.17. A *state-based lens* consists of functions $f\colon A \to B$ and $p\colon A \times B \to A$ satisfying the following three axioms:

(i) $fp(a, b) = b$,

(ii) $p(a, fa) = a$,

(iii) $p(p(a, b), b') = p(a, b')$.

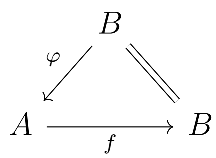
State-based lenses are equivalent to (delta) lenses between codiscrete categories [JR16].

Now consider some examples of lenses which are not split opfibrations.

Example 2.18. A monoid homomorphism with a chosen section is equivalent to a lens between categories with a single object. The monoid homomorphism is a called *Schreier split epimorphism* if and only if the corresponding lens is a split opfibration [Bou21, Section 3.2].

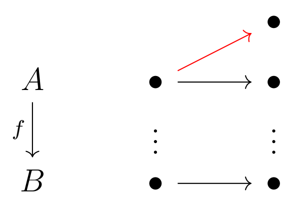
The above example may be generalised further to give examples of lenses between ordinary categories that are not (in general) split opfibrations.

Example 2.19. A bijective-on-objects functor has a lens structure if and only if it has a chosen section. These correspond to commutative diagrams (2.3) of the form:



All of the examples considered so far are large classes of lenses. What is the smallest example which is not in any of the above classes?

Example 2.20. Consider the categories $A = \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ and $B = \{\bullet \rightarrow \bullet\}$. Then there is a lens $(f, \varphi)\colon A \to B$ given by,



where the red arrow is the non-trivial chosen lift.

Note that while this functor has two possible lens structures up to *equality*, it has a unique lens structure up to *isomorphism*. Of course, while this is intuitively clear, a precise definition of what it means to have a unique lens structure up to isomorphism will be stated in the next chapter.

Chapter 3

The double category of lenses

This chapter forms the foundation for the thesis by introducing the double category of lenses. Section 3.1 begins by defining the double category of cofunctors, and it is observed that lenses appear as certain cells in this double category. Using this observation, Section 3.2 studies the right-connected completion of a double category and several of its properties. Section 3.3 constructs the double category of lenses as the right-connected completion of the double category of cofunctors, one of the central results of this chapter. In Section 3.4, the double category of cofunctors is shown to be strongly span representable, yielding a diagrammatic approach to cofunctors and lenses. The chapter concludes with a diagrammatic characterisation of the double category of split opfibrations using the décalage construction in Section 3.5. The relevant background material on double categories and notational conventions for this chapter may be found in Appendix A.

3.1 The double category of cofunctors

Functors and cofunctors are both useful kinds of morphisms between categories, and together they can be assembled into a double category.

Definition 3.1. Let $\mathbb{C}\mathrm{of}$ be the *double category of cofunctors*, whose objects are (small) categories, whose horizontal morphisms are functors, whose vertical morphisms are cofunctors, and whose cells with boundary given by,

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (g,\gamma)} \\ B & \xrightarrow[k]{} & D \end{array} \tag{3.1}$$

are such that $gh_0 = k_0 f$ and $h\varphi(a, u) = \gamma(ha, ku)$ for all pairs $(a \in A, u \colon fa \to b \in B)$.

The first condition $gh_0 = k_0 f$ states that the underlying object assignments of the functors and cofunctors commute, while the second condition $h\varphi(a, u) = \gamma(ha, ku)$ states that the functors commute with the lifting operations on morphisms.

The double category $\mathbb{C}\mathrm{of}$ is a *strict* double category, as its horizontal composition (of functors) and vertical composition (of cofunctors) are both unital and associative. It is also a *flat* double category, as the cells in $\mathbb{C}\mathrm{of}$ are determined by their boundary. Let $\mathcal{C}\mathrm{of}$ be the category whose objects are cofunctors and whose morphisms are cells (3.1). The double category $\mathbb{C}\mathrm{of}$ corresponds to an *internal category* in $\mathcal{C}\mathrm{AT}$ given by the diagram:

$$\mathcal{C}\mathrm{of} \times_{\mathcal{C}\mathrm{at}} \mathcal{C}\mathrm{of} \xrightarrow{\text{comp}} \mathcal{C}\mathrm{of} \; \substack{\xrightarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xrightarrow[\text{cod}]{}} \; \mathcal{C}\mathrm{at}$$

There are a number of properties of cofunctors introduced in Chapter 2 which may be understood in terms of the double category $\mathbb{C}\mathrm{of}$. For example, the statement that every bijective-on-objects functor or discrete opfibration induces a unique cofunctor (Proposition 2.11) may be understood in terms of *companions* (Definition A.11) and *conjoints* (Definition A.12).

Proposition 3.2. *A functor has a vertical conjoint in* $\mathbb{C}\mathrm{of}$ *if and only if it is bijective-on-objects.*

Proof. Suppose that a functor $f\colon A \to B$ has a vertical conjoint $(g, \gamma)\colon B \to A$ in $\mathbb{C}\mathrm{of}$. Then there are cells in $\mathbb{C}\mathrm{of}$ of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (g,\gamma)} \\ A & = & A \end{array} \qquad\qquad \begin{array}{ccc} B & = & B \\ {\scriptstyle (g,\gamma)}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B \end{array}$$

These cells imply that $f_0 g = 1_{B_0}$ and $g f_0 = 1_{A_0}$, and thus f is bijective-on-objects.

Conversely, if $f\colon A \to B$ is bijective-on-objects, then there is a cofunctor consisting of the function $f_0^{-1}\colon B_0 \to A_0$ and the lifting operation which sends a pair $(b \in B, u\colon f_0^{-1} b \to a \in A)$ to the morphism $fu \in B$. It is straightforward to check that this cofunctor is a conjoint of f. □

Proposition 3.3. *A functor has a vertical companion in* $\mathbb{C}\mathrm{of}$ *if and only if it is a discrete opfibration.*

Proof. Suppose that a functor $f\colon A \to B$ has a vertical companion $(g, \gamma)\colon A \to B$ in $\mathbb{C}\mathrm{of}$. Then there are cells in $\mathbb{C}\mathrm{of}$ of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle (g,\gamma)}\downarrow & & \downarrow{\scriptstyle 1} \\ B & = & B \end{array} \qquad\qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (g,\gamma)} \\ A & \xrightarrow[f]{} & B \end{array}$$

These cells imply that $f_0 = g$, as well as that $f\gamma(a, u) = u$ and $\gamma(a, fw) = w$ hold for all pairs $(a \in A, u\colon fa \to b \in B)$ and morphisms $w\colon a \to a' \in A$. Since for all pairs $(a \in A, u\colon fa \to b \in B)$ there exists a unique morphism $\gamma(a, u)$ such that $f\gamma(a, u) = u$, the functor f is a discrete opfibration.

Conversely, if $f\colon A \to B$ is a discrete opfibration, then there is a cofunctor consisting of the function $f_0\colon A_0 \to B_0$ and the lifting operation which sends a pair $(a \in A, u\colon fa \to b \in B)$ to its corresponding unique lift. It is straightforward to check that this cofunctor is a companion to f. □

A double category is *horizontally invariant* if every invertible horizontal morphism has a conjoint, or equivalently, a companion. The invertible horizontal morphisms in $\mathbb{C}\mathrm{of}$ are precisely the isomorphisms of categories; these are simultaneously bijective-on-objects functors and discrete opfibrations. Using either of the above propositions gives the following statement.

Corollary 3.4. *The double category* $\mathbb{C}\mathrm{of}$ *is horizontally invariant.*

Another property of cofunctors explored in Chapter 2 was that every cofunctor $(f, \varphi)\colon A \to B$ determines a category $\Lambda(f, \varphi)$. This may now be understood in terms of a certain double-categorical limit (Definition A.19).

Proposition 3.5. *The double category* $\mathbb{C}\mathrm{of}$ *has tabulators.*

Proof. Using Proposition 2.10, given a cofunctor $(f, \varphi)\colon A \to B$ there is a cell in $\mathbb{C}\mathrm{of}$ given by:

$$\begin{array}{ccc} \Lambda(f,\varphi) & \xrightarrow{\varphi} & A \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle (f,\varphi)} \\ \Lambda(f,\varphi) & \xrightarrow[\overline{f}]{} & B \end{array}$$

To show that this cell has the universal property of the tabulator, consider a cell:

$$\begin{array}{ccc} X & \xrightarrow{h} & A \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle (f,\varphi)} \\ X & \xrightarrow[k]{} & B \end{array}$$

Its existence implies that $fh_0 = k_0$ and $hw = \varphi(hx, kw)$ for all $w\colon x \to x' \in X$. Then there exists a unique functor,

$$X \xrightarrow{j} \Lambda(f,\varphi)$$

$$\begin{array}{ccc} x & \cdots & hx \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle (hx,kw)} \\ x' & \cdots & hx' \end{array}$$

such that $\varphi j = h$ and $\overline{f} j = k$, so the required universal property holds. □

The following straightforward result implies that $\mathbb{C}\mathrm{of}$ is *unit-pure* (Definition A.17).

Lemma 3.6. *The tabulator of a vertical identity in* $\mathbb{C}\mathrm{of}$ *is a vertical identity cell.*

While there are several interesting properties of the double category of cofunctors, many of these will be easier to prove in later chapters once more robust tools have been developed. A final result which will be useful for this section relates conjoints, companions, and tabulators through the property of *strong tabulators* (Definition A.23).

Proposition 3.7. *The double category* $\mathbb{C}\mathrm{of}$ *has strong tabulators.*

Proof. To show $\mathbb{C}\mathrm{of}$ has strong tabulators, it is required that for every cofunctor $(f, \varphi) \colon A \to B$, the following cell is horizontally invertible,

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle 1} \\ \Lambda(f,\varphi) & \xrightarrow{\varphi} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (f,\varphi)} \\ \Lambda(f,\varphi) & \xrightarrow[\overline{f}]{} & B \\ {\scriptstyle \overline{f}_*}\downarrow & & \downarrow{\scriptstyle 1} \\ B & = & B \end{array} \tag{3.2}$$

where φ^* and $\overline{f}_*$ are the conjoint and companion of φ and $\overline{f}$, respectively. Using the construction of the conjoint and companion of a functor in Proposition 3.2 and Proposition 3.3, it is straightforward to prove that the vertical composite of φ^* followed by $\overline{f}_*$ is equal to (f, φ). Since $\mathbb{C}\mathrm{of}$ is a flat double category, the cell (3.2) must be a horizontal identity cell, and is therefore horizontally invertible. □

The primary consequence of this result is that every cofunctor has a canonical factorisation, up to isomorphism, into a conjoint (a bijective-on-objects cofunctor) followed by a companion (a discrete opfibration). In Section 3.4, this will be used to construct a double category equivalent to $\mathbb{C}\mathrm{of}$ whose vertical morphisms are certain spans of functors.

Recall that the goal of introducing the double category of cofunctors was that it could be used to construct the double category of lenses. This task will be undertaken in the next two sections, and is based on the following important observation.

Lemma 3.8. *A pair consisting of a functor $f\colon A \to B$ and a cofunctor $(f_0, \varphi)\colon A \to B$ is a lens if and only if there is a cell in $\mathbb{C}\mathrm{of}$ of the form:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle (f_0,\varphi)}\downarrow & & \downarrow{\scriptstyle 1} \\ B & = & B \end{array}$$

3.2 The right-connected completion

The goal of this section is to study a certain property of double categories called *right-connectedness*, which was first introduced by Bourke and Garner [BG16a].

Definition 3.9. A double category $\mathbb{D}$ is called *right-connected* if its identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is right adjoint to its codomain map $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$.

Dually, a double category $\mathbb{D}$ is called *left-connected* if its identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is left adjoint to its domain map $\mathrm{dom}\colon \mathcal{D}_1 \to \mathcal{D}_0$.

Every right-connected double category has cotabulators, (Definition A.20) and every left-connected double category has tabulators (Definition A.19).

In more detail, a double category is right-connected if for every vertical morphism $f\colon A \to\!\!\!\!\!\bullet\; B$ there is a specified cell,

$$\begin{array}{ccc} A & \xrightarrow{\widehat{f}} & B \\ {\scriptstyle f}\downarrow\!\bullet & \rho_f & \bullet\!\downarrow{\scriptstyle 1} \\ B & = & B \end{array}$$

such that for every cell α below, there is a unique factorisation:

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle 1} \\ B & \xrightarrow[k]{} & X \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{\widehat{f}} & B & \xrightarrow{k} & X \\ {\scriptstyle f}\downarrow\!\bullet & \rho_f & \bullet\!\downarrow & 1_k & \bullet\!\downarrow{\scriptstyle 1} \\ B & = & B & \xrightarrow[k]{} & X \end{array} \tag{3.3}$$

Each right-connected double category shares a close relationship with the double category of squares of its underlying category of objects.

Proposition 3.10. *Given a right-connected double category $\mathbb{D}$, there is a canonical strict double functor $\mathbb{D} \to \mathbb{S}\mathrm{q}(\mathcal{D}_0)$ with action on cells given by:*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle \widehat{g}} \\ B & \xrightarrow[k]{} & D \end{array}$$

Proof. An abstract proof is outlined in [BG16a, Section 3.2]. An explicit proof follows from direct application of the universal property (3.3). □

Let $\mathcal{R}\mathrm{cDBL}$ be the category of right-connected double categories and *unitary* double functors, that is to say, double functors which *strictly* preserve vertical identities. Let $(-)_0\colon \mathcal{R}\mathrm{cDBL} \to \mathcal{C}\mathrm{at}$ be the functor sending each double category $\mathbb{D}$ to its category of objects $\mathcal{D}_0$, and let $\mathbb{S}\mathrm{q}\colon \mathcal{C}\mathrm{at} \to \mathcal{R}\mathrm{cDBL}$ be the functor sending each category $\mathcal{C}$ to its (right-connected) double category of squares $\mathbb{S}\mathrm{q}(\mathcal{C})$.

Proposition 3.11. *The functor* $(-)_0\colon \mathcal{R}\mathrm{cDBL} \to \mathcal{C}\mathrm{at}$ *is left adjoint left inverse to the functor* $\mathbb{S}\mathrm{q}\colon \mathcal{C}\mathrm{at} \to \mathcal{R}\mathrm{cDBL}$*.*

Proof. It is immediate that the composite $(-)_0 \circ \mathbb{S}\mathrm{q}$ is the identity functor on $\mathcal{C}\mathrm{at}$, while the components of the unit of the adjunction are described by Proposition 3.10. It is straightforward to show these components form a natural transformation, and that the triangle identities of an adjunction hold. □

Given any double category, it is natural to wonder if it can be *completed* to a double category with a specified property. This question will now be answered for the property of right-connectedness.

Definition 3.12. The *right-connected completion* of a double category $\mathbb{D}$ is a double category $\Gamma(\mathbb{D})$ whose objects and horizontal arrows are those of $\mathbb{D}$, whose vertical arrows are triples (f, α, f') given by cells in $\mathbb{D}$ of the form,

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ {\scriptstyle f}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle 1} \\ B & = & B \end{array}$$

and whose cells are given by those cells θ in $\mathbb{D}$ which satisfy the following condition:

$$\begin{array}{ccccc} A & \xrightarrow{h} & C & \xrightarrow{g'} & D \\ {\scriptstyle f}\big\downarrow\!\bullet & \theta & {\scriptstyle g}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle 1} \\ B & \xrightarrow[k]{} & D & = & D \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{f'} & B & \xrightarrow{k} & D \\ {\scriptstyle f}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle 1} & 1_k & \bullet\!\big\downarrow{\scriptstyle 1} \\ B & = & B & \xrightarrow[k]{} & D \end{array}$$

The identity vertical morphism on an object A in $\Gamma(\mathbb{D})$ is the cell in $\mathbb{D}$ given by,

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\big\downarrow\!\bullet & 1_A & \bullet\!\big\downarrow{\scriptstyle 1} \\ A & = & A \end{array}$$

while composition of vertical arrows in $\Gamma(\mathbb{D})$ is the composite cell in $\mathbb{D}$ given by:

$$
\begin{array}{ccccc}
A & \xrightarrow{f'} & B & \xrightarrow{g'} & C \\
{\scriptstyle f}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle 1} & & \bullet\!\downarrow{\scriptstyle 1} \\
B & = & B & \xrightarrow{g'} & C \\
{\scriptstyle g}\downarrow\!\bullet & & \bullet\!\downarrow{\scriptstyle g} & \beta & \bullet\!\downarrow{\scriptstyle 1} \\
C & = & C & = & C
\end{array}
$$

Horizontal composition and identities correspond to those in $\mathbb{D}$.

The following result is evident from the construction of $\Gamma(\mathbb{D})$.

Lemma 3.13. *The right-connected completion of a double category is right-connected.*

There is a close relationship between a double category $\mathbb{D}$ and its right-connected completion $\Gamma(\mathbb{D})$ given by a faithful double functor.

Lemma 3.14. *Given a double category $\mathbb{D}$, there is a canonical strict double functor $\Gamma(\mathbb{D}) \to \mathbb{D}$ with action on cells given by:*

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (f,\alpha,f')}\downarrow\!\bullet & \theta & \bullet\!\downarrow{\scriptstyle (g,\beta,g')} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow\!\bullet & \theta & \bullet\!\downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
$$

Let $\mathcal{D}\mathrm{bl}_{\mathrm{unit}}$ be the category of double categories and unitary double functors. There is a fully faithful functor $\mathcal{R}\mathrm{cDBL} \to \mathcal{D}\mathrm{bl}_{\mathrm{unit}}$ which includes the right-connected double categories. The following result justifies $\Gamma(\mathbb{D})$ as the right-connected *completion* of $\mathbb{D}$.

Theorem 3.15. *The fully faithful functor $\mathcal{R}\mathrm{cDBL} \to \mathcal{D}\mathrm{bl}_{\mathrm{unit}}$ has a right adjoint given by the right-connected completion Γ.*

Proof. It is required to show that given a unitary double functor $F\colon \mathbb{C} \to \mathbb{D}$, where $\mathbb{C}$ is right-connected, and the canonical double functor $\Gamma(\mathbb{D}) \to \mathbb{D}$ defined in Lemma 3.14,

$$
\begin{array}{ccc}
 & & \Gamma(\mathbb{D}) \\
 & {\scriptstyle \overline{F}}\nearrow & \downarrow \\
\mathbb{C} & \xrightarrow[F]{} & \mathbb{D}
\end{array}
\tag{3.4}
$$

there exists a unique unitary double functor $\overline{F}$ such that the diagram commutes.

For the diagram (3.4) to commute, the functor $\overline{F}$ must agree with F on objects and arrows. Since $\mathbb{C}$ is right-connected and F is unitary, given a vertical morphism

$f\colon A \to\!\!\!\!\bullet\; B$ in $\mathbb{C}$ there is an assignment on cells,

$$
\begin{array}{ccc}
A & \xrightarrow{\hat{f}} & B \\
{\scriptstyle f}\big\downarrow\!\bullet & \rho_f & \bullet\!\big\downarrow{\scriptstyle 1} \\
B & = & B
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
FA & \xrightarrow{F\hat{f}} & FB \\
{\scriptstyle Ff}\big\downarrow\!\bullet & F\rho_f & \bullet\!\big\downarrow{\scriptstyle 1} \\
FB & = & FB
\end{array}
$$

that determines $\overline{F}(f) = (Ff, F\rho_f, Ff')$ on vertical morphisms. On cells $\overline{F}(\theta) = F(\theta)$ which is well-defined since $\mathbb{C}$ is right-connected. It is straightforward to show that the assignment $\overline{F}$ extends to a unitary double functor which makes the diagram (3.4) commute. □

The above theorem establishes the universal property of $\Gamma(\mathbb{D})$ as the right-connected completion of a double category $\mathbb{D}$. If $\mathbb{D}$ is already right-connected, then $\Gamma(\mathbb{D}) \cong \mathbb{D}$.

There are a number of convenient properties that $\Gamma(\mathbb{D})$ inherits from $\mathbb{D}$ which are collected below. Given their straightforward and elementary nature, the proofs are omitted.

Lemma 3.16. *If $\mathbb{D}$ is a flat double category, then $\Gamma(\mathbb{D})$ is a flat double category.*

Lemma 3.17. *A morphism has a vertical companion in $\Gamma(\mathbb{D})$ if and only it has a vertical companion in $\mathbb{D}$.*

Lemma 3.18. *Let $\mathbb{D}$ be a horizontally invariant double category. A morphism has a vertical conjoint in $\Gamma(\mathbb{D})$ if and only if it is horizontally invertible. Moreover, $\Gamma(\mathbb{D})$ is horizontally invariant.*

Lemma 3.19. *If $\mathbb{D}$ is a unit-pure double category with tabulators, then $\Gamma(\mathbb{D})$ is a unit-pure double category with tabulators.*

While this section has focused on right-connected double categories, all of the results hold similarly for left-connected double categories. In Chapter 4, the left-connected completion of the double category $\mathbb{S}\mathrm{pan}(\mathbb{S}\mathrm{et})$ will play an important role.

3.3 The double category of lenses

It is now time to introduce the main mathematical structure of the thesis.

Definition 3.20. Let $\mathbb{L}\mathrm{ens}$ be the *double category of lenses* whose objects are (small) categories, whose horizontal morphisms are functors, whose vertical morphisms are lenses, and whose cells,

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (f,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (g,\gamma)} \\
B & \xrightarrow[k]{} & D
\end{array}
\tag{3.5}
$$

are such that $gh = kf$ and $h\varphi(a, u) = \gamma(ha, ku)$ for all pairs $(a \in A, u\colon fa \to b \in B)$.

The double category $\mathbb{L}\mathrm{ens}$ is a *strict* double category, as its horizontal composition (of functors) and vertical composition (of lenses) are both unital and associative. Let $\mathcal{L}\mathrm{ens}$ be the category whose objects are lenses and whose morphisms are cells (3.5). The double category $\mathbb{L}\mathrm{ens}$ corresponds to an internal category in $\mathcal{CAT}$ given by the diagram:

$$\mathcal{L}\mathrm{ens} \times_{\mathcal{C}\mathrm{at}} \mathcal{L}\mathrm{ens} \xrightarrow{\text{comp}} \mathcal{L}\mathrm{ens} \underset{\text{cod}}{\overset{\text{dom}}{\substack{\longrightarrow \\ \xleftarrow{\ \text{id}\ } \\ \longrightarrow}}} \mathcal{C}\mathrm{at}$$

The following statement is the main result of the chapter, and is the basis on which generalisations of lenses in future chapters depends.

Theorem 3.21. *The double category of lenses* $\mathbb{L}\mathrm{ens}$ *is the right-connected completion of the double category of cofunctors* $\mathbb{C}\mathrm{of}$*.*

Proof. Using Definition 3.12, the right-connected completion $\Gamma(\mathbb{C}\mathrm{of})$ has categories as objects and functors as horizontal morphisms. The vertical morphisms in $\Gamma(\mathbb{C}\mathrm{of})$ are cells in $\mathbb{C}\mathrm{of}$, ensuring by definition that $f\varphi(a, u) = u$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle (f_0, \varphi)}\downarrow & & \downarrow{\scriptstyle 1} \\ B & = & B \end{array}$$

Thus by Proposition 2.4, the vertical morphisms are precisely lenses. Furthermore, the cells in $\Gamma(\mathbb{C}\mathrm{of})$ given by,

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f, \varphi)}\downarrow & & \downarrow{\scriptstyle (g, \gamma)} \\ B & \xrightarrow[k]{} & D \end{array}$$

are given by cells in $\mathbb{C}\mathrm{of}$,

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle (f_0, \varphi)}\downarrow & & \downarrow{\scriptstyle (g_0, \gamma)} \\ B & \xrightarrow[k]{} & D \end{array}$$

between their underlying cofunctors such that $kf = gh$. Finally, the composition and identities clearly correspond, therefore $\Gamma(\mathbb{C}\mathrm{of}) \cong \mathbb{L}\mathrm{ens}$. □

Conceptually, this theorem not only reaffirms that lenses consist of two parts, functors and cofunctors, but it also demonstrates that lenses are constructed from these parts in a universal way via the right-connected completion. Indeed, there are forgetful double functors:

$$\mathbb{C}\mathrm{of} \longleftarrow \mathbb{L}\mathrm{ens} \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$$

Moreover, Theorem 3.21 shows that many properties of the double category $\mathbb{L}\mathrm{ens}$ are inherited from the double category $\mathbb{C}\mathrm{of}$, or arise as properties which hold for any right-connected completion. This perspective will be utilised in future chapters where $\mathbb{C}\mathrm{of}$ is replaced by a more general, yet similar, double category $\mathbb{D}$, and the theory of $\Gamma(\mathbb{D})$ is studied.

Some of the immediate properties of $\mathbb{L}\mathrm{ens}$ include:

- $\mathbb{L}\mathrm{ens}$ is a flat double category (Lemma 3.16);
- A functor has a companion in $\mathbb{L}\mathrm{ens}$ if and only if it is a discrete opfibration (Lemma 3.17);
- A functor has a conjoint in $\mathbb{L}\mathrm{ens}$ if and only if it is an isomorphism (Lemma 3.18);
- $\mathbb{L}\mathrm{ens}$ is a horizontally invariant double category (Lemma 3.18);
- $\mathbb{L}\mathrm{ens}$ is a unit-pure double category with tabulators (Lemma 3.19).

Further properties of $\mathbb{L}\mathrm{ens}$ will established in the following chapters.

3.4 Span representability

A double category $\mathbb{D}$ is *span representable* if it has tabulators, its category of objects $\mathcal{D}_0$ has pullbacks, and the canonical lax double functor $\mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$, which sends a vertical morphism to the span constructed by its tabulator, is faithful [GP17].

Proposition 3.22. *The double category* $\mathbb{C}\mathrm{of}$ *is span representable.*

Proof. By Proposition 3.5, $\mathbb{C}\mathrm{of}$ has tabulators, and its category of objects is given by $\mathcal{C}\mathrm{at}$ which has pullbacks. Finally, since $\mathbb{C}\mathrm{of}$ is a flat double category, the lax double functor $\mathbb{C}\mathrm{of} \to \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$ is faithful. □

The goal of this section is to show that $\mathbb{C}\mathrm{of}$ satisfies a particularly strong version of span representability, defined as follows.

Definition 3.23. A double category $\mathbb{D}$ is *strongly span representable* if it has tabulators, its category of objects $\mathcal{D}_0$ has pullbacks, and the canonical lax double functor $\mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$ is strong and fully faithful.

Previous work of Niefield [Nie12] has shown that if a span representable double category has all companions and conjoints, then the lax double functor $\mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$ has a unitary oplax left adjoint. The thesis of Aleiferi [Ale18] studies further conditions on $\mathbb{D}$ for this adjunction to form an equivalence of double categories. The idea behind strongly span representable double categories $\mathcal{D}$ is that they are equivalent to full double subcategories of $\mathbb{S}\mathrm{pan}(\mathcal{D}_0)$, simply by restricting to the essential image of the canonical double functor $\mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$.

Theorem 3.24. *The double category* $\mathbb{C}\mathrm{of}$ *is strongly span representable.*

Proof. Consider the canonical double functor $\mathbb{C}\mathrm{of} \to \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$ whose assignment on cells is constructed using tabulators:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (f,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle (g,\gamma)} \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
A & \xrightarrow{\quad h \quad} & C \\
{\scriptstyle \varphi}\big\uparrow & & \big\uparrow{\scriptstyle \gamma} \\
\Lambda(f,\varphi) & \xrightarrow{\Lambda_{h,k}} & \Lambda(g,\gamma) \\
{\scriptstyle \overline{f}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{g}} \\
B & \xrightarrow[\quad k \quad]{} & D
\end{array}
\tag{3.6}
$$

Given the identity cofunctor $(1_{A_0}, \pi)$ on a category A, the category $\Lambda(1_{A_0}, \pi)$ is canonically isomorphic to A via the assignment-on-morphisms $\pi(a, w \colon a \to a') = w$. Thus there is a horizontally invertible cell:

$$
\begin{array}{ccc}
A & = \!= \!= \!= & A \\
{\scriptstyle \pi}\big\uparrow & & \big\uparrow{\scriptstyle 1_A} \\
\Lambda(1_{A_0}, \pi) & \xrightarrow{\pi} & A \\
{\scriptstyle \overline{1_{A_0}}}\big\downarrow & & \big\downarrow{\scriptstyle 1_A} \\
A & = \!= \!= \!= & A
\end{array}
$$

Let $(h, \psi) \colon A \to C$ denote the composite of $(f, \varphi) \colon A \to B$ and $(g, \gamma) \colon B \to C$, and consider the category $X := \Lambda(f, \varphi) \times_B \Lambda(g, \gamma)$ given by the pullback,

$$
\begin{array}{ccccccc}
 & & & X & & & \\
 & & \swarrow{\scriptstyle \pi_1} & \lrcorner & {\scriptstyle \pi_2}\searrow & & \\
 & \Lambda(f,\varphi) & & & & \Lambda(g,\gamma) & \\
{\scriptstyle \varphi}\swarrow & & \searrow{\scriptstyle \overline{f}} & & {\scriptstyle \gamma}\swarrow & & \searrow{\scriptstyle \overline{g}} \\
A & & & B & & & C
\end{array}
$$

whose objects may be taken to be those of A and whose morphisms are pairs of pairs

$((a, \gamma(fa, u)), (fa, u))$ with respect to the following illustration:

$$
\begin{array}{ccccc}
A & \qquad & a & \xrightarrow{\varphi(a, \gamma(fa, u))} & a' \\
{\scriptstyle (f,\varphi)}\downarrow & & \vdots & & \vdots \\
B & & fa & \xrightarrow{\gamma(fa, u)} & b \\
{\scriptstyle (g,\gamma)}\downarrow & & \vdots & & \vdots \\
C & & gfa & \xrightarrow{u} & c
\end{array}
$$

On the other hand, the category $\Lambda(h, \psi)$ has the same objects as A and morphisms given by pairs $(a, u\colon gfa \to c)$. There is a canonical isomorphism $X \cong \Lambda(h, \psi)$ with assignment-on-morphisms $((a, \gamma(fa, u)), (fa, u)) \mapsto (a, u)$. Thus there is a horizontally invertible cell:

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle \varphi\pi_1}\uparrow & & \uparrow{\scriptstyle \psi} \\
X & \xrightarrow{\cong} & \Lambda(h, \psi) \\
{\scriptstyle \overline{g}\pi_2}\downarrow & & \downarrow{\scriptstyle \overline{h}} \\
C & = & C
\end{array}
$$

This shows that the canonical lax double functor $\mathbb{C}\mathrm{of} \to \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$ is a pseudo double functor.

The double functor $\mathbb{C}\mathrm{of} \to \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$ is full if for every commutative diagram,

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \gamma} \\
\Lambda(f, \varphi) & \xrightarrow{j} & \Lambda(g, \gamma) \\
{\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\
B & \xrightarrow[k]{} & D
\end{array}
\tag{3.7}
$$

there is a cell in $\mathbb{C}\mathrm{of}$ of the form:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (f,\varphi)}\downarrow & & \downarrow{\scriptstyle (g,\gamma)} \\
B & \xrightarrow[k]{} & D
\end{array}
$$

Commutativity of (3.7) determines the functor j, since necessarily $j_0 = h_0$ and $j(a, u) = (ha, ku)$ for each morphism $(a, u) \in \Lambda(f, \varphi)$. Furthermore, since $\gamma j = h\varphi$, it follows that $\gamma(ha, ku) = h\varphi(a, u)$. Therefore (3.7) determines a corresponding cell in $\mathbb{C}\mathrm{of}$. $\square$

What is the essential image of the fully faithful double functor $\mathbb{C}\mathrm{of} \to \mathbb{S}\mathrm{pan}(\mathcal{C}\mathrm{at})$? Suppose $\mathcal{W}$ and $\mathcal{M}$ are classes of functors which contain the isomorphisms, are closed under composition, and stable under pullback. Let $\mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$ be the full double subcategory of $\mathbb{S}\text{pan}(\mathcal{C}\text{at})$ determined by spans whose left leg is in $\mathcal{W}$ and whose right leg is in $\mathcal{M}$.

Proposition 3.25. *There is an equivalence of double categories,*

$$\mathbb{C}\text{of} \simeq \mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$$

where $\mathcal{W}$ is the class of bijective-on-objects functors, and $\mathcal{M}$ is the class of discrete opfibrations.

Proof. There is a pseudo double functor $\mathbb{C}\text{of} \to \mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$ defined by (3.6), and it is fully faithful by Theorem 3.24. Moreover, it is essentially surjective on vertical morphisms by Proposition 2.11. Therefore, $\mathbb{C}\text{of} \simeq \mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$ as required. □

The central benefit of this result is that it facilitates an entirely diagrammatic approach to cofunctors and the double category $\mathbb{C}\text{of}$, utilising basic properties of bijective-on-objects functors and discrete opfibrations. It is worth noting that although $\mathbb{C}\text{of}$ is a strict double category, $\mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$ is only a pseudo double category. The equivalence in Proposition 3.25 may be made an isomorphism of double categories through taking isomorphism classes of spans in $\mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M})$, however this overly-strict approach will not be used in this thesis.

What does Proposition 3.25 reveal about the double category of lenses? Consider the right-connected completion $\Gamma(\mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M}))$ whose vertical morphisms are commutative triangles of functors,

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\varphi} & & \searrow^{f\varphi} & \\ A & & \xrightarrow[f]{} & & B \end{array}$$

where φ is a bijective-on-objects functor and $f\varphi$ is a discrete opfibration, and whose cells are commutative diagrams of the form:

$$\begin{array}{ccccccc} & & A & \xrightarrow{h} & C & & \\ & \nearrow^{\varphi} & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & \nwarrow^{\gamma} & \\ X & & \longrightarrow & \longrightarrow & & & Y \\ & \searrow_{f\varphi} & \downarrow & & \downarrow & \swarrow_{g\gamma} & \\ & & B & \xrightarrow[k]{} & D & & \end{array} \tag{3.8}$$

Theorem 3.26. *There is an equivalence of double categories,*

$$\mathbb{L}\text{ens} \simeq \Gamma(\mathbb{S}\text{pan}(\mathcal{C}\text{at}, \mathcal{W}, \mathcal{M}))$$

where $\mathcal{W}$ is the class of bijective-on-objects functors, and $\mathcal{M}$ is the class of discrete opfibrations.

Proof. Follows immediately from Theorem 3.21 and Proposition 3.25. □

This theorem is the cornerstone for almost every major result in this thesis, and is the basis for a generalised diagrammatic approach to lenses in Chapter 6. Given the ubiquity of these results, the equivalences in Proposition 3.25 and Theorem 3.26 will often be used without remark and treated like identities.

3.5 Split opfibrations via décalage

Split opfibrations are perhaps the most important example of lenses, and it is interesting to find new ways of characterising when a lens is a split opfibration. The standard definition of split opfibrations uses *opcartesian morphisms.* The goal of this section is to characterise split opfibrations as lenses satisfying a certain diagrammatic property with respect to the *décalage* construction.

Definition 3.27. The *décalage* construction is an endofunctor $D \colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ which assigns each category A to the coproduct of its slice categories:

$$DA = \sum_{a \in A} A/a$$

In detail, given a category A the décalage DA is a category whose objects are morphisms in A, and whose morphisms $u \colon w \to v$ are commutative triangles:

$$\begin{array}{ccc} x & \xrightarrow{\;u\;} & y \\ & {\scriptstyle w}\searrow \quad \swarrow{\scriptstyle v} & \\ & a & \end{array} \tag{3.9}$$

There is a canonical natural transformation which makes the décalage construction into a copointed endofunctor:

$$\mathcal{C}\mathrm{at} \underset{1}{\overset{D}{\rightrightarrows}} \mathcal{C}\mathrm{at} \quad \Downarrow \varepsilon$$

The component functors $\varepsilon_A \colon DA \to A$ assign each object in DA, given by a morphism in A, to its domain. A morphism (3.9) in DA is sent by ε_A to the morphism $u \in A$.

Given a functor $f \colon A \to B$ which has the structure of a lens, under what conditions does the functor $Df \colon DA \to DB$ have the structure of a lens? Consider the following commutative diagram of functors,

$$\begin{array}{ccccc} X \times_A DA & \xrightarrow{\pi_2} & DA & \xrightarrow{Df} & DB \\ \downarrow{\scriptstyle \pi_1} & \lrcorner & \downarrow{\scriptstyle \varepsilon_A} & & \downarrow{\scriptstyle \varepsilon_B} \\ X & \xrightarrow[\varphi]{} & A & \xrightarrow[f]{} & B \end{array} \tag{3.10}$$

where φ is bijective-on-objects and $f\varphi$ is a discrete opfibration, therefore yielding a lens $(f, \varphi)\colon A \to B$. Note that since bijective-on-objects functors are stable under pullback, the functor π_2 is bijective-on-objects. By Proposition 2.13, the only obstruction to Df being a lens is that the composite functor $Df \circ \pi_2$ may not be a discrete opfibration. Surprisingly, this is the same obstruction to the lens (f, φ) being a split opfibration.

Theorem 3.28. *A lens $(f, \varphi)\colon A \to B$ is a split opfibration if and only if the functor $Df \circ \pi_2 \colon X \times_A DA \to DB$ in (3.10) is a discrete opfibration.*

Proof. Without loss of generality, assume that φ is identity-on-objects. Up to isomorphism, the category $X \times_A DA$ has the same objects as DA given by morphisms in A. The morphisms in $X \times_A DA$ are given by commutative diagrams (3.9) such that u is a chosen lift of the lens (f, φ).

The functor $Df \circ \pi_2$ is a discrete opfibration if and only if given a morphism $w\colon a \to a''$ in A (an object of $X \times_A DA$) and a pair of morphisms $u\colon fa \to b$ and $v\colon b \to fa''$ in B such that $fw = v \circ u$ (a morphism of DB with domain $(Df \circ \pi_2)(w)$) as shown below,

$$\begin{array}{ccc} a & \xrightarrow{\varphi(a,u)} & p(a,u) \\ & {\scriptstyle w}\searrow \quad \swarrow{\scriptstyle \hat{v}} & \\ & a'' & \end{array} \qquad\qquad \begin{array}{ccc} fa & \xrightarrow{u} & b \\ & {\scriptstyle fw}\searrow \quad \swarrow{\scriptstyle v} & \\ & fa'' & \end{array}$$

there is a unique morphism $\hat{v}\colon p(a, u) \to a''$ such that $\hat{v} \circ \varphi(a, u) = w$ and $f\hat{v} = v$ (a unique lift). However, this condition holds if and only if each $\varphi(a, u)$ is an opcartesian morphism. Since a lens (f, φ) is a split opfibration if and only if each chosen lifting $\varphi(a, u)$ is opcartesian, this completes the proof. $\square$

This characterisation of split opfibrations is completely diagrammatic, in the sense that it utilises closure properties of certain classes of functors in $\mathfrak{Cat}$. The following result answers the question of when the décalage of a functor is a lens.

Corollary 3.29. *If* $(f, \varphi)\colon A \to B$ *is a split opfibration, then* $Df\colon DA \to DB$ *has the structure of a lens, and there is a cell in* $\mathbb{L}\mathrm{ens}$ *given by:*

$$\begin{array}{ccc} DA & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle (Df, \pi_2)}\big\downarrow & & \big\downarrow{\scriptstyle (f, \varphi)} \\ DB & \xrightarrow[\varepsilon_B]{} & B \end{array} \tag{3.11}$$

Using the characterisation in Theorem 3.28, it may be shown that split opfibrations compose by just using basic properties of discrete opfibrations. Let $\mathbb{S}\mathrm{Opf}$ denote the full double subcategory of $\mathbb{L}\mathrm{ens}$ on the vertical morphisms which are split opfibrations, and let $\mathcal{S}\mathrm{Opf}$ be its category of morphisms. The décalage construction $D\colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ extends to a functor $D\colon \mathcal{S}\mathrm{Opf} \to \mathcal{L}\mathrm{ens}$ with a natural transformation to the inclusion $\mathcal{S}\mathrm{Opf} \hookrightarrow \mathcal{L}\mathrm{ens}$ whose components are the cells (3.11). Chapter 4 further studies the relationship between lenses and split opfibrations from the perspective of the Grothendieck construction.

Chapter 4

Lenses as lax double functors

The aim of this chapter is to establish an equivalence between lenses *into* a category B, and lax double functors *out of* a double category $\mathbb{V}(B)$. More formally, there is an equivalence of categories whose objects are lenses and lax double functors, respectively.

Let $\mathcal{L}\mathrm{ens}_B$ be the fibre over B of the codomain map $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \to \mathfrak{C}\mathrm{at}$ of the double category of lenses. Let $[\mathbb{C}, \mathbb{D}]_{\mathrm{lax}}$ be the category of lax double functors from $\mathbb{C}$ to $\mathbb{D}$ and horizontal transformations. Let $\mathbb{V}(B)$ be the vertical double category of B (Example A.10). The main theorem of this chapter (Theorem 4.14) establishes an equivalence of categories,

$$\mathcal{L}\mathrm{ens}_B \simeq [\mathbb{V}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$$

between the category $\mathcal{L}\mathrm{ens}_B$ and the category of lax double functors from $\mathbb{V}(B)$ into the double category $\mathbb{S}\mathrm{Mult}$ of *split multi-valued functions*.

The proof of the main result first involves introducing the double category $\mathbb{S}\mathrm{Mult}$, and understanding its universal property as the left-connected completion of the double category of spans. Section 4.1 begins the chapter with a review of the left-connected completion as the formal dual to the right-connected completion (first introduced in Section 3.2). For a double category equipped with a functorial choice of companions, the left-connected completion is shown to have both a new 1-dimensional universal property (Theorem 4.4) and a new 2-dimensional universal property (Theorem 4.6) with respect to globular transformations. In Section 4.2, the left-connected completion is applied to the double category $\mathbb{S}\mathrm{pan}$, which has a functorial choice of companions, to yield the double category of split multi-valued functions. Using the universal property of $\mathbb{S}\mathrm{Mult}$, the proof of the main result (Theorem 4.14) is completed in Section 4.3.

The chapter concludes with an application of the main result to understanding several kinds of lenses. Section 4.4 focuses on a new characterisation of split opfibrations as lax double functors with a simple property, while Section 4.5 investigates various classes of lenses which arise naturally from a fibred perspective.

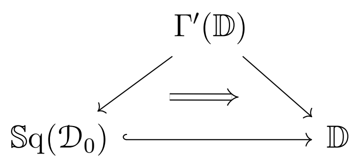
4.1 Companions and the left-connected completion

The goal of this section is to study the left-connected completion of a double category equipped with a functorial choice of companions.

Companions (Definition A.11) are one of the most useful concepts in double category theory, and many double categories enjoy the property that every horizontal morphism has a vertical companion. A double category $\mathbb{D}$ equipped with a *functorial choice of companions* expresses this property via structure, and induces a strict double functor $\mathbb{S}\text{q}(\mathcal{D}_0) \to \mathbb{D}$ as shown in Lemma 4.1. In comparison, *left-connectness* is a property of a double category (Definition 3.9) which induces a strict double functor $\mathbb{D} \to \mathbb{S}\text{q}(\mathcal{D}_0)$ without any additional choices (Proposition 3.10). Perhaps surprisingly, these two properties of a double category are closely related (Proposition 4.2) via an adjunction:

$$\mathbb{S}\text{q}(\mathcal{D}_0) \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} \mathbb{D} \tag{4.1}$$

The *left-connected completion* (Definition 4.3) completes a double category $\mathbb{D}$ to a double category $\Gamma'(\mathbb{D})$ under the property of left-connectedness. As a formal dual to right-connected completion which was first introduced in Section 3.2, the left-connected completion enjoys an analogous universal property (Theorem 3.15) and comes equipped with a strict double functor $\Gamma'(\mathbb{D}) \to \mathbb{D}$ (Lemma 3.14). In the case that $\mathbb{D}$ has a functorial choice of companions, whiskering the counit of the adjunction (4.1) with the double functor $\Gamma'(\mathbb{D}) \to \mathbb{D}$ yields a globular transformation of double functors:



The main result of this section is to show that the above transformation, constructed from the left-connected completion of a double category equipped with a functorial choice of companions, enjoys both a 1-*dimensional* universal property (Theorem 4.4) and a 2-*dimensional* universal property (Theorem 4.6). These universal properties are stronger than the standard universal property of the left-connected completion which does not assume a functorial choice of companions (Proposition 4.5), and are essential for proving the main result of the chapter in Section 4.3.

Without further ado, recall (Definition A.11) that a double category is equipped with a *functorial choice of companions* if every horizontal morphism $f\colon A \to B$ has a chosen vertical companion $f_*\colon A \mathrel{-\!\!\bullet\!\!\to} B$, and the equations $(1_A)_* = 1_A$ and $(gf) = g_* f_*$ hold.

Lemma 4.1. *If $\mathbb{D}$ is a double category equipped with a functorial choice of companions, then there is a strict double functor $(-)_*\colon \mathbb{S}\mathrm{q}(\mathcal{D}_0) \to \mathbb{D}$ with an assignment on cells given by:*

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccccc}
A & \xrightarrow{h} & A & = & C \\
{\scriptstyle 1}\downarrow & 1_h & \downarrow{\scriptstyle 1} & \diamondsuit & \downarrow{\scriptstyle g_*} \\
A & \xrightarrow[h]{} & C & \xrightarrow[g]{} & D \\
{\scriptstyle 1}\downarrow & & 1 & & \downarrow{\scriptstyle 1} \\
A & \xrightarrow{f} & B & \xrightarrow{k} & D \\
{\scriptstyle f_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} & 1_k & \downarrow{\scriptstyle 1} \\
B & = & B & \xrightarrow[k]{} & D
\end{array}
\tag{4.2}
$$

Proof. Follows from routine application of the pasting conditions for companions. □

Recall that a double category $\mathbb{D}$ is *left-connected* (Definition 3.9) if its identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is left adjoint to its domain map $\mathrm{dom}\colon \mathcal{D}_1 \to \mathcal{D}_0$. In elementary terms, a double category is left-connected if for every vertical morphism $f\colon A \mathrel{-\!\!\bullet\!\!\to} B$ there exists a cell,

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle 1}\downarrow & \lambda_f & \downarrow{\scriptstyle f} \\
A & \xrightarrow[\widehat{f}]{} & B
\end{array}
$$

such that for every cell α there is a unique factorisation:

$$
\begin{array}{ccc}
X & \xrightarrow{h} & A \\
{\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle f} \\
X & \xrightarrow[k]{} & B
\end{array}
\quad = \quad
\begin{array}{ccccc}
X & \xrightarrow{h} & A & = & A \\
{\scriptstyle 1}\downarrow & 1_h & \downarrow{\scriptstyle 1} & \lambda_f & \downarrow{\scriptstyle f} \\
X & \xrightarrow[h]{} & A & \xrightarrow[\widehat{f}]{} & B
\end{array}
\tag{4.3}
$$

By the dual of Proposition 3.10, every left-connected double category admits a strict double functor $\widehat{(-)}\colon \mathbb{D} \to \mathbb{S}\mathrm{q}(\mathcal{D}_0)$ with an assignment on cells given by:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle \widehat{g}} \\
B & \xrightarrow[k]{} & D
\end{array}
$$

There is an interesting similarity between double categories with companions, and left-connected double categories. Both involve cells $\diamondsuit$ or λ, linking a vertical morphism with a horizontal morphism, which satisfy a universal property (A.1) or (4.3). The following result explains the relationship between these properties of double categories.

Proposition 4.2. *Given a left-connected double category $\mathbb{D}$ with a functorial choice of companions, there is an adjunction of double categories:*

$$\mathbb{S}\mathrm{q}(\mathcal{D}_0) \underset{(-)_*}{\overset{\widehat{(-)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbb{D}$$

Proof. Given a horizontal morphism $f\colon A \to B$ with a vertical companion $f_*\colon A \multimap B$, one may observe from the universal property of left-connectedness (4.3) applied to the cell $\Diamond$ that:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \lambda_{f_*} & \downarrow{\scriptstyle f_*} \\ A & \xrightarrow[\widehat{f_*}]{} & B \end{array} \quad = \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \Diamond & \downarrow{\scriptstyle f_*} \\ A & \xrightarrow[f]{} & B \end{array}$$

Therefore, the composite of $(-)_*$ followed by $\widehat{(-)}$ is the identity functor on $\mathbb{S}\mathrm{q}(\mathcal{D}_0)$, and the unit of the adjunction is taken to be the identity transformation.

Given a vertical morphism $g\colon A \multimap B$, the universal property of the companion (A.1) applied to the cell λ_g determines the component of the counit ϵ:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \lambda_g & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\widehat{g}]{} & B \end{array} \quad = \quad \begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle 1}\downarrow & \Diamond & \downarrow{\scriptstyle \widehat{g}_*} & \epsilon_g & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\widehat{g}]{} & B & = & B \end{array}$$

It is straightforward to show that this defines an adjunction of double categories. □

Note that the counit ϵ is a *globular transformation* of double functors, since its components are globular cells. This is important for characterising the left-connected completion of a double category with companions in Theorem 4.4.

Definition 4.3. The *left-connected completion* of a double category $\mathbb{D}$ is the double category $\Gamma'(\mathbb{D})$ whose objects and horizontal morphisms are those of $\mathbb{D}$, whose vertical morphisms $(f, \alpha, f')\colon A \multimap B$ are cells in $\mathbb{D}$ of the form,

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f']{} & B \end{array}$$

and whose cells θ are those of $\mathbb{D}$ which satisfy the following condition:

$$\begin{array}{ccccc} A & = & A & \xrightarrow{h} & C \\ {\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle f} & \theta & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f']{} & B & \xrightarrow[k]{} & D \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{h} & C & = & C \\ {\scriptstyle 1}\downarrow & 1_h & \downarrow{\scriptstyle 1} & \beta & \downarrow{\scriptstyle g} \\ A & \xrightarrow[h]{} & C & \xrightarrow[g']{} & D \end{array}$$

The vertical identities and composition are analogous to those in Definition 3.12.

The left-connected completion of a double category $\mathbb{D}$ is equipped with a canonical strict double functor $R\colon \Gamma'(\mathbb{D}) \to \mathbb{D}$ by duality with the right-connected completion (Lemma 3.14). It is also easy to see that if $\mathbb{D}$ is equipped with a functorial choice of companions, then $\Gamma'(\mathbb{D})$ is also equipped with a functorial choice of companions. Whiskering with the counit ϵ of the adjunction in Proposition 4.2 with the double functor R yields a globular transformation $\phi \coloneqq \epsilon R$ of strict double functors:

$$\begin{tikzcd} & \Gamma'(\mathbb{D}) \arrow[dl, "L"'] \arrow[dr, "R"] \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[rr, hook, "(-)_*"'] & & \mathbb{D} \end{tikzcd} \tag{4.4}$$

The following theorem shows that the cell (4.4), constructed from the left-connected completion of any double category equipped with a functorial choice of companions, enjoys a 1-dimensional universal property with respect to globular transformations between lax double functors.

Theorem 4.4. *Let $\mathbb{D}$ be a double category with a functorial choice of companions. Given a globular transformation between lax double functors,*

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\psi}{\Longrightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[rr, hook, "(-)_*"'] & & \mathbb{D} \end{tikzcd} \tag{4.5}$$

there exists a unique lax double functor $F\colon \mathbb{C} \to \Gamma'(\mathbb{D})$ such that $\phi F = \psi$, $LF = F_1$ and $RF = F_2$.

Proof. Since ψ is a globular transformation and $(-)_*$ is the identity on objects and horizontal morphisms, the lax double functors F_1 and F_2 agree on objects and horizontal morphisms; denote this common assignment by F. For a vertical morphism $f\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ in $\mathbb{C}$, define vertical morphism in $\Gamma'(\mathbb{D})$ by the cell:

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "1"'] & FA \arrow[r, equal] \arrow[d, "(F_1 f)_*" description] & FA \arrow[d, "F_2 f"] \\ FA \arrow[r, "F_1 f"'] \arrow[ur, phantom, "\Diamond"] & FB \arrow[r, equal] \arrow[ur, phantom, "\psi_f"] & FB \end{tikzcd}$$

Given a cell α in $\mathbb{C}$, naturality of ψ yields the equation:

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d, "(F_1 f)_*"'] & FA \arrow[r, "Fh"] \arrow[d, "F_2 f" description] & FC \arrow[d, "F_2 g"] \\ FB \arrow[r, equal] \arrow[ur, phantom, "\psi_f"] & FB \arrow[r, "Fk"'] \arrow[ur, phantom, "F_2\alpha"] & FD \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "(F_1 f)_*"'] & FC \arrow[r, equal] \arrow[d, "(F_1 g)_*" description] & FC \arrow[d, "F_2 g"] \\ FB \arrow[r, "Fk"'] \arrow[ur, phantom, "(F_1\alpha)_*"] & FD \arrow[r, equal] \arrow[ur, phantom, "\psi_g"] & FD \end{tikzcd}$$

Pasting the cell $\diamondsuit$ for the companion of $F_1 f$ on both sides of the equation above, and simplifying by noting that $(F_1\alpha)_*$ is of the form (4.2), yields the following equation:

$$\begin{array}{ccccccc} FA & = & FA & = & FA & \xrightarrow{Fh} & FC \\ {\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle (F_1 f)_*} & \psi_f & \downarrow{\scriptstyle F_2 f} & F_2\alpha & \downarrow{\scriptstyle F_2 g} \\ FA & \xrightarrow[F_1 f]{} & FB & = & FB & \xrightarrow[Fk]{} & FD \end{array}$$

$$= \quad \begin{array}{ccccccc} FA & \xrightarrow{Fh} & FC & = & FC & = & FC \\ {\scriptstyle 1}\downarrow & 1_{Fh} & \downarrow{\scriptstyle 1} & \diamondsuit & \downarrow{\scriptstyle (F_1 g)_*} & \psi_g & \downarrow{\scriptstyle F_2 g} \\ FA & \xrightarrow[Fh]{} & FC & \xrightarrow[F_1 g]{} & FD & = & FD \end{array}$$

This shows that there is a well-defined assignment on cells defined by sending a cell α in $\mathbb{C}$ to the cell $F_2\alpha$ in $\Gamma'(\mathbb{D})$.

Altogether, the data above define a lax double functor $F\colon \mathbb{C} \to \Gamma'(\mathbb{D})$, whose laxity comparison cells are inherited from F_2. While it is tricky to prove concisely in the our chosen notation, the universal property of companions can be used to show that $\phi F = \psi$, while $LF = F_1$ and $RF = F_2$ by the construction of F. □

The universal property of (4.4) presented in Theorem 4.4 completely characterises the left-connected completion among double categories with companions. The following is the analogous statement to Proposition 3.15 for this setting.

Proposition 4.5. *Let $\mathbb{C}$ and $\mathbb{D}$ be double categories with a functorial choice of companions. If $\mathbb{C}$ is left-connected, then for every unitary double functor $F\colon \mathbb{C} \to \mathbb{D}$ there exists a unique unitary double functor $\overline{F}\colon \mathbb{C} \to \Gamma'(\mathbb{D})$ such that $R\overline{F} = F$.*

Proof. There is a commutative square of unitary double functors given by:

$$\begin{array}{ccc} \mathbb{S}\mathrm{q}(\mathcal{C}_0) & \xhookrightarrow{(-)_*} & \mathbb{C} \\ {\scriptstyle \mathbb{S}\mathrm{q}(F_0)}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) & \xhookrightarrow[(-)_*]{} & \mathbb{D} \end{array}$$

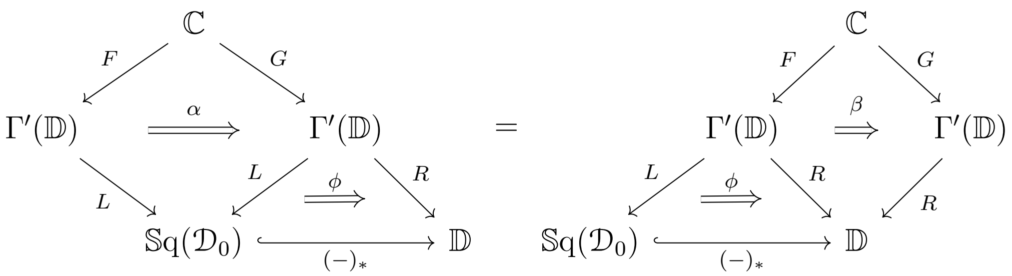
Globular transformations are closed under whiskering with unitary double functors, and pasting the above square with the counit ϵ of the adjunction in Proposition 4.2 applied to $\mathbb{C}$ yields a globular transformation:

$$\begin{array}{ccc} & \mathbb{C} & \\ \swarrow & \overset{F\epsilon}{\Longrightarrow} & \searrow{\scriptstyle F} \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) & \xhookrightarrow[(-)_*]{} & \mathbb{D} \end{array} \tag{4.6}$$

Finally, applying Theorem 4.4 yields the desired result. □

In addition to the 1-dimensional universal property presented in Theorem 4.4, the globular transformation (4.4) also has a 2-dimensional universal property with respect to horizontal transformations between lax double functors. The proof is very similar to Theorem 4.4 and is omitted.

Theorem 4.6. *Let $\mathbb{D}$ be a double category equipped with a functorial choice of companions. Given lax double functors F and G, and horizontal transformations α and β such that,*



(4.7)

there is a unique horizontal transformation $\delta\colon F \Rightarrow G$ such that $L\delta = \alpha$ and $R\delta = \beta$.

4.2 Split multi-valued functions

The goal of this section is to introduce the double category of split multi-valued functions (Definition 4.8), and show that it is isomorphic to the the left-connected completion of the double category of spans (Proposition 4.9). Although only the universal property of the induced globular transformation (Corollary 4.10) is required for the main result in Section 4.3, the section begins with a detailed examination of split multi-valued functions as they are used for applications in Section 4.4 and Section 4.5.

Definition 4.7. A *split multi-valued function* is span of functions whose left leg has a chosen section (or right inverse):

$$A \rightleftarrows X \longrightarrow B \tag{4.8}$$

A *multi-valued function* is a span of functions whose left leg is an epimorphism.

Remark. There are two important ways in which the definition of (split) multi-valued function is more general than one might expect. First, it is not required that the underlying span be jointly monic, so not every multi-valued function is a relation. Second, isomorphism classes of spans are not used, so composition of multi-valued functions is not strictly associative.

The identity split multi-valued function on a set A is given by the diagram:

$$A \underset{1_A}{\overset{1_A}{\rightleftarrows}} A \xrightarrow{1_A} A$$

Given a composable pair of split multi-valued functions $A \nrightarrow B$ and $B \nrightarrow C$, their composite is given by pullback:

$$\begin{array}{ccccccccc} & & & & X \times_B Y & & & & \\ & & & {\scriptstyle\langle 1,\gamma t\rangle}\nearrow\swarrow{\scriptstyle\pi_X} & \lrcorner & \searrow{\scriptstyle\pi_Y} & & & \\ & & X & & & & Y & & \\ & {\scriptstyle\varphi}\nearrow\swarrow{\scriptstyle s} & & \searrow{\scriptstyle t} & & {\scriptstyle\gamma}\nearrow\swarrow{\scriptstyle p} & & \searrow{\scriptstyle q} & \\ A & & & & B & & & & C \end{array}$$

Composition is well-defined since split epimorphisms are closed under composition and stable under pullback.

Let $\mathbb{S}\mathrm{pan}$ denote the double category of sets, functions, and spans (Example A.4). A split multi-valued function (4.8) is the same as a particular cell in $\mathbb{S}\mathrm{pan}$ of the form:

$$\begin{array}{ccc} A & = & A \\ \| & & \uparrow \\ A & \rightarrowtail & X \\ \| & & \downarrow \\ A & \longrightarrow & B \end{array} \qquad (4.9)$$

Definition 4.8. Let $\mathbb{S}\mathrm{Mult}$ be the *double category of split multi-valued functions* whose objects are sets, horizontal morphisms are functions, vertical morphisms are split multi-valued functions (4.8) and cells are commutative diagrams of functions,

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle\varphi} & & {\scriptstyle p}\uparrow\downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{j} & Y \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[k]{} & D \end{array} \qquad (4.10)$$

where $s\varphi = 1_A$, $p\gamma = 1_C$, $pj = hs$, $qj = kt$, and $\gamma h = j\varphi$. Horizontal identities and composition are given by pasting these commutative diagrams, while vertical identities and composition are defined as above.

The following result shows that the double category of split multi-valued functions is the left-connected completion of the double category of spans.

Proposition 4.9. *There is an isomorphism of double categories* $\mathbb{S}\mathrm{Mult} \cong \Gamma'(\mathbb{S}\mathrm{pan})$.

Proof. The objects and horizontal morphisms of both double categories are the same, and from (4.9) a split multi-valued function is the same as a vertical morphism in $\Gamma'(\mathbb{S}\mathrm{pan})$. By Definition 4.3, a cell in the left-connected completion $\Gamma'(\mathbb{S}\mathrm{pan})$ is given by a cell in $\mathbb{S}\mathrm{pan}$ satisfying the equation,

$$\begin{array}{ccccc} A & = & A & \xrightarrow{h} & C \\ \| & & \uparrow s & & \uparrow p \\ A & \xrightarrow{\varphi} & X & \xrightarrow{j} & Y \\ \| & & \downarrow t & & \downarrow q \\ A & \xrightarrow[t\varphi]{} & B & \xrightarrow[k]{} & D \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{h} & C & = & C \\ \| & & \| & & \uparrow p \\ A & \xrightarrow{h} & C & \xrightarrow{\gamma} & Y \\ \| & & \| & & \downarrow q \\ A & \xrightarrow[h]{} & C & \xrightarrow[q\gamma]{} & D \end{array} \tag{4.11}$$

which is the same as a cell (4.10). This demonstrates that the $\mathbb{S}\mathrm{Mult}$ and $\Gamma'(\mathbb{S}\mathrm{pan})$ have the same objects, horizontal morphisms, vertical morphisms, and cells. It is straight-forward to show that the identities and composition also coincide. □

The double category of spans is equipped with a functorial choice of companions (Example A.13), and therefore its left-connected completion induces a globular cell (4.4) whose universal properties will be used in the proof of main result of this chapter.

Corollary 4.10. *There is a globular transformation of strict double functors,*

$$\begin{array}{ccc} & \mathbb{S}\mathrm{Mult} & \\ {}^{L}\swarrow & \overset{\phi}{\Longrightarrow} & \searrow^{R} \\ \mathbb{S}\mathrm{q}(\mathbb{S}\mathrm{et}) & \xhookrightarrow[(-)_*]{} & \mathbb{S}\mathrm{pan} \end{array} \tag{4.12}$$

whose component at a split multi-valued function is given by the cell:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1_A}\uparrow & & \uparrow s \\ A & \xrightarrow{\varphi} & X \\ {\scriptstyle t\varphi}\downarrow & & \downarrow t \\ B & = & B \end{array} \tag{4.13}$$

4.3 The Grothendieck construction for lenses

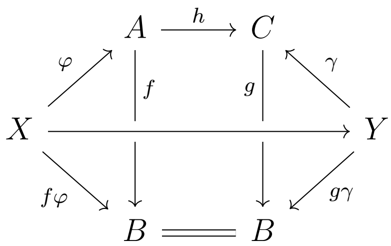
The goal of this section to the prove an equivalence of categories,

$$\mathcal{L}\mathrm{ens}_B \simeq [\mathbb{V}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$$

between the category $\mathcal{L}\text{ens}_B$ and the category of lax double functors from $\mathbb{V}(B)$ into the double category $\mathbb{S}\text{Mult}$ of split multi-valued functions (Theorem 4.14). The right-to-left functor is called the *Grothendieck construction for lenses*, as it generalises the classical Grothendieck construction for split opfibrations.

The proof utilises both the 1-dimensional and 2-dimensional universal properties of $\mathbb{S}\text{Mult}$ as the left-connected completion of the double category $\mathbb{S}\text{pan}$, together with two classical variations of the Grothendieck construction for discrete opfibrations (Lemma 4.11) and ordinary functors (Lemma 4.12). The final piece of the proof is the observation that globular transformations between lax double functors from $\mathbb{V}(B)$ into $\mathbb{S}\text{pan}$ correspond to identity-on-objects functors (Lemma 4.13), and linking this to the representation of lenses as commutative diagrams of functors (Proposition 2.12).

To prepare for the proof of the main result, recall that the category $\mathcal{L}\text{ens}_B$ is defined as the fibre over B of the codomain functor $\text{cod}\colon \mathcal{L}\text{ens} \to \mathcal{C}\text{at}$ of the double category of lenses. Therefore, the objects in $\mathcal{L}\text{ens}_B$ are lenses with codomain B, and morphisms are commutative diagrams (3.8) of the form:



Also recall from Example A.10 that given a category B, there is a double category $\mathbb{V}(B)$ whose objects and vertical morphisms are those of B, and whose horizontal morphisms and cells are identities.

There are two variants of the Grothendieck construction involving double categories which are required for the main proof.

The first is known classically as the *category of elements* construction, which demonstrates an equivalence between discrete opfibrations and $\mathcal{S}\text{et}$-valued functors. Let the category $\mathcal{D}\text{Opf}_B$ be the fibre over B of the codomain functor $\text{cod}\colon \mathcal{D}\text{Opf} \to \mathcal{C}\text{at}$ of the double category $\mathbb{D}\text{Opf}$ of discrete opfibrations. More plainly, $\mathcal{D}\text{Opf}_B$ is the full subcategory of $\mathcal{C}\text{at}/B$ determined by the discrete opfibrations. The following lemma is a simple restatement of the category of elements construction using double categories.

Lemma 4.11. *For each small category B, there is an equivalence of categories,*

$$\mathcal{D}\text{Opf}_B \simeq [\mathbb{V}(B), \mathbb{S}\text{q}(\mathcal{S}\text{et})]$$

between the category of discrete opfibrations over B, and the category of strict double functors from $\mathbb{V}(B)$ into the double category of squares $\mathbb{S}\text{q}(\mathcal{S}\text{et})$.

The second variant involves an equivalence between ordinary functors and $\mathbb{S}\text{pan}$-valued lax double functors. While the result is certainly classical, it is difficult to ascertain exactly where it was first proved. It is likely that the version for bicategories was known to Jean Bénabou; some details of the double-categorical version are present in the paper by Paré [Par11].

Lemma 4.12. *For each small category B, there is an equivalence of categories,*

$$\mathcal{C}\text{at}/B \simeq [\mathbb{V}(B), \mathbb{S}\text{pan}]_{\text{lax}}$$

between the slice category of functors over B, and the category of lax double functors from $\mathbb{V}(B)$ into the double category of spans $\mathbb{S}\text{pan}$. Let $\int \colon [\mathbb{V}(B), \mathbb{S}\text{pan}]_{\text{lax}} \to \mathcal{C}\text{at}/B$ denote the right-to-left direction of this equivalence.

Given a lax double functor $F \colon \mathbb{V}(B) \to \mathbb{S}\text{pan}$, the functor $\int F \to B$ is a discrete opfibration if and only if F factors through the inclusion $(-)_* \colon \mathbb{S}\text{q}(\mathcal{S}\text{et}) \to \mathbb{S}\text{pan}$ as depicted below.

$$\begin{array}{ccc} & \mathbb{V}(B) & \\ \swarrow & & \searrow^{F} \\ \mathbb{S}\text{q}(\mathcal{S}\text{et}) & \xrightarrow[(-)_*]{} & \mathbb{S}\text{pan} \end{array} \tag{4.14}$$

There is a final lemma which is required before proving the main result.

Lemma 4.13. *Globular transformations between lax double functors,*

$$\mathbb{V}B \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{S}\text{pan} \quad (\varphi \colon F \Rightarrow G)$$

are equivalent to commutative diagrams in $\mathcal{C}\text{at}$,

$$\begin{array}{ccc} \int F & \xrightarrow{\int \varphi} & \int G \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

such that $\int \varphi$ is an identity-on-objects functor.

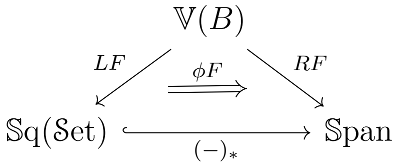
Proof. Follows directly from restricting Lemma 4.12 to globular transformations. □

Theorem 4.14. *For each small category B, there is an equivalence of categories,*

$$\mathcal{L}\text{ens}_B \simeq [\mathbb{V}(B), \mathbb{S}\text{Mult}]_{\text{lax}}$$

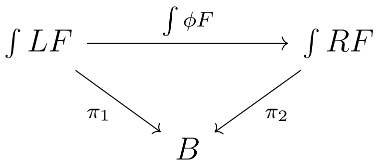
between the category of lenses over B, and the category of lax double functors from $\mathbb{V}(B)$ into the double category of split multi-valued functions $\mathbb{S}\text{Mult}$.

Proof. By the 1-dimensional universal property (Theorem 4.4) of the globular transformation (4.12) induced by $\mathbb{S}\mathrm{Mult}$ as the left-connected completion of $\mathbb{S}\mathrm{pan}$, there is an equivalence between lax double functors $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ and globular transformations of the form:



(4.15)

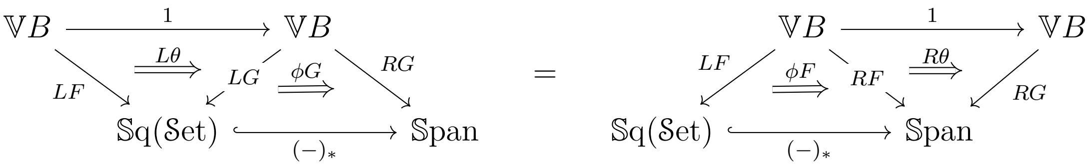
By Lemma 4.11, the lax double functor $LF\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et})$, is equivalent to a discrete opfibration over B. Post-composing LF by $(-)_*$ is likewise equivalent to a discrete opfibration by (4.14). Therefore, by Lemma 4.13, the transformation ϕF is equivalent to a commutative diagram of functors,



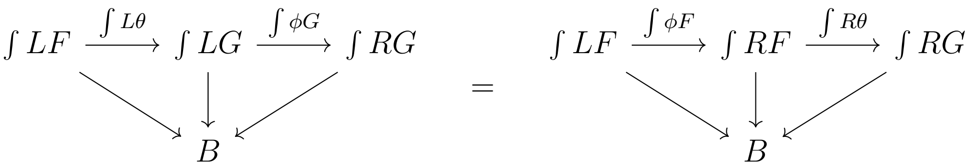
(4.16)

where $\int \phi F$ is identity-on-objects and π_1 is a discrete opfibration. By Proposition 2.13, the diagram (4.16) is equivalent to a lens over B.

By the 2-dimensional universal property (Theorem 4.6) of the globular transformation (4.12) induced by $\mathbb{S}\mathrm{Mult}$ as the left-connected completion, given lax double functors $F, G\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$, there is an equivalence between horizontal transformations $\theta\colon F \Rightarrow G$ and horizontal transformations $L\theta\colon LF \Rightarrow LG$ and $R\theta\colon RF \Rightarrow RG$ such that:



By Lemma 4.12, this is equivalent to the following morphisms in $\mathcal{C}\mathrm{at}/B$:



By Theorem 3.26, the diagram above is equivalent to a morphism of lenses over B. □

Conceptually, the central idea of this theorem is that lenses may be understood as generalised fibrations. Given a lens $(f, \varphi)\colon A \to B$, for every morphism $u\colon x \to y$

in B, there is a split multi-valued function between the fibres. The splitting of the multi-valued function is capturing the information of the chosen lifts of the lens.

Another key benefit is the new perspective this fibred approach to lenses provides. Rather than beginning with a functor and attempting to build a lens structure on it (which may not exist), one may start with a category and build a lens into it via a "lax diagram" into $\mathbb{S}\mathrm{Mult}$. In a sense, $\mathbb{S}\mathrm{Mult}$ is a classifying object for lenses.

Section 3.5 provided a new characterisation of split opfibrations in terms of lenses using the décalage construction. In the next section, a new characterisation of split opfibrations in terms of lax double functors into $\mathbb{S}\mathrm{Mult}$ is studied.

4.4 Split opfibrations as lax double functors

The classical Grothendieck construction is the right-to-left direction of the well-known equivalence of categories,

$$\mathcal{S}\mathrm{Opf}_B \simeq [B, \mathcal{C}\mathrm{at}]$$

between the category of split opfibrations over B and the category of functors from B to $\mathcal{C}\mathrm{at}$. However, split opfibrations are also lenses with a certain property, so it is natural to ask how this property might transfer under the equivalence in Theorem 4.14. The goal of this section is to show that lax double functors $\mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ satisfying a local invertibility requirement are equivalent to split opfibrations (Proposition 4.15).

Consider a lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$. Using the components of the globular transformation (4.15) and the composition comparison cell μ for F, there exists a cell,

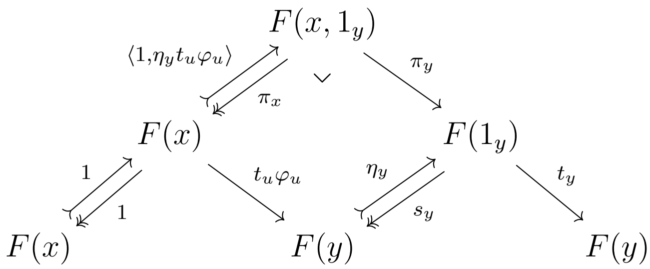
$$\begin{array}{ccccc}
F(x) & = & F(x) & = & F(x) \\
{\scriptstyle (LF(u))_*}\downarrow & \epsilon_u & \downarrow{\scriptstyle F(u)} & & \\
F(y) & = & F(y) & \mu_{u,y} & \downarrow{\scriptstyle F(u)} \\
{\scriptstyle F(1_y)}\downarrow & 1 & \downarrow{\scriptstyle F(1_y)} & & \\
F(y) & = & F(y) & = & F(y)
\end{array} \tag{4.17}$$

in $\mathbb{S}\mathrm{Mult}$ which may be constructed for all morphisms $u\colon x \to y \in B$. The cell ϵ_u above is defined by the following morphism in $\mathbb{S}\mathrm{Mult}$:

$$\begin{array}{ccc}
F(x) & = & F(x) \\
{\scriptstyle 1}\uparrow\downarrow{\scriptstyle 1} & & {\scriptstyle s_u}\uparrow\downarrow{\scriptstyle \varphi_u} \\
F(x) & \xrightarrow{\varphi_u} & F(u) \\
{\scriptstyle t_u\varphi_u}\downarrow & & \downarrow{\scriptstyle t_u} \\
F(y) & = & F(y)
\end{array}$$

Proposition 4.15. *A lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ is equivalent to a split opfibration if and only if the cell (4.17) is horizontally invertible.*

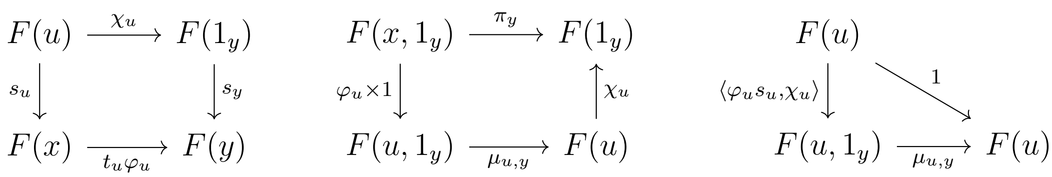
Proof. Consider the composite of the vertical morphisms $(LF(u))_*\colon F(x) \to F(y)$ and $F(1_y)\colon F(y) \to F(y)$ in $\mathbb{S}\mathrm{Mult}$:



The cell (4.17) is invertible if and only if the function,

$$\mu_{u,y}(\varphi_u \times 1)\colon F(x, 1_y) \to F(u)$$

is a bijection, which holds if and only if there exists a function $\chi_u\colon F(u) \to F(1_y)$ rendering the following three diagrams commutative:



Altogether, these diagrams are equivalent to stating that for each $w\colon a \to a' \in F(u)$ there exists a unique $\chi_u(w) \in F(1_y)$ such that $\chi_u(w) \circ \varphi_u(a) = w$. This states that each morphism $\varphi_u(a)$ is *weakly opcartesian*. However, by the composition coherence condition for the lax double functor F, these morphisms are closed under composition, and thus each morphism $\varphi_u(a)$ is *opcartesian*. Since each lift is opcartesian, the lax double functor F corresponds to a split opfibration. □

Although the complete proof of the above result is quite technical, the main benefit is another perspective on how split opfibrations arise as lenses. This particular view of split opfibrations is considered again in Proposition 5.29 where it is stated in terms of a strict factorisation system on the domain of a lens.

4.5 Characterising classes of lenses

The goal of this section is to demonstrate the way in which many classes of lenses may be characterised as lax double functors with a specified property. The basic idea is that given a lens corresponding to a lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$, the lens belongs to a specified class if there exists a specified double functor $\mathbb{D} \to \mathbb{S}\mathrm{Mult}$ such that F factorises through it:

$$\begin{array}{ccc} & \mathbb{V}(B) & \\ \swarrow^{\text{-}\,\text{-}} & & \searrow^{F} \\ \mathbb{D} & \longrightarrow & \mathbb{S}\mathrm{Mult} \end{array}$$

Example 4.16. A lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ corresponds to a *discrete opfibration* if and only if it factors through the inclusion $\mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) \to \mathbb{S}\mathrm{Mult}$, whose image on vertical morphisms are split multi-valued functions of the form:

$$A \underset{1_A}{\overset{1_A}{\rightleftarrows}} A \xrightarrow{f} B$$

Example 4.17. By Proposition 4.2, there is an adjunction of double categories:

$$\mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} \mathbb{S}\mathrm{Mult}$$

Restricting this adjunction by pre-composing with a lax double functor $\mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ and applying the Grothendieck construction for lenses (Theorem 4.14) yields, for each category B, an adjunction:

$$\mathcal{D}\mathrm{Opf}_B \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} \mathcal{L}\mathrm{ens}_B$$

Thus the category of discrete opfibrations over B is a coreflective subcategory of the category of lenses over B. Lenses in the image of the left adjoint correspond to discrete opfibrations as in Example 2.15.

However, since $\mathcal{S}\mathrm{et}$ has products there is also an adjunction of double categories,

$$\mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) \underset{\longleftarrow}{\overset{\hookrightarrow}{\top}} \mathbb{S}\mathrm{Mult}$$

whose right adjoint sends vertical morphisms in $\mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et})$ to split multi-valued functions of the form:

$$A \underset{\pi_A}{\overset{\langle 1_A, f\rangle}{\rightleftarrows}} A \times B \xrightarrow{\pi_B} B$$

Restricting this adjunction by pre-composing with a lax double functor $\mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ and applying the Grothendieck construction for lenses yields, for each category B, an adjunction:

$$\mathcal{D}\mathrm{Opf}_B \underset{\longleftarrow}{\overset{\hookrightarrow}{\top}} \mathcal{L}\mathrm{ens}_B$$

Thus the category of discrete opfibrations over B is a reflective subcategory of the category of lenses over B. Lenses in the image of the right adjoint correspond to lenses whose underlying functor is fully faithful as in Example 2.16.

Example 4.18. A category enriched in a monoidal category $\mathcal{V}$ may be defined as a lax functor of bicategories from a codiscrete category into $\mathcal{V}$, considered as a bicategory with a single object. Enrichment in a *double category* $\mathbb{D}$ may be similarly defined as lax double functor from $\mathbb{V}(B) \to \mathbb{D}$ for B a codiscrete category. Therefore, if B is a codiscrete category, then lax double functors $\mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ correspond to categories *enriched in the double category* $\mathbb{S}\mathrm{Mult}$. They are also equivalent to state-based lenses as in Example 2.17.

Example 4.19. The full double subcategory $\mathbb{P}(\mathcal{S}\mathrm{et})$ of $\mathbb{S}\mathrm{Mult}$ on a singleton set $\{*\}$, whose vertical morphisms are split multi-valued functions of the form,

$$\{*\} \overset{x}{\rightleftarrows} X \longrightarrow \{*\}$$

corresponds to the cartesian monoidal category of pointed sets. A lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ corresponds to a *bijective-on-objects lens* if and only if it factors through the inclusion $\mathbb{P}(\mathcal{S}\mathrm{et}) \to \mathbb{S}\mathrm{Mult}$. These are also equivalent to bijective-on-objects functors with a chosen section as in Example 2.19.

Example 4.20. A lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ corresponds to a *surjective-on-objects lens* if and only if $F(x)$ is non-empty for all objects $x \in B$.

Dually, a lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ corresponds to an *injective-on-objects lens* if and only if $F(x)$ is the empty set or a singleton set for all objects $x \in B$.

Example 4.21. The double category $\mathbb{S}\mathrm{Mono}$ of functions with a chosen retraction is a full double subcategory of $\mathbb{S}\mathrm{Mult}$ on split multi-valued functions of the form:

$$A \underset{\rho}{\overset{\sigma}{\rightleftarrows}} B \xrightarrow{1_B} B$$

A lax double functor $F\colon \mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$ corresponds to a lens structure on a *discrete fibration* if and only if it factors through the inclusion $\mathbb{S}\mathrm{Mono} \to \mathbb{S}\mathrm{Mult}$. The lens structure determines exactly a choice of retraction to each function between the fibres defined by the discrete fibration.

This final example is interesting, as the theory of lenses usually involves working with discrete *opfibrations* rather than discrete *fibrations*. It also demonstrates how Theorem 4.14 may be used to generate new examples of lenses as well as unifying existing ones.

Chapter 5

Lenses as monad morphisms

The aim of this chapter is to generalise from lenses as morphisms between *categories* to lenses as morphisms between *monads*. The key idea is that for a double category $\mathbb{D}$ equipped with a functorial choice of companions, there is a double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ whose objects are monads, such that the double category $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ of *lenses between monads* may be defined as its right-connected completion:

$$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D}) := \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}))$$

Specialising this definition to the double category $\mathbb{D} = \mathbb{S}\mathrm{pan}$ recovers the familiar double category of lenses (between categories),

$$\mathbb{L}\mathrm{ens} \cong \mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{S}\mathrm{pan})$$

while the case $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$, for a category $\mathcal{E}$ with pullbacks, yields a notion of *internal lens* which is used to study new characterisations of internal split opfibrations.

Section 5.1 begins with a review of the formal theory of monads in double categories, and the double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of monads in a double category $\mathbb{D}$ is constructed. In Section 5.2, the notion of *retrocell* in a double category with companions is used to define *monad retromorphisms*, and $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is obtained as the full double subcategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ determined by these vertical morphisms. Section 5.3 focuses on the case $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$, where monads are internal categories, monad morphisms are internal functors, and monad retromorphisms are *internal cofunctors*, thus generalising the results of Chapter 3 where $\mathcal{E} = \mathcal{S}\mathrm{et}$.

The definition of lenses between monads is introduced in Section 5.4, where they are defined as vertical morphisms in the right-connected completion of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, and internal lenses are obtained in the case $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$. The main application of the theory developed in this chapter is presented in Section 5.5, with two new characterisations of *internal split opfibrations* via properties of internal lenses.

5.1 Monads in double categories

The goal of this section is to review some elementary notions from the formal theory of monads in double categories [FGK11; FGK12], based on the formal theory of monads in 2-categories [Str72; LS02]. The double category $\mathbb{M}\mathrm{nd}(\mathbb{D})$ of monads, horizontal monad morphisms, and vertical monad morphisms in a double category $\mathbb{D}$ is defined, and is shown to have conjoints whenever $\mathbb{D}$ has conjoints (Proposition 5.1).

The main purpose of this section is to establish notational conventions, and to prepare for the definition of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ as a full double subcategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ in Section 5.2, and thus it does not contain any new material.

A *monad* (A, t, η, μ) in $\mathbb{D}$ consists of an object A, a vertical morphism $t\colon A \mathrel{\bullet\!\!\to} A$, and cells called the unit and multiplication, respectively,

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle 1_A}\downarrow & \eta & \downarrow{\scriptstyle t} \\
A & = & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & = & A \\
{\scriptstyle t}\downarrow & & \\
A & \mu & \downarrow{\scriptstyle t} \\
{\scriptstyle t}\downarrow & & \\
A & = & A
\end{array}
$$

which satisfy the standard unit and associativity laws.

A *horizontal monad morphism* $(u, \bar{u})$ from (A, t) to (A', t') consists of a horizontal morphism $u\colon A \to A'$ and a cell,

$$
\begin{array}{ccc}
A & \xrightarrow{u} & A' \\
{\scriptstyle t}\downarrow & \bar{u} & \downarrow{\scriptstyle t'} \\
A & \xrightarrow[u]{} & A'
\end{array}
$$

satisfying the following conditions:

$$
\begin{array}{ccccc}
A & = & A & \xrightarrow{u} & A' \\
{\scriptstyle 1_A}\downarrow & \eta & {\scriptstyle t}\downarrow & \bar{u} & \downarrow{\scriptstyle t'} \\
A & = & A & \xrightarrow[u]{} & A'
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{u} & A' & = & A' \\
{\scriptstyle 1_A}\downarrow & 1_u & \downarrow & \eta & \downarrow{\scriptstyle t'} \\
A & \xrightarrow[u]{} & A' & = & A'
\end{array}
$$

$$
\begin{array}{ccccc}
A & = & A & \xrightarrow{u} & A' \\
{\scriptstyle t}\downarrow & & & & \\
A & \mu & \downarrow{\scriptstyle t} & \bar{u} & \downarrow{\scriptstyle t'} \\
{\scriptstyle t}\downarrow & & & & \\
A & = & A & \xrightarrow[u]{} & A'
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{u} & A' & = & A' \\
{\scriptstyle t}\downarrow & \bar{u} & {\scriptstyle t'}\downarrow & & \\
A & \longrightarrow & A' & \mu & \downarrow{\scriptstyle t'} \\
{\scriptstyle t}\downarrow & \bar{u} & {\scriptstyle t'}\downarrow & & \\
A & \xrightarrow[u]{} & A' & = & A'
\end{array}
$$

A *vertical monad morphism* (f, φ) from (A, t) to (B, s) consists of a vertical morphism $f \colon A \to B$ and a cell,

$$\begin{array}{ccc}
A & = & A \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle t} \\
B & \varphi & A \\
{\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
B & = & B
\end{array} \tag{5.1}$$

satisfying the following conditions:

$$\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle f}\big\downarrow & 1_f & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle t} \\
B & = & B & \varphi & A \\
{\scriptstyle 1_B}\big\downarrow & \eta & {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
B & = & B & = & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & = & A \\
{\scriptstyle 1_A}\big\downarrow & \eta & \big\downarrow{\scriptstyle t} \\
A & = & A \\
{\scriptstyle f}\big\downarrow & 1_f & \big\downarrow{\scriptstyle f} \\
B & = & B
\end{array}$$

$$\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle f}\big\downarrow & 1_f & {\scriptstyle f}\big\downarrow & & \Big\downarrow{\scriptstyle t} \\
B & = & B & & \\
{\scriptstyle s}\big\downarrow & & \Big\downarrow{\scriptstyle s} & \varphi & A \\
B & \mu & & & \\
{\scriptstyle s}\big\downarrow & & & & \big\downarrow{\scriptstyle f} \\
B & = & B & = & B
\end{array}
\quad = \quad
\begin{array}{ccccccc}
A & = & A & = & A & = & A \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle t}\big\downarrow & 1_t & \big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t} \\
B & \varphi & A & = & A & \mu & \\
{\scriptstyle s}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle t} & & \\
B & = & B & \varphi & A & = & A \\
{\scriptstyle s}\big\downarrow & 1_s & {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle f} & 1_f & \big\downarrow{\scriptstyle f} \\
B & = & B & = & B & = & B
\end{array}$$

Let $\mathbb{M}\mathrm{nd}(\mathbb{D})$ denote the double category of monads, horizontal monad morphisms, and vertical monad morphisms, with cells in $\mathbb{M}\mathrm{nd}(\mathbb{D})$ on the left below given by cells in $\mathbb{D}$ on the right below,

$$\begin{array}{ccc}
(A, t) & \xrightarrow{(h, \bar{h})} & (A', t') \\
{\scriptstyle (f,\varphi)}\big\downarrow & \alpha & \big\downarrow{\scriptstyle (g,\gamma)} \\
(B, s) & \xrightarrow[(k, \bar{k})]{} & (B', s')
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{h} & A' \\
{\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & B'
\end{array}$$

satisfying the following condition:

$$\begin{array}{ccccc}
A & = & A & \xrightarrow{h} & A' \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle t}\big\downarrow & \bar{h} & \big\downarrow{\scriptstyle t'} \\
B & \varphi & A & \longrightarrow & A' \\
{\scriptstyle s}\big\downarrow & & {\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} \\
B & = & B & \xrightarrow[k]{} & B'
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{h} & A' & = & A' \\
{\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle t'} \\
B & \longrightarrow & B' & \gamma & A' \\
{\scriptstyle s}\big\downarrow & \bar{k} & \big\downarrow{\scriptstyle s'} & & \big\downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & B' & = & B'
\end{array} \tag{5.2}$$

Remark. Note that underlying vertical bicategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ is precisely the bicategory of monads, monad functors, and monad functor transformations defined by Street [Str72]. In more modern terminology, *monad functors* are often called *lax monad morphisms.*

Remark. If a double category $\mathbb{D}$ has *local coequalisers*, then there is also a double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of monads, horizontal monad morphisms, and *bimodules* [Shu08, Theorem 11.5], and there is a double functor $\mathbb{M}\mathrm{nd}(\mathbb{D}) \to \mathbb{M}\mathrm{od}(\mathbb{D})$ which assigns each vertical monad morphism to a corresponding bimodule.

Recall (Definition A.12) that a horizontal morphism $f\colon A \to B$ has a vertical *conjoint* $f^*\colon B \mathrel{-\!\!\bullet\!\!\rightarrow} A$ if there are cells,

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle f^*}\big\downarrow & \clubsuit & \big\downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1}\big\downarrow & \spadesuit & \big\downarrow{\scriptstyle f^*} \\ A & = & A \end{array} \tag{5.3}$$

such that the following pasting conditions hold:

$$\spadesuit \mid \clubsuit = 1_f \qquad \text{and} \qquad \frac{\clubsuit}{\spadesuit} = 1_{f^*}$$

Proposition 5.1 ([FGK11, Lemma 3.4])**.** *If $\mathbb{D}$ is a double category with conjoints, then $\mathbb{M}\mathrm{nd}(\mathbb{D})$ has conjoints.*

Proof. Suppose that $\mathbb{D}$ has conjoints, and consider a horizontal monad morphism $(f, \bar{f})\colon (A, t) \to (B, s)$ together with a conjoint $f^*\colon B \mathrel{-\!\!\bullet\!\!\rightarrow} A$ to f with cells (5.3). Then there exists a vertical monad morphism $(B, s) \mathrel{-\!\!\bullet\!\!\rightarrow} (A, t)$ consisting of the vertical morphism $f^*\colon B \mathrel{-\!\!\bullet\!\!\rightarrow} A$ and the cell,

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle f^*}\big\downarrow & \clubsuit & \big\downarrow{\scriptstyle 1_B} \\ A & \longrightarrow & B \\ {\scriptstyle t}\big\downarrow & \bar{f} & \big\downarrow{\scriptstyle s} \\ A & \longrightarrow & B \\ {\scriptstyle 1_A}\big\downarrow & \spadesuit & \big\downarrow{\scriptstyle f^*} \\ A & = & A \end{array}$$

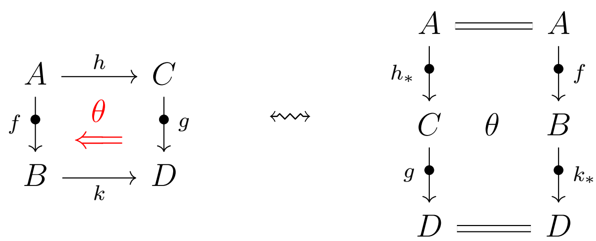
which is conjoint to $(f, \bar{f})\colon (A, t) \to (B, s)$ using the cells (5.3). □

The above result shows that every horizontal monad morphism is, in the presence of conjoints, a special case of a vertical monad morphism. Recall [GP99, Proposition 1.4] that in a double category $\mathbb{D}$ with all companions and conjoints, for a horizontal morphism $f\colon A \to B$, the companion $f_*\colon A \,\mathrlap{\bullet}\to B$ is left adjoint to the conjoint $f^*\colon B \,\mathrlap{\bullet}\to A$ in the vertical bicategory of $\mathbb{D}$. In this way, companions and conjoints are dual to each other. In the next section, companions will be used to define a special case of vertical monad morphisms which are, in a sense, dual to horizontal monad morphisms.

5.2 Monad retromorphisms

The goal of this section is to introduce a morphism between monads called a *monad retromorphism* (Definition 5.2) which generalises cofunctors between categories. The full double subcategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ determined by the monad retromorphisms is given by $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, and several basic properties of this double category are investigated including companions (Proposition 5.4), conjoints (Proposition 5.7), horizontal invariance (Corollary 5.5) and flatness (Proposition 5.8).

The key idea motivating monad retromorphisms is the notion of retrocells [Par19]. In a double category $\mathbb{D}$ equipped with a functorial choice of companions, a *retrocell* on the left below is defined by a globular cell in $\mathbb{D}$ on the right below:

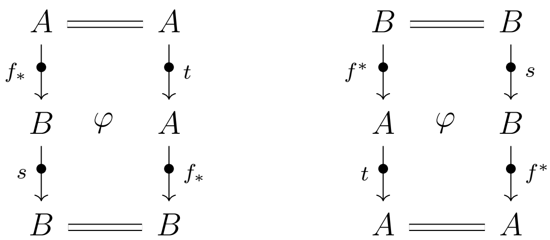


(5.4)

The similarity between a retrocell and the underlying cell (5.1) of a vertical monad morphism encourages the following definition.

Definition 5.2. A *monad retromorphism* from (A, t) to (B, s) is a vertical monad morphism $(f_*, \varphi)\colon A \,\mathrlap{\bullet}\to B$ such that $f_*\colon A \,\mathrlap{\bullet}\to B$ is the companion of a horizontal morphism $f\colon A \to B$.

The duality between monad retromorphisms and horizontal monad morphisms is inherited from the duality between companions and conjoints, which is demonstrated by comparing their corresponding vertical monad morphisms:



On the left-hand side, a monad retromorphism is a vertical monad morphism defined using a *companion* (that is, a retrocell), while on the right-hand side, a horizontal monad morphism corresponds to a vertical monad morphism constructed using a *conjoint* (Proposition 5.1).

Remark. Although monad retromorphisms don't strictly require the notion of retrocells for their definition, the notion is closely related to *horizontal comonad morphisms* in a double category $\mathbb{D}^{\mathrm{ret}}$ constructed by Paré using retrocells. The concept of retrocells was crucial to arriving at the definition of monad retromorphism, and is included here to provide context for the name (which is due to Matthew Di Meglio) and the idea.

Given two kinds of morphisms which are dual to each other, it is often useful to assemble them into a double category to study their relationship. Examples include the double category of small double categories, lax double functors, and oplax double functors [GP04], and the double category of model categories, right Quillen functors, and left Quillen functors [Shu11]. These examples motivate the following definition.

Definition 5.3. For a double category $\mathbb{D}$ with companions, let $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ be the full double subcategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ determined by the monad retromorphisms.

In detail, $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is the double category whose objects are monads in $\mathbb{D}$, whose horizontal morphisms are (horizontal) monad morphisms, whose vertical morphisms are monad retromorphisms, and whose cells are the same as those in $\mathbb{M}\mathrm{nd}(\mathbb{D})$. From the remarks above, these horizontal and vertical morphisms share a kind of duality, and the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ provides a natural setting to study their relationship.

The remainder of this section studies some of the basic properties of the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, namely, companions, conjoints, horizontal invariance, and flatness. The aim is to show how these properties, which were proven for $\mathbb{C}\mathrm{of}$, actually follow from the level of monads rather than categories. Henceforth, horizontal monad morphisms will be simply called *monad morphisms* in the context of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$.

Proposition 5.4. *Let $\mathbb{D}$ be a double category with a functorial choice of companions. A monad morphism $(f, \bar{f})\colon (A, t) \to (B, s)$ has a companion in* $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ *if and only if the cell,*

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1_A}\downarrow & \diamondsuit & \downarrow{\scriptstyle f_*} \\ A & \longrightarrow & B \\ {\scriptstyle t}\downarrow & \bar{f} & \downarrow{\scriptstyle s} \\ A & \longrightarrow & B \\ {\scriptstyle f_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1_B} \\ B & = & B \end{array} \tag{5.5}$$

(where f_* is a companion of f) is invertible.

Proof. Suppose that $(f, \bar{f})\colon (A, t) \to (B, s)$ is a horizontal monad morphism such that the cell (5.5) has an inverse φ. Then it is straightforward to verify that the pair (f_*, φ) is a monad retromorphism which is a companion of $(f, \bar{f})$.

Conversely, suppose that $(f, \bar{f})\colon (A, t) \to (B, s)$ is a horizontal monad morphism with a companion (g_*, γ) in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ defined using the cells:

$$
\begin{array}{ccc}
(A,t) & = & (A,t) \\
{\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle (g_*,\gamma)} \\
(A,t) & \xrightarrow[(f,\bar{f})]{} & (B,s)
\end{array}
\qquad\qquad
\begin{array}{ccc}
(A,t) & \xrightarrow{(f,\bar{f})} & (B,s) \\
{\scriptstyle (g_*,\gamma)}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\
(B,s) & = & (B,s)
\end{array}
$$

Applying the identities (5.2) to the cells $\diamondsuit$ and $\heartsuit$ above, it is straightforward to prove that the cell γ is inverse to (5.5) as required. □

Recall (Definition A.15) that a double category is horizontally invariant if every horizontal isomorphism has a companion.

Corollary 5.5. *If $\mathbb{D}$ is a double category equipped with a functorial choice of companions, then $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is horizontally invariant.*

Proof. For every horizontal monad morphism $(f, \overline{f})$ which is invertible, the cell (5.5) is invertible. Therefore $(f, \overline{f})$ has a companion, and the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is horizontally invariant □

Recall (Definition A.17) that a double category $\mathbb{D}$ is *unit-pure* if the identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is fully faithful. This means that for every cell whose vertical boundary morphisms are identities,

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
{\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle 1} \\
A & \xrightarrow[k]{} & B
\end{array}
$$

then it necessarily holds that $h = k$ and $\alpha = 1_h = 1_k$.

Lemma 5.6. *Let $\mathbb{D}$ be a unit-pure double category equipped with a functorial choice of companions, and consider horizontal morphisms $f\colon A \to B$ and $g\colon B \to A$. Then $f_* = g^*$ if and only if $f = g^{-1}$.*

Proof. Suppose $f_* = g^*$ and consider the following pasting of cells:

$$
\begin{array}{ccccc}
B & \xrightarrow{g} & A & \xrightarrow{f} & B \\
{\scriptstyle 1}\downarrow & \spadesuit & \downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\
B & = & B & = & B
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle 1}\downarrow & \diamondsuit & \downarrow & \clubsuit & \downarrow{\scriptstyle 1} \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & A
\end{array}
$$

Then by unit-purity, $fg = 1_B$ and $gf = 1_A$. Conversely, suppose $f = g^{-1}$. Since $(fg)_* = f_* g_* = 1_B$ and $(gf)_* = g_* f_* = 1_A$, it follows that f_* is adjoint (indeed, inverse) to g_* in the vertical bicategory of $\mathbb{D}$, and therefore $g^* = f_*$. □

Proposition 5.7. *Let $\mathbb{D}$ be a unit-pure double category equipped with a functorial choice of companions. A horizontal monad morphism $(f, \bar{f})\colon (A, t) \to (B, s)$ has a conjoint in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ if and only if $f\colon A \to B$ is invertible.*

Proof. Suppose $(f, \bar{f})\colon (A, t) \to (B, s)$ has a conjoint $(g, \gamma)\colon (B, s) \to (A, t)$ together with cells:

$$\begin{array}{ccc} (A,t) & \xrightarrow{(f,\bar{f})} & (B,s) \\ {\scriptstyle 1}\downarrow & \spadesuit & \downarrow{\scriptstyle (g_*,\gamma)} \\ (A,t) & = & (A,t) \end{array} \qquad\qquad \begin{array}{ccc} (B,s) & \xrightarrow{1} & (B,s) \\ {\scriptstyle (g_*,\gamma)}\downarrow & \clubsuit & \downarrow{\scriptstyle 1} \\ (A,t) & \xrightarrow[(f,\bar{f})]{} & (B,s) \end{array}$$

By construction, $f^* = g_*$, therefore $f = g^{-1}$ by Lemma 5.6.

Conversely, suppose $(f, \overline{f})\colon (A, t) \to (B, s)$ such that $f\colon A \to B$ has an inverse $g\colon B \to A$. Then f has a conjoint $f^* = g_*$ by Lemma 5.6. Thus there is a cell,

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle g_*}\downarrow & \clubsuit & \downarrow{\scriptstyle 1} \\ A & \longrightarrow & B \\ {\scriptstyle t}\downarrow & \bar{f} & \downarrow{\scriptstyle s} \\ A & \longrightarrow & B \\ {\scriptstyle 1}\downarrow & \spadesuit & \downarrow{\scriptstyle g_*} \\ B & = & B \end{array}$$

which yields a monad retromorphism $(B, s) \to (A, t)$. This monad retromorphism is conjoint to $(f, \overline{f})$; the cells ♠ and ♣ are used to construct corresponding cells in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, from which it is straightforward to check that the pasting laws for conjoints hold. □

Proposition 5.8. *If $\mathbb{D}$ is a unit-pure double category equipped with a functorial choice of companions, then $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is flat.*

Proof. Consider a pair of cells α and β in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ with the same boundary. The vertical boundary morphisms of the underlying cells in $\mathbb{D}$ are companions, since vertical morphisms in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ are monad retromorphisms. Pasting with the binding cells for companions yields a pair of cells which are equal to a vertical identity cell by unit-purity:

$$
\begin{array}{ccccccc}
A & = & A & \xrightarrow{h} & C & \xrightarrow{g} & D \\
{\scriptstyle 1}\big\downarrow\!\bullet & \Diamond & {\scriptstyle f_*}\big\downarrow\!\bullet & \alpha & \big\downarrow\!\bullet{\scriptstyle g_*} & \heartsuit & \big\downarrow\!\bullet{\scriptstyle 1} \\
A & \xrightarrow[f]{} & B & \xrightarrow[k]{} & D & = & D
\end{array}
\quad = \quad
\begin{array}{ccccccc}
A & = & A & \xrightarrow{h} & C & \xrightarrow{g} & D \\
{\scriptstyle 1}\big\downarrow\!\bullet & \Diamond & {\scriptstyle f_*}\big\downarrow\!\bullet & \beta & \big\downarrow\!\bullet{\scriptstyle g_*} & \heartsuit & \big\downarrow\!\bullet{\scriptstyle 1} \\
A & \xrightarrow[f]{} & B & \xrightarrow[k]{} & D & = & D
\end{array}
$$

Pasting the above equation with the alternative binding cells and applying the identities for companions yields $\alpha = \beta$, and therefore $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is flat (Definition A.2). □

5.3 The double category of internal cofunctors

This section studies the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ in the case of $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks. The central motivation is that $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$ is a double category whose objects are internal categories, whose monad morphisms are internal functors, and whose monad retromorphisms are internal cofunctors. Specialising to $\mathcal{E} = \mathcal{S}\mathrm{et}$ yields the usual double category of cofunctors:

$$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{S}\mathrm{et})) = \mathbb{C}\mathrm{of}$$

The benefit is that many of the results of Chapter 3 may be understood as properties of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, while also revealing cofunctors as a fundamental kind of morphism between categories.

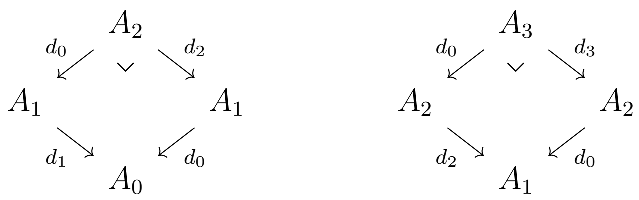
It is well-known that monads in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ are precisely internal categories, and that (horizontal) monad morphisms are internal functors [FGK11]. The definition of internal category and internal functor are now recalled to set notation for the rest of the chapter.

Remark. The notation for an internal category and internal functor used in this chapter follows the usage in [Str17], where internal categories are treated as truncated simplicial objects. The advantage is that notation is concise, at the expense of immediate familiarity for some readers. To avoid distracting from the essential part of each definition, the reader should also be aware, particularly in Definition 5.13, that some morphisms appear in commutative diagrams before they are defined at the end of the definition.

Definition 5.9. An *internal category* A is a diagram in a category $\mathcal{E}$ with pullbacks,

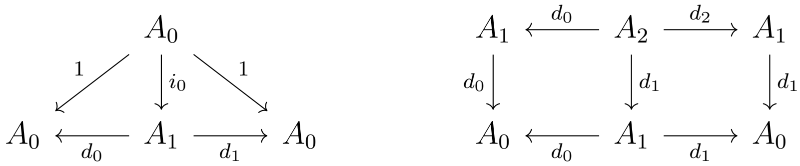
$$
A_0 \;
\begin{array}{c} \xleftarrow{\;d_0\;} \\ \xrightarrow{\;i_0\;} \\ \xleftarrow{\;d_1\;} \end{array}
\; A_1 \;
\begin{array}{c} \xleftarrow{\;d_0\;} \\ \xrightarrow{\;i_0\;} \\ \xleftarrow{\;d_1\;} \\ \xrightarrow{\;i_1\;} \\ \xleftarrow{\;d_2\;} \end{array}
\; A_2 \;
\begin{array}{c} \xleftarrow{\;d_0\;} \\ \xleftarrow{\;d_1\;} \\ \xleftarrow{\;d_2\;} \\ \xleftarrow{\;d_3\;} \end{array}
\; A_3
$$

where the objects A_2 and A_3 are defined by the pullbacks,



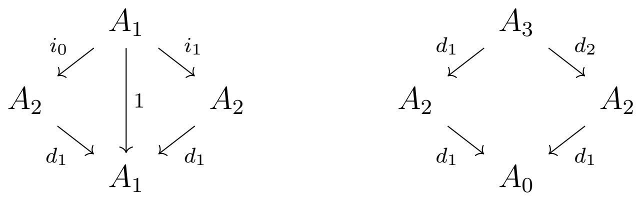
(5.6)

where the *identity map* $i_0 \colon A_0 \to A_1$ and the *composition map* $d_1 \colon A_2 \to A_1$ satisfy the commutative diagrams,



(5.7)

and satisfy the *unitality* and *associativity* axioms given by the commutative diagrams:



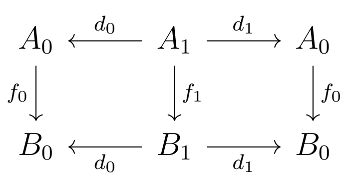
(5.8)

An internal category is often depicted by its underlying directed graph consisting of the *object of objects* A_0, the *object of morphisms* A_1, the *domain map* $d_0 \colon A_1 \to A_0$, and the *codomain map* $d_1 \colon A_1 \to A_0$. The morphisms $i_0, i_1 \colon A_1 \to A_2$ and $d_1, d_2 \colon A_3 \to A_2$ appearing in (5.8) are defined using the universal property of the pullback.

Definition 5.10. An *internal functor* $f \colon A \to B$ consists of a pair of morphisms,

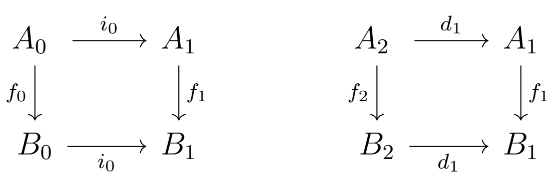
$$f_0 \colon A_0 \longrightarrow B_0 \qquad\qquad f_1 \colon A_1 \longrightarrow B_1$$

satisfying the commutative diagrams with respect to the domain and codomain maps,



(5.9)

and which respect the identity and composition maps:



(5.10)

The morphism $f_2 \colon A_2 \to B_2$ is defined using the universal property of the pullback.

Together, internal categories and internal functors form a category $\mathcal{C}\mathrm{at}(\mathcal{E})$.

In Chapter 3 it was shown that bijective-on-objects functors and discrete opfibrations were the conjoints and companions, respectively, in the double category of cofunctors. The notions readily generalise to the internal category setting.

Definition 5.11. An internal functor $f\colon A \to B$ is called *isomorphism-on-objects* if the morphism f_0 is an isomorphism. In particular, if $f_0 = 1$ the internal functor is called *identity-on-objects*.

Definition 5.12. An *internal discrete opfibration* is an internal functor $f\colon A \to B$ such that the following commutative diagram, appearing in (5.9), is a pullback:

$$\begin{array}{ccccc} & & A_1 & & \\ & {}^{d_0}\swarrow & & \searrow^{f_1} & \\ A_0 & & & & B_1 \\ & {}_{f_0}\searrow & & \swarrow_{d_0} & \\ & & B_0 & & \end{array} \tag{5.11}$$

Cofunctors as morphisms between categories were actually first introduced in the setting of internal category. The following definition of internal cofunctor adapts the original version introduced by Aguiar [Agu97].

Definition 5.13. An *internal cofunctor* $(f_0, \varphi_1)\colon A \to B$ consists of a pair of morphisms,

$$f_0\colon A_0 \longrightarrow B_0 \qquad\qquad \varphi_1\colon \Lambda_1 \longrightarrow A_1$$

with the objects Λ_1 and Λ_2 defined by the pullbacks,

$$\begin{array}{ccccc} & & \Lambda_1 & & \\ & {}^{d_0}\swarrow & \lrcorner & \searrow^{\overline{f}_1} & \\ A_0 & & & & B_1 \\ & {}_{f_0}\searrow & & \swarrow_{d_0} & \\ & & B_0 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & \Lambda_2 & & \\ & {}^{d_0}\swarrow & \lrcorner & \searrow^{\overline{f}_2} & \\ \Lambda_1 & & & & B_2 \\ & {}_{\overline{f}_1}\searrow & & \swarrow_{d_0} & \\ & & B_1 & & \end{array} \tag{5.12}$$

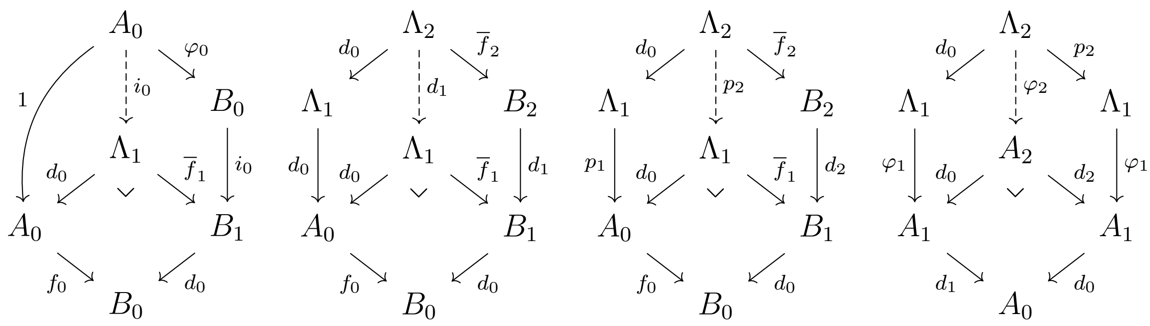
satisfying commutative diagrams with respect to the domain and codomain maps,

$$\begin{array}{ccccccc} & & \Lambda_1 & \xrightarrow{\quad\overline{f}_1\quad} & & & B_1 \\ & {}^{d_0}\swarrow & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle d_1} \\ A_0 & \xleftarrow[d_0]{} & A_1 & \xrightarrow[d_1]{} & A_0 & \xrightarrow[f_0]{} & B_0 \end{array} \tag{5.13}$$

and with respect to the identity and composition maps:

$$\begin{array}{ccc} A_0 & \xrightarrow{i_0} & \Lambda_1 \\ \| & & \downarrow{\scriptstyle \varphi_1} \\ A_0 & \xrightarrow[i_0]{} & A_1 \end{array} \qquad\qquad \begin{array}{ccc} \Lambda_2 & \xrightarrow{d_1} & \Lambda_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ A_2 & \xrightarrow[d_1]{} & A_1 \end{array} \tag{5.14}$$

It is useful to define the morphism $p_1 = d_1\varphi_1 \colon \Lambda_1 \to A_0$. The morphisms $i_0 \colon A_0 \to \Lambda_1$, $d_1 \colon \Lambda_2 \to \Lambda_1$, and $\varphi_2 \colon \Lambda_2 \to A_2$ are defined using the universal property of the pullback:



Remark. The definition of internal cofunctor uses the notation Λ_1 for the pullback $A_0 \times_{B_0} B_1$ as it is more compact (likewise for Λ_2). However, it is also chosen to align with the notation for the tabulator of a cofunctor as in Proposition 2.6; later it will be shown that the pair (A_0, Λ_1) forms the tabulator of an internal cofunctor. For a similar reason, the notation $d_0 \colon \Lambda_1 \to A_0$ and $\overline{f}_1 \colon \Lambda_1 \to B_1$ is chosen for the pullback projections $\pi_0 \colon A_0 \times_{B_0} B_1 \to A_0$ and $\pi_1 \colon A_0 \times_{B_0} B_1 \to B_1$, respectively, as this will be particularly useful in the following results.

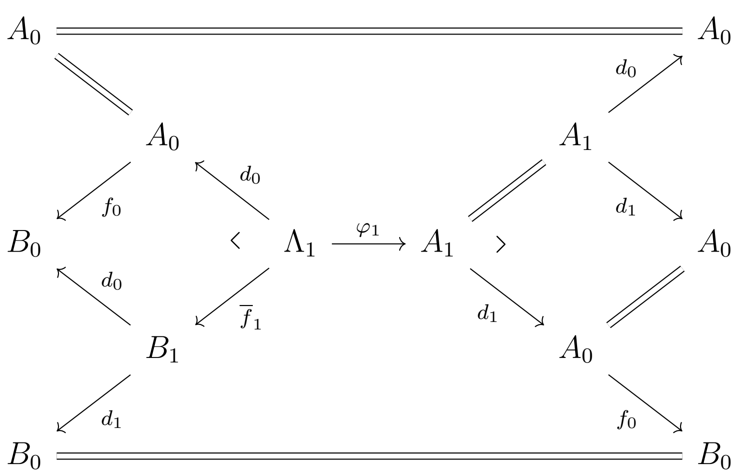
Recall (Example A.13) that the double category $\mathbb{S}\mathrm{pan}(\mathcal{E})$ is equipped with a functorial choice of companions. The companion of a horizontal morphism $f \colon A \to B$ is given by the span:

$$A \xleftarrow{1_A} A \xrightarrow{f} B$$

The following result provides the main motivation for studying (internal) cofunctors.

Proposition 5.14. *A monad retromorphism between monads in* $\mathbb{S}\mathrm{pan}(\mathcal{E})$ *is precisely an internal cofunctor.*

Proof. A monad retromorphism in $\mathbb{S}\mathrm{pan}(\mathcal{E})$ between internal categories A and B consists of a morphism $f_0 \colon A_0 \to B_0$ together with a cell in $\mathbb{S}\mathrm{pan}$ given by:



This agrees with the data of a cofunctor $(f_0, \varphi_1)\colon A \to B$ satisfying (5.13). It is straightforward to prove that the diagrams (5.14) correspond exactly to the compatibility of a monad retromorphism with the unit and multiplication cells. □

It has now been shown that the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$ has internal categories as objects, internal functors as horizontal morphisms, and internal cofunctors as vertical morphisms. By Proposition 5.8, this double category is flat. The cells with boundary,

$$\begin{array}{ccc} A & \xrightarrow{(h_0, h_1)} & C \\ {\scriptstyle (f_0,\varphi_1)}\downarrow & & \downarrow{\scriptstyle (g_0,\gamma_1)} \\ B & \xrightarrow[(k_0,k_1)]{} & D \end{array}$$

satisfy the commutative diagrams:

$$\begin{array}{ccc} A_0 & \xrightarrow{h_0} & C_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle g_0} \\ B_0 & \xrightarrow[k_0]{} & D_0 \end{array} \qquad \begin{array}{ccc} A_0 \times_{B_0} B_1 & \xrightarrow{h_0 \times k_1} & C_0 \times_{D_0} D_1 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \gamma_1} \\ A_1 & \xrightarrow[h_1]{} & C_1 \end{array} \tag{5.15}$$

When $\mathcal{E} = \mathcal{S}\mathrm{et}$, the conditions in (5.15) correspond to the conditions on cells in the double category of cofunctors $\mathbb{C}\mathrm{of}$ stated in Definition 3.1. Together with Proposition 5.14, this yields the following result.

Theorem 5.15. *The double category of monads, monad morphisms, and monad retromorphisms in* $\mathbb{S}\mathrm{pan}$ *is equivalent to the double category of categories, functors, and cofunctors:*

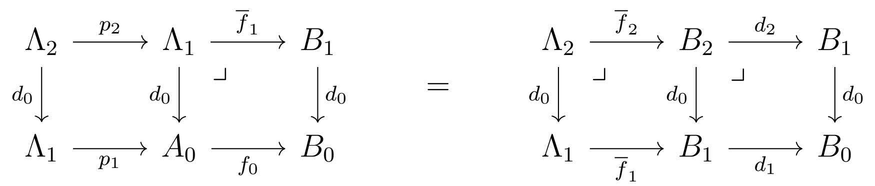
$$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}) \simeq \mathbb{C}\mathrm{of}$$

An immediate implication of this result is that many of the properties of the double category of cofunctors are inherited directly from the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$. This includes that the companions are discrete opfibrations, that the conjoints are bijective-on-objects functors, and that $\mathbb{C}\mathrm{of}$ is horizontally invariant and flat. One of the important properties however which doesn't automatically arise from this level of generality is that of tabulators. However, in the case of $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$, these are also quite easy to compute.

Lemma 5.16. *Given a cofunctor* $(f_0, \varphi_1)\colon A \to B$*, there is an internal category* Λ *defined by the diagram:*

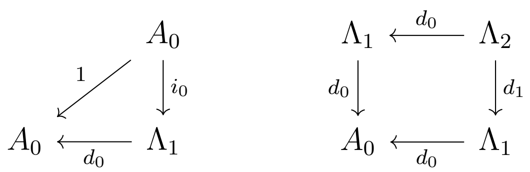
$$A_0 \;\begin{array}{c} \xleftarrow{\;d_0\;} \\ \xrightarrow{\;i_0\;} \\ \xleftarrow{\;p_1\;} \end{array}\; \Lambda_1 \;\begin{array}{c} \xleftarrow{\;d_0\;} \\ \xleftarrow{\;d_1\;} \\ \xleftarrow{\;p_2\;} \end{array}\; \Lambda_2$$

Proof. First, to show Λ_2 is the pullback of the morphisms $d_0, p_1 \colon \Lambda_1 \to A_0$, consider the following diagrams which are equal by the construction of $p_2 \colon \Lambda_2 \to \Lambda_1$ in Definition 5.13:

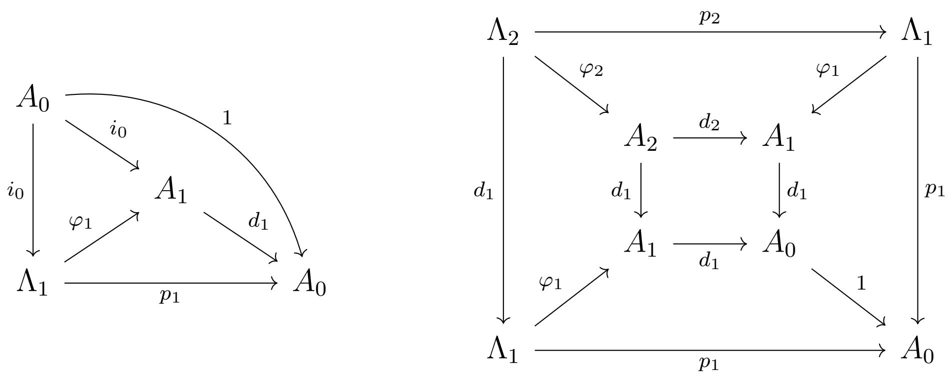


Using the pullback pasting lemma, the remaining square must be a pullback.

To show the identity map $i_0 \colon A_0 \to \Lambda_1$ and the composition map $d_1 \colon \Lambda_2 \to \Lambda_1$ are well-defined, first notice by their construction in Definition 5.13 that the following diagrams commute:



The counterparts to the diagrams immediately above are obtained by pasting as follows:



Thus axiom (5.7) is satisfied as required. The details required to prove (5.8) are not difficult, but involve many diagrams to construct the appropriate morphisms and check commutativity using the universal property of the pullback. □

Proposition 5.17. *The double category* $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$ *has tabulators.*

Proof. Given an internal cofunctor $(f_0, \varphi_0) \colon A \to B$, there is a cell in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$ given by:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (f_0, \varphi_1)} \\ \Lambda & \xrightarrow[\overline{f}]{} & B \end{array} \tag{5.16}$$

The internal fucntor $\overline{f}\colon \Lambda \to B$ is determined by the pair of morphisms $(\varphi_0, \overline{f}_1)$ which satisfy the commutative diagrams:

$$\begin{tikzcd} A_0 \arrow[d, "f_0"'] & \Lambda_1 \arrow[l, "d_0"'] \arrow[r, "p_1"] \arrow[d, "\overline{f}_1"] \arrow[dl, phantom, "\llcorner", very near start] & A_0 \arrow[d, "f_0"] \\ B_0 & B_1 \arrow[l, "d_0"] \arrow[r, "d_1"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "i_0"] \arrow[d, "f_0"'] & \Lambda_1 \arrow[d, "\overline{f}_1"] \\ B_0 \arrow[r, "i_0"'] & B_1 \end{tikzcd} \qquad \begin{tikzcd} \Lambda_2 \arrow[r, "d_1"] \arrow[d, "\overline{f}_2"'] & \Lambda_1 \arrow[d, "\overline{f}_1"] \\ B_2 \arrow[r, "d_1"'] & B_1 \end{tikzcd}$$

The internal functor $\varphi\colon \Lambda \to A$ is determined by the pair of morphisms $(1_{A_0}, \varphi_1)$ which satisfy the commutative diagrams:

$$\begin{tikzcd} A_0 \arrow[d, "1"'] & \Lambda_1 \arrow[l, "d_0"'] \arrow[r, "p_1"] \arrow[d, "\varphi_1"] & A_0 \arrow[d, "1"] \\ A_0 & A_1 \arrow[l, "d_0"] \arrow[r, "d_1"'] & A_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "i_0"] \arrow[d, "1"'] & \Lambda_1 \arrow[d, "\varphi_1"] \\ A_0 \arrow[r, "i_0"'] & A_1 \end{tikzcd} \qquad \begin{tikzcd} \Lambda_2 \arrow[r, "d_1"] \arrow[d, "\varphi_2"'] & \Lambda_1 \arrow[d, "\varphi_1"] \\ A_2 \arrow[r, "d_1"'] & A_1 \end{tikzcd}$$

These commutative diagrams all appear in Definition 5.13. Using the universal property of the pullback in $\mathcal{E}$, it may be shown that (5.16) has the universal property of a tabulator. □

Finally, note that the functor $\overline{f}\colon \Lambda \to B$ is an internal discrete opfibration, while $\varphi\colon \Lambda \to B$ is clearly identity-on-objects. It may also be shown that $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$ has strong tabulators, meaning that every cofunctor has a unique factorisation, up to isomorphism, into a conjoint (isomorphism-on-objects functor) followed by a companion (discrete opfibration). Therefore, all the properties of interest of the double category $\mathbb{C}\mathrm{of}$ arise more generally as properties of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E}))$.

5.4 Lenses between monads

In the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ the horizontal morphisms generalise functors while the vertical morphisms generalise cofunctors. In Chapter 3, it was shown that the double category of lenses, $\mathbb{L}\mathrm{ens}$, was equivalent to the right-connected completion of the double category of cofunctors, $\mathbb{C}\mathrm{of}$. The goal of this section is to define the double category $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ of *lenses between monads* in a double category $\mathbb{D}$, and to introduce *internal lenses* in the case $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$.

Assumption. Throughout this section $\mathbb{D}$ is assumed to be a unit-pure double category equipped with a functorial choice of companions.

Definition 5.18. A *lens between monads* is a vertical morphism in the right-connected completion of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$. Let $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D}) \coloneqq \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}))$ denote the double category of lenses between monads.

By construction, the objects and horizontal morphisms in $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ are monads and (horizontal) monad morphisms, respectively.

Proposition 5.19. *Consider a vertical morphism in* $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ *given by a monad morphism* $(f, \bar{f})\colon (A, t) \to (B, s)$ *and a monad retromorphism* $(g, \gamma)\colon (A, t) \to\!\!\!\bullet\!\!\!\to (B, s)$. *Then the cell,*

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle f_*} \\ A & \longrightarrow & B \\ {\scriptstyle t}\downarrow & \bar{f} & \downarrow{\scriptstyle s} \\ A & \longrightarrow & B \\ {\scriptstyle f_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\ B & = & B \end{array} \tag{5.17}$$

has a chosen horizontal section given by the cell γ.

Proof. The vertical morphisms in $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ are cells in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ of the form:

$$\begin{array}{ccc} (A, t) & \xrightarrow{(f, \bar{f})} & (B, s) \\ {\scriptstyle (g_*, \gamma)}\downarrow & \alpha & \downarrow{\scriptstyle 1} \\ (B, s) & \xrightarrow[1]{} & (B, s) \end{array}$$

In detail, there is a cell,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g_*}\downarrow & \alpha & \downarrow{\scriptstyle 1_B} \\ B & = & B \end{array} \tag{5.18}$$

satisfying the following condition:

$$\begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ {\scriptstyle g_*}\downarrow & & {\scriptstyle t}\downarrow & \bar{f} & \downarrow{\scriptstyle s} \\ B & \gamma & A & \longrightarrow & B \\ {\scriptstyle s}\downarrow & & {\scriptstyle g_*}\downarrow & \alpha & \downarrow{\scriptstyle 1} \\ B & = & B & = & B \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{f} & B & = & B \\ {\scriptstyle g_*}\downarrow & \alpha & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle s} \\ B & = & B & & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 1} \\ B & = & B & = & B \end{array} \tag{5.19}$$

Pasting (5.18) with the companion binding cell yields a cell,

$$\begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & \diamondsuit & {\scriptstyle g_*}\downarrow & \alpha & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[g]{} & B & = & B \end{array} \tag{5.20}$$

which implies that $f = g$ and $g_* = f_*$, since $\mathbb{D}$ is assumed to be unit-pure. By properties of the right-connected double categories with a functorial choice of companions, it follows that (5.18) is equal to the binding cell $\heartsuit$. Moreover, pasting (5.19) with the binding cell $\diamondsuit$ for the companion f yields the following:

$$
\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle f_*} \\
A & = & A & \longrightarrow & B \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle t}\downarrow & \bar{f} & \downarrow{\scriptstyle s} \\
B & \gamma & A & \longrightarrow & B \\
{\scriptstyle s}\downarrow & & {\scriptstyle f_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\
B & = & B & = & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & = & A \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
B & 1 & B \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
B & = & B
\end{array}
$$

Therefore γ is a section to the cell (5.17) as required. □

The main benefit of the above result is a deeper conceptual understanding of lenses as *generalised section-retraction pairs*. While it is intuitively understood that ordinary lenses between categories lift morphisms, and thus behave like a split epimorphism with a chosen section, this is not formally true. However, from the perspective of lenses between monads, Proposition 5.19 demonstrates the sense in which chosen sections play a formal role. It is also interesting to note that when this section-retraction pair is an isomorphism, the lens between monads is equivalent to a companion in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$.

Lenses between monads may be denoted by triples $(f, \overline{f}, \varphi) \colon (A, t) \to (B, s)$. Recall that since $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is flat (Proposition 5.8), its right-connected completion $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ is also flat (Lemma 3.16). The following result provides a characterisation of the cells in the double category of lenses between monads.

Proposition 5.20. *A cell in* $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ *with boundary given by,*

$$
\begin{array}{ccc}
(A, t) & \xrightarrow{(h, \bar{h})} & (A', t') \\
{\scriptstyle (f, \bar{f}, \varphi)}\downarrow & & \downarrow{\scriptstyle (g, \bar{g}, \gamma)} \\
(B, s) & \xrightarrow[(k, \bar{k})]{} & (B', s')
\end{array}
\tag{5.21}
$$

is equivalent to a cell in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ *given by,*

$$
\begin{array}{ccc}
(A, t) & \xrightarrow{(h, \bar{h})} & (A', t') \\
{\scriptstyle (f, \varphi)}\downarrow & & \downarrow{\scriptstyle (g, \gamma)} \\
(B, s) & \xrightarrow[(k, \bar{k})]{} & (B', s')
\end{array}
\tag{5.22}
$$

and a commuting diagram of horizontal monad morphisms:

$$\begin{array}{ccc} (A,t) & \xrightarrow{(h,\bar{h})} & (A',t') \\ {\scriptstyle (f,\bar{f})}\downarrow & & \downarrow{\scriptstyle (g,\bar{g})} \\ (B,s) & \xrightarrow[(k,\bar{k})]{} & (B',s') \end{array} \tag{5.23}$$

Proof. By definition of the right-connected completion, cell (5.21) in $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ is given by a cell (5.22) in $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ such that the following condition holds:

$$\begin{array}{ccccc} (A,t) & \xrightarrow{(h,\bar{h})} & (A',t') & \xrightarrow{(g,\bar{g})} & (B',s') \\ {\scriptstyle (f,\varphi)}\downarrow & & \downarrow{\scriptstyle (g,\gamma)} & & \downarrow{\scriptstyle 1} \\ (B,s) & \xrightarrow[(k,\bar{k})]{} & (B',s') & = & (B',s') \end{array} \quad = \quad \begin{array}{ccccc} (A,t) & \xrightarrow{(f,\bar{f})} & (B,s) & \xrightarrow{(k,\bar{k})} & (B',s') \\ {\scriptstyle (f,\varphi)}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ (B,s) & = & (B,s) & \xrightarrow[(k,\bar{k})]{} & (B',s') \end{array}$$

However, since $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is flat, this condition contains no information on cells, and is therefore equivalent to the condition (5.23) on the boundary of monad morphisms. □

Taken together, Proposition 5.17 and Proposition 5.20 provide simple a characterisation of the vertical morphisms and cells in $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$. This section now concludes with the case of $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{E})$, where vertical morphisms in $\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$ are called internal lenses.

Definition 5.21. Let A and B be internal categories. An *internal lens* $(f,\varphi)\colon A \to B$ consists of an internal functor $f\colon A \to B$ and an internal cofunctor $(\varphi_0,\varphi_1)\colon A \to B$ satisfying the commutative diagrams:

$$\begin{array}{ccc} & A_0 & \\ /\!/ & & \searrow{\scriptstyle \varphi_0} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array} \qquad\qquad \begin{array}{ccc} & \Lambda_1 & \\ {\scriptstyle \varphi_1}\swarrow & & \searrow{\scriptstyle \overline{\varphi}_1} \\ A_1 & \xrightarrow[f_1]{} & B_1 \end{array} \tag{5.24}$$

By the universal property of the pullback, the right-hand diagram in (5.24) is equivalent to the commutative diagram:

$$\begin{array}{ccc} & \Lambda_1 & \\ {\scriptstyle \varphi_1}\swarrow & & \searrow{\scriptstyle 1} \\ A_1 & \xrightarrow[\langle d_0, f_1\rangle]{} & \Lambda_1 \end{array}$$

This section-retraction may be understood as an instantiation of Proposition 5.19.

The following result provides a version of Proposition 2.12 for the internal category setting.

Proposition 5.22. *An internal lens* $(f, \varphi)\colon A \to B$ *is equivalent to a commutative diagram of internal functors,*

$$\begin{array}{ccc} & \Lambda & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\overline{f}} \\ A & \xrightarrow[f]{} & B \end{array} \tag{5.25}$$

where φ *is identity-on-objects and* $\overline{f}$ *is an internal discrete opfibration.*

Proof. Follows directly from the span representation of internal cofunctors implicit in Proposition 5.17 together with (5.24). □

Lenses between categories are a special case of internal lenses between internal categories, as summarised by the following theorem.

Theorem 5.23. *The right-connected completion of the double category of monads, monad morphisms, and monad retromorphisms in* $\mathbb{S}\mathrm{pan}$ *is equivalent to the double category of categories, functors, and lenses:*

$$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{S}\mathrm{pan}) = \mathbb{L}\mathrm{ens}$$

Proof. Follows immediately from Theorem 5.15 and Theorem 3.21. □

5.5 Characterising internal split opfibrations

The goal of this section is to provide two characterisations of *internal split opfibrations* as internal lenses with a certain property.

Recall that an internal discrete opfibration is an internal functor $f\colon A \to B$ such that the following commutative square is a pullback:

$$\begin{array}{ccccc} & & A_1 & & \\ & {\scriptstyle d_0}\swarrow & & \searrow{\scriptstyle f_1} & \\ A_0 & & & & B_1 \\ & {\scriptstyle f_0}\searrow & & \swarrow{\scriptstyle d_0} & \\ & & B_0 & & \end{array}$$

There are two key aspects of this definition:

1. It involves a simple pullback condition on a commutative diagram;

2. It equips the object of morphisms A_1 of the domain with a universal property.

The characterising *property* of an internal discrete opfibration may equivalently be expressed via *unique structure*. An internal discrete opfibration is equivalent to an internal functor $f\colon A \to B$ equipped with a morphism,

$$\varphi_1\colon A_0 \times_{B_0} B_1 \longrightarrow A_1$$

making the following diagrams commute:

$$\begin{array}{ccccc} & & A_0 \times_{B_0} B_1 & & \\ & \swarrow^{\pi_0} & \downarrow{\scriptstyle \varphi_1} & \searrow^{\pi_1} & \\ A_0 & \xleftarrow[d_0]{} & A_1 & \xrightarrow[f_1]{} & B_1 \end{array} \qquad \begin{array}{ccc} & A_0 \times_{B_0} B_1 & \\ \nearrow^{\langle d_0, f_1 \rangle} & & \searrow^{\varphi_1} \\ A_1 & \xrightarrow[1]{} & A_1 \end{array}$$

The two characterisations of internal split opfibrations as internal lenses presented in this section involve:

1. A simple pullback condition on a commutative diagram;
2. A universal property on the object of morphisms A_1 of the domain.

Furthermore, the above properties are expressed equivalently via unique structure. These characterisations completely mirror the definition of an internal discrete opfibration, and provide a deeper insight into lenses between internal categories.

Internal split opfibrations via décalage

In Section 3.5, split opfibrations were characterised using the décalage construction. There is a corresponding construction on internal categories which may be used to define internal split opfibrations.

Definition 5.24. The *internal décalage construction* is a copointed endofunctor D on $\mathcal{C}\mathrm{at}(\mathcal{E})$ which assigns $f\colon A \to B$ to an internal functor $Df\colon DA \to DB$ given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{d_2} & A_1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_1 & \xleftarrow[d_1]{} & B_2 & \xrightarrow[d_2]{} & B_1 \end{array} \tag{5.26}$$

The natural transformation $\varepsilon\colon D \Rightarrow 1$ assigns each internal category A to an internal functor $\varepsilon_A\colon DA \to A$ given by:

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{d_2} & A_1 \\ {\scriptstyle d_0}\downarrow & & {\scriptstyle d_0}\downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ A_0 & \xleftarrow[d_0]{} & A_1 & \xrightarrow[d_1]{} & A_0 \end{array} \tag{5.27}$$

Remark. In general, the internal décalage construction is defined on simplicial objects A in a category $\mathcal{E}$ by forgetting the object A_0 and each morphism labelled d_0 and i_0. From the presentation of an internal category as truncated simplicial object (Definition 5.9) we may recover a simplicial object in $\mathcal{E}$, called the *nerve*, by taking pullbacks of certain morphisms. In this sense, the décalage of a internal category is well-defined, as we may take the décalage of its nerve then truncate.

Consider an internal lens $(f, \varphi)\colon A \to B$ together with its representation (5.25) as a commutative triangle of internal functors, and construct the following commutative diagram in $\mathcal{C}\mathrm{at}(\mathcal{E})$:

$$\begin{array}{ccccc} \Lambda \times_A DA & \xrightarrow{\pi_2} & DA & \xrightarrow{Df} & DB \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon_A} & & \downarrow{\scriptstyle \varepsilon_B} \\ \Lambda & \xrightarrow{\varphi} & A & \xrightarrow{f} & B \end{array} \qquad \overline{f} = f \circ \varphi \colon \Lambda \to B \tag{5.28}$$

The identity-on-objects internal functor $\pi_2\colon \Lambda \times_A DA \to DA$ is given by,

$$\begin{array}{ccccc} A_1 & \xleftarrow{d_1\pi_2} & \Lambda_1 \times_{A_1} A_2 & \xrightarrow{d_2\pi_2} & A_1 \\ {\scriptstyle 1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle 1} \\ A_1 & \xleftarrow[d_1]{} & A_2 & \xrightarrow[d_2]{} & A_1 \end{array}$$

where the object of morphisms $\Lambda_1 \times_{A_1} A_2$ is defined by the pullback:

$$\begin{array}{ccccc} & & \Lambda_1 \times_{A_1} A_2 & & \\ & \swarrow^{\pi_1} & \vee & {}^{\pi_2}\searrow & \\ \Lambda_1 & & & & A_2 \\ & {}_{\varphi_1}\searrow & & \swarrow_{d_0} & \\ & & A_1 & & \end{array} \tag{5.29}$$

The internal functor $Df \circ \pi_2\colon \Lambda \times_A DA \to DB$ is an internal discrete opfibration if and only if the commutative square,

$$\begin{array}{ccccc} & & \Lambda_1 \times_{A_1} A_2 & & \\ & \swarrow^{d_1\pi_2} & & {}^{f_2\pi_2}\searrow & \\ A_1 & & & & B_2 \\ & {}_{f_1}\searrow & & \swarrow_{d_1} & \\ & & B_1 & & \end{array} \tag{5.30}$$

is a pullback. The characterisation of split opfibrations in terms of the décalage construction in Theorem 3.28 motivates the following working definition of internal split opfibration.

Definition 5.25. An *internal split opfibration* is an internal lens $(f, \varphi): A \to B$ such that the following commutative square is a pullback:

$$\begin{array}{ccccc} & & \Lambda_1 \times_{A_1} A_2 & & \\ & {}^{d_1\pi_2}\swarrow & & \searrow^{f_2\pi_2} & \\ A_1 & & & & B_2 \\ & {}_{f_1}\searrow & & \swarrow_{d_1} & \\ & & B_1 & & \end{array}$$

To unpack this definition of an internal split opfibration consider the pullback:

$$\begin{array}{ccccc} & & A_1 \times_{B_1} B_2 & & \\ & {}^{\pi_0}\swarrow & \lrcorner & \searrow^{\pi_1} & \\ A_1 & & & & B_2 \\ & {}_{f_1}\searrow & & \swarrow_{d_1} & \\ & & B_1 & & \end{array} \tag{5.31}$$

The following result characterises internal split opfibrations in terms of unique structure on an internal lens. This characterisation is essentially the same as given in [AU17], except that it is formulated using internal category theory rather than directed containers.

Proposition 5.26. *An internal lens $(f, \varphi): A \to B$ is an internal split opfibration if and only if there exists a morphism,*

$$\psi: A_1 \times_{B_1} B_2 \longrightarrow A_1$$

satisfying the following four commutative diagrams:

- *This diagram specifies a "domain condition":*

$$\begin{array}{ccccc} A_1 \times_{B_1} B_2 & & \xrightarrow{\quad\psi\quad} & & A_1 \\ {\scriptstyle d_0 \times d_0}\downarrow & & & & \downarrow{\scriptstyle d_0} \\ \Lambda_1 & \xrightarrow[\varphi_1]{} & A_1 & \xrightarrow[d_1]{} & A_0 \end{array} \tag{5.32}$$

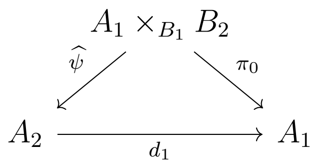
- *This diagram specifies a "uniqueness condition":*

$$\begin{array}{ccccc} \Lambda_1 \times_{A_1} A_2 & \xrightarrow{\pi_1} & A_2 & \xrightarrow{\langle d_1, f_2\rangle} & A_1 \times_{B_1} B_2 \\ {\scriptstyle \pi_1}\downarrow & & & & \downarrow{\scriptstyle \psi} \\ A_2 & & \xrightarrow[d_2]{\qquad\qquad} & & A_1 \end{array} \tag{5.33}$$

- *This diagram specifies a "lifting condition":*

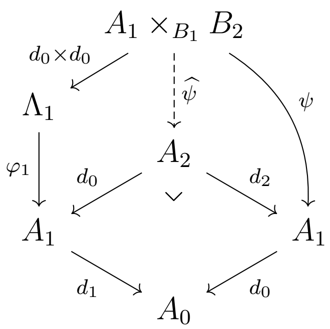
$$\begin{array}{ccc} A_1 \times_{B_1} B_2 & \xrightarrow{\psi} & A_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_2 & \xrightarrow[d_2]{} & B_1 \end{array} \tag{5.34}$$

◇ *This diagram specifies a "composition condition":*



(5.35)

The pullback $\Lambda_1 \times_{A_1} A_2$ *is from* (5.29)*, the pullback* $A_1 \times_{B_1} B_2$ *is from* (5.31)*, and the morphism* $\widehat{\psi}$ *is defined using* (5.32) *via the universal property of the pullback:*



Proof. First notice from Definition 5.25 that an internal lens $(f, \varphi)\colon A \to B$ is an internal split opfibration if and only if there is an isomorphism $\Lambda_1 \times_{A_1} A_2 \cong A_1 \times_{B_1} B_1$. For any internal lens the diagram (5.30) commutes, thus following composite morphism exists by the universal property of the pullback:

$$\Lambda_1 \times_{A_1} A_2 \xrightarrow{\pi_1} A_2 \xrightarrow{\langle d_1, f_2 \rangle} A_1 \times_{B_1} B_2 \tag{5.36}$$

Therefore an internal lens is an internal split opfibration if and only if (5.36) has an inverse. Given a morphism $\psi\colon A_1 \times_{B_1} B_2 \to A_1$ satisfying axiom (5.32) there exists a morphism,

$$\langle d_0 \times d_0, \widehat{\psi} \rangle \colon A_1 \times_{B_1} B_2 \longrightarrow \Lambda_1 \times_{A_1} A_2$$

which is the required inverse if axioms (5.33), (5.34), and (5.35) are satisfied. Conversely, an inverse to (5.36) exists exactly when there is a morphism ψ which satisfies the axioms in the statement of Proposition 5.26. □

Interpreting the above proposition for $\mathcal{E} = \mathcal{S}\mathrm{et}$, the morphism $\psi\colon A_1 \times_{B_1} B_2 \to A_1$ is assigning the *unique filler morphisms* determining the universal property of the chosen lifts of an internal lens to be opcartesian.

Internal split opfibrations via strict factorisation systems

It is well known that a split opfibration factorises each morphism in its domain into a chosen opcartesian lift followed by a vertical morphism with respect to the functor.

This was the main idea behind the characterisation of split opfibrations in Section 4.4, where this process is carried out fibre-wise. There is a corresponding characterisation for internal split opfibrations which uses strict factorisation systems [RW02].

Definition 5.27. A *strict factorisation system* (E, M) on an internal category A consists of a pair of injective-on-morphisms, identity-on-objects internal functors,

$$e\colon E \longrightarrow A \qquad\qquad m\colon M \longrightarrow A$$

together with the pullback,

$$\begin{array}{ccccc} & & E_1 \times_{A_0} M_1 & & \\ & \swarrow^{\pi_0} & & \searrow^{\pi_1} & \\ E_1 & & & & M_1 \\ & \searrow_{d_1} & & \swarrow_{d_0} & \\ & & A_0 & & \end{array} \tag{5.37}$$

such that the morphism $d_1(e_1 \times m_1)\colon E_1 \times_{A_0} M_1 \to A_2 \to A_1$ is an isomorphism.

Given an internal lens $(f, \varphi)\colon A \to B$, the goal is to give a strict factorisation system (Λ, V) on the internal category A, where the internal functor $\varphi\colon \Lambda \to A$ provides the left class, and the right class $j\colon V \to A$ is defined as follows.

Definition 5.28. Given an internal functor $f\colon A \to B$, there is an injective-on-morphisms, identity-on-objects internal functor $j\colon V \to A$ constructed by the following pullback,

$$\begin{array}{ccccc} & & V & & \\ & \swarrow^{j} & & \searrow^{\widehat{f}} & \\ A & & & & B_0 \\ & \searrow_{f} & & \swarrow_{i} & \\ & & B & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} & & V_1 & & \\ & \swarrow^{j_1} & & \searrow^{\widehat{f}_1} & \\ A_1 & & & & B_0 \\ & \searrow_{f_1} & & \swarrow_{i_0} & \\ & & B_1 & & \end{array} \tag{5.38}$$

where $i\colon B_0 \to B$ is the inclusion of the discrete category of objects, and V is the *internal category of vertical morphisms* for an internal functor f. Explicitly, the internal functor $j\colon V \to A$ is given by:

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_0} & V_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle 1} \\ A_0 & \xleftarrow[d_0]{} & A_1 & \xrightarrow[d_1]{} & A_0 \end{array} \tag{5.39}$$

To define the strict factorisation system (Λ, V) on A for an internal lens given by

$(f, \varphi)\colon A \to B$, first construct the following pullback:

$$\begin{tikzcd}[column sep=small] & \Lambda_1 \times_{A_0} V_1 \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ \Lambda_1 \arrow[dr, "p_1"'] & & V_1 \arrow[dl, "d_0"] \\ & A_0 & \end{tikzcd} \tag{5.40}$$

The following result provides the unique structure such that $d_1(\varphi_1 \times j_1)\colon \Lambda_1 \times_{A_0} V_1 \to A_1$ is an isomorphism.

Proposition 5.29. *If $(f, \varphi)\colon A \to B$ is an internal lens, then (Λ, V) is a strict factorisation system on A consisting of the internal functors $\varphi\colon \Lambda \to A$ and $j\colon V \to A$ if and only if there exists an endomorphism,*

$$\chi\colon A_1 \longrightarrow A_1$$

satisfying the following four commutative diagrams:

- *This diagram specifies a "domain condition":*

$$\begin{tikzcd} A_1 \arrow[rr, "\chi"] \arrow[d, "\langle d_0, f_1\rangle"'] & & A_1 \arrow[d, "d_0"] \\ \Lambda_1 \arrow[r, "\varphi_1"'] & A_1 \arrow[r, "d_1"'] & A_0 \end{tikzcd} \tag{5.41}$$

- *This diagram specifies a "uniqueness condition":*

$$\begin{tikzcd} \Lambda_1 \times_{A_0} V_1 \arrow[r, "\varphi_1 \times j_1"] \arrow[d, "\pi_1"'] & A_2 \arrow[r, "d_1"] & A_1 \arrow[d, "\chi"] \\ V_1 \arrow[rr, "j_1"'] & & A_1 \end{tikzcd} \tag{5.42}$$

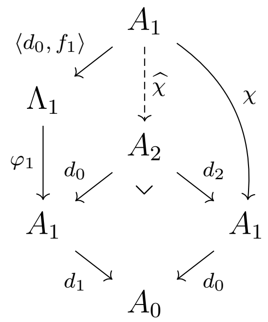
- *This diagram specifies a "fibre condition":*

$$\begin{tikzcd} A_1 \arrow[rr, "\chi"] \arrow[d, "f_1"'] & & A_1 \arrow[d, "f_1"] \\ B_1 \arrow[r, "d_1"'] & B_0 \arrow[r, "i_0"'] & B_1 \end{tikzcd} \tag{5.43}$$

- *This diagram specifies a "composition condition":*

$$\begin{tikzcd} & A_1 \arrow[dl, "\widehat{\chi}"'] \arrow[dr, "1"] & \\ A_2 \arrow[rr, "d_1"'] & & A_1 \end{tikzcd} \tag{5.44}$$

The pullback $\Lambda_1 \times_{A_0} V_1$ is defined in (5.40), and the morphism $\widehat{\chi}$ is defined using (5.41) via the universal property of the pullback:



Proof. By Definition 5.27 the pair (Λ, V) is a strict factorisation system on A if and only if the morphism $d_1(\varphi_1 \times j_1)\colon \Lambda_1 \times_{A_0} V_1 \to A_1$ has an inverse.

Given a morphism $\chi\colon A_1 \to A_1$ satisfying axioms (5.41) and (5.43) there is a morphism,

$$\Big\langle \langle d_0, f_1 \rangle, \langle \chi, d_1 f_1 \rangle \Big\rangle \colon A_1 \longrightarrow \Lambda_1 \times_{A_0} V_1$$

which is the required inverse if axioms (5.42) and (5.44) are satisfied. Conversely, an inverse to $d_1(\varphi_1 \times j_1)\colon \Lambda_1 \times_{A_0} V_1 \to A_1$ exists exactly when there is a morphism χ which satisfies the axioms in the statement of Proposition 5.29. □

The important aspect of the above strict factorisation is that it equips the object of morphisms A_1 of an internal lens with the universal property of a pullback. The following theorem shows that this result may be used to characterise internal split opfibrations.

Theorem 5.30. *An internal lens $(f, \varphi)\colon A \to B$ is an internal split opfibration if and only if the pair (Λ, V) is a strict factorisation system on A.*

Proof. Given an internal split opfibration $(f, \varphi)\colon A \to B$ equipped with a morphism $\psi\colon A_1 \times_{A_0} B_2 \to A_1$ as in Proposition 5.26, define a morphism $\chi'\colon A_1 \to A_1$ as the following composite:

$$A_1 \xrightarrow{\langle 1, i_0 f_1 \rangle} A_1 \times_{B_1} B_2 \xrightarrow{\quad \psi \quad} A_1 \tag{5.45}$$

It may be shown that χ' satisfies the axioms of Proposition 5.29 and thus yields a strict factorisation system (Λ, V) on A.

Conversely, given an internal lens $(f, \varphi)\colon A \to B$ equipped with an endomorphism $\chi\colon A_1 \to A_1$ as in Proposition 5.29, define a morphism $\psi'\colon A_1 \times_{B_1} B_2 \to A_1$ as the following composite:

$$A_1 \times_{B_1} B_2 \xrightarrow{\langle \varphi_1 p_2 \langle d_0 \times d_0, \pi_1 \rangle, \chi \pi_0 \rangle} A_2 \xrightarrow{\quad d_1 \quad} A_1 \tag{5.46}$$

It may be shown that ψ' satisfies the axioms of Proposition 5.26 and thus yields an internal split opfibration. □

In summary, internal split opfibrations admit two characterisations as internal lenses with a certain property:

1. A simple pullback condition on a commutative diagram, using the décalage construction;

2. A universal property on the object of morphisms A_1 of the domain, using strict factorisation systems.

The above properties may equivalently be expressed via unique structure, and are entirely analogous to the way in which internal discrete opfibrations are defined.

Chapter 6

Lenses as algebras and coalgebras

The aim of this chapter is to characterise lenses as both algebras for a monad and coalgebras for a comonad. Formally, this amounts to proving that the forgetful functor $\mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{of}$, assigning a lens to its underlying cofunctor, is comonadic, and that the forgetful functor $\mathcal{L}\mathrm{ens} \to \mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$, assigning a lens to its underlying functor, is monadic.

The question of whether lenses are algebras or coalgebras has long been asked. For the classical state-based lenses (Example 2.17), Johnson, Rosebrugh, and Wood demonstrated a characterisation as algebras [JRW10], and independently, O'Connor determined a characterisation as coalgebras [OCo11]. These two approaches were later shown to be closely related [GJ12], and the results were generalised to state-based lenses between *objects* in a suitable category rather than merely between *sets*.

Instead of taking a concrete approach to characterising (delta) lenses as algebras and coalgebras, Section 6.1 first introduces a setting $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ in which *generalised lenses* between objects are defined. The main example specialises to $\mathcal{C} = \mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations to yield the diagrammatic formulation of lenses. By gradually adding axioms to the setting $(\mathcal{C}, \mathcal{W}, \mathcal{M})$, each of which are satisfied by the main example, clear and precise sufficient conditions are identified for (generalised) lenses to arise as algebras or coalgebras.

Section 6.2 identifies two basic axioms on $(\mathcal{C}, \mathcal{W}, \mathcal{M})$, from which it is shown that (generalised) lenses are coalgebras for a comonad (Theorem 6.15), and the cofree lens on a cofunctor is constructed. Section 6.3 introduces three additional axioms on $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ such that the corresponding forgetful functor for generalised lenses has a left adjoint (Theorem 6.24). In the main example, this forgetful functor is then proved to be monadic (Theorem 6.30), and the free lens on a functor is constructed.

The chapter concludes with several interesting results arising from the study of generalised lenses. In Section 6.4, it is shown that the double category $\mathbb{L}\mathrm{ens}$ corresponds to an algebraic weak factorisation system on $\mathcal{C}\mathrm{at}$, while in Section 6.5, the notions of change of base and generalised split opfibrations are explored.

6.1 Preliminaries

For a category $\mathcal{C}$ with pullbacks, let $\mathbb{S}\mathrm{pan}(\mathcal{C})$ be the double category of spans. Suppose that $\mathcal{W}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{C}$ which each satisfy the following properties:

(A1) The class contains the isomorphisms;

(A2) The class is closed under composition;

(A3) The class is stable under pullback along morphisms in $\mathcal{C}$.

Notation 6.1. The following decorations are used to denote the arrows in these classes:

$$\bullet \xrightarrow{\sim} \bullet \ \in \mathcal{W} \qquad\qquad \bullet \rightarrowtail \bullet \ \in \mathcal{M}$$

Under additional axioms to be introduced in the following sections, $\mathcal{W}$ is shown to be a class of *weak equivalences* (that is, additionally satisfies *2-out-of-3*), and $\mathcal{M}$ will be the *right class* of an orthogonal factorisation system.

Definition 6.2. Let $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ be the full double subcategory of $\mathbb{S}\mathrm{pan}(\mathcal{C})$ determined by the collection of spans whose left leg is in the class $\mathcal{W}$ and whose right leg is in the class $\mathcal{M}$, as depicted below.

$$\bullet \xleftarrow{\sim} \bullet \rightarrowtail \bullet$$

The corresponding internal category presentation is given by:

$$\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \times_{\mathcal{C}\mathrm{at}} \mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \xrightarrow{\text{comp}} \mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \underset{\text{cod}}{\overset{\text{dom}}{\underset{\leftarrow \text{id} -}{\rightrightarrows}}} \mathcal{C}$$

The vertical composition of spans in the double category $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is well-defined by axioms (A2) and (A3), and is horizontally invariant by axiom (A1) as every horizontal isomorphism has a companion and a conjoint. Indeed, the classes of conjoints and companions are given by $\mathcal{W}$ and $\mathcal{M}$, respectively; this can be seen by direct comparison with conjoints and companions in $\mathbb{S}\mathrm{pan}(\mathcal{C})$ in Example A.13. Apart from flatness, the double category $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ has many of the properties that were proven of $\mathbb{C}\mathrm{of}$ in Chapter 3.

Example 6.3. For the category $\mathcal{C} = \mathcal{C}\mathrm{at}$, let $\mathcal{W}$ be the class of bijective-on-objects functors and let $\mathcal{M}$ be the class of discrete opfibrations. These classes are easily shown to satisfy axioms (A1)–(A3). By Theorem 3.24, it follows that $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \simeq \mathbb{C}\mathrm{of}$.

In Theorem 3.21, the double category of lenses is shown to be the right-connected completion of the double category of cofunctors. This result motivates the following definition for generalised lenses.

Definition 6.4. Let the double category $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ of *generalised lenses* be the right-connected completion of $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$. The corresponding internal category presentation is given by:

$$\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \times_{\mathcal{C}\mathrm{at}} \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \xrightarrow{\text{comp}} \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} \mathcal{C} \quad (\xleftarrow{\text{id}})$$

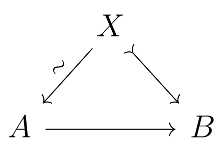
By construction of the right-connected completion (Definition 3.12), the vertical morphisms in $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ are given by commutative diagrams in $\mathcal{C}$ of the form,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \wr\uparrow & & \| \\ X & \rightarrowtail & B \\ \downarrow & & \| \\ B & = & B \end{array} \tag{6.1}$$

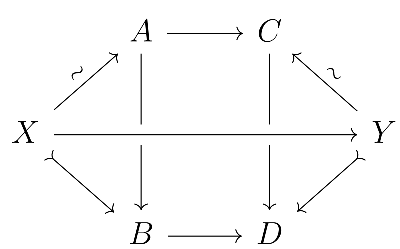
while cells in $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ are given by cells in $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ such that the following condition holds:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & D \\ \wr\uparrow & & \| & & \| \\ X & \rightarrowtail & B & \longrightarrow & D \\ \downarrow & & \| & & \| \\ B & = & B & \longrightarrow & D \end{array} \quad = \quad \begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & D \\ \wr\uparrow & & \wr\uparrow & & \| \\ X & \longrightarrow & Y & \rightarrowtail & D \\ \downarrow & & \downarrow & & \| \\ B & \longrightarrow & D & = & D \end{array}$$

The vertical morphisms in $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ will be called (*generalised*) *lenses*, and are often simply depicted as a commutative triangle in $\mathcal{C}$ of the form:



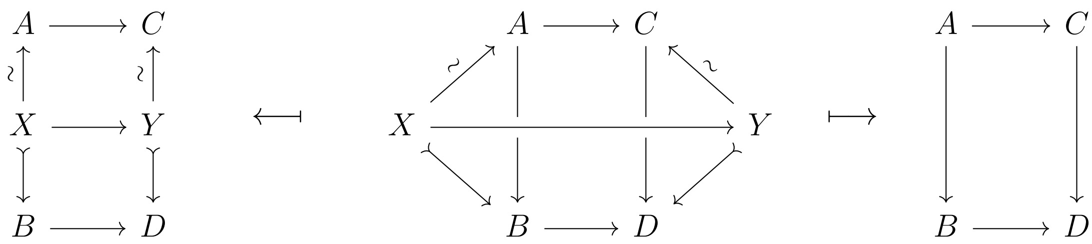
Similarly, cells in $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ can be depicted in the following way:



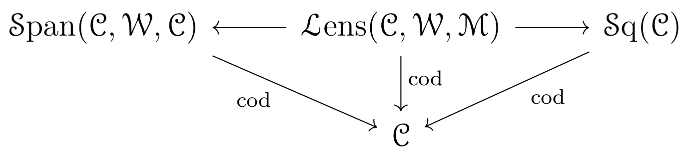
Example 6.5. For the category $\mathcal{C} = \mathcal{C}\text{at}$, let $\mathcal{W}$ be the class of bijective-on-objects functors and let $\mathcal{M}$ be the class of discrete opfibrations. Then generalised lenses are precisely lenses by Proposition 2.13, and there is an equivalence of double categories $\mathbb{L}\text{ens} \simeq \mathbb{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ by Theorem 3.26.

The classes of companions and conjoints are given by $\mathcal{M}$ and $\text{Iso}(\mathcal{C})$, respectively, by Lemma 3.17 and Lemma 3.18. Tabulators are inherited from $\mathbb{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ by Lemma 3.19. Furthermore, since $\mathbb{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is right-connected, the identity map $\text{id} \colon \mathcal{C} \to \mathbb{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ has a left adjoint, and therefore all cotabulators exist.

The construction of $\mathbb{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ as a right-connected completion yields forgetful double functors to $\mathbb{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ and $\mathbb{S}\text{q}(\mathcal{C})$ with assignment on cells given by:



Underlying these double functors are ordinary functors over the category $\mathcal{C}$ given by:



The aim of the following two sections is to show that, under certain conditions, these forgetful functors are comonadic and monadic, respectively, in the slice over $\mathcal{C}$. Thus (generalised) lenses may be understood as coalgebras for a comonad, and algebras for monad.

6.2 Lenses as coalgebras for a comonad

The goal of this section is to show that the forgetful functor,

$$\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \longrightarrow \mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$$

is *comonadic* over $\mathcal{C}$ with respect to the codomain functors. However, it is not possible to establish this result for generalised lenses with just the axioms (A1)–(A3) introduced in Section 6.1. To prove this comonadicity, two additional conditions (which are satisfied for the $\mathcal{C} = \mathcal{C}\text{at}$ case) are required on the class of morphisms $\mathcal{W}$:

(A4) The canonical inclusion functor $\mathcal{S}\text{q}(\mathcal{C}, \mathcal{W}) \to \mathcal{S}\text{q}(\mathcal{C})$ creates pullbacks.

(A5) The functor $\mathrm{dom}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W}) \to \mathcal{C}$ has a right adjoint right inverse.

Axiom (A4) is a limit closure property on the class of morphisms $\mathcal{W}$, which asserts that given a diagram in $\mathcal{C}$ of the form,

$$\begin{array}{ccccc} A & \longrightarrow & X & \longleftarrow & C \\ \wr\downarrow & & \downarrow\wr & & \downarrow\wr \\ B & \longrightarrow & Y & \longleftarrow & D \end{array}$$

the unique morphism $A \times_X C \to B \times_Y D$ is in the class of morphisms $\mathcal{W}$.

Axiom (A5) introduces additional structure on the category $\mathcal{C}$ which behaves well with the class of morphisms $\mathcal{W}$. Having a right inverse means that for every morphism $A \to B$ there is a commutative square in $\mathcal{C}$ denoted:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \wr\downarrow & & \downarrow\wr \\ A_\infty & \longrightarrow & B_\infty \end{array} \tag{6.2}$$

Being a right adjoint means that for every morphism $X \to A$ in $\mathcal{W}$, there exists a unique morphism $A \to X_\infty$ such that the following diagram commutes:

$$\begin{array}{ccc} X & = \!= & X \\ \wr\downarrow & & \downarrow\wr \\ A & \underset{\exists!}{\dashrightarrow} & X_\infty \end{array} \tag{6.3}$$

This diagram describes the component of the unit of the adjunction at $X \to A$. An object A in $\mathcal{C}$ is called *codiscrete* if if $A \cong X_\infty$.

Example 6.6. For the category $\mathcal{C} = \mathcal{C}\mathrm{at}$, the class $\mathcal{W}$ of bijective-on-objects functors satisfies the axioms (A4) and (A5). The objects X_∞ are precisely the codiscrete categories.

Proposition 6.7. *Let* $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *be a triple satisfying axioms (A1)-(A5). Then the codomain map* $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *has a right adjoint right inverse, and for each object* B *in* $\mathcal{C}$*, the fibre with respect to the codomain map* $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *has finite products.*

Proof. Using (A5) and (6.2), the right inverse $\mathcal{C} \to \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ assigns to each morphism $A \to B$ a morphism given by:

$$\begin{array}{ccc} A_\infty & \longrightarrow & B_\infty \\ \wr\uparrow & & \uparrow\wr \\ A & \longrightarrow & B \\ \| & & \| \\ A & \longrightarrow & B \end{array}$$

Using (6.3), the component of the unit for the adjunction at a span $A \leftarrow X \rightarrow B$ is the cell in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ defined by:

$$\begin{array}{ccc} A & \longrightarrow & B_\infty \\ \wr\uparrow & & \uparrow\wr \\ X & \rightarrowtail & B \\ \downarrow & & \| \\ B & = & B \end{array} \quad = \quad \begin{array}{ccccc} A & \overset{\exists!}{\dashrightarrow} & X_\infty & \longrightarrow & B_\infty \\ \wr\uparrow & & \uparrow\wr & & \uparrow\wr \\ X & = & X & \rightarrowtail & B \\ \downarrow & & \downarrow & & \| \\ B & = & B & = & B \end{array}$$

The above diagram demonstrates that the fibre $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ over an object B with respect to the codomain functor has a terminal object.

Moreover, binary products in $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ are computed by pullback over the terminal object. For a cospan of morphisms in $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by,

$$\begin{array}{ccccc} A & \longrightarrow & B_\infty & \longleftarrow & C \\ \wr\uparrow & & \uparrow\wr & & \uparrow\wr \\ X & \rightarrowtail & B & \leftarrowtail & Y \\ \downarrow & & \| & & \downarrow \\ B & = & B & = & B \end{array}$$

using (A4) their pullback (that is, their binary product) is given by:

$$\begin{array}{ccccc} A & \longleftarrow & A \times_{B_\infty} C & \longrightarrow & C \\ \wr\uparrow & & \uparrow\wr & & \uparrow\wr \\ X & \leftarrowtail & X \times_B Y & \rightarrowtail & Y \\ \downarrow & & \| & & \downarrow \\ B & = & B & = & B \end{array}$$

Therefore each fibre $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ has finite products. □

Collection of minor properties

This subsection collects a number of interesting properties of the class $\mathcal{W}$ which follow from axioms (A4) and (A5), but which are not required for the main result of this section.

Lemma 6.8. *Let $\mathcal{C}$ be a category with a class of morphisms $\mathcal{W}$ satisfying axioms (A4) and (A5). Then each fibre of the domain functor* $\mathrm{dom}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W}) \to \mathcal{C}$ *has finite limits.*

Proof. By axiom (A4), each fibre has pullbacks. By axiom (A5), each fibre has a terminal object; for example, the universal property of the terminal object for the fibre over X is exhibited in (6.3). Therefore, each fibre has finite limits. □

Lemma 6.9. *Let $\mathcal{C}$ be a category with a class of morphisms $\mathcal{W}$ satisfying axioms (A4) and (A5). Then $\mathcal{W}$ is the left class of an orthogonal factorisation system on $\mathcal{C}$.*

Proof. Given a morphism $A \to B$ in $\mathcal{C}$, consider the cospan in $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W})$ given by the following commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & B & = & B \\ {\scriptstyle\wr}\downarrow & & \downarrow{\scriptstyle\wr} & & \| \\ A_\infty & \longrightarrow & B_\infty & \longleftarrow & B \end{array}$$

Then there is a factorisation given by the universal property of the pullback whose left factor is in $\mathcal{W}$ by (A4):

$$\begin{array}{ccccc} A & = & A & \longrightarrow & B \\ {\scriptstyle\wr}\downarrow & & \downarrow{\scriptstyle\wr} & & \| \\ A_\infty & \xleftarrow{\sim} & A_\infty \times_{B_\infty} B & \longrightarrow & B \end{array} \tag{6.4}$$

The right class of the factorisation system is characterised by those morphisms for which the square (6.2) is a pullback. It is not difficult to show that these classes of morphisms are orthogonal, and therefore form an orthogonal factorisation system. □

Lemma 6.10. *Let $\mathcal{C}$ be a category with a class of morphisms $\mathcal{W}$ satisfying axioms (A4) and (A5). Then the class $\mathcal{W}$ satisfies the two-out-of-three property.*

Proof. Consider morphisms $f\colon A \to B$ and $g\colon B \to C$. If $f, g \circ f \in \mathcal{W}$, then $g \in \mathcal{W}$ since $\mathcal{W}$ is the left class of an orthogonal factorisation system. If $g, g \circ f \in \mathcal{W}$, then $f \in \mathcal{W}$ using (A4). □

Remark. Since $\mathcal{W}$ contains the isomorphisms, by (A1), and satisfies two-out-of-three, by Lemma 6.10, it is a class of *weak equivalences*. This justifies the decoration for the arrows which is traditionally used to denote the weak equivalences in a model structure.

Lemma 6.11. *Let $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ be a triple satisfying axioms (A1)-(A5). Then the functor* $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is a fibration.*

Proof. Given an object $C \leftarrow Y \to D$ in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ and a morphism $B \to D$ in $\mathcal{C}$, there is a morphism in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by,

$$\begin{array}{ccc} Q & \longrightarrow & C \\ {\scriptstyle\wr}\uparrow & \text{o.f.s.} & \uparrow{\scriptstyle\wr} \\ B \times_D Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

where Q is the image with respect to the orthogonal factorisation system in Lemma 6.9. It is straightforward to verify that this is a cartesian lift. □

Constructing the right adjoint

This subsection constructs a right adjoint to the forgetful functor,

$$\begin{array}{ccc} \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) & \longrightarrow & \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \\ & {\scriptstyle \mathrm{cod}} \searrow \quad \swarrow {\scriptstyle \mathrm{cod}} & \\ & \mathcal{C} & \end{array} \tag{6.5}$$

in the slice 2-category $\mathcal{C}\mathrm{AT}/\mathcal{C}$. Assume throughout that $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is a triple which satisfies the axioms (A1)–(A5).

In Proposition 6.7 it was shown that for each object B, the category $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ has finite products constructed by taking the pullback over the terminal object. The main idea in constructing the right adjoint is that for each object in $\mathcal{S}\mathrm{pan}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$ one takes the product with the identity span over B to yield a generalised lens over B.

In detail, consider an object in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by,

$$A \xleftarrow{\sim} X \rightarrowtail B$$

together with the canonical cospan of morphisms in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by the commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & B_\infty & \xleftarrow{\sim} & B \\ \wr\uparrow & & \uparrow\wr & & \| \\ X & \rightarrowtail & B & = & B \\ \downarrow & & \| & & \| \\ B & = & B & = & B \end{array}$$

By Proposition 6.7, taking the pullback of this cospan yields a span of morphisms in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by:

$$\begin{array}{ccccc} A & \xleftarrow{\sim} & A \times_{B_\infty} B & \longrightarrow & B \\ \wr\uparrow & & \uparrow\wr & & \| \\ X & = & X & \rightarrowtail & B \\ \downarrow & & \downarrow & & \| \\ B & = & B & = & B \end{array} \tag{6.6}$$

The right-hand side of this diagram is of the form (6.1), and therefore is a lens with codomain B.

Lemma 6.12. *There is a functor* $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *with an assignment on objects given by:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle\wr}\swarrow & & \searrow \\ A & & B \end{array} \quad \longmapsto \quad \begin{array}{ccc} & X & \\ {\scriptstyle\wr}\swarrow & & \searrow \\ A \times_{B_\infty} B & \longrightarrow & B \end{array}$$

Proof. The assignment on objects is constructed in (6.6), and using the universal property of the pullback it is straightforward to show this extends to a functorial assignment on morphisms. □

Proposition 6.13. *The functor* $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *has a right adjoint.*

Proof. The right adjoint $\mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is defined in Lemma 6.12. The component of the counit at an object in $\mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is constructed in (6.6) from the left projection:

$$\begin{array}{ccc} A \times_{B_\infty} B & \xrightarrow{\sim} & A \\ \wr\uparrow & & \wr\uparrow \\ X & = & X \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

The component of the unit at an object in $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is constructed using the universal property of the pullback:

$$\begin{array}{ccccc} & A & \overset{\sim}{\dashrightarrow} & A \times_{B_\infty} B & \\ {\scriptstyle\sim}\nearrow & | & & | & \nwarrow{\scriptstyle\sim} \\ X & = & = & = & X \\ \searrow & \downarrow & & \downarrow & \swarrow \\ & B & = & B & \end{array}$$

It is not difficult to show that components of the unit and counit are natural and satisfy the triangle identities for an adjunction. □

Note that whiskering the unit natural transformation with the codomain map $\text{cod}\colon \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ yields the identity natural transformation; similarly for the counit. The following result is immediate.

Corollary 6.14. *The morphism in* $\mathcal{C}\text{AT}/\mathcal{C}$ *shown below has a right adjoint.*

$$\begin{array}{ccc} \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) & \longrightarrow & \mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \\ & {\scriptstyle\text{cod}}\searrow \quad \swarrow{\scriptstyle\text{cod}} & \\ & \mathcal{C} & \end{array}$$

Comonadicity

Theorem 6.15. *The forgetful functor* $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *is comonadic.*

Proof. A coalgebra for the comonad induced by the adjunction in Proposition 6.13 is given by a section to the component of the counit in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$:

$$\begin{array}{ccccc} A & \overset{\sim}{\dashrightarrow} & A \times_{B_\infty} B & \overset{\sim}{\longrightarrow} & A \\ \wr\uparrow & & \wr\uparrow & & \uparrow\wr \\ X & = & X & = & B \\ \downarrow & & \downarrow & & \downarrow \\ B & = & B & = & B \end{array}$$

However, by the universal property of the pullback, the coalgebra map above is equivalent to the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \wr\uparrow & & \| \\ X & \rightarrowtail & B \\ \downarrow & & \| \\ B & = & B \end{array}$$

This is precisely an object in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$. Moreover, it can be shown that compatibility of the coalgebra with the comultiplication of the comonad adds no further conditions. Likewise, there is a correspondence between morphisms of coalgebras and morphisms in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$. Therefore, $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is equivalent to the category of coalgebras and the forgetful functor is comonadic. □

The cofree lens on a cofunctor

Consider a cofunctor $(f, \varphi)\colon A \to B$ as an object in the category,

$$\mathcal{C}\mathrm{of} = \mathcal{S}\mathrm{pan}(\mathcal{C}\mathrm{at}, \mathcal{W}, \mathcal{M})$$

where $\mathcal{W}$ is the class of bijective-on-objects functors and $\mathcal{M}$ is the class of discrete opfibrations.

Example 6.16. The *cofree lens on a cofunctor* $(f, \varphi)\colon A \to B$ is a lens with:

- domain category $A \times_{B_\infty} B$ which has the same objects as A and morphisms given by pairs $(w\colon a \to a' \in A, u\colon fa \to fa' \in B)$;
- functor component given by the projection $\pi\colon A \times_{B_\infty} B \to B$ which sends a morphism above to $u\colon fa \to fa'$;
- lifting operation given by $(a, u\colon fa \to b) \longmapsto (\varphi(a, u), u)$.

Intuitively, the reason why a cofunctor is not a lens is that it does not "know" how to send morphisms from the domain to codomain. The cofree lens solves this problem universally through constructing a functor with the same underlying object assignment as the cofunctor, whose assignment on morphisms is given trivially by projection, while essentially keeping the same lifting operation.

6.3 Lenses as algebras for a monad

The goal of this section is to show that the forgetful functor,

$$\begin{array}{ccc} \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) & \xrightarrow{\;U\;} & \mathcal{S}\mathrm{q}(\mathcal{C}) \\ & {}_{\mathrm{cod}}\searrow \quad \swarrow_{\mathrm{cod}} & \\ & \mathcal{C} & \end{array} \tag{6.7}$$

has a left adjoint in the slice 2-category $\mathcal{C}\mathrm{AT}/\mathcal{C}$, and is moreover *monadic* in the case where $\mathcal{C} = \mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations. In other words, lenses are characterised as algebras for a monad. A useful corollary is the construction of the free lens on a functor over a fixed codomain.

There are three conditions on the triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ which will be required in order to prove that the left adjoint exists.

(A6) The functor $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W}) \to \mathcal{C}$ has a left adjoint right inverse.

(A7) There is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$, where the arrows in the left class $\mathcal{E}$ are denoted:

$$\bullet \longrightarrow\!\!\!\!\!\rightarrow \bullet \quad \in \mathcal{E}$$

(A8) Pushouts of morphisms in $\mathcal{W}$ along morphisms in $\mathcal{E}$ exist.

Let us consider each of these conditions in more detail before progressing to the main results.

Axiom (A6) introduces additional structure on the category $\mathcal{C}$ which behaves well with the class of morphisms $\mathcal{W}$. The functor $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W}) \to \mathcal{C}$ having a right inverse means that for every morphism $f\colon A \to B$ in $\mathcal{C}$, there is a morphism in $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W})$ denoted:

$$\begin{array}{ccc} A_0 & \xrightarrow{\;f_0\;} & B_0 \\ \wr\downarrow & & \downarrow\wr \\ A & \xrightarrow[\;f\;]{} & B \end{array} \tag{6.8}$$

Moreover, being a left adjoint means that for every morphism $X \to A$ in $\mathcal{W}$, that is, an object in $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{W})$, there exists a unique morphism $A_0 \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} A_0 & \overset{\exists!}{\dashrightarrow} & X \\ {\scriptstyle\wr}\downarrow & & \downarrow{\scriptstyle\wr} \\ A & = & A \end{array} \tag{6.9}$$

This diagram describes the component of the counit of the adjunction at $X \to A$.

Axiom (A7) means that every morphism $A \to B$ in $\mathcal{C}$ has a unique factorisation (up to unique isomorphism) into a morphism in $\mathcal{E}$ followed by a morphism in $\mathcal{M}$,

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow\!\!\!\!\searrow \quad \nearrow & \\ & \bullet & \end{array}$$

and for each solid commutative square in $\mathcal{C}$ of the form,

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle e}\downarrow\!\!\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle m} \\ B & \xrightarrow[g]{} & D \end{array} \tag{6.10}$$

there exists a unique morphism h such that $f = h \circ e$ and $g = m \circ h$.

The left class $\mathcal{E}$ and the right class $\mathcal{M}$ of an orthogonal factorisation system enjoy several nice properties. If $g \circ f \in \mathcal{E}$ and $f \in \mathcal{E}$, then $g \in \mathcal{E}$; dually, if $g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$, then $f \in \mathcal{M}$. The left class $\mathcal{E}$ is stable under pushout along morphisms in $\mathcal{C}$, while the right class $\mathcal{M}$ is stable under pullback along morphisms in $\mathcal{C}$, a condition already required of $\mathcal{M}$ by (A3).

Axiom (A8) means that every cospan as follows has a pushout:

$$\bullet \twoheadleftarrow \bullet \xrightarrow{\sim} \bullet$$

Moreover, since both $\mathcal{E}$ and $\mathcal{W}$ are left classes of a factorisation system (see Lemma 6.9) they are stable under pushout, so we obtain a commutative square:

$$\begin{array}{ccc} \bullet & \xrightarrow{\sim} & \bullet \\ \downarrow\!\!\downarrow & \ulcorner & \downarrow\!\!\downarrow \\ \bullet & \xrightarrow[\sim]{} & \bullet \end{array}$$

We now check that these three conditions hold for our main example.

Example 6.17. Let $\mathcal{C} = \mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations.

- Axiom (A6): For each category A there is a *discrete category* A_0, together with an identity-on-objects functor $A_0 \to A$. By functoriality this assignment satisfies (6.8), while (6.9) is precisely the universal property for bijective-on-objects functors.

- Axiom (A7): Discrete opfibrations are the right class of the *comprehensive factorisation system* [SW73], whose left class $\mathcal{E}$ is precisely the class of *initial functors* (that is, those functors $f\colon A \to B$ such that for each $b \in B$ the comma category f/b is connected).

- Axiom (A8): All pushouts in $\mathcal{C}\mathrm{at}$ exist.

Assumption. For the remainder of this chapter, assume that the axioms (A1)–(A8) hold unless otherwise stated.

We now consider a couple of statements which may be proven using these axioms, and which will be essential in constructing the left adjoint to the functor U in (6.7).

Proposition 6.18. *The functor* $\mathrm{dom}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *has a left adjoint right inverse.*

Proof. Given a morphism $f\colon A \to B$ in $\mathcal{C}$, there is a morphism in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ \wr\uparrow & & \uparrow\wr \\ A_0 & \xrightarrow{\;f_0\;} & B_0 \\ \| & & \| \\ A_0 & \xrightarrow[\;f_0\;]{} & B_0 \end{array} \tag{6.11}$$

This assignment is functorial and right inverse to the domain map using (6.8), and is moreover left adjoint using (6.9), with the counit of the adjunction given by the commutative diagram:

$$\begin{array}{ccc} A & = & A \\ \wr\uparrow & & \uparrow\wr \\ A_0 & \overset{\exists!}{\dashrightarrow} & X \\ \| & & \downarrow \\ A_0 & \longrightarrow & B \end{array} \tag{6.12}$$

Therefore the functor $\mathrm{dom}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ has a left adjoint right inverse. □

Corollary 6.19. *For each morphism $f\colon A \to B$, there is a morphism in* $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *given by:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\wr}\uparrow & & \Vert \\ A_0 & \longrightarrow & B \\ \Vert & & \Vert \\ A_0 & \longrightarrow & B \end{array} \tag{6.13}$$

Proof. Apply the counit (6.12) to the identity span on B, and then compose with the morphism (6.11) in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$. □

In Lemma 6.11 it was shown that the codomain map $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is a fibration. The theorem below shows that this functor is also an opfibration.

Proposition 6.20. *The functor* $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is an opfibration.*

Proof. We demonstrate the existence and universal property of weakly opcartesian lifts, and note that these morphisms are closed under composition, from which it follows that all opcartesian lifts exist.

Given an object $A \leftarrow X \to B$ in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ and a morphism $B \to D$ in $\mathcal{C}$, there is a morphism in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ constructed from (A7) and (A8) as follows,

$$\begin{array}{ccc} A & \longrightarrow & A +_X J \\ {\scriptstyle\wr}\uparrow & \lrcorner & \uparrow{\scriptstyle\wr} \\ X & \twoheadrightarrow & J \\ \downarrow & \text{o.f.s.} & \downarrow \\ B & \longrightarrow & D \end{array}$$

where J is the image of the composite morphism $X \to B \to D$ with respect to the $(\mathcal{E}, \mathcal{M})$-factorisation.

Given any other morphism over $B \to D$, there is a unique factorisation below using the universal properties of the orthogonal factorisation system (6.10) and pushouts:

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle\wr}\uparrow & & \uparrow{\scriptstyle\wr} \\ X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \quad = \quad \begin{array}{ccccc} A & \longrightarrow & A +_X J & \overset{\exists!}{\dashrightarrow} & C \\ {\scriptstyle\wr}\uparrow & \lrcorner & \uparrow{\scriptstyle\wr} & & \uparrow{\scriptstyle\wr} \\ X & \twoheadrightarrow & J & \overset{\exists!}{\dashrightarrow} & Y \\ \downarrow & \text{o.f.s.} & \downarrow & & \downarrow \\ B & \longrightarrow & D & = & D \end{array} \tag{6.14}$$

Finally, by the pushout pasting property and the uniqueness of factorisations, weakly opcartesian morphisms for the codomain map are closed under composition. Therefore, the functor $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is an opfibration. □

Corollary 6.21. *The functor* cod: $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is a bifibration.*

Proof. Follows immediately from Lemma 6.11 and Proposition 6.20. □

Using Proposition 6.20 and Proposition 6.18 (in particular, Corollary 6.19), we may construct the left adjoint to the functor (6.7). In preparation we introduce some useful notation, which is chosen to provide some consistency with the next section where algebraic weak factorisation systems [BG16a] will be studied.

Notation 6.22. Given a morphism $f \colon A \to B$, denote the factorisation (6.14) of the cell (6.13) as follows:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\wr\uparrow & & \| \\
A_0 & \longrightarrow & B \\
\| & & \| \\
A_0 & \longrightarrow & B
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{Lf} \!\!\!\!\rightarrow & Ef & \overset{Rf}{\dashrightarrow} & B \\
\wr\uparrow & \lrcorner & \uparrow\wr & & \| \\
A_0 & \xrightarrow{e_f} \!\!\!\!\rightarrow & J_f & \overset{m_f}{\rightarrowtail} & B \\
\| & \text{o.f.s.} & \downarrow{\scriptstyle m_f} & & \| \\
A_0 & \longrightarrow & B & = & B
\end{array}
$$

Iterating this construction with the morphism $Rf \colon Ef \to B$ produces the following:

$$
\begin{array}{ccc}
Ef & \xrightarrow{Rf} & B \\
\wr\uparrow & & \| \\
(Ef)_0 & \longrightarrow & B \\
\| & & \| \\
(Ef)_0 & \longrightarrow & B
\end{array}
\quad = \quad
\begin{array}{ccccc}
Ef & \xrightarrow{LRf} \!\!\!\!\rightarrow & ERf & \overset{R^2 f}{\dashrightarrow} & B \\
\wr\uparrow & \lrcorner & \uparrow\wr & & \| \\
(Ef)_0 & \xrightarrow{e_{Rf}} \!\!\!\!\rightarrow & J_{Rf} & \overset{m_{Rf}}{\rightarrowtail} & B \\
\| & \text{o.f.s.} & \downarrow{\scriptstyle m_{Rf}} & & \| \\
(Ef)_0 & \longrightarrow & B & = & B
\end{array}
$$

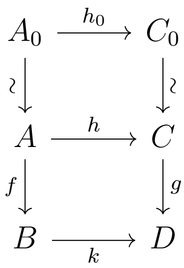
Before proving the universal property of a left adjoint, let us show that there is a suitable functor F which is well-defined by the following proposition.

Proposition 6.23. *There is a functor* $F \colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ *with assignment on objects given by:*

$$
A \xrightarrow{f} B \quad \longmapsto \quad
\begin{array}{ccc}
 & J_f & \\
\overset{\wr}{\swarrow} & & \searrow{\scriptstyle m_f} \\
Ef & \xrightarrow[Rf]{} & B
\end{array}
\tag{6.15}
$$

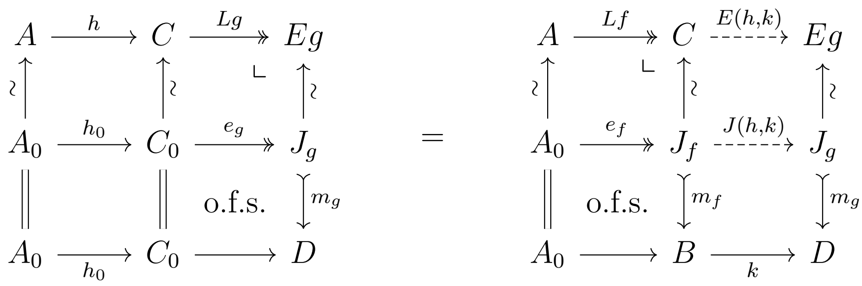
Proof. The assignment on objects above is constructed in the first displayed diagram in Notation 6.22. For the assignment on morphisms, consider a morphism in $\mathcal{S}\mathrm{q}(\mathcal{C})$

and construct the following commutative diagram:

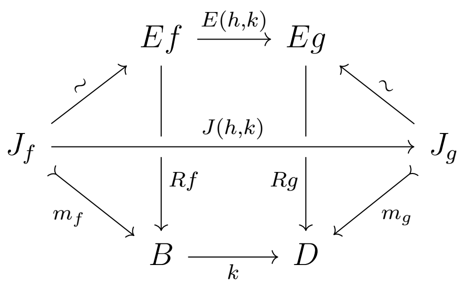


(6.16)

Then since the functor cod: $\mathcal{S}\text{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is an opfibration, there exists the following factorisation:



This yields a morphism in $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by:



It is straightforward to show that this assignment on morphisms is functorial using universal properties. □

Note also that post-composing the above functor with cod: $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ yields the codomain map cod: $\mathcal{S}\text{q}(\mathcal{C}) \to \mathcal{C}$ as desired. We are now able to prove the first main result of this section: showing that the functor F defined in Proposition 6.23 is a left adjoint to the forgetful functor U.

Theorem 6.24. *The functor* $U\colon \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\text{q}(\mathcal{C})$ *has a left adjoint given by* $F\colon \mathcal{S}\text{q}(\mathcal{C}) \to \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$.

Proof. We construct the components of the unit and counit natural transformations, and omit the straightforward details establishing naturality and the triangle identities.

For an object $f\colon A \to B$ in $\mathcal{S}\mathrm{q}(\mathcal{C})$, the component of the unit is given by:

$$\begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & = & B \end{array} \tag{6.17}$$

The component of the counit at an object in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ is given by the morphism,

$$\begin{array}{ccccc} & Ef & \xrightarrow{\hat{p}} & A & \\ \nearrow^{\sim} & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle f} & \nwarrow^{\sim} \\ J_f & & \rightarrowtail & & X \\ \searrow_{m_f} & & & & \swarrow_{\bar{f}} \\ & B & = & B & \end{array} \tag{6.18}$$

which is constructed from applying the factorisation (6.14) to the morphism (6.12):

$$\begin{array}{ccc} A & = & A \\ \wr\uparrow & & \uparrow\wr \\ A_0 & \xrightarrow{\sim} & X \\ \| & & \downarrow{\scriptstyle \bar{f}} \\ A_0 & \longrightarrow & B \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{Lf} & Ef & \xrightarrow{\hat{p}} & A \\ \wr\uparrow & & \uparrow\wr & & \uparrow\wr \\ A_0 & \xrightarrow{e_f} & J_f & \xrightarrow{\exists !} & X \\ \| & \text{o.f.s.} & \downarrow{\scriptstyle m_f} & & \downarrow{\scriptstyle \bar{f}} \\ A_0 & \longrightarrow & B & = & B \end{array}$$

This completes the proof. □

A key benefit to the existence of the left adjoint F is the construction of the *free lens* (6.15) on a morphism in $\mathcal{C}$. In particular, when $\mathcal{C} = \mathcal{C}\mathrm{at}$ we are able to construct the free delta lens on a functor, which without diagrammatic reasoning would seem to be very difficult to achieve. The underlying reason is that the construction of pushouts and the image of the comprehensive factorisation system in $\mathcal{C}\mathrm{at}$ are typically difficult to realise explicitly due to quotients. However, as the following lemma shows, some instances are easier to understand.

Lemma 6.25. *Given a functor $f\colon A \to B$, there is an isomorphism of categories:*

$$J_f \cong \sum_{a \in A_0} fa/B$$

Proof. Since the factorisation is unique, up to unique isomorphism, it is enough to verify that the functor $e_f\colon A_0 \to J_f$ with an assignment on objects $a \mapsto (a, 1_{fa})$ is initial, and that the functor $m_f\colon J_f \to B$ with assignment on objects $(a, u\colon fa \to b) \mapsto b$ is a discrete opfibration. The conditions in both cases are easy to check. □

The initial functor $e_f\colon A_0 \to J_f$ is a particularly interesting case, as it chooses an *initial object* in each *connected component* of its codomain (also known as local initial objects). Therefore, constructing the pushout of e_f along the identity-on-objects functor $A_0 \to A$ essentially involves no quotients, instead merely gluing the morphisms in A between the local initial objects in J_f and closing under composition.

Example 6.26. Consider category $B = \{\bullet \to \bullet\}$. The free lens $Ef \to B$ on the identity functor 1_B is given by Example 2.20.

On the other hand, not every lens is free on a functor. For example, a discrete opfibration is a free lens on a functor if and only if its domain is discrete.

Constructing the algebras

Theorem 6.24 has shown that the forgetful functor $U\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\mathrm{q}(\mathcal{C})$ has a left adjoint under some reasonable conditions on the triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$. The goal of this subsection is to show that, in the case where $\mathcal{C} = \mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations, the functor U is monadic.

We begin by constructing the algebras for the monad $UF\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{S}\mathrm{q}(\mathcal{C})$ induced by the adjunction, then provide a simpler presentation which makes use of the universal property of the pushout. Finally we show how the category of algebras $\mathcal{A}\mathrm{lg}(UF)$ is equivalent to $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ in the case of $\mathcal{C} = \mathcal{C}\mathrm{at}$.

Consider the induced monad UF on $\mathcal{S}\mathrm{q}(\mathcal{C})$. The unit for the monad at an object $f\colon A \to B$ is given by (6.17):

$$\begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Rf} \\ B & = & B \end{array}$$

To construct the component of the multiplication for the monad, consider the following factorisation:

$$\begin{array}{ccc} Ef & = & Ef \\ \wr\big\uparrow & & \big\uparrow\wr \\ (Ef)_0 & \overset{\exists!}{\underset{\sim}{\dashrightarrow}} & J_f \\ \big\| & & \big\downarrow{\scriptstyle m_f} \\ (Ef)_0 & \longrightarrow & B \end{array} \quad = \quad \begin{array}{ccccc} Ef & \xrightarrow{LRf} & ERf & \overset{\mu_f}{\dashrightarrow} & Ef \\ \wr\big\uparrow & \lrcorner & \big\uparrow\wr & & \big\uparrow\wr \\ (Ef)_0 & \xrightarrow{e_{Rf}} & J_{Rf} & \overset{\nu_f}{\rightarrowtail} & J_f \\ \big\| & \text{o.f.s.} & \big\downarrow{\scriptstyle m_{Rf}} & & \big\downarrow{\scriptstyle m_f} \\ (Ef)_0 & \longrightarrow & B & = & B \end{array} \tag{6.19}$$

Post-composing the above morphisms in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ with the morphism for the free lens,

$$\begin{array}{ccc} Ef & \xrightarrow{Rf} & B \\ {\scriptstyle \wr}\uparrow & & \| \\ J_f & \overset{m_f}{\rightarrowtail} & B \\ {\scriptstyle m_f}\downarrow & & \| \\ B & = & B \end{array}$$

and applying the universal property of the pushout yields the component of the multiplication μ at $f\colon A \to B$ given by:

$$\begin{array}{ccc} ERf & \xrightarrow{\mu_f} & Ef \\ {\scriptstyle R^2 f}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & = & B \end{array} \tag{6.20}$$

Example 6.27. An algebra for the monad $UF\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{S}\mathrm{q}(\mathcal{C})$ consists of an object $f\colon A \to B$ and a morphism in $\mathcal{S}\mathrm{q}(\mathcal{C})$ given by,

$$\begin{array}{ccc} Ef & \xrightarrow{\hat{p}} & A \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle f} \\ B & = & B \end{array} \tag{6.21}$$

which is compatible with the unit,

$$\begin{array}{ccccc} A & \overset{Lf}{\twoheadrightarrow} & Ef & \overset{\hat{p}}{\twoheadrightarrow} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle f} \\ B & = & B & = & B \end{array} \quad = \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & = & B \end{array} \tag{6.22}$$

and compatible with the multiplication:

$$\begin{array}{ccccc} ERf & \overset{\mu_f}{\twoheadrightarrow} & Ef & \overset{\hat{p}}{\twoheadrightarrow} & A \\ {\scriptstyle R^2 f}\downarrow & & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle f} \\ B & = & B & = & B \end{array} \quad = \quad \begin{array}{ccccc} ERf & \overset{E(\hat{p},1)}{\twoheadrightarrow} & Ef & \overset{\hat{p}}{\twoheadrightarrow} & A \\ {\scriptstyle R^2 f}\downarrow & & \downarrow{\scriptstyle Rf} & & \downarrow{\scriptstyle f} \\ B & = & B & = & B \end{array} \tag{6.23}$$

Note that by the closure properties of the left class of a factorisation system, the algebra map $\hat{p}$ is necessarily in $\mathcal{E}$.

A morphism of algebras $(f, \hat{p}) \to (g, \hat{q})$ consists of a morphism in $\mathcal{S}\mathrm{q}(\mathcal{C})$ compatible with the algebra maps:

$$\begin{array}{ccccc} Ef & \xrightarrow{\hat{p}} & A & \xrightarrow{h} & C \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & = & B & \xrightarrow[k]{} & D \end{array} \quad = \quad \begin{array}{ccccc} Ef & \overset{E(h,k)}{\twoheadrightarrow} & Eg & \overset{\hat{q}}{\twoheadrightarrow} & C \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Rg} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D & = & D \end{array}$$

Let $\mathcal{A}\mathrm{lg}(UF)$ denote the category of algebras for the monad UF.

There is a lot of redundancy in the full presentation of algebras for the monad UF. Aside from (6.21), only the top row of the diagrams for compatibility with the unit and multiplication contain information. More importantly, an algebra map $\hat{p}\colon Ef \to A$ is a morphism out of a pushout, and utilising its universal property provides a simpler presentation as follows.

Proposition 6.28. *An algebra* $(f\colon A \to B, \hat{p}\colon Ef \to A)$ *for the monad* UF *is equivalent to a pair,*

$$(f\colon A \to B, p\colon J_f \to A)$$

which satisfies the following commutative diagrams:

$$\begin{array}{ccc} J_f & \xrightarrow{p} & A \\ {\scriptstyle m_f}\downarrow & & \downarrow{\scriptstyle f} \\ B & = & B \end{array} \qquad \begin{array}{ccc} A_0 & \xrightarrow{e_f} & J_f \\ {\scriptstyle \wr}\downarrow & & \downarrow{\scriptstyle p} \\ A & = & A \end{array} \qquad \begin{array}{ccc} J_{Rf} & \xrightarrow{J(p_0,1)} & J_f \\ {\scriptstyle \nu_f}\downarrow & & \downarrow{\scriptstyle p} \\ J_f & \xrightarrow{p} & A \end{array} \tag{6.24}$$

Proof. Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Consider the middle square in (6.24) and take the pushout of $e_f\colon A_0 \to J_f$ along the weak equivalence $A_0 \to A$ to yield a universal morphism $[1,p]\colon Ef \to A$. It is not difficult to show that this is an algebra for the monad UF.

Conversely, given an algebra (6.21), precompose with the morphism $J_f \to Ef$ to obtain a morphism $J_f \to A$ which satisfies the commutative diagrams (6.24).

These two processes are mutually inverse, and therefore the two presentations of algebras for the monad UF are equivalent. □

When we say an algebra for the monad UF, we will mean either of its presentations above.

Corollary 6.29. *There is a functor* $Q\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{A}\mathrm{lg}(UF)$ *with an assignment on objects determined by the components of the unit* (6.18) *as follows:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle \wr}\swarrow & & \searrow{\scriptstyle \overline{f}} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} & Ef & \\ {\scriptstyle \hat{p}}\swarrow & & \searrow{\scriptstyle Rf} \\ A & \xrightarrow[f]{} & B \end{array}$$

The main theorem of this section is to show that the functor Q is an equivalence of categories in the case of $\mathcal{C} = \mathcal{C}\mathrm{at}$. The proof is based on the presentation of lenses as algebras for a semi-monad by Johnson and Rosebrugh [JR13].

Theorem 6.30. *Let* $\mathcal{C} = \mathcal{C}\mathrm{at}$ *together with the class* $\mathcal{W}$ *of bijective-on-objects functors and the class* $\mathcal{M}$ *of discrete opfibrations. Then functor* $U\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\mathrm{q}(\mathcal{C})$ *is monadic, that is, there is an equivalence of categories* $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \simeq \mathcal{A}\mathrm{lg}(UF)$.

Proof. By Proposition 6.28 and Lemma 6.25, an algebra for the monad UF consists of a functor $f\colon A \to B$ together with a functor,

$$
\begin{array}{ccc}
J_f & \xrightarrow{\quad p \quad} & A \\[1em]
(a,u) & \cdots & p(a,u) \\
{\scriptstyle\langle a,v\rangle}\downarrow & & \downarrow{\scriptstyle p\langle a,v\rangle} \\
(a, v \circ u) & \cdots & p(a, v \circ u)
\end{array}
\tag{6.25}
$$

where $u\colon fa \to b$ and $v\colon b \to b'$ are morphisms in B. Compatibility with the unit means that $p(a, 1_{fa}) = a$. Therefore, for each morphism $\langle a, u\rangle\colon (a, 1_{fa}) \to (a, u)$, define a lifting operation:

$$\varphi(a, u\colon fa \to b) = p\langle a, u\rangle\colon a \to p(a, u)$$

Then the axioms (6.24) imply the axioms (L1), (L2), and (L3) for the lifting operation are satisfied. Therefore every algebra (f, p) corresponds to a lens (f, φ). It is straightforward to show that this correspondence extends functorially to morphisms, and provides an inverse to the inclusion $Q\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{A}\mathrm{lg}(UF)$. □

Remark. This proof essentially follows that of [JR13, Proposition 3]. The key observation is that the presentation of algebras in Proposition 6.28 for the *monad* UF is precisely (up to change of notation) the same as the algebras for a *semi-monad* in the cited paper (that is, the semi-monad which assigns $f\colon A \to B$ to $m_f\colon J_f \to B$ in the notation above). Thus the most important aspect of Theorem 6.30 lies in constructing the left adjoint (Theorem 6.24) and simplifying the presentation of the algebras for the induced monad (Proposition 6.28).

6.4 Algebraic weak factorisation systems and lenses

In this section, we show that lenses are the R-algebras for an algebraic weak factorisation system [BG16a] on $\mathcal{C}\mathrm{at}$. The corresponding L-coalgebras for this AWFS are constructed, and we demonstrate the lifting "property" exhibited by lenses with respect to these coalgebras.

Recall from Bourke and Garner [BG16a, Section 3.3] that a right-connected double category $\mathbb{D}$ is called *monadic* if the canonical functor $\mathcal{D}_1 \to \mathcal{S}\mathrm{q}(\mathcal{D}_0)$ is strictly monadic.

Lemma 6.31. *The double category* $\mathbb{L}\mathrm{ens}$ *is a monadic right-connected double category.*

Proof. By Theorem 3.21, the double category of lenses is the right-connected completion of $\mathbb{C}$of, and is therefore right-connected. The functor $\mathcal{L}$ens $\to \mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ is monadic by Theorem 6.30, and therefore the double category of lenses is monadic. □

Using this lemma, the main theorem of this section follows immediately.

Theorem 6.32. *The double category* $\mathbb{L}$ens *of lenses is isomorphic the double category* R-$\mathbb{A}$lg *of* R*-algebras for an algebraic weak factorisation system.*

Proof. This follows from Lemma 6.31 and [BG16a, Proposition 11]. □

Recall [BG16a, Section 2.2] that an *algebraic weak factorisation system* (L, R) on $\mathcal{C}$ consists of a monad R on $\mathcal{S}\mathrm{q}(\mathcal{C})$ over the codomain map $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$ and a comonad L on $\mathcal{S}\mathrm{q}(\mathcal{C})$ over the domain map $\mathrm{dom}\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$ which are compatible via a distributive law and form a functorial factorisation (L, E, R) on $\mathcal{C}$. Algebraic weak factorisation systems (AWFS) generalise orthogonal factorisation systems (OFS). Analogous to how an OFS has unique filler morphisms (6.10), an AWFS has a *canonical choice* of filler morphisms (Proposition 6.35).

Since lenses are R-algebras for the monad $R = UF$ on $\mathcal{S}\mathrm{q}(\mathcal{C})$, we now wish to construct the comonad L and the L-coalgebras, and state the relationship that they share with R-algebras.

Notation 6.33. Given the morphism $Lf\colon A \to Ef$ constructed in Notation 6.22, construct the following factorisation:

$$
\begin{array}{ccc}
A & \xrightarrow{Lf} & Ef \\
\uparrow \wr & & \| \\
A_0 & \longrightarrow & Ef \\
\| & & \| \\
A_0 & \longrightarrow & Ef
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow{L^2 f} & ELf & \overset{RLf}{\dashrightarrow} & Ef \\
\uparrow \wr & \llcorner & \uparrow \wr & & \| \\
A_0 & \xrightarrow{e_{Lf}} & J_{Lf} & \xrightarrow{m_{Lf}} & Ef \\
\| & \text{o.f.s.} & \downarrow m_{Lf} & & \| \\
A_0 & \longrightarrow & Ef & = & Ef
\end{array}
$$

Now using the universal property of the orthogonal factorisation system to obtain a morphism,

$$
\begin{array}{ccc}
A_0 & \xrightarrow{e_{Lf}} & J_{Lf} \\
{\scriptstyle e_f}\downarrow & \nearrow & \downarrow{\scriptstyle m_{Lf}} \\
J_f & \xrightarrow[\sim]{} & Ef
\end{array}
$$

we can construct a morphism $\Delta_f\colon Ef \to ELf$ using the universal property of the pushout:

$$\begin{array}{ccccc} A_0 & \xrightarrow{\sim} & A & = & A \\ {\scriptstyle e_f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle L^2 f} \\ J_f & \xrightarrow[\sim]{} & Ef & \xrightarrow[\Delta_f]{} & ELf \end{array} \quad = \quad \begin{array}{ccccc} A_0 & = & A_0 & \xrightarrow{\sim} & A \\ {\scriptstyle e_f}\downarrow & & \downarrow{\scriptstyle e_{Lf}} & & \downarrow{\scriptstyle L^2 f} \\ J_f & \longrightarrow & J_{Lf} & \xrightarrow[\sim]{} & ELf \end{array} \tag{6.26}$$

Proposition 6.34. *There is a comonad* $L\colon \mathcal{Sq}(\mathcal{C}) \to \mathcal{Sq}(\mathcal{C})$ *with assignment on objects* $f \mapsto Lf$. *Given an object* $f\colon A \to B$, *the component of the counit is given by,*

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ Ef & \xrightarrow[Rf]{} & B \end{array} \tag{6.27}$$

while the component of the comultiplication was constructed in (6.26) *and is:*

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle L^2 f} \\ Ef & \xrightarrow[\Delta_f]{} & ELf \end{array} \tag{6.28}$$

A coalgebra for the comonad L consists of a morphism $f\colon A \to B$ and a morphism $q\colon B \to Ef$ which is compatible with the counit,

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[q]{} & Ef & \xrightarrow[Rf]{} & B \end{array} \quad = \quad \begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & = & B \end{array} \tag{6.29}$$

and compatible with the comultiplication:

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle L^2 f} \\ B & \xrightarrow[q]{} & Ef & \xrightarrow[E(1,q)]{} & ELf \end{array} \quad = \quad \begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle L^2 f} \\ B & \xrightarrow[q]{} & Ef & \xrightarrow[\Delta_f]{} & ELf \end{array} \tag{6.30}$$

Proposition 6.35 ([BG16a, Section 2.4])**.** *Given a commutative square,*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

such that (f, q) *is a* L*-coalgebra and* (g, p) *is a* R*-algebra, there is a canonical morphism* $\psi = p \circ E(h, k) \circ q\colon B \to C$ *such that* $g \circ \psi = k$ *and* $\psi \circ f = h$.

When the left-hand side of the above square is the special case of a cofree coalgebra, this canonical choice of morphism can be illustrated as follows.

Proposition 6.36. *Let $f\colon A \to B$ be a functor arising from a pushout of the form:*

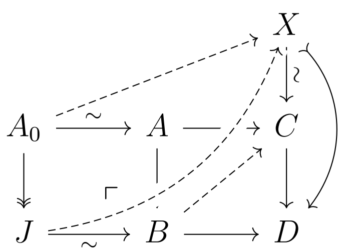
$$\begin{array}{ccc} A_0 & \xrightarrow{\sim} & A \\ \downarrow & \ulcorner & \downarrow f \\ J & \xrightarrow[\sim]{} & B \end{array}$$

Then for every commutative square,

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & \overset{\psi}{\nearrow} & \downarrow g \\ B & \xrightarrow[k]{} & D \end{array}$$

such that g has a lens structure, there is a canonical choice of morphism $\psi\colon B \to C$ such that $g \circ \psi = k$ and $\psi \circ f = h$.

Proof. Consider the solid diagram below where the morphism $C \to D$ has the structure of a generalised lens:



We describe the construction of each of the dashed morphisms in three steps.

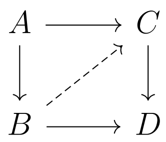
By the universal property of discrete objects (6.9) there exists a unique morphism $A_0 \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} A_0 & \dashrightarrow & X \\ \wr\downarrow & & \downarrow\wr \\ A & \longrightarrow & C \end{array}$$

By the universal property (6.10) of the orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$, there exists a unique morphism $J \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} A_0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ J & \longrightarrow & D \end{array}$$

Finally, by the universal property of the pushout there exists a morphism $B \to C$ such that the following diagram commutes:



Although the construction of this morphism is canonical, it may not be the only morphism such that the lower-right triangle above commutes. □

6.5 Change of base for lenses and generalised split opfibrations

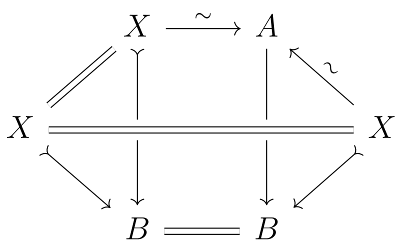
In this section, we demonstrate two interesting uses of a diagrammatic approach to (generalised) lenses. The first is the notion of change of base, through the statement that the codomain functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is a *bifibration*, yielding an adjunction between the fibres for every morphism $A \to B$ in $\mathcal{C}$. The second is exploration of how split opfibrations may be transferred to this diagrammatic setting in the presence of a copointed endofunctor on $\mathcal{C}$.

Change of base

Let $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$ be the restriction of $\mathbb{S}\mathrm{q}(\mathcal{C})$ to the class of morphisms $\mathcal{M}$ as defined in Example A.8. There is a strict double functor $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \to \mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ which sends a morphism in $\mathcal{M}$ to its companion. There is an adjunction of categories,

$$\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \tag{6.31}$$

whose unit is the identity transformation, and whose counit component at a lens is given by the cell:

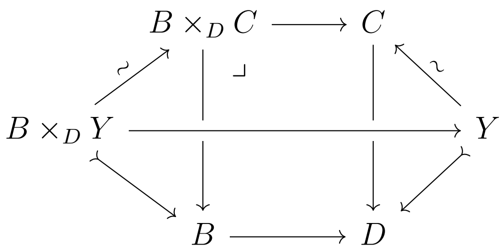


(6.32)

Furthermore, this is an adjunction in the slice over $\mathcal{C}$ between the codomain maps $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \to \mathcal{C}$ and $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$. Since $\mathcal{C}$ has pullbacks and the class $\mathcal{M}$ satisfies (A3), the former map, $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \to \mathcal{C}$, is a fibration. Since $\mathcal{W}$ satisfies (A3), we also have the following result for the latter.

Proposition 6.37. *The functor* $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is a fibration.*

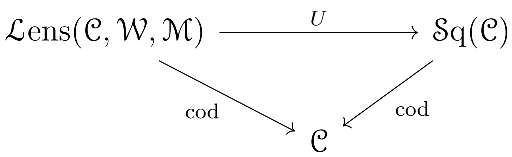
Proof. Given a lens from C to D and a morphism $B \to D$, there is a morphism in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by:



Using the universal property of the pullback in $\mathcal{C}$, it is straightforward to verify that this a cartesian lift. □

Since the cartesian lifts with respect to the fibration $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \to \mathcal{C}$ are given pullback squares, the following result is immediate consequence of Proposition 6.37.

Corollary 6.38. *The forgetful functor,*

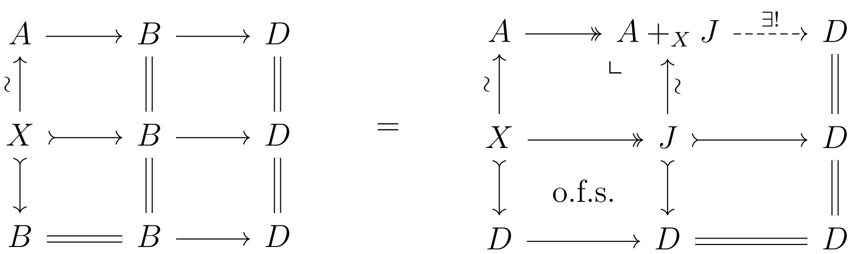


preserves cartesian lifts.

When $\mathcal{M}$ satisfies (A7), that is, when there is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$, the codomain map $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) \to \mathcal{C}$ is an opfibration (Proposition 6.20). When $\mathcal{W}$ satisfies (A8), we also have the following result for lenses.

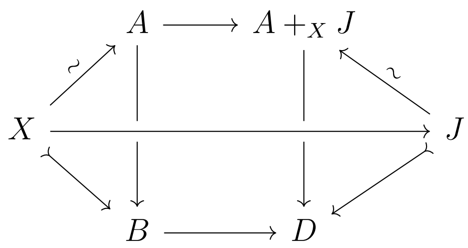
Proposition 6.39. *The functor* $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is an opfibration.*

Proof. Given a lens from A to B and a morphism $B \to D$, there is a factorisation in $\mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ using (6.14) given by:



(6.33)

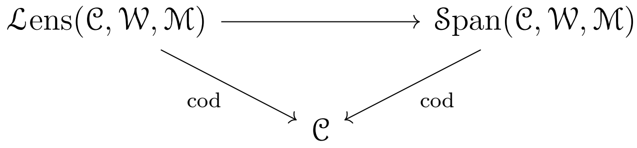
Rearranging the diagram, we may see this as a morphism in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ given by:



Using the universal property of the opcartesian lift for $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$, it is straightforward to verify that this is an opcartesian lift in $\mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ with respect to the codomain functor. □

Since the opcartesian lifts of the opfibration $\mathrm{cod}\colon \mathcal{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ are constructed in exactly the same way, the following result is an immediate consequence of Proposition 6.39.

Corollary 6.40. *The forgetful functor,*



preserves opcartesian lifts.

We now arrive at the main result of this subsection.

Theorem 6.41. *The functor* $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ *is a bifibration.*

Proof. Follows directly from Proposition 6.37 and Proposition 6.39. □

From this result, we obtain a notion of change of base for lenses over an object.

Proposition 6.42. *For every morphism $f\colon A \to B$ there is an adjunction,*

$$\mathcal{L}\mathrm{ens}_A(\mathcal{C}, \mathcal{W}, \mathcal{M}) \underset{\Delta_f}{\overset{\Sigma_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{L}\mathrm{ens}_B(\mathcal{C}, \mathcal{W}, \mathcal{M})$$

between the fibres of the codomain map $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$.

Proof. The right adjoint Δ_f is given by taking the cartesian lift of the morphism f, while Σ_f is given by taking the opcartesian lift of the morphism f. Since the codomain map $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$ is a bifibration, it follows immediately that these functors are adjoint. □

In the case where $\mathcal{C} = \mathcal{C}\text{at}$, if f is a discrete opfibration then Σ_f is given by post-composition of a lens into A by f . Therefore, Σ_f more generally is describing how one "composes" a lens with a functor. On the other hand, Δ_f is describing the pullback of a lens along a functor, and if f has a lens structure, this is the "pullback" of lenses as studied in the work of Johnson and Rosebrugh [JR17a].

Generalised split opfibrations

In Section 3.5, split opfibrations were characterised as lenses satisfying a certain diagrammatic property with respect to the décalage construction. The goal of this subsection is to extend this to the setting of generalised lenses for a triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ satisfying (A1)-(A3).

The key ingredient in the generalisation is the additional structure of a copointed endofunctor on $\mathcal{C}$:

$$\mathcal{C} \underset{1}{\overset{D}{\rightrightarrows}} \mathcal{C} \quad \Downarrow \varepsilon$$

A natural transformation is called *cartesian* if all the naturality squares are pullbacks. Of course, the natural transformation ε need not be cartesian in general, but one may still consider the morphisms in $\mathcal{C}$ for which the naturality square is a pullback. For that purpose, we introduce the following definition.

Definition 6.43. A morphism $f\colon A \to B$ in a category $\mathcal{C}$ with a copointed endofunctor (D, ε) is called *D-universal* if the naturality square,

$$\begin{array}{ccc} DA & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle Df}\downarrow & & \downarrow{\scriptstyle f} \\ DB & \xrightarrow[\varepsilon_B]{} & B \end{array}$$

is a pullback square.

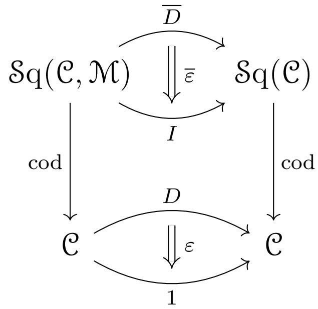
The class of D-universal morphisms in $\mathcal{C}$ is closed under composition, contains the isomorphisms, and is stable under pullback. Therefore, the class of D-universal morphisms satisfies the axioms (A1)–(A3). Recall (Definition 3.27) that the décalage construction is a copointed endofunctor on $\mathcal{C}\text{at}$.

Example 6.44. The class of discrete opfibrations in $\mathcal{C}\text{at}$ is precisely the class of D-universal morphisms for the décalage construction.

If $\mathcal{M}$ denotes the class of D-universal morphisms, then there is a full subcategory,

$$I\colon \mathcal{S}\text{q}(\mathcal{C}, \mathcal{M}) \longrightarrow \mathcal{S}\text{q}(\mathcal{C})$$

and the copointed endofunctor lifts to this full subcategory, in the sense that there is a diagram in the 2-category $\mathcal{C}\mathrm{AT}$ given by,



(6.34)

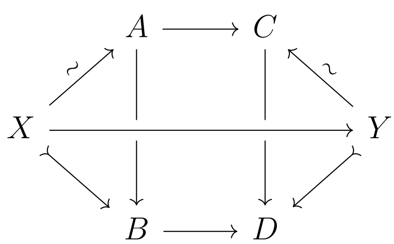
such that $\varepsilon \cdot \mathrm{cod} = \mathrm{cod} \cdot \overline{\varepsilon}$. The assignment of $\overline{D}$ on objects is defined by the assignment on D on morphisms. Furthermore, the components of the natural transformation $\overline{\varepsilon}$ at an object in $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$ are pullback squares in $\mathcal{C}$, which in turn are precisely cartesian lifts with respect to the codomain functor $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$.

Proposition 6.45. *The category* $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$ *is the largest full subcategory of* $\mathcal{S}\mathrm{q}(\mathcal{C})$ *such that the lifting* $\overline{\varepsilon}$ *of the natural transformation* ε *along the functor* $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$ *has components which are cartesian lifts.*

In a sense, this result describes the universal property characterising the lifting $\overline{\varepsilon}$ a copointed endofunctor (D, ε). It also leads to a nice coincidence of the term *cartesian*. In summary, a copointed endofunctor (D, ε) on a category $\mathcal{C}$ is cartesian if and only if $\mathcal{S}\mathrm{q}(\mathcal{C}, \mathcal{M}) = \mathcal{S}\mathrm{q}(\mathcal{C})$, which of course holds if and only if $\mathcal{M} = \mathcal{C}$.

There are a few aspects of Proposition 6.45 that require further explanation. First, note that the functor $\mathrm{cod}\colon \mathcal{S}\mathrm{q}(\mathcal{C}) \to \mathcal{C}$ is a fibration, thus all cartesian lifts exist. Second, it is possible to replace cod with an arbitrary functor and ask for the universal lift of a copointed endofunctor along it. Generalising these aspects allows one to define a suitable replacement for split opfibrations in the setting of generalised lenses.

Given a copointed endofunctor (D, ε) on $\mathcal{C}$, there is an induced copointed endofunctor on $\mathcal{S}\mathrm{q}(\mathcal{C})$, which through an abuse of notation will also be denoted (D, ε). The forgetful functor $U\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{S}\mathrm{q}(\mathcal{C})$ is not a fibration in general, but we can describe its opcartesian lifts as morphisms,

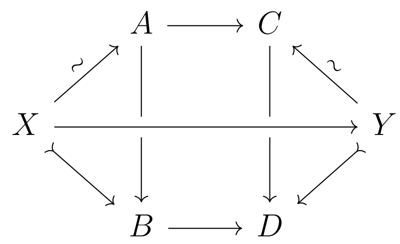


such that the commutative square below is a pullback:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\wr}\downarrow & & \downarrow{\scriptstyle\wr} \\ A & \longrightarrow & C \end{array}$$

Definition 6.46. Let $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}, D)$ be the full subcategory of $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ determined by lenses which admit a cartesian lift, with respect to the forgetful functor U, over a naturality square for copointed endofunctor (D, ε) on $\mathcal{S}\text{q}(\mathcal{C})$.

In other words, the objects in $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}, D)$ are precisely those lenses $C \to D$ such that the following morphism in $\mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$,



is a cartesian lift of the functor $U \colon \mathcal{L}\text{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$.

The following example of split opfibrations is the main reason for this abstract definition.

Example 6.47. The category $\mathcal{S}\text{Opf}$ is the full subcategory of $\mathcal{L}\text{ens}$ determined by the lenses which admit a cartesian lift over naturality squares for the décalage construction with respect to the functor $U \colon \mathcal{L}\text{ens} \to \mathcal{S}\text{q}(\mathcal{C}\text{at})$.

Chapter 7

Conclusion

In this thesis, we have demonstrated the important role that double categories play in unifying the theory of lenses. The motivation for this research grew out of a desire to better understand lenses as mathematical structures, while also building a strong foundation for their continued study using category theory.

Although there are now many kinds of lenses studied by the computer science and applied category theory community, one of the reasons why *delta lenses* [DXC11] remain of interest to the category theorist, more than ten years since their introduction, is that they are morphisms between *categories*. Johnson and Rosebrugh [JR13] showed that lenses capture the underlying structure of *split opfibrations*, while Ahman and Uustalu [AU17] demonstrated how lenses integrate the notions of functor and *cofunctor* [Agu97].

The main goal of this thesis was to explore how lenses between categories could be characterised using universal properties.

In Chapter 3, we introduced the right-connected completion of a double category, and proved that the double category of lenses is the right-connected completion of the double category of cofunctors. The construction induces a canonical span of double functors,

$$\mathbb{C}\mathrm{of} \xleftarrow{U_1} \mathbb{L}\mathrm{ens} \xrightarrow{U_2} \mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$$

and for every unitary double functor $W\colon \mathbb{D} \to \mathbb{C}\mathrm{of}$, there exists a unique unitary double functor $V\colon \mathbb{D} \to \mathbb{L}\mathrm{ens}$, such that $W = U_1 \circ V$. Therefore the double category $\mathbb{L}\mathrm{ens}$ is completely determined by the universal property of the right-connected completion.

In Chapter 4, we introduced the left-connected completion of a double category equipped with a functorial choice of companions, and constructed $\mathbb{S}\mathrm{Mult}$, the double category of split multi-valued functions, as the left-connected completion of $\mathbb{S}\mathrm{pan}$, the double category of spans. Lenses into a category B were characterised as lax double functors $\mathbb{V}(B) \to \mathbb{S}\mathrm{Mult}$, which by the universal property of the left-connected completion, correspond uniquely to globular transformations:

$$\begin{array}{ccc} & \mathbb{V}(B) & \\ \swarrow & \Longrightarrow & \searrow \\ \mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) & \xrightarrow[(-)_*]{} & \mathbb{S}\mathrm{pan} \end{array}$$

In Chapter 5, we introduced the flat double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ of monads, monad morphisms, and monad retromorphisms in a double category $\mathbb{D}$ equipped with a functorial choice of companions. The double category of cofunctors was shown to arise as from the formal theory of monads via an equivalence of categories:

$$\mathbb{C}\mathrm{of} \simeq \mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan})$$

Using the right-connected completion, this construction provided a natural setting for introducing lenses between more general structures such as monads and internal categories.

In Chapter 6, we introduced the double category of lenses $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ in a category $\mathcal{C}$ equipped with suitable classes of morphisms $\mathcal{W}$ and $\mathcal{M}$. When $\mathcal{C} = \mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations, it was shown that the forgetful functor $\mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{of}$ is comonadic, and that the forgetful functor $\mathcal{L}\mathrm{ens} \to \mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ is monadic. Therefore every lens is both a coalgebra for a comonad and an algebra for a monad, unifying previous work of Ahman and Uustalu [AU16] and Johnson and Rosebrugh [JR13]. Moreover, since $\mathbb{L}\mathrm{ens}$ is both right-connected and monadic over $\mathbb{S}\mathrm{q}(\mathcal{C}\mathrm{at})$, it corresponds to an algebraic weak factorisation system, and every lens admits a canonical lift against the class of coalgebras for a certain comonad on $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$.

A significant application of this work has been towards new characterisations of split opfibrations, which have long been important in the study of lenses [JRW12]. At the most basic level, split opfibrations are lenses whose chosen lifts are opcartesian, and one of the core challenges has been translating this property into each new context where lenses were studied. New approaches to split opfibrations were found using the décalage construction, using strict factorisation systems, and via lax double functors satisfying a simple property.

The results presented in each chapter focus on a certain aspect of lenses using both new constructions and familiar tools in category theory. Collectively, they illustrate the utility of using double categories to study lenses, as well as the richness of $\mathbb{L}\mathrm{ens}$ as a double category itself.

While the primary focus of this thesis is towards the theory of lenses, it has also made important contributions to double category theory. These include introducing the right-connected completion and the left-connected completion, formalising the relationship between left-connected double categories and double categories with companions, defining a stronger notion of span representable double categories, and developing new applications of the formal theory of monads for double categories and retrocells.

Category theory has long been used to study lenses, and the elevation of this study to the level of double category theory represents a natural progression in the journey towards understanding their mathematical structure. It is hoped that the research presented in this thesis helps inspire new work on both lenses and double categories.

Future work and open questions

There are many possibilities for future work arising from the research presented in this thesis. The following explores a collection of ideas and open questions which may be avenues for further study.

Enriched lenses Given a construction in category theory, it is often interesting to consider how it generalises to the setting on internal categories or enriched categories. For a category $\mathcal{E}$ with pullbacks, internal categories and internal functors are precisely monads and monad morphisms in the double category $\mathbb{S}\mathrm{pan}(\mathcal{E})$. In Chapter 5, it was shown that *internal cofunctors* are precisely *monad retromorphisms* in the double category $\mathbb{S}\mathrm{pan}(\mathcal{E})$. The double category of internal lenses is then defined as the right-connected completion of the double category of internal cofunctors:

$$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{S}\mathrm{pan}(\mathcal{E})) = \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{S}\mathrm{pan}(\mathcal{E})))$$

In ongoing joint work with Matthew Di Meglio, we show that the framework of lenses between monads may also be used to define *enriched lenses*. A *$\mathcal{V}$-matrix* from A to B is a functor $A \times B \to \mathcal{V}$ where A and B are discrete categories (or sets). For a distributive monoidal category $\mathcal{V}$, enriched categories and enriched functors are monads and monad morphisms in the double category $\mathbb{M}\mathrm{at}(\mathcal{V})$ of sets, functions, and $\mathcal{V}$-matrices. An *enriched cofunctor* may be defined as a monad retromorphism in the double category $\mathbb{M}\mathrm{at}(\mathcal{V})$. In other words, a $\mathcal{V}$-enriched cofunctor $(F, \Phi)\colon \mathcal{C} \to \mathcal{D}$ consists of a function $F\colon \mathcal{C}_0 \to \mathcal{D}_0$ between underlying sets of objects, together with, for each pair of objects $(c, d) \in \mathcal{C}_0 \times \mathcal{D}_0$, a lifting operation given by a morphism in $\mathcal{V}$,

$$\Phi_{c,d}\colon \mathcal{D}(Fc, d) \longrightarrow \sum_{x \in X} \mathcal{C}(c, x)$$

where $X = F^{-1}(d)$ is the fibre of the function F over the object d. The lifting operation is also subject to two conditions related to identities and composition that are notably more complex than the corresponding conditions for enriched functors. The double category of enriched lenses may then be defined as the right-connected completion of the double category of enriched cofunctors:

$$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{M}\mathrm{at}(\mathcal{V})) = \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{M}\mathrm{at}(\mathcal{V})))$$

One example of the utility of enriched lenses comes from recent work of Perrone [Per21] which introduces the notion of *weighted lens*. Let the interval $[0, \infty]$ be a category with morphisms $a \geq b$, and a monoidal structure given by point-wise addition. The distributive monoidal category $\mathrm{w}\mathcal{S}\mathrm{et}$ of *weighted sets* arises as the free coproduct completion of $[0, \infty]$. Categories enriched in $\mathrm{w}\mathcal{S}\mathrm{et}$ are called *weighted categories*. It may be shown that weighted lenses are exactly enriched lenses in the category $\mathrm{w}\mathcal{S}\mathrm{et}$. Future work aims to uncover further examples of enriched lenses, as well as study their mathematical properties.

Internal lenses (again!) Suppose $\mathcal{W}$ is a monoidal category with equalisers, such that, for each pair of objects A and B, the functor $A \otimes (-) \otimes B \colon \mathcal{W} \to \mathcal{W}$ preserves them. Let $\mathbb{C}\mathrm{omod}(\mathcal{W})$ be the double category of comonoids, comonoid homomorphisms, and (two-sided) comodules in $\mathcal{W}$. The monads and monad morphisms in $\mathbb{C}\mathrm{omod}(\mathcal{W})$ are precisely internal categories and internal functors in $\mathcal{W}$ as defined by Aguiar [Agu97]. Internal cofunctors were also originally introduced in this monoidal category setting, and although the definition was not framed in terms of monads, it is not difficult to see that an internal cofunctor corresponds to a monad retromorphism in $\mathbb{C}\mathrm{omod}(\mathcal{W})$. The double category of internal lenses in $\mathcal{W}$ may then be defined as the right-connected completion of the double category of internal cofunctors in $\mathcal{W}$:

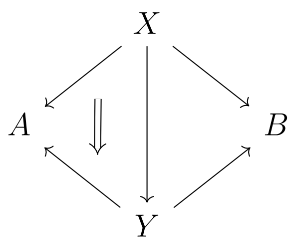
$$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{C}\mathrm{omod}(\mathcal{W})) = \Gamma(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{C}\mathrm{omod}(\mathcal{W})))$$

A *cartesian* monoidal category with equalisers is the same as a category with finite limits, and in this setting the two notions of internal category coincide. However there are at least two reasons why the monoidal setting is interesting. The first reason is that it encompasses new examples, such as for $\mathcal{W}$ the category of vectors spaces together with the usual tensor product. The second reason is that it opens the possibility of using a string diagram calculus [Chi11] for cofunctors and lenses, which may be useful both for proving new results and for drawing comparisons to other kinds of lenses based on monoidal structures.

Transformations between monad retromorphisms An unsatisfying aspect of the double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is that it is flat whenever $\mathbb{D}$ is a unit-pure double category equipped with a functorial choice of companions. This means that the vertical bicategory $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, whose objects are monads and whose 1-cells are monad retromorphisms, has only trivial 2-cells. However, in the case $\mathbb{D} = \mathbb{C}\mathrm{omod}(\mathcal{W})$, Aguiar defines natural transformations between cofunctors. How can we understand these transformations as arising naturally from the formal theory of monads?

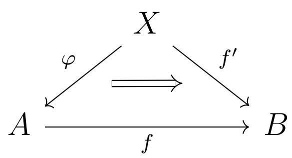
For every double category $\mathbb{D}$ there is a double category $\mathbb{EM}(\mathbb{D})$ which has the same objects, horizontal morphisms, and vertical morphisms as $\mathbb{M}\mathrm{nd}(\mathbb{D})$, but with different cells. While the definition of this double category does not appear in the work of Fiore, Gambino, and Kock [FGK11; FGK12] on monads, its definition is analogous to the free completion of a 2-category under Eilenberg-Moore objects [LS02]. Despite the similarities in definition, it is currently unknown whether $\mathbb{EM}(\mathbb{D})$ is also the free completion of $\mathbb{D}$ under Eilenberg-Moore objects, as there are subtle details to check with regards to strictness and the appropriate direction of transformation between double functors.

Restricting the double category $\mathbb{EM}(\mathbb{D})$ to monad retromorphisms yields a double category $\mathbb{EM}_{\mathrm{ret}}(\mathbb{D})$ whose underlying vertical bicategory contains non-trivial 2-cells. When $\mathbb{D} = \mathbb{S}\mathrm{pan}$, monad retromorphisms correspond to spans of functors whose left leg is bijective-on-objects and whose right leg is a discrete opfibration (that is, cofunctors), and the 2-cells correspond to diagrams in $\mathcal{C}\mathrm{at}$ given by:



These 2-cells correspond exactly to the transformations of cofunctors due to Aguiar, thus demonstrating how they arise naturally from the formal theory of monads.

Lax lenses The double category $\mathbb{EM}_{\mathrm{ret}}(\mathbb{D})$ contains within it all of the information of $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$, and therefore may be understood as a direct generalisation. Vertical morphisms in the right-connected completion of $\mathbb{EM}_{\mathrm{ret}}(\mathbb{D})$ may be appropriately called *lax lenses*, as they correspond to diagrams in $\mathcal{C}\mathrm{at}$ of the form,



where φ is bijective-on-objects and f' is a discrete opfibration. It is conjectured that lax lenses share a close relationship to *delta lenses with amendment* [DKL19; Dis20] and this will be the subject of further study.

The category of cofunctors is coreflective For any double category $\mathbb{D}$ with a functorial choice of companions, we may define monad retromorphisms together with the fully faithful inclusion of double categories:

$$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}) \longrightarrow \mathbb{M}\mathrm{nd}(\mathbb{D})$$

When $\mathbb{D} = \mathbb{S}\mathrm{pan}$, the corresponding fully faithful functor between the categories of morphisms,

$$\mathcal{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D}) \longrightarrow \mathcal{M}\mathrm{nd}(\mathbb{D})$$

admits a right adjoint, and thus $\mathcal{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ is a coreflective subcategory. This provides another reason why monad retromorphisms are special among all vertical monad morphisms in the case of $\mathbb{D} = \mathbb{S}\mathrm{pan}$. It is an open question as to what conditions on $\mathbb{D}$ are required for this property to hold in general.

Strong tabulators and monad retromorphisms In Section 3.4 it was shown that the double category of cofunctors has strong tabulators and is strongly span representable, forming the basis for replacing $\mathbb{C}\mathrm{of}$ with the strongly span representable double category $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ in Chapter 6. It is natural to ask: what are necessary conditions on $\mathbb{D}$ to ensure that that double category $\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$ has strong tabulators, and moreover is strongly span representable? In other words, when are monad retromorphisms equivalent to spans of monad morphisms? It is conjectured that $\mathbb{D}$ must be equivalent to the double category $\mathbb{S}\mathrm{pan}(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks. The potential for studying enriched cofunctors as spans of enriched functors is dependent on a solution to this conjecture.

The right-connected completion Right-connected double categories were first defined by Bourke and Garner [BG16a; BG16b] in the study of algebraic weak factorisation systems (AWFS). A right-connected double category $\mathbb{D}$ such that the forgetful functor $\mathcal{D}_1 \to \mathcal{S}\mathrm{q}(\mathcal{D}_0)$ is monadic is equivalent to an AWFS on the category $\mathcal{D}_0$.

The central construction of this thesis is the construction of the right-connected completion $\Gamma(\mathbb{D})$ of a double category $\mathbb{D}$, which comes equipped with a canonical span of double functors:

$$\mathbb{D} \longleftarrow \Gamma(\mathbb{D}) \longrightarrow \mathbb{S}\mathrm{q}(\mathcal{D}_0)$$

In Chapter 6, it was shown that when $\mathbb{D} = \mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$, the corresponding functor between categories of morphisms $\Gamma(\mathbb{D})_1 \to \mathcal{D}_1$ is comonadic, and when $\mathbb{D} = \mathbb{C}\mathrm{of}$, the corresponding functor between categories of morphisms $\Gamma(\mathbb{D})_1 \to \mathcal{S}\mathrm{q}(\mathcal{D}_0)$ is monadic. It is reasonable to wonder if these results could be proven entirely at the level of a general double category $\mathbb{D}$ with conditions, rather than for specific double categories as considered above.

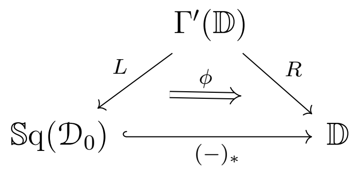
Conjecture 7.1. The component of the double functor $\Gamma(\mathbb{D}) \to \mathbb{D}$ between categories of morphisms is comonadic if the category $\mathcal{D}_1$ has pullbacks and the codomain functor $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$ has a right adjoint right inverse.

Conjecture 7.2. The component of the double functor $\Gamma(\mathbb{D}) \to \mathbb{S}\mathrm{q}(\mathcal{D}_0)$ between categories of morphisms has a left adjoint if the functor $\mathrm{dom}\colon \mathcal{D}_1 \to \mathcal{D}_0$ has a left adjoint right inverse and the functor $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$ is an opfibration.

Conjecture 7.2 raises a further question: under what conditions on $\mathbb{D}$ does the right-connected completion correspond to an AWFS on $\mathcal{D}_0$?

In many senses, the right-connected completion presents a generalised theory of lenses in a double category, however further examples are needed to solidify this intuition. For example, if $\mathbb{D}$ is the double category of quintets for a 2-category $\mathcal{K}$, vertical morphisms in the right-connected completion $\Gamma(\mathbb{D})$ are simply 2-cells in $\mathcal{K}$, which seems to be very far from a lens. There is also the question of what it means for a category (such as $\mathcal{L}\mathrm{ens}$) to be both comonadic over a category (such as $\mathcal{C}\mathrm{of}$) and monadic over another category (such as $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$).

Lax limits and restricted classes of 2-cells In Section 4.1, the left-connected completion of a double category equipped with a functorial choice of companions was shown to induce a 2-cell,



which *almost* has the universal properties of a lax limit of the morphism $(-)_*$ in the 2-category $\mathcal{DBL}_{\mathrm{lax}}$ of double categories, lax double functors, and horizontal transformations. The reason this isn't precisely the lax limit is that the 1-dimensional universal property only holds for *globular transformations*, a restricted class of 2-cells in $\mathcal{DBL}_{\mathrm{lax}}$. What is the general theory of such lax limits?

The Grothendieck construction for cofunctors The double category $\mathbb{S}\mathrm{pan}$ *classifies* functors, in the sense that functors into a category B are in bijection with lax double functors $\mathbb{V}(B) \to \mathbb{S}\mathrm{pan}$. Similarly, the double category $\mathbb{S}\mathrm{Mult} = \Gamma'(\mathbb{S}\mathrm{pan})$ classifies lenses. Let $\mathcal{C}\mathrm{of}_B$ be the fibre of the codomain functor $\mathrm{cod}\colon \mathcal{C}\mathrm{of} \to \mathcal{C}\mathrm{at}$. Is there a double category $\mathbb{D}$ such that there is an equivalence of categories,

$$\mathcal{C}\mathrm{of}_B \simeq [\mathbb{V}(B), \mathbb{D}]_{lax}$$

between the category of cofunctors over B and the category of lax double functors from $\mathbb{V}(B)$ into $\mathbb{D}$?

Dependent lenses Consider a double category $\mathbb{D}\mathrm{epLens}$ whose objects are functions, whose horizontal morphisms are commutative squares of functions, and whose vertical morphisms are *dependent lenses*, that is, pairs $(f, f^\sharp)\colon g \to h$ such that the following diagram of functions commutes:

$$\begin{array}{ccccc} A & \xleftarrow{f^\sharp} & \bullet & \longrightarrow & C \\ & {\scriptstyle g}\searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ & & B & \xrightarrow[f]{} & D \end{array}$$

With a suitable definition of cells for $\mathbb{D}\mathrm{epLens}$, there is a double functor $\mathbb{C}\mathrm{of} \to \mathbb{D}\mathrm{epLens}$ which assigns:

- every category A, considered as an internal category in $\mathcal{S}\mathrm{et}$, to its domain map $d_0\colon A_1 \to A_0$;
- every functor $f\colon A \to B$ to the commutative square of functions:

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

- every cofunctor $(f_0, \varphi_1)\colon A \to B$ to the dependent lens:

$$\begin{array}{ccccc} A_1 & \xleftarrow{\varphi_1} & \Lambda_1 & \longrightarrow & B_1 \\ & {\scriptstyle d_0}\searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ & & A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

Does this double functor have a left or a right adjoint? What can the theory of cofunctors tell us about the theory of dependent lenses? Is it interesting to consider the right-connected completion of $\mathbb{D}\mathrm{epLens}$?

Symmetric lenses In the paper [Cla21a] by the author, a diagrammatic approach to *symmetric lenses* [Dis+11; JR17a; JR17b] is developed. Without explaining the key details of the paper, there are two comments on its relation to this thesis. Firstly, although the constructions developed in that paper used bicategories, it is possible to generalise to the setting of double categories. Second, although the diagrammatic approach worked entirely in the category $\mathcal{C}\mathrm{at}$ together with the class $\mathcal{W}$ of bijective-on-objects functors and the class $\mathcal{M}$ of discrete opfibrations, every result in the paper may be generalised to a triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ which satisfies the same axioms as in Chapter 6. The study of symmetric lenses in this general setting will be the subject of future work.

Properties of lenses In the paper [Cho+21] by the author and colleagues, properties of the category of (small) categories and lenses are studied. Key results characterised initial objects, terminal objects, monomorphims and epimorphisms, equalisers, and coproducts. A distributive monoidal product, a proper factorisation system, and the property of extensivity were also established. Several other properties of this double category were studied by Di Meglio [DiM21]. Many of the results utilise a diagrammatic approach to lenses, and it is natural to ask if they also generalise to a triple $(\mathcal{C}, \mathcal{W}, \mathcal{M})$ under certain conditions, and furthermore, if they may be framed in terms of properties of the double category $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$? For example, monomorphisms may be characterised as certain cartesian lifts for the codomain functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M}) \to \mathcal{C}$, and dually for epimorphisms. The systematic study of properties of the double category $\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ will be the subject of future work.

The work presented in this thesis, and the way it seems to lead to promising new questions, shows the utility of taking a double categorical approach to the study of lenses and their generalisations. In some cases, the results can be understood or even obtained without double categories, but the double categorical approach nicely orders the material required such as when defining lenses as monad morphisms, or showing that they are algebras and coalgebras. In other cases, such as for the Grothendieck construction for lenses, the use of the double categorical approach is essential and has provided interesting and fruitful insights. It is hoped that this thesis leads to exciting new directions in the theory of lenses and double categories.

Appendix A

Review of double categories

The aim of this appendix is to review the basic notions of double category theory for the reader's quick reference and to avoid cluttering the exposition of new work with well known material. The main reference is Grandis and Paré [GP99; GP04], and most of the notation used here is chosen to agree with their work (for example, denoting vertical morphisms with a decorated $\relbar\!\!\bullet\!\!\!\rightarrow$ arrow to distinguish them from horizontal morphisms). Definitions or results taken from other places in the literature are cited as they occur.

Definition A.1. A *double category* $\mathbb{D}$ is a (pseudo) internal category in $\mathcal{C}\mathrm{AT}$,

$$\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \xrightarrow{\mathrm{comp}} \mathcal{D}_1 \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\underset{\longleftarrow\,\mathrm{id}\,\longrightarrow}{\longrightarrow}}} \mathcal{D}_0$$

with *category of objects* $\mathcal{D}_0$ and *category of morphisms* $\mathcal{D}_1$.

Using standard terminology for an internal category, $\mathrm{dom}\colon \mathcal{D}_1 \to \mathcal{D}_0$ is the *domain map*, $\mathrm{cod}\colon \mathcal{D}_1 \to \mathcal{D}_0$ is the *codomain map*, $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is the *identity map*, and $\mathrm{comp}\colon \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \to \mathcal{D}_1$ is the *composition map* of a double category $\mathbb{D}$.

Unpacking the formal definition, a double category $\mathbb{D}$ has a collection of *objects* (the objects of $\mathcal{D}_0$), collections of *horizontal morphisms* (the morphisms of $\mathcal{D}_0$) and *vertical morphisms* (the objects of $\mathcal{D}_1$), and a collection of *cells* (the morphisms of $\mathcal{D}_1$). A typical cell α in a double category is denoted as follows:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow\!\!\bullet & \alpha & \bullet\!\!\big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

Such a cell is called *globular* if both h and k are identities. Cells may be composed (or pasted) both horizontally and vertically, and these compositions are compatible via an interchange law. Identity cells on a vertical morphism $f\colon A \mathrel{\mathrlap{\rightarrow}\mkern-3mu\bullet}\!\!\rightarrow B$, a horizontal morphism $h\colon A \to C$, and an object A are denoted, respectively, as follows:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f}\,\downarrow\!\bullet & 1_f & \bullet\!\downarrow\,{\scriptstyle f} \\ B & = & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle 1}\,\downarrow\!\bullet & 1_h & \bullet\!\downarrow\,{\scriptstyle 1} \\ A & \xrightarrow[h]{} & C \end{array} \qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\,\downarrow\!\bullet & 1_A & \bullet\!\downarrow\,{\scriptstyle 1} \\ A & = & A \end{array}$$

In this thesis, the horizontal composition of a double category is always taken to be *strictly* associative and unital, while vertical composition may only be *weakly* associative and unital up to comparison isocells (which are omitted in practice); this is precisely what is meant by *(pseudo) internal category* in Definition A.1). A double category is called *unitary* if vertical composition is strictly unital, and *strict* if vertical composition is both strictly unital and associative.

Assumption. A double category is always understood to be *unitary* unless stated otherwise.

Definition A.2. A double category is called *flat* if its cells are determined by their boundary morphisms.

Example A.3. Given a category $\mathcal{C}$, let $\mathbb{S}\mathrm{q}(\mathcal{C})$ denote the *double category of squares* in $\mathcal{C}$, whose objects are objects in $\mathcal{C}$, whose horizontal and vertical morphisms are morphisms in $\mathcal{C}$, and whose cells are commutative squares in $\mathcal{C}$.

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\,\downarrow\!\bullet & \alpha & \bullet\!\downarrow\,{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \quad := \quad \begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\,\downarrow & & \downarrow\,{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

Composition and identities in $\mathbb{S}\mathrm{q}(\mathcal{C})$ are given by those in $\mathcal{C}$. The category of morphisms is denoted $\mathcal{S}\mathrm{q}(\mathcal{C})$ and is commonly called the *arrow category* of $\mathcal{C}$. Thus the corresponding internal category presentation is given by:

$$\mathcal{S}\mathrm{q}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{S}\mathrm{q}(\mathcal{C}) \xrightarrow{\text{comp}} \mathcal{S}\mathrm{q}(\mathcal{C}) \;\begin{array}{c} \xrightarrow{\text{dom}} \\ \xleftarrow{\text{id}} \\ \xrightarrow[\text{cod}]{} \end{array}\; \mathcal{C}$$

The double category of squares is a flat strict double category.

Example A.4. Given a category $\mathcal{C}$ with pullbacks, let $\mathbb{S}\mathrm{pan}(\mathcal{C})$ denote the *double category of spans* in $\mathcal{C}$, whose objects are objects in $\mathcal{C}$, whose horizontal morphisms are morphisms in $\mathcal{C}$, whose vertical morphisms are spans of morphisms in $\mathcal{C}$, and whose cells are given by commutative diagrams in $\mathcal{C}$ of the form:

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle (s,X,t)}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle (u,Y,v)} \\
B & \xrightarrow[k]{} & D
\end{array}
\qquad \rightsquigarrow \qquad
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle s}\big\uparrow & & \big\uparrow{\scriptstyle u} \\
X & \xrightarrow{\alpha} & Y \\
{\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle v} \\
B & \xrightarrow[k]{} & D
\end{array}
$$

Vertical composition of spans is given by pullback, while horizontal composition is given by composition in $\mathcal{C}$. Let $\mathcal{S}\mathrm{pan}(\mathcal{C})$ denote the category of morphisms, whose objects are spans and whose morphisms are cells as above. The corresponding internal category presentation is given by:

$$\mathcal{S}\mathrm{pan}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{S}\mathrm{pan}(\mathcal{C}) \xrightarrow{\mathrm{comp}} \mathcal{S}\mathrm{pan}(\mathcal{C}) \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\substack{\longrightarrow \\ \xleftarrow{\ \mathrm{id}\ } \\ \longrightarrow}}} \mathcal{C}$$

The double category of spans in $\mathcal{C}$ is a non-flat non-strict double category, however pullbacks in $\mathcal{C}$ are chosen such that composition in $\mathbb{S}\mathrm{pan}(\mathcal{C})$ in unitary. As a convenient shorthand, the double category of spans in $\mathcal{S}\mathrm{et}$ is denoted $\mathbb{S}\mathrm{pan}$.

Notation A.5. Note that blackboard bold initial letters (e.g. $\mathbb{D}$) are used to denote double categories, and are replaced with calligraphic initial letters (e.g. $\mathcal{D}$) to denote their categories of morphisms and objects (except if they are better known by other names, like $\mathcal{S}\mathrm{et}$ or $\mathcal{C}\mathrm{at}$).

Definition A.6. A *full double subcategory* of a double category $\mathbb{D}$ is determined by a subset of vertical morphisms which is closed under vertical composition.

Example A.7. The double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ is the full double subcategory of $\mathbb{S}\mathrm{pan}(\mathcal{C})$ on the class of spans whose left leg (i.e. leg adjacent to the domain) is an identity morphism.

Example A.8. Given a category $\mathcal{C}$, let $\mathcal{M}$ be a class of morphisms which contains the identities and is closed under composition. Let $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$ denote the full double subcategory of $\mathbb{S}\mathrm{q}(\mathcal{C})$ determined by the class of morphisms $\mathcal{M}$.

Example A.9. Given a category $\mathcal{C}$ with pullbacks, let $\mathcal{W}$ and $\mathcal{M}$ be classes of morphisms which contain the identities, are closed under composition, and are stable under pullback. Let $\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$ denote the full double subcategory of $\mathbb{S}\mathrm{pan}(\mathcal{C})$ determined by the class of spans whose left leg (i.e. leg adjacent to the domain) is in the class $\mathcal{W}$ and whose right leg (i.e. leg adjacent to the codomain) is in the class $\mathcal{M}$.

Example A.10. Given a category $\mathcal{C}$, let $\mathbb{V}(\mathcal{C})$ denote the *vertical double category* of $\mathcal{C}$, whose objects are objects of $\mathcal{C}$, whose horizontal morphisms are identity morphisms in $\mathcal{C}$, whose vertical morphisms are morphisms in $\mathcal{C}$, and whose cells are horizontal identities.

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & = & B \end{array}$$

This is a flat strict double category. Note that $\mathbb{V}(\mathcal{C})$ is *not* a full double subcategory of $\mathbb{S}\mathrm{q}(\mathcal{C})$.

Definition A.11. A horizontal morphism $f\colon A \to B$ has a *companion* $f_*\colon A \twoheadrightarrow B$ if there are cells,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\ B & = & B \end{array} \qquad\qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle f_*} \\ A & \xrightarrow[f]{} & B \end{array}$$

which are called *binding cells*, such that the following pasting conditions hold:

$$\diamondsuit \mid \heartsuit = 1_f \qquad \text{and} \qquad \frac{\diamondsuit}{\heartsuit} = 1_{f_*}$$

A double category is equipped with a *functorial choice of companions* if every horizontal morphism f has a companion f_*, and the equations $(1_A)_* = 1_A$ and $(gf) = g_* f_*$ hold.

In a unitary double category, companions may be expressed via a universal property. A horizontal morphism $f\colon A \to B$ has a vertical *companion* $f_*\colon A \twoheadrightarrow B$ if for every cell α below, there is a cell $\diamondsuit$ as above, such that there is a unique factorisation:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array} \quad = \quad \begin{array}{ccccc} A & = & A & \xrightarrow{h} & C \\ {\scriptstyle 1}\downarrow & \diamondsuit & \downarrow{\scriptstyle f_*} & \alpha' & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B & = & B \end{array} \tag{A.1}$$

Definition A.12. A horizontal morphism $f\colon A \to B$ has a *conjoint* $f^*\colon B \twoheadrightarrow A$ if there are cells (also called binding cells),

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle f^*}\downarrow & \clubsuit & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & \spadesuit & \downarrow{\scriptstyle f^*} \\ A & = & A \end{array}$$

such that the following pasting conditions hold:

$$\spadesuit \mid \clubsuit = 1_f \qquad \text{and} \qquad \frac{\clubsuit}{\spadesuit} = 1_{f^*}$$

A double category is equipped with a *functorial choice of conjoints* if every horizontal morphism f has a conjoint f^*, and the equations $(1_A)^* = 1_A$ and $(gf)^* = f^* g^*$ hold.

Example A.13. A horizontal morphism $f\colon A \to B$ in the double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has a companion $f_*\colon A \mathrel{-\!\!\bullet\!\!\to} B$ given by the span,

$$A \xleftarrow{1_A} A \xrightarrow{f} B$$

and a conjoint $f^*\colon B \mathrel{-\!\!\bullet\!\!\to} A$ given by the span:

$$B \xleftarrow{f} A \xrightarrow{1_A} A$$

Therefore $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has a functorial choice of companions and conjoints.

Example A.14. The double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ has a functorial choice of companions. However, a horizontal morphism has a conjoint if and only if it is an isomorphism.

A double category may not have all companions or conjoints (as the above example shows). However, in most "nice" double categories, the *horizontal isomorphisms* have companions and conjoints.

Definition A.15. A double category is called *horizontally invariant* if every horizontal isomorphism has a companion, or equivalently, if every horizontal isomorphism has a conjoint.

The reason this property is called *horizontal invariance* is because it allows the transport of vertical morphisms along horizontal isomorphisms.

Example A.16. In the double category $\mathbb{S}\mathrm{q}(\mathcal{C})$, every horizontal isomorphism has a conjoint given by its inverse. Therefore $\mathbb{S}\mathrm{q}(\mathcal{C})$ is a horizontally invariant double category. The double category $\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$ is horizontally invariant if and only if the class of morphisms $\mathcal{M}$ contains the isomorphisms.

Definition A.17 ([Ale18])**.** A double category $\mathbb{D}$ is called *unit-pure* if the identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ is fully faithful.

Unpacking the definition, a double category is unit-pure if for every cell whose vertical boundary morphisms are identities,

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle 1}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle 1} \\ A & \xrightarrow[k]{} & B \end{array}$$

then it necessarily holds that $h = k$ and $\alpha = 1_h = 1_k$.

Example A.18. The double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ is unit-pure.

Definition A.19. A vertical morphism $f\colon A \mathrel{\bullet\!\!\to} B$ has a *tabulator* if there is a cell,

$$\begin{array}{ccc} \top f & \xrightarrow{\pi_A} & A \\ {\scriptstyle 1}\big\downarrow\bullet & \pi_f & \bullet\big\downarrow{\scriptstyle f} \\ \top f & \xrightarrow[\pi_B]{} & B \end{array} \tag{A.2}$$

such that for every cell α below there is a unique factorisation:

$$\begin{array}{ccc} X & \xrightarrow{h} & A \\ {\scriptstyle 1}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle f} \\ X & \xrightarrow[k]{} & B \end{array} \quad = \quad \begin{array}{ccccc} X & \xrightarrow{g} & \top f & \xrightarrow{\pi_A} & A \\ {\scriptstyle 1}\big\downarrow\bullet & 1_g & \bullet\big\downarrow & \pi_f & \bullet\big\downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & \top f & \xrightarrow[\pi_B]{} & B \end{array}$$

A double category $\mathbb{D}$ has all tabulators if the identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ has a right adjoint.

Definition A.20. A vertical morphism $f\colon A \mathrel{\bullet\!\!\to} B$ has a *cotabulator* if there is a cell,

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \bot f \\ {\scriptstyle f}\big\downarrow\bullet & \iota_f & \bullet\big\downarrow{\scriptstyle 1} \\ B & \xrightarrow[\iota_B]{} & \bot f \end{array}$$

satisfying a universal property dual to that of tabulators. A double category $\mathbb{D}$ has all tabulators if the identity map $\mathrm{id}\colon \mathcal{D}_0 \to \mathcal{D}_1$ has a left adjoint.

Note that a double category is unit-pure if and only if the tabulator of each vertical identity morphism is an identity cell if and only if the cotabulator of vertical identity morphism is an identity cell.

Example A.21. The double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ has all tabulators and cotabulators. The tabulator of a vertical morphism $f\colon A \to B$ is given by the commuting square:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

Example A.22. The double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has all tabulators. The tabulator of a span is given by the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{s} & A \\ {\scriptstyle 1_X}\big\uparrow & & \big\uparrow{\scriptstyle s} \\ X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow[t]{} & B \end{array}$$

The double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has all cotabulators if and only if $\mathcal{C}$ has pushouts.

Definition A.23 ([Ale18])**.** A vertical morphism $f\colon A \to\!\!\!\!\!\bullet\; B$ has a *strong tabulator* if it has a tabulator (A.2) such that π_A has a conjoint, π_B has a companion, and the composite cell,

$$\begin{array}{ccc}
A & = & A \\
{\scriptstyle (\pi_A)^*}\downarrow & \clubsuit & \downarrow{\scriptstyle 1} \\
\top f & \xrightarrow{\pi_A} & A \\
{\scriptstyle 1}\downarrow & \pi_f & \downarrow{\scriptstyle f} \\
\top f & \xrightarrow[\pi_B]{} & B \\
{\scriptstyle (\pi_B)_*}\downarrow & \heartsuit & \downarrow{\scriptstyle 1} \\
B & = & B
\end{array}$$

is horizontally invertible.

In a double category with strong tabulators, every vertical morphism is isomorphic to a conjoint followed by a companion

Example A.24. The double categories $\mathbb{S}\mathrm{q}(\mathcal{C})$ and $\mathbb{S}\mathrm{pan}(\mathcal{C})$ have strong tabulators.

Definition A.25. A *lax double functor* $F\colon \mathbb{C} \to \mathbb{D}$ between double categories consists of an assignment,

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
{\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\
B & \xrightarrow[k]{} & D
\end{array}
\quad\longmapsto\quad
\begin{array}{ccc}
FA & \xrightarrow{Fh} & FC \\
{\scriptstyle Ff}\downarrow & F\alpha & \downarrow{\scriptstyle Fg} \\
FB & \xrightarrow[Fk]{} & FD
\end{array}$$

together with identity and composition *comparison cells* (where $f \otimes g$ denotes the vertical composite of f followed by g, and $\frac{\alpha}{\beta}$ similarly denotes the vertical composite of the cell α followed by the cell β),

$$\begin{array}{ccc}
FA & = & FA \\
{\scriptstyle 1_{FA}}\downarrow & \eta_A & \downarrow{\scriptstyle F(1_A)} \\
FA & = & FA
\end{array}
\qquad
\begin{array}{ccc}
FA & = & FA \\
{\scriptstyle Ff}\downarrow & & \\
FB & \mu_{f,g} & \downarrow{\scriptstyle F(f\otimes g)} \\
{\scriptstyle Fg}\downarrow & & \\
FC & = & FC
\end{array}$$

which satisfying the following naturality conditions,

$$\begin{array}{ccccc}
FA & \xrightarrow{Fh} & FC & = & FC \\
{\scriptstyle 1_{FA}}\downarrow & 1_{Fh} & \downarrow & \eta_C & \downarrow{\scriptstyle F(1_C)} \\
FA & \xrightarrow[Fh]{} & FC & = & FC
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & = & FA & \xrightarrow{Fh} & FC \\
{\scriptstyle 1_{FA}}\downarrow & \eta_A & \downarrow & F(1_h) & \downarrow{\scriptstyle F(1_C)} \\
FA & = & FA & \xrightarrow[Fh]{} & FC
\end{array}$$

$$\begin{array}{ccccc} FA & = & FA & \longrightarrow & FA' \\ {\scriptstyle Ff}\big\downarrow & & \big| & & \big| \\ FB & \mu_{f,g} & \big| & F\binom{\alpha}{\beta} & \big| {\scriptstyle F(f'\otimes g')} \\ {\scriptstyle Fg}\big\downarrow & & \big\downarrow & & \big\downarrow \\ FC & = & FC & \longrightarrow & FC' \end{array} \quad = \quad \begin{array}{ccccc} FA & \longrightarrow & FA' & = & FA' \\ {\scriptstyle Ff}\big\downarrow & F\alpha & \big\downarrow & & \big| \\ FB & \longrightarrow & FB' & \mu_{f',g'} & \big| {\scriptstyle F(f'\otimes g')} \\ {\scriptstyle Fg}\big\downarrow & F\beta & \big\downarrow & & \big\downarrow \\ FC & \longrightarrow & FC' & = & FC' \end{array}$$

as well as a coherence condition with the associativity comparison cells.

Definition A.26. A (*pseudo*) double functor is a lax double functor whose comparison cells η and μ are horizontally invertible. A lax double functor is *unitary* if the comparison cells η are identities. A *strict* double functor is a lax double functor whose comparison cells η and μ are both identities.

Remark. A double functor is always understood to be a pseudo double functor unless stated otherwise. The terms *strong* double functor and pseudo double functor are synonymous, and strong double functor is occasionally used to emphasise strengthening of the notion of lax double functor.

Example A.27. A small category is the same as a lax double functor $* \to \mathbb{S}\mathrm{pan}$, where $*$ denotes the double category with a single object and no non-identity morphisms or cells. A functor with codomain $\mathcal{C}$ is the same as a lax double functor $\mathbb{V}(\mathcal{C}) \to \mathbb{S}\mathrm{pan}$.

Example A.28 ([Nie12])**.** Suppose $\mathbb{D}$ is a double category with tabulators such that $\mathcal{D}_0$ has pullbacks. There is a canonical lax double functor $\mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathcal{D}_0)$ given by the assignment:

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle \pi_A}\big\uparrow & & \big\uparrow{\scriptstyle \pi_C} \\ \top f & \xrightarrow{\top\alpha} & \top g \\ {\scriptstyle \pi_B}\big\downarrow & & \big\downarrow{\scriptstyle \pi_D} \\ B & \xrightarrow[k]{} & D \end{array} \tag{A.3}$$

If a double category is *unit-pure*, then this double functor is unitary.

A double functor $F\colon \mathbb{C} \to \mathbb{D}$ is equivalent to a pair of functors $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ between the categories of objects and morphisms, respectively, satisfying the conditions in Definition A.25.

Definition A.29. A double functor $F\colon \mathbb{C} \to \mathbb{D}$ is called:

- *faithful* if the functor $F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ is faithful
- *full* if the functor $F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ is full;

- *fully faithful* if it is both full and faithful;
- *representative* if the functor $F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ is essentially surjective-on-objects;
- *locally trivial* if the functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ is the identity functor.

Definition A.30 ([GP17])**.** A double category $\mathbb{D}$ is called *span representable* if the double functor (A.3) exists and is faithful.

Definition A.31. A pair of horizontally invariant double categories are *equivalent* if there is a fully faithful representative double functor between them.

Definition A.32. Consider a pair of lax double functors $F, G\colon \mathbb{C} \to \mathbb{D}$. A *horizontal transformation* $\Psi\colon F \Rightarrow G$ consists of, for each vertical morphism $f\colon A \mathrel{-\!\!\bullet\!\!\to} B$, a cell,

$$
\begin{array}{ccc}
FA & \xrightarrow{\Psi_A} & GA \\
{\scriptstyle Ff}\big\downarrow\!\bullet & \Psi_f & \bullet\!\big\downarrow{\scriptstyle Gf} \\
FB & \xrightarrow[\Psi_B]{} & GB
\end{array}
$$

which satisfies, for each cell α, the naturality condition,

$$
\begin{array}{ccccc}
FA & \xrightarrow{Fh} & FC & \xrightarrow{\Psi_C} & GC \\
{\scriptstyle Ff}\big\downarrow\!\bullet & F\alpha & \bullet\!\big\downarrow & \Psi_g & \bullet\!\big\downarrow{\scriptstyle Gg} \\
FB & \xrightarrow[Fk]{} & FD & \xrightarrow[\Psi_D]{} & GD
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & \xrightarrow{\Psi_A} & GA & \xrightarrow{Gh} & GC \\
{\scriptstyle Ff}\big\downarrow\!\bullet & \Psi_f & \bullet\!\big\downarrow & G\alpha & \bullet\!\big\downarrow{\scriptstyle Gg} \\
FB & \xrightarrow[\Psi_B]{} & GB & \xrightarrow[Gk]{} & GD
\end{array}
$$

and the following coherence conditions with respect to the identity and composition comparison cells (which by an abuse of notation, are denoted η and μ, respectively, for both F and G):

$$
\begin{array}{ccccc}
FA & \xrightarrow{\Psi_A} & GA & = & GA \\
{\scriptstyle 1_{FA}}\big\downarrow\!\bullet & 1_{\Psi_A} & \bullet\!\big\downarrow & \eta_A & \bullet\!\big\downarrow{\scriptstyle G(1_A)} \\
FA & \xrightarrow[\Psi_A]{} & GA & = & GA
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & = & FA & \xrightarrow{\Psi_A} & GA \\
{\scriptstyle 1_{FA}}\big\downarrow\!\bullet & \eta_A & \bullet\!\big\downarrow & \Psi_{1_A} & \bullet\!\big\downarrow{\scriptstyle G(1_A)} \\
FA & = & FA & \xrightarrow[\Psi_A]{} & GA
\end{array}
$$

$$
\begin{array}{ccccc}
FA & = & FA & \xrightarrow{\Psi_A} & GA \\
{\scriptstyle Ff}\big\downarrow\!\bullet & & & & \\
FB & \mu_{f,g} & \bullet\!\big\downarrow & \Psi_{f\otimes g} & \bullet\!\big\downarrow{\scriptstyle G(f\otimes g)} \\
{\scriptstyle Fg}\big\downarrow\!\bullet & & & & \\
FC & = & FC & \xrightarrow[\Psi_C]{} & GC
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & \xrightarrow{\Psi_A} & GA & = & GA \\
{\scriptstyle Ff}\big\downarrow\!\bullet & \Psi_f & \bullet\!\big\downarrow & & \\
FB & \longrightarrow & GB & \mu_{f,g} & \bullet\!\big\downarrow{\scriptstyle G(f\otimes g)} \\
{\scriptstyle Fg}\big\downarrow\!\bullet & \Psi_g & \bullet\!\big\downarrow & & \\
FC & \xrightarrow[\Psi_C]{} & GC & = & GC
\end{array}
$$

Definition A.33. A *globular transformation* is a horizontal transformation $\Psi\colon F \Rightarrow G$ such that $\Psi_A = 1_A$ for every object A. It necessarily follows that $F_0 = G_0$.

Definition A.34. Define the following 2-categories:

- $\mathcal{D}\mathrm{bl}$ of double categories, double functors, and horizontal transformations with hom-categories $[\mathbb{C}, \mathbb{D}]$
- $\mathcal{D}\mathrm{bl}_{\mathrm{lax}}$ of double categories, lax double functors, and horizontal transformations with hom-categories $[\mathbb{C}, \mathbb{D}]_{\mathrm{lax}}$
- $\mathcal{D}\mathrm{bl}_{\mathrm{unit}}$ of double categories, unitary double functors, and globular transformations with hom-categories $[\mathbb{C}, \mathbb{D}]_{\mathrm{unit}}$

Example A.35 ([Par11])**.** There are equivalences of categories,

$$\mathcal{C}\mathrm{at} \cong [*, \mathbb{S}\mathrm{pan}]_{\mathrm{lax}} \qquad\qquad \mathcal{C}\mathrm{at}/B \cong [\mathbb{V}(B), \mathbb{S}\mathrm{pan}]_{\mathrm{lax}}$$

where B is a small category. Both of these equivalences play an important role in Chapter 4.

Definition A.36. A pair of double functors $F \colon \mathbb{C} \to \mathbb{D}$ and $G \colon \mathbb{D} \to \mathbb{C}$ are *adjoint* if they form an adjunction internal to the 2-category $\mathcal{D}\mathrm{bl}$.

There are some additional specific uses of known concepts in double category theory that are only needed in a small part of the thesis, and are introduced when required. These include right-connected double categories (Section 3.2), left-connected completion (Section 4.1), monads in double categories (Section 5.1), and algebraic weak factorisation systems (Section 6.4).

List of Notation

The following is a list of notation for important categories and double categories, in the order in which they were introduced or used in the body of the thesis.

$\mathcal{S}\mathrm{et}$	The category of sets and functions
$\mathcal{C}\mathrm{at}$	The category of small categories and functors
$\mathcal{C}\mathrm{AT}$	The category of locally small categories and functors
B_∞	The codiscrete category on the set of objects B_0 of a category B
$\mathbb{D}$	A (unitary pseudo) double category
$\mathcal{D}_0$	The category of objects of a double category $\mathbb{D}$
$\mathcal{D}_1$	The category of morphisms of a double category $\mathbb{D}$
$\Gamma(\mathbb{D})$	The right-connected completion of a double category $\mathbb{D}$
$\Gamma'(\mathbb{D})$	The left-connected completion of a double category $\mathbb{D}$
$\mathbb{L}\mathrm{ens}$	The double category of lenses
$\mathcal{L}\mathrm{ens}$	The category of morphisms of $\mathbb{L}\mathrm{ens}$
$\mathcal{L}\mathrm{ens}_B$	The fibre of the functor $\mathrm{cod}\colon \mathcal{L}\mathrm{ens} \to \mathcal{C}\mathrm{at}$ over a category B
$\mathbb{C}\mathrm{of}$	The double category of cofunctors
$\mathcal{C}\mathrm{of}$	The category of morphisms of $\mathbb{C}\mathrm{of}$
$\mathbb{S}\mathrm{q}(\mathcal{C})$	The double category of squares in a category $\mathcal{C}$
$\mathcal{S}\mathrm{q}(\mathcal{C})$	The arrow category of $\mathcal{C}$; also the category of morphisms of $\mathbb{S}\mathrm{q}(\mathcal{C})$
$\mathcal{D}\mathrm{Opf}$	The full subcategory of $\mathcal{S}\mathrm{q}(\mathcal{C}\mathrm{at})$ on the class of discrete opfibrations

$\mathcal{S}\mathrm{Opf}$	The full subcategory of $\mathcal{L}\mathrm{ens}$ on the class of split opfibrations
$\mathbb{S}\mathrm{pan}$	The double category of spans in $\mathcal{S}\mathrm{et}$
$\mathcal{S}\mathrm{pan}$	The category of morphisms of $\mathbb{S}\mathrm{pan}$
$\mathbb{S}\mathrm{pan}(\mathcal{C})$	The double category of spans in a category $\mathcal{C}$ with pullbacks
$\mathbb{V}(B)$	The vertical double category on a category B
$\mathbb{S}\mathrm{Mult}$	The double category of split multi-valued functions
$\mathbb{M}\mathrm{nd}(\mathbb{D})$	The double category of monads in a double category $\mathbb{D}$
$\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{D})$	The full double subcategory of $\mathbb{M}\mathrm{nd}(\mathbb{D})$ on the class of monad retromorphisms
$\mathbb{M}\mathrm{nd}_{\mathrm{lens}}(\mathbb{D})$	The double category of lenses between monads
$(\mathcal{C}, \mathcal{W}, \mathcal{M})$	A category $\mathcal{C}$ equipped with classes of morphisms $\mathcal{W}$ and $\mathcal{M}$
$\mathbb{S}\mathrm{q}(\mathcal{C}, \mathcal{M})$	The full double subcategory of $\mathbb{S}\mathrm{q}(\mathcal{C})$ on a class of morphisms $\mathcal{M}$
$\mathbb{S}\mathrm{pan}(\mathcal{C}, \mathcal{W}, \mathcal{M})$	The full double subcategory of $\mathbb{S}\mathrm{pan}(\mathcal{C})$ on the class of spans whose left leg is in the class $\mathcal{W}$ and whose right leg is in the class $\mathcal{M}$
$\mathbb{L}\mathrm{ens}(\mathcal{C}, \mathcal{W}, \mathcal{M})$	The double category of generalised lenses

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