

# Emergent feedback loops

John C. Baez<sup>1,2</sup> Aditya Chaudhuri\*<sup>3,4,5</sup>

<sup>1</sup>School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, UK EH9 3FD

<sup>2</sup>Department of Mathematics, University of California, Riverside, CA, USA 92521

<sup>3</sup>Institute of Computer Science, University of Rostock, Albert-Einstein-Str. 22, 18059 Rostock, Germany (affiliation at the time of work)

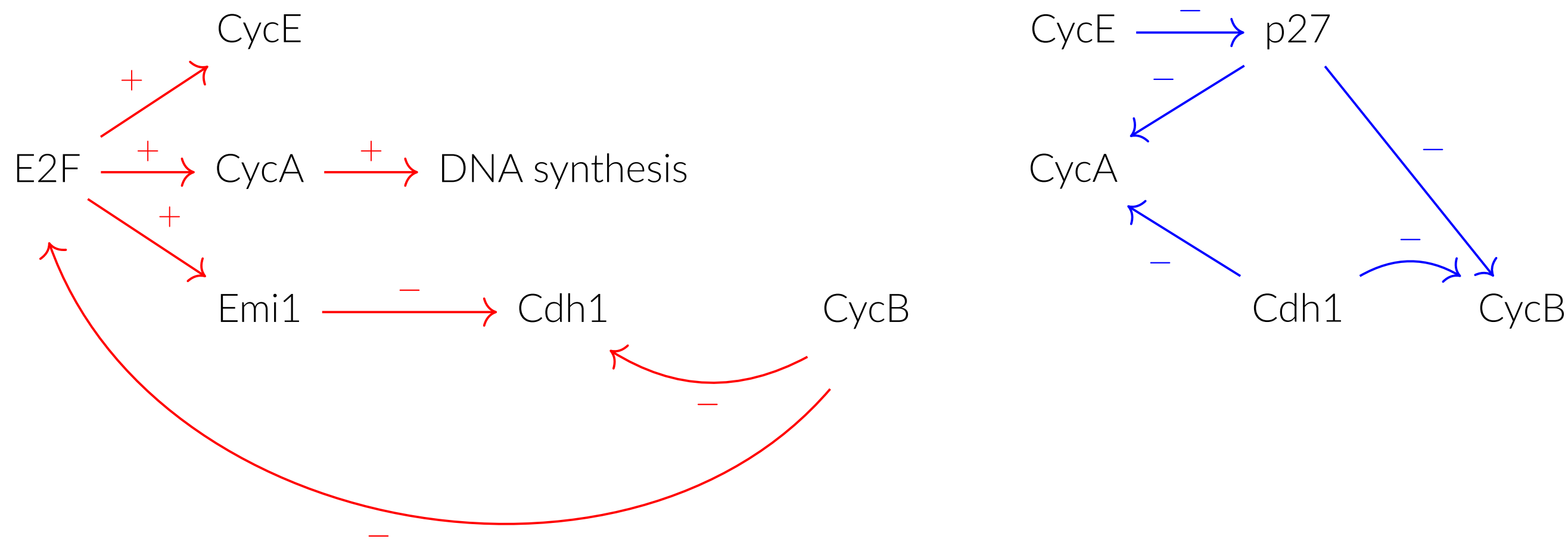
<sup>4</sup>Statistics and Mathematics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata 700108, India (affiliation at the time of work)

<sup>5</sup>Machine Intelligence Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata 700108, India (current affiliation)

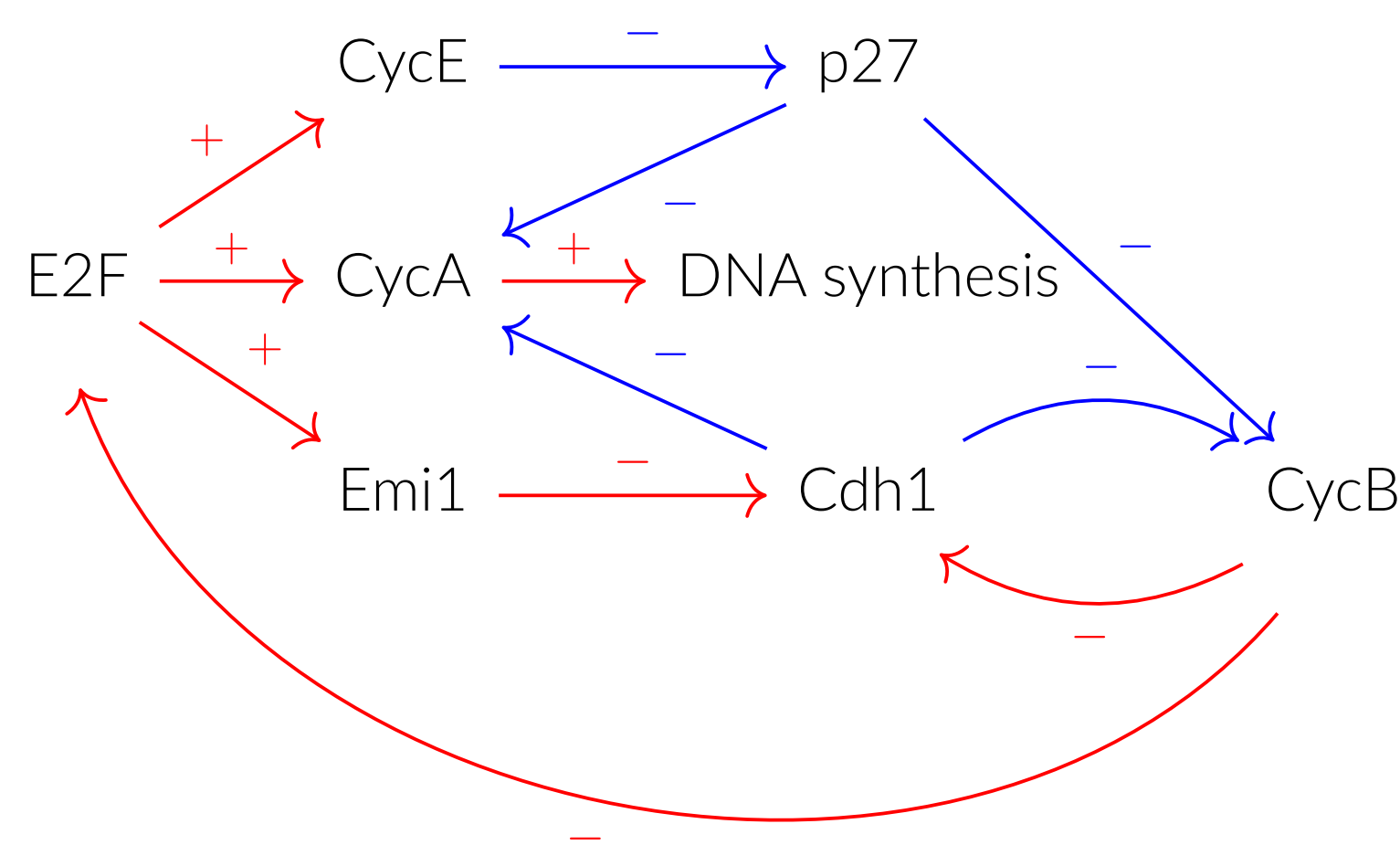
## Motivation

Directed graphs with signed edges model biological networks, where nodes represent biological entities and signed directed edges capture activation or inhibition. Multiplying signs along paths reveals indirect effects, and loops define feedback—crucial for system behavior—with often *positive loops* driving switch-like transitions, and *negative loops* promoting homeostasis.

Let us consider the following pair of regulatory interactions within a cell, which are crucial for DNA synthesis.



Observe that there are no feedback loops present in each of them. However, when we compose them along the common nodes viz. proteins CycE, CycA, Cdh1 and CycB, we obtain feedback loops, which were absent before, as is evident from the diagram below.



Note that in the above diagram, every directed loop is composed of mixed colors (red and blue), and there is not a single monochromatic directed cycle present. We call these new feedback loops **emergent feedback loops**, which are expected to give rise to *new emergent behaviour* in the combined system of networks.

**Question:** How to formalize the notion of emergent feedback loops?

We approach the question in two different ways:

- (1) Monoid-grading of the free category generated by a graph.
- (2) Developing graph homology with coefficients in a commutative monoid.

## Monoid-grading of the free category generated by a graph

Let  $\mathbf{Gph}$  be the category of graphs which is equivalent to the functor category  $\mathbf{Set}^{\mathbf{G}}$ , where  $\mathbf{G}$  is the category freely generated by two parallel morphisms  $e \rightrightarrows v$ . Explicitly, objects of  $\mathbf{Gph}$  are tuples  $G = (s, t: E \rightarrow V)$  called **graphs** which consists of a set of **vertices**  $V$  and a set of **edges**  $E$  together with a **source map**  $s: E \rightarrow V$  and a **target map**  $t: E \rightarrow V$ . Let  $\text{disc}: \mathbf{Set} \rightarrow \mathbf{Gph}$  be the functor which takes a set  $S$  to the unique discrete graph with vertex set  $S$  and no edges.

Suppose  $X$  and  $Y$  are graphs whose intersection  $X \cap Y$  is a discrete graph  $\text{disc}(B)$  for a set  $B$  (as in the case of the example illustrated above). Let  $o: \text{disc}(B) \rightarrow X$  and  $i: \text{disc}(B) \rightarrow Y$  be a pair of homomorphism of graphs which picks out the common vertices along which  $X$  and  $Y$  intersect. Now, define  $X \cup Y := X +_{o, \text{disc}(B), i} Y$ .

**Example:** If  $X$  is the red graph,  $Y$  is the blue graph, and if  $B := \{1, 2, 3, 4\}$ ,

$$o: \text{disc}(B) \rightarrow X, 1 \mapsto \text{CycE}, 2 \mapsto \text{CycA}, 3 \mapsto \text{Cdh1}, 4 \mapsto \text{CycB}$$

and

$$i: \text{disc}(B) \rightarrow Y, 1 \mapsto \text{CycE}, 2 \mapsto \text{CycA}, 3 \mapsto \text{Cdh1}, 4 \mapsto \text{CycB},$$

then  $X \cup Y$  is the bicolored combined graph shown above consisting of both red and blue edges.

Applying the free functor  $\mathbf{Free}: \mathbf{Gph} \rightarrow \mathbf{Cat}$  to  $X \cup Y$ , we get the category

$$\mathbf{C} = \mathbf{Free}(X) +_{\text{Disc}(B)} \mathbf{Free}(Y),$$

where  $\text{Disc}(B) \cong \mathbf{Free}(\text{disc}(B))$ .

Then the set of paths from a vertex  $v$  of the pushout graph  $X \cup Y$  to the vertex  $w$  is the homset  $\mathbf{C}(v, w)$ . We would like to understand which of these paths are 'emergent', i.e., not already paths in  $X$  or  $Y$ .

Let  $M\{x, y\}$  be the free monoid on two generators  $x$  and  $y$ . Then the category  $\mathbf{C}$  has a unique  $M\{x, y\}$ -grading where each edge of  $X$  has grade  $x$  and each edge of  $Y$  has grade  $y$ . Let  $I\{x, y\}$  be the quotient of the monoid  $M\{x, y\}$  by relations saying that  $x$  and  $y$  are idempotent:

$$x^2 = x, \quad y^2 = y.$$

The quotient map  $M\{x, y\} \rightarrow I\{x, y\}$  lets us push forward the  $M\{x, y\}$ -grading on  $\mathbf{C}$  to obtain an  $I\{x, y\}$ -grading on this category.

This has a simplifying effect: Now the grade of a path records only how it goes back and forth between the graphs  $X$  and  $Y$ .

**Example:** If we apply the above grading to our bicolored combined graph  $X \cup Y$  illustrated above, we get the grading of  $\text{Cdh1} \xrightarrow{-} \text{CycB} \xrightarrow{-} \text{Cdh1}$  as  $xy$  and the grading of  $\text{E2F} \xrightarrow{+} \text{CycE} \xrightarrow{-} \text{p27} \xrightarrow{-} \text{CycB} \xrightarrow{-} \text{E2F}$  as  $xyx$ . This grading gives us a measure of the 'amount of emergence' involved in these loops arising as a consequence of composing the red and the blue graph along the nodes CycE, CycA, Cdh1 and CycB.

## Homology for detecting directed loops

For any graph  $G = (s, t: E \rightarrow V)$  and commutative monoid  $C$ , we consider

$$C_1(G, C) \begin{array}{c} \xrightarrow{C[s]} \\ \xrightarrow{C[t]} \end{array} C_0(G, C),$$

where

$$C_0(G, C) = C[V], \quad C_1(G, C) = C[E],$$

the set of formal finite linear combinations of elements of respectively, vertex set  $V$  and the edge set  $E$  with coefficients in  $C$ . We call  $C_0(G, C)$  (resp.  $C_1(G, C)$ ) the commutative monoid of **0-chains** (resp. **1-chains**) on  $G$  with coefficients in  $C$ . Taking the equalizer and coequalizer of  $C[s]$  and  $C[t]$ , we get

$$H_1(G, C) \xrightarrow{i} C_1(G, C) \begin{array}{c} \xrightarrow{C[s]} \\ \xrightarrow{C[t]} \end{array} C_0(G, C) \xrightarrow{p} H_0(G, C),$$

where we call the equalizer  $H_1(G, C)$  and the coequalizer  $H_0(G, C)$  as the **first** and **zeroth homology of  $G$  with coefficients in  $C$** .

**Coefficients in  $\mathbb{N}$ :**

For any loop

$$\gamma = (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} v_{n-1} \xrightarrow{e_n} v_0)$$

in a graph  $G$ , we define  $[\gamma] = e_1 + e_2 + \dots + e_n \in C_1(G, \mathbb{N})$ . We say two loops  $\gamma$  and  $\delta$  **homologous** if  $[\gamma] = [\delta]$ , and call  $[\gamma] \in H_1(G, \mathbb{N})$ , the **homology class** of  $\gamma$  and  $\delta$ .

We call a loop

$$\gamma = (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} v_{n-1} \xrightarrow{e_n} v_0)$$

in a graph  $G$  to be **simple** if all the vertices  $v_0, \dots, v_{n-1}$  are distinct.

**Proposition:** For any graph  $G$ ,  $H_1(G, \mathbb{N})$  is generated by the homology classes of simple loops in  $G$ .

**Directed loop vs undirected loop:**

$$G = u \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} v, \quad G' = u \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} v$$

$H_1(G, \mathbb{N}) \cong \mathbb{N}$ , while  $H_1(G', \mathbb{N}) \cong \{0\}$ . On the other hand, we have  $H_1(G, \mathbb{Z}) \cong H_1(G', \mathbb{Z}) \cong \mathbb{Z}$ . We have  $H_0(G, \mathbb{N}) \cong H_0(G', \mathbb{N}) \cong \{0\}$  and similarly with  $\mathbb{Z}$  coefficients, since both graphs are connected.

## Emergent directed loops via Mayer-Vietoris

To answer our **question**, we proved the following version of Mayer-Vietoris for directed graph homology.

**Theorem:** If  $C$  is a cancellative commutative monoid and  $X, Y$  are subgraphs of a graph  $X \cup Y$  whose intersection  $X \cap Y$  is a discrete graph, then the following is an equalizer diagram in the category of commutative monoids:

$$H_1(X, C) \oplus H_1(Y, C) \xrightarrow{\iota} H_1(X \cup Y, C) \begin{array}{c} \xrightarrow{qC[s]p_X} \\ \xrightarrow{qC[t]p_X} \end{array} H_0(X \cap Y, C),$$

where

$$\iota: H_1(X, C) \oplus H_1(Y, C) \rightarrow H_1(X \cup Y, C), \quad (c_X, c_Y) \mapsto c_X + c_Y.$$

$$p_X: C_1(X \cup Y, C) \rightarrow C_1(X, C) \text{ by}$$

$$p_X(c) = c_X,$$

$$q: C[s], C[t]: C_1(X, C) \rightarrow C_0(X, C) \text{ induced from source and target maps,}$$

$$q: C_0(X, C) \rightarrow H_0(X \cap Y, C) \text{ induced by}$$

$$q': C_0(X, C) \rightarrow C_0(X \cap Y, C),$$

given as

$$q' \left( \sum_{v \in \{\text{vertices of } X\}} c_v v \right) = \sum_{v \in \{\text{vertices of } X \cap Y\}} c_v v.$$

By the symmetry between  $X$  and  $Y$ , the analogous diagram with  $p_X$  replaced by  $p_Y$  is also an equalizer.

**Example:** In the context of our example graphs illustrated in the left column, the associated monomorphism  $\iota: H_1(X, C) \oplus H_1(Y, C) \rightarrow H_1(X \cup Y, C)$ ,  $(c_X, c_Y) \mapsto c_X + c_Y$  is not an isomorphism.

## Feedback around a loop

For any commutative monoid  $C$ , and any  $C$ -labeled graph  $(G, \ell: E \rightarrow C)$ , there is a map

$$C_1(G, \mathbb{N}) \rightarrow C$$

defined by

$$\sum_{e \in E} n_e e \mapsto \sum_{e \in E} n_e \ell(e),$$

which restricts to a homomorphism of commutative monoids that we call  $\tilde{\ell}: H_1(G, \mathbb{N}) \rightarrow C$ . Given a cycle  $c \in H_1(G, \mathbb{N})$ , we call  $\tilde{\ell}(c)$  the **feedback of  $(G, \ell)$  around  $c$** .

**Observation:** For any commutative monoid  $C$  and any  $C$ -labeled graph  $(G, \ell)$ , the homomorphism  $\tilde{\ell}: H_1(G, \mathbb{N}) \rightarrow C$  is determined by its values on the homology classes of simple loops.

**Example:** The feedback around the emergent feedback loops  $\text{Cdh1} \xrightarrow{-} \text{CycB} \xrightarrow{-} \text{Cdh1}$  is '+', and along  $\text{E2F} \xrightarrow{+} \text{CycE} \xrightarrow{-} \text{p27} \xrightarrow{-} \text{CycB} \xrightarrow{-} \text{E2F}$  is '-', as  $C = (\mathbb{Z}/2, \times)$ .

## This presentation is based on some results in our paper

John C. Baez and Aditya Chaudhuri

Motifs and Emergent Feedback in Labeled Graphs, 2025

arXiv:2506.23375