Chapter 3: Linear Topics

The Differential Theory of Qualitative Equations

To denote lists of propositions and detail their components, we use notations like

$\mathbf{a} = <a, b, c>, \quad \mathbf{p} = <p, q, r>, \quad \mathbf{x} = <x, y, z>,$

or, in more complicated situations,

$\mathbf{x} = <x1, x2, x3>, \; \mathbf{y} = <y1, y2, y3>, \; \mathbf{z} = <z1, z2, z3>.$

In a universe where some region is ruled by a proposition, it is natural to ask whether we can change the value of that proposition by changing the features of a current state.

Given a Venn diagram with a shaded region and starting from any cell in the universe, what sequences of feature changes, what traverses of cell walls, will take us from shaded to unshaded areas, or the reverse?

In order to discuss questions of this type, it is useful to define several *operators* on functions. An operator is nothing more than a function between sets that happen to have functions as members. A typical operator F takes us from thinking about a given function f to thinking about another function g. To express the fact that g can be obtained by applying F to f, we write g = Ff.

The first operator, E, associates with a function f: A –> B, another function Ef, where Ef: A x A –> B is defined by

$Ef(x, y) = f(x + y).$

E is called a shift operator because it takes us from contemplating the value of f at a place x to considering the value of f at a shift of y away. It tells us the absolute effect on f of changing its argument from x by an amount y.

The second operator, D, associates with a function f: A –> B another function Df, where Df: A x A –> B is defined by

$Df(x, y) = Ef(x, y) - f(x),$

or, equivalently,

$Df(x, y) = f(x + y) - f(x).$

D is called a difference operator because it gives us the relative change in the value of f along the shift from x to x + y.

In practice, one of the variables x or y is usually considered to be "less variable" than the other, being fixed in the context of a concrete discussion. Thus we may we see any one of the following idioms:

1. Df: A x A –> B
   Df(c, x) = f(c + x) – f(c)

Here, c is held constant and Df(c, x) is regarded mainly as a function of the second variable x, giving the relative change in f at various distances x from a center c.

2. Df: A x A –> B
   Df(x, h) = f(x + h) – f(x)

Here, h is either a constant (usually 1) in discrete contexts, or a variably "small" amount (near to 0) over which a limit is taken in continuous contexts. Df(x, h) is regarded mainly as a function of the first variable x, giving the differences in the value of f between x and a neighbor h away as x ranges over various locations.

3. Df: A x A —> B
    Df(x, dx) = f(x + dx) – f(x)

This is another variant of the previous form, with dx denoting the small changes contemplated in x.

Imagine that we are sitting in one of the cells of a Venn diagram, contemplating the walls. There are N of them, one for each positive feature x1, ..., xN in our universe of discourse. Our particular cell is described by a concatenation of N signed assertions, positive or negative, regarding each of these features. Are we locked into this interpretation?

With respect to each edge x of the cell we consider a test proposition dx, to determine our decision to differ on x. If dx is true we decide to cross over edge x at some point in the future. To reckon the effect of several such decisions on the value of the reigning proposition, we transform that proposition by making the following set of substitutions everywhere in its expression:

```
Substitute ( x1 , dx1 ) for x1,
Substitute ( x2 , dx2 ) for x2,
   ... ,
Substitute ( xN , dxN ) for xN.
```
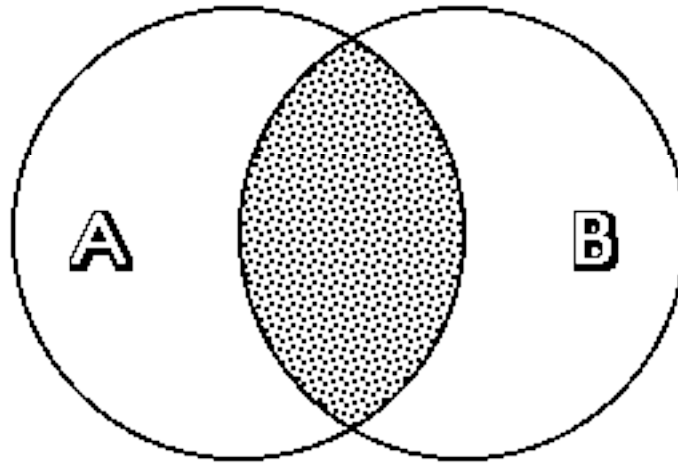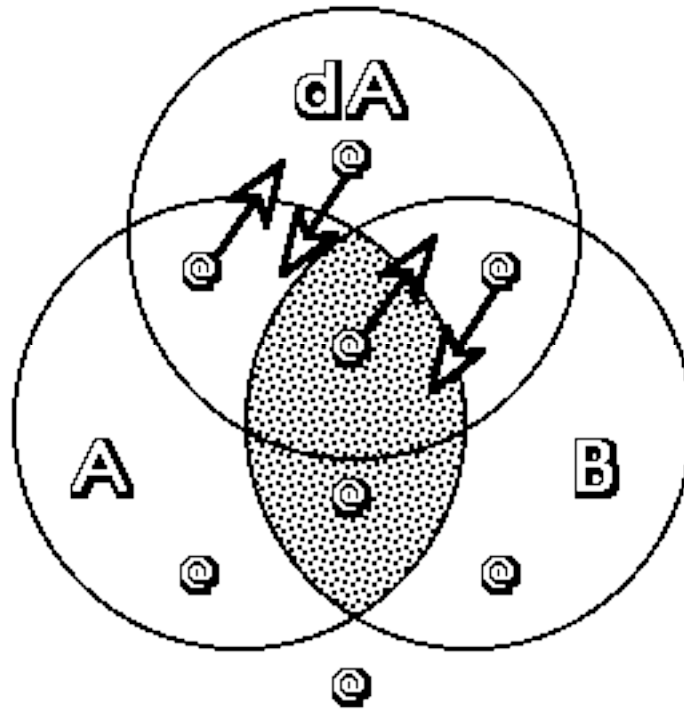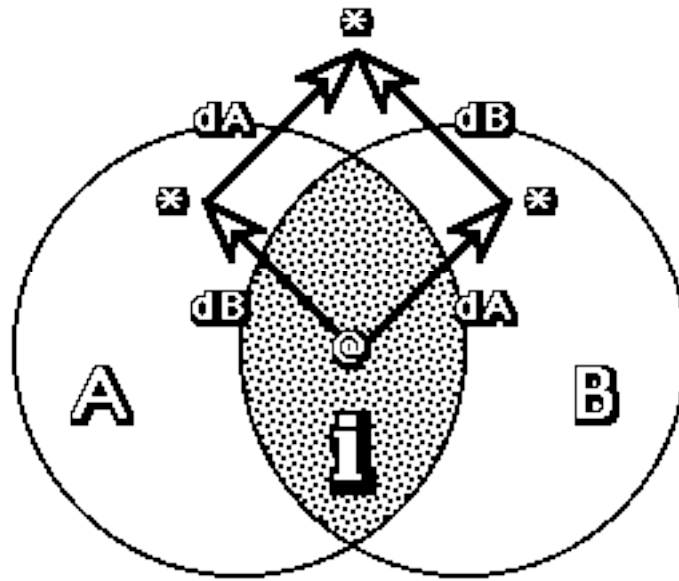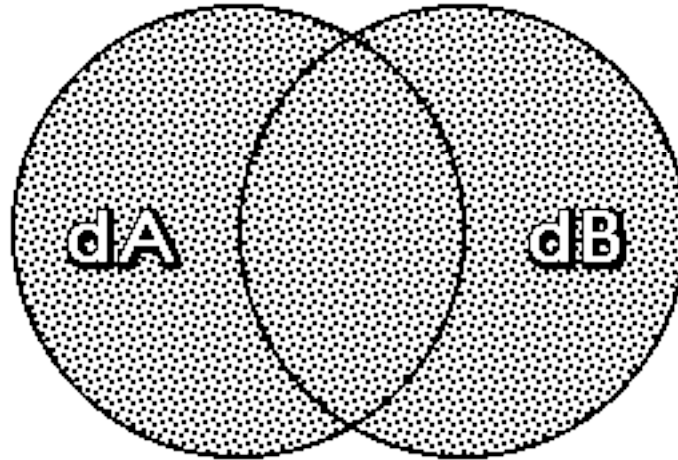
For a more complex example, consider the polymorphous set T of Example 1 and focus on the central cell described by the features A B C. The proposition or truth function T′ describing T is:

```
(( A B )( B C )( C A ))
```

Conjoining the query that specifies the center cell yields:

```
(( A B )( B C )( C A )) A B C
```

and we know the value of the interpretation by whether this expression issues in a model.

The result of the shift transformation on the original proposition is:

```
( (     ( A ,  dA ) ( B ,  dB )
) (     ( B ,  dB ) ( C ,  dC )
) (     ( C ,  dC ) ( A ,  dA )
) )
```

Conjoining a query on the center cell yields:

```
( (     ( A ,  dA ) ( B ,  dB )
) (     ( B ,  dB ) ( C ,  dC )
) (     ( C ,  dC ) ( A ,  dA )
) )

A  B  C
```

and the models of this expression tell us which feature changes will take us from our current interpretation (the center cell) to a true value under the proposition.

The application of the difference operator to the original proposition, conjoined with a query on the center cell, gives:

```
(
      ( (     ( A ,  dA ) ( B ,  dB )
      ) (     ( B ,  dB ) ( C ,  dC )
      ) (     ( C ,  dC ) ( A ,  dA )
      ) )
    ,
      ( (     A  B
      ) (     B  C
      ) (     C  A
      ) )
)

A  B  C
```

Applying Study to this last query results in the following output:

```
A
 B
  C
    dA
      dB  *
      (dB )
       dC  *
       (dC )  -
     (dA )
      dB
       dC  *
       (dC )  -
      (dB )  -
    (C )  -
   (B )  -
  (A )  -
```
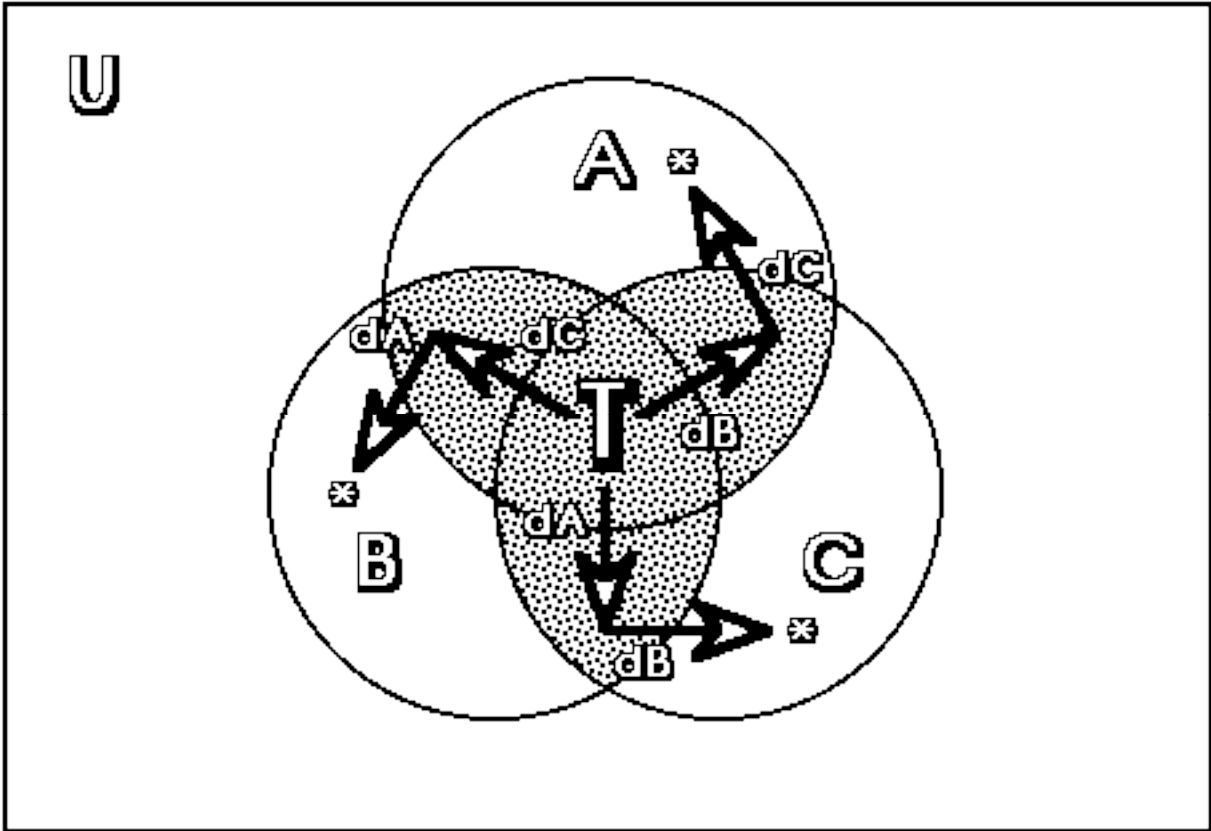
from which we pick out the following model paths:

```
<1>    A    B    C    dA     dB           *
<2>    A    B    C    dA    (dB ) dC      *
<3>    A    B    C    (dA ) dB    dC      *
```

This tells us that changing any two of the three features will take us from the center cell to outside the shaded region of the corresponding Venn diagram.

Another way of looking at this is by letting dA, dB, dC be taken as the features of another universe of discourse, called the tangent space of U with respect to the given interpretation i. Then the difference function Δ = Df(i, x) corresponds to the shaded region in the next figure.