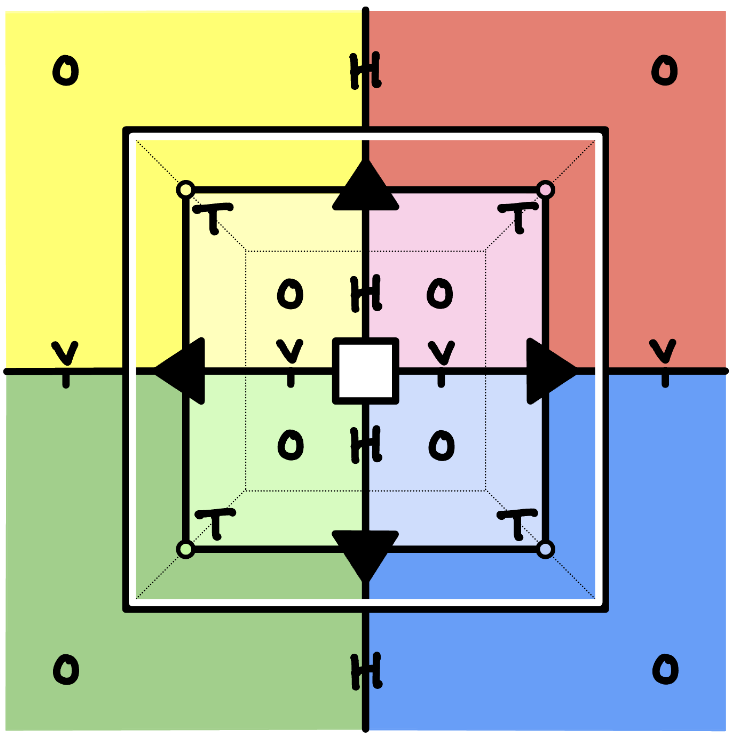
Monads in three dimensions

[Chapter 4, Section 3]

Let $\mathcal{E}$ be a triple category, with dimensions H, V, and T. Our main example is MatCat, as defined in the previous section. We define $\mathrm{Mnd}(\mathcal{E})$, the triple category of pseudomonads. Applied to MatCat, this constructs Logos, the triple category of logics.

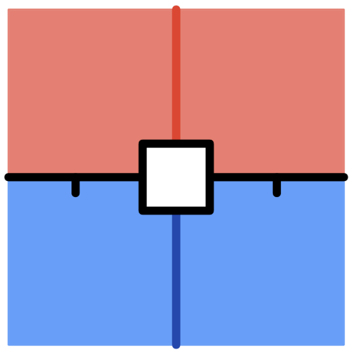
The terminology for each dimension is as follows.

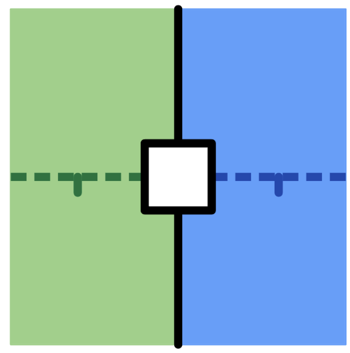


$\mathcal{E}$	MatCat	$\to$	$\mathrm{Mnd}(\mathrm{MatCat})$	$=$	Logos
0	category	$\to$	fibrant double category	:	logic
V	profunctor	$\to$	vertical profunctor	:	metaterm
H	fibered category	$\to$	horizontal profunctor	:	metajudgement
VH	fibered profunctor	$\to$	double profunctor	:	metainference
T	functor	$\to$	pseudo double functor	:	system
VT	transformation	$\to$	vertical transformation	:	term system
HT	fibered functor	$\to$	horizontal transformation	:	judgement system
3	fibered transformation	$\to$	double transformation	:	inference system

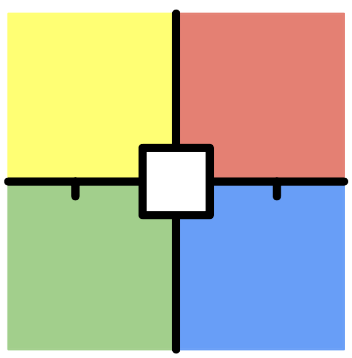
In MatCat, dimension T is functors, V is profunctors, and H is matrix categories (ref). "Pseudo"monads in H are *logics*, the two-dimensional structures we have explored, also known as *fibrant double categories*.

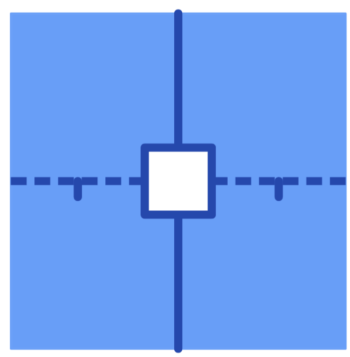
Hence there are *two* kinds of "relations" between logics, *metaterms* in dimension V and *metajudgements* in dimension H. They are exactly how they look: metaterms give "terms between logics" and metajudgements give "judgements between logics".





Connecting a pair of H-relations via a pair of V-relations is a *metainference*. The two-dimensional "hom" of a logic is its identity metainference.





By contrast, dimension T is "functional": a *flow* between logics is a pseudofunctor of fibrant double categories, a way to transform the inferences of one logic into another.

the dim of blank

WHAT ARE PSMNDS

how is an FDC a pseudomonad, "as we will see in theorem blank, ..."

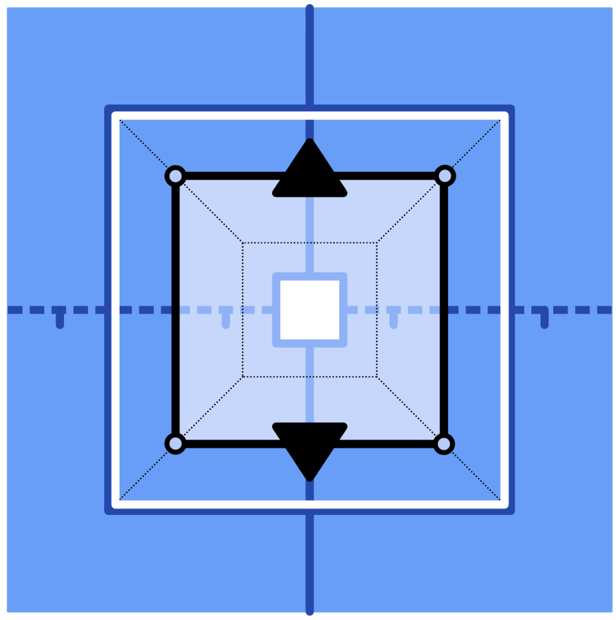
previous section and here: relation between the dimensions

we will think of FDCs as logics

i.e. profunctors

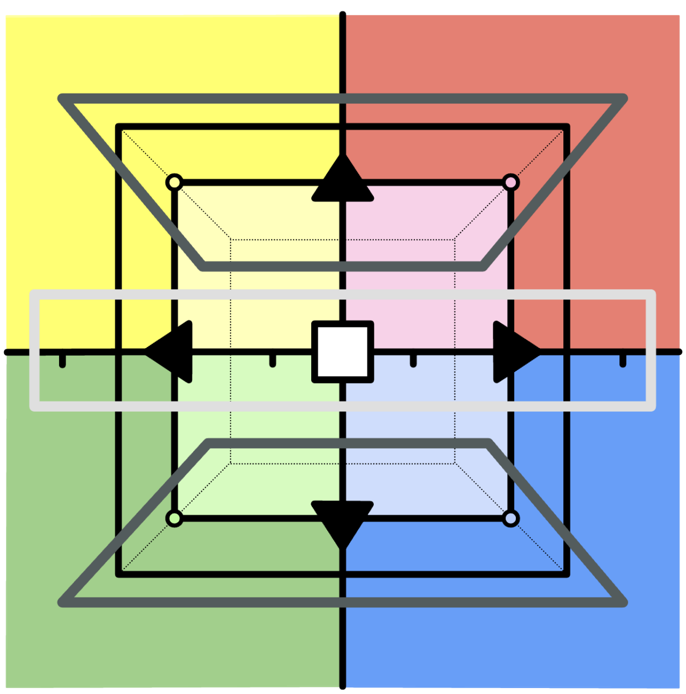
vertical morphism between...

categorifying "term between types"



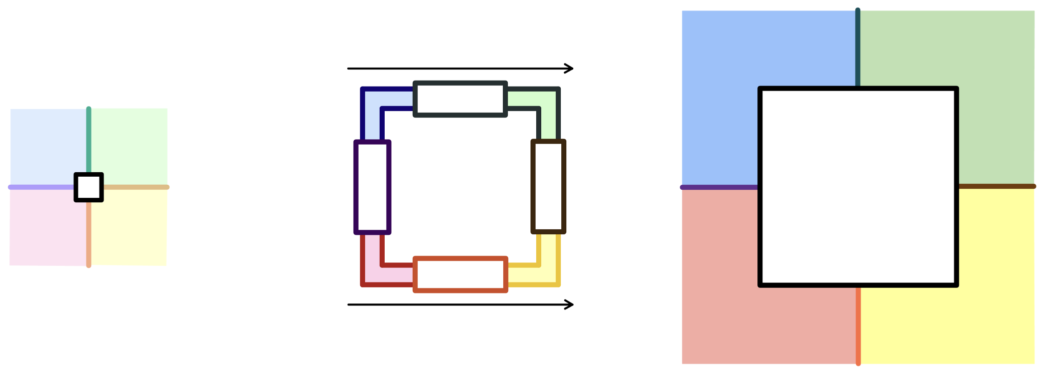
Morphisms of pseudomonads generalize in two opposite ways, “lax” and “colax”; these vastly expand the concept of monad and comonad which we have explored (ref).

We can organize the construction $\mathrm{Mnd}(-)$ as follows. Dimensions T and H, the top and bottom faces of the cube, form the double category of (pseudo)monads and functors, and (pseudo)modules and transformations: just like the two-dimensional monads and modules construction (ref), but weakened up to coherent 3-isomorphism.

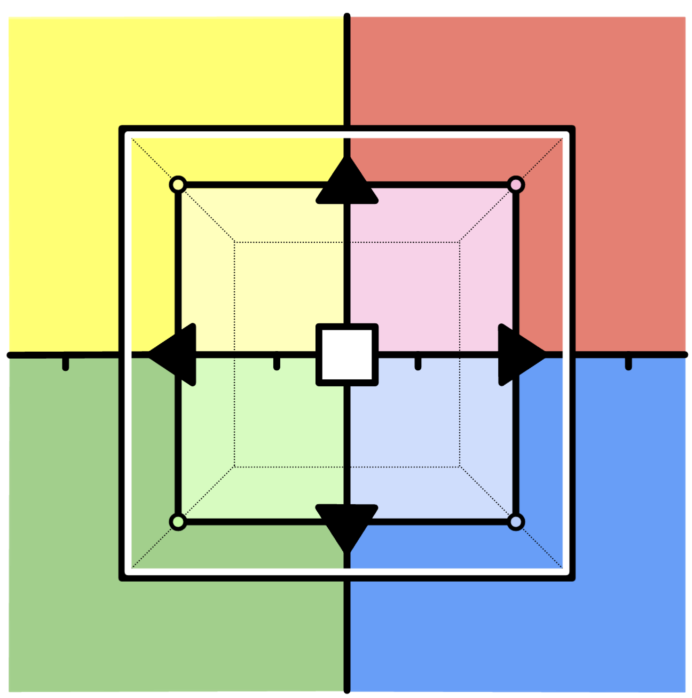


The "term dimension" V, the middle gray box above, is distinct because profunctors in MatCat are "already polarized"; they connect categories which act on them. Hence in V we form "polarized" monads: profunctors of terms between logics, with parallel composition and nullary introduction. Modulo this new aspect, in dimension V the construction is exactly the same as monads in two dimensions (ref).

A cube is an *inference system*, a natural map of metainferences; in category theory this is known as a modification (cite). An inference system can be visualized as a cube which transforms an *inner* metainference to an *outer* metainference, i.e. as the following process.



We now define the triple category of pseudomonads for a general triple category $\mathcal{E}$. Yet as one may see in the visualization, the case of fibrant double categories is canonical, and will serve as a guiding intuition for each definition.

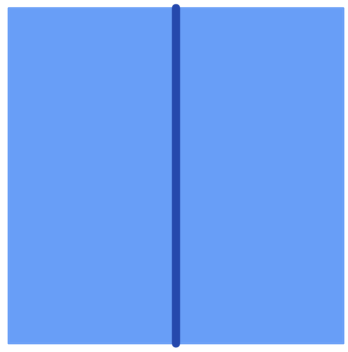


Definition 1. A **pseudomonad** in a triple category $\mathcal{E}$ is an object with an H-morphism

object

$\mathbb{A}_0$

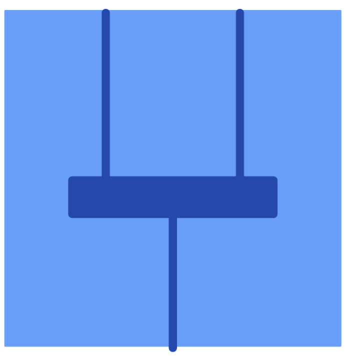
monad



$\mathbb{A}_0 \xrightarrow{\mathbb{A}} \mathbb{A}_0$

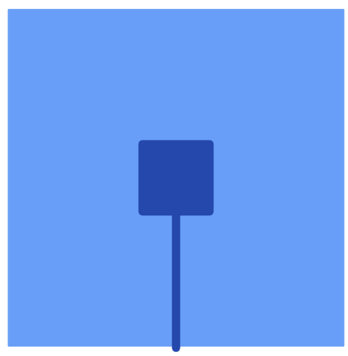
with HT-morphisms

join



$\mu_{\mathbb{A}} : \mathbb{A} \circ \mathbb{A} \Rightarrow \mathbb{A}$

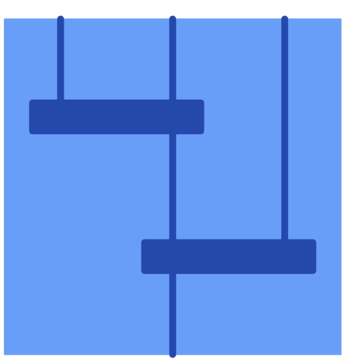
unit



$\eta : 1_{\mathbb{A}_0} \Rightarrow \mathbb{A}$

and invertible 3-morphisms

associator



$(\mu\mathbb{A})\mu \quad \overset{\sim}{\underset{\sim}{\xrightarrow{\alpha}}} \quad (\mathbb{A}\mu)\mu$

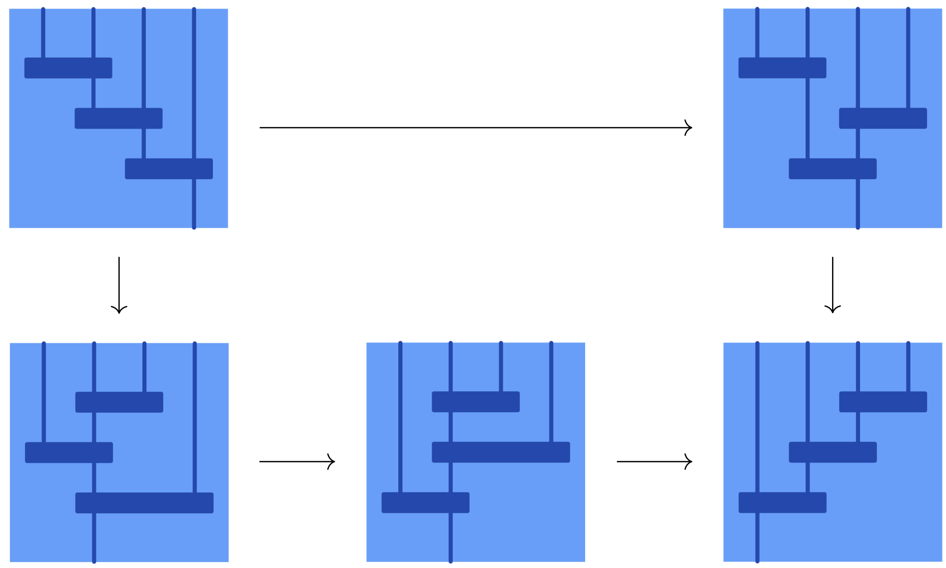
left unitor

$$(\eta\mathbb{A})\mu \overset{\lambda}{\underset{\sim}{\to}} \mathbb{A}$$

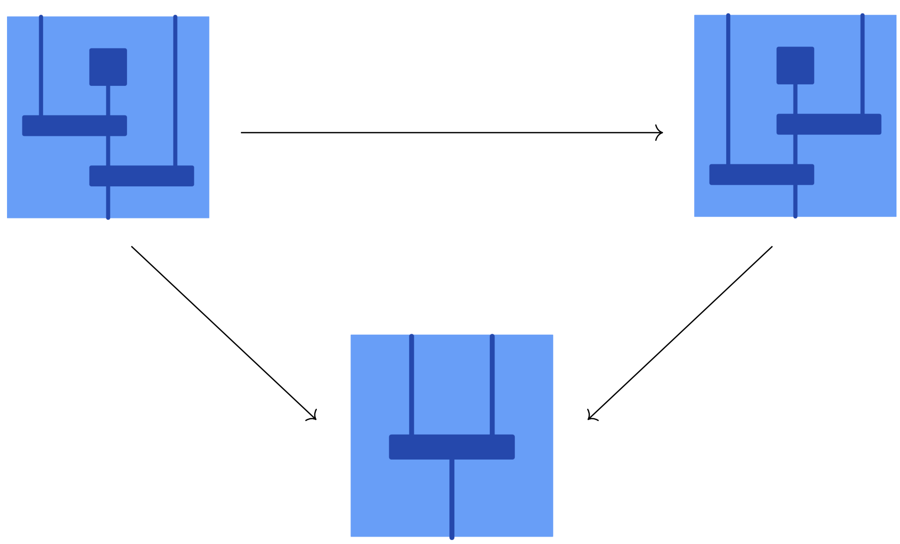
right unitor

$$(\mathbb{A}\eta)\mu \overset{\rho}{\underset{\sim}{\to}} \mathbb{A}$$

satisfying the following coherence.



[M.4J] Associator Equation

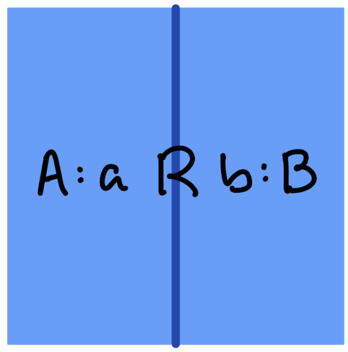


[M.4U] Unitor Equation

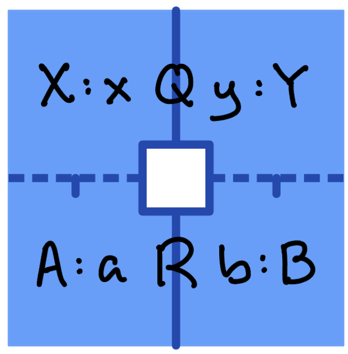
In MatCat, a pseudomonad is a fibrant double category (ref), which we call a **logic**. The object is the category of types and terms, and the monad is the fibered category of judgements and inferences.

THEOREM: PSMNDS ARE FDCS

Yet so far, the color and string only visually represent types and judgements.



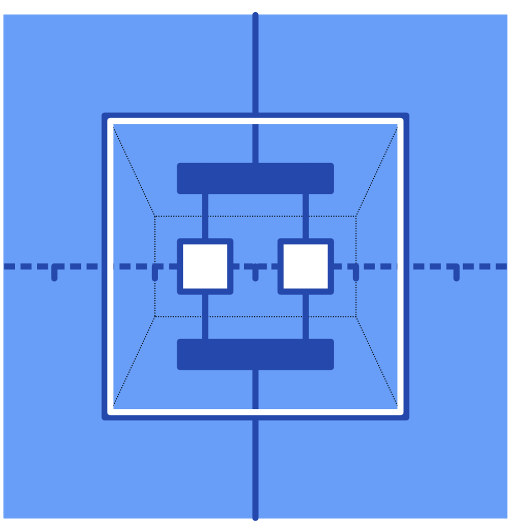
To make terms and inferences explicit, we can think of $\mathbb{A}_0$ as its identity profunctor $\mathbb{A}_0(-,-) : \mathbb{A}_0 \nrightarrow \mathbb{A}_0$, and $\mathbb{A}$ as its identity fibered profunctor $\mathbb{A}(-,-) : \mathbb{A} \nrightarrow \mathbb{A}$.

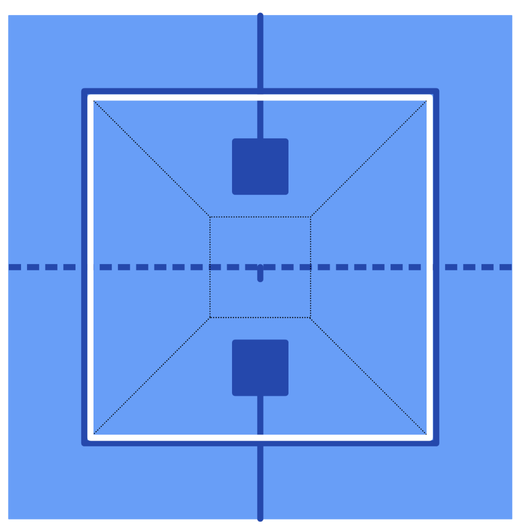


This is the logic as its identity *metaterm*, which we define next. The "true hom" of a logic is this two-dimensional structure, the fibered profunctor of all inferences.

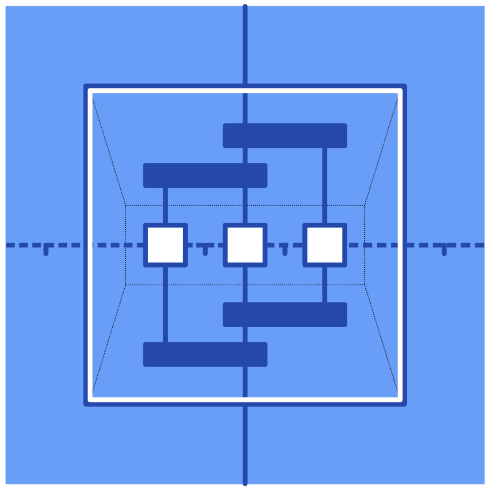
Drawing a logic in this way, we can see the action of join and unit on inferences.

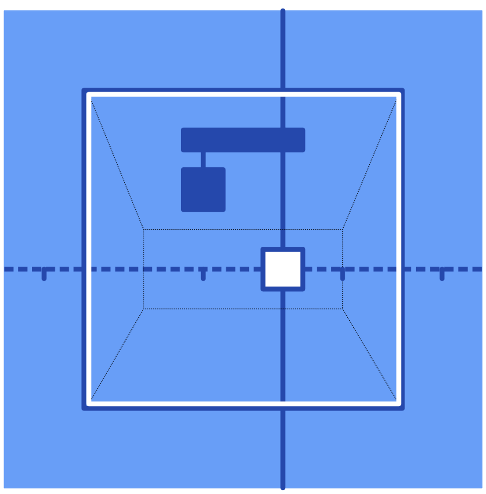
footnote: variables as identities

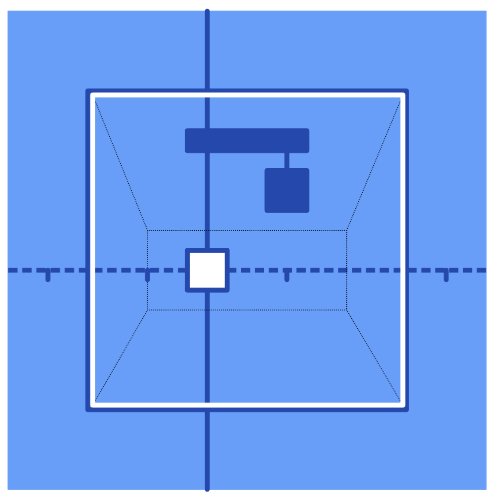




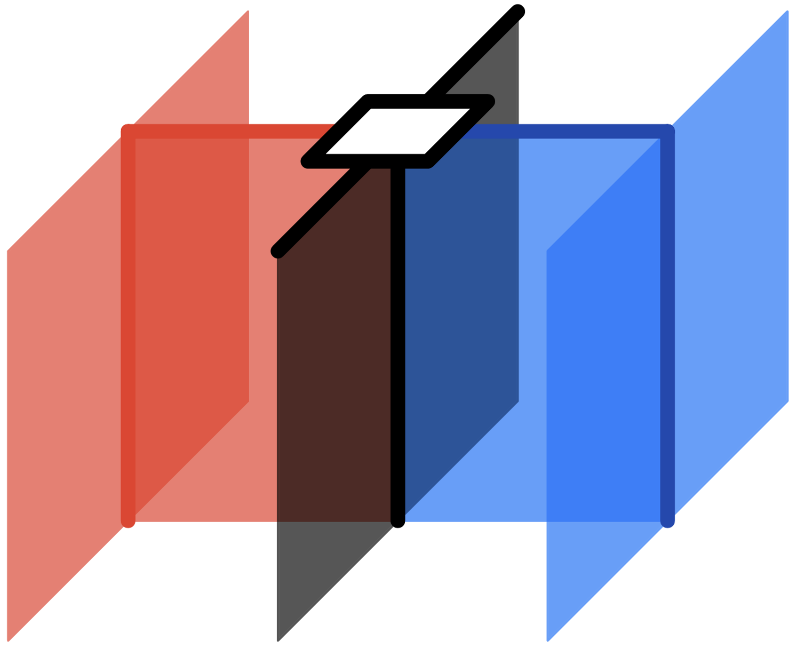
The associator and unitors are each equivalences between a pair of inference-constructors; as cubes, they can be viewed as “mirrors” in the vertical direction.



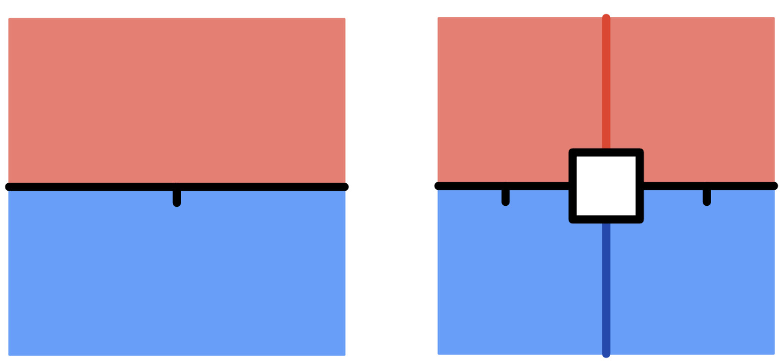




We now define the general concept of metaterm, or vertical relation of logics. The concept is *exactly the same* as that of logic, except "polarized" in the vertical direction. We see this by viewing the concept both in three dimensions and sliced vertically.

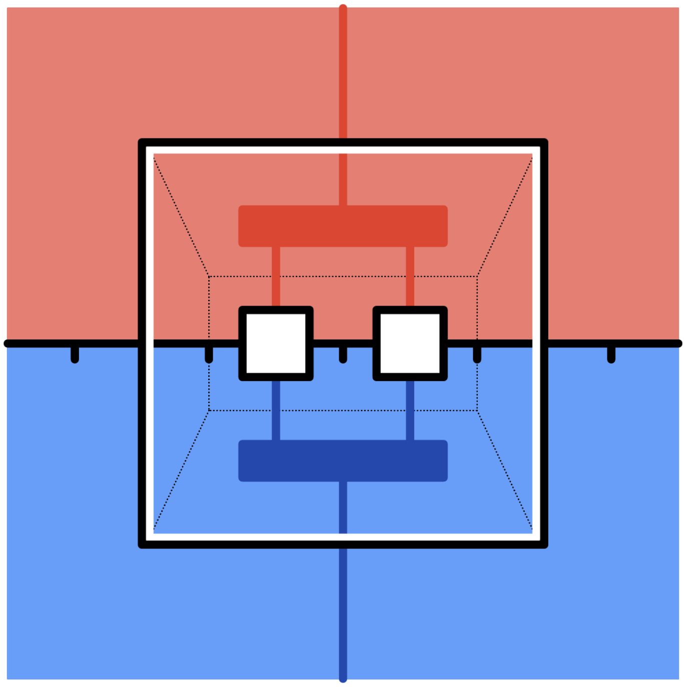


Definition 2. Let $\mathbb{A}, \mathbb{B}$ by pseudomonads in a triple category $\mathcal{E}$. A **vertical relation** $\mathtt{t} : [\mathbb{A} \mid \mathbb{B}]$ is a V-morphism and an HV-morphism

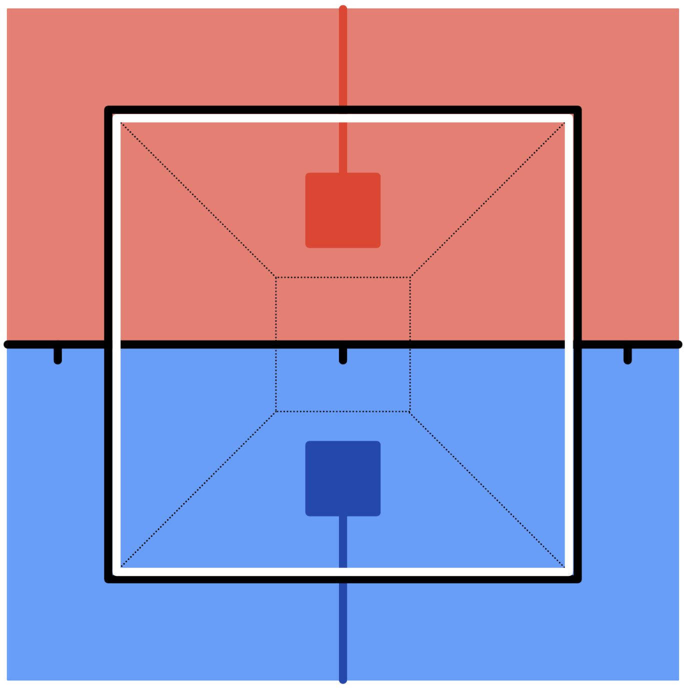


$\underline{\mathtt{t}} : [\mathbb{A}_0 \mid \mathbb{B}_0]$ $\qquad$ $\mathtt{t} : [\mathbb{A} \,\|\, \mathbb{B}](\underline{\mathtt{t}}, \underline{\mathtt{t}})$

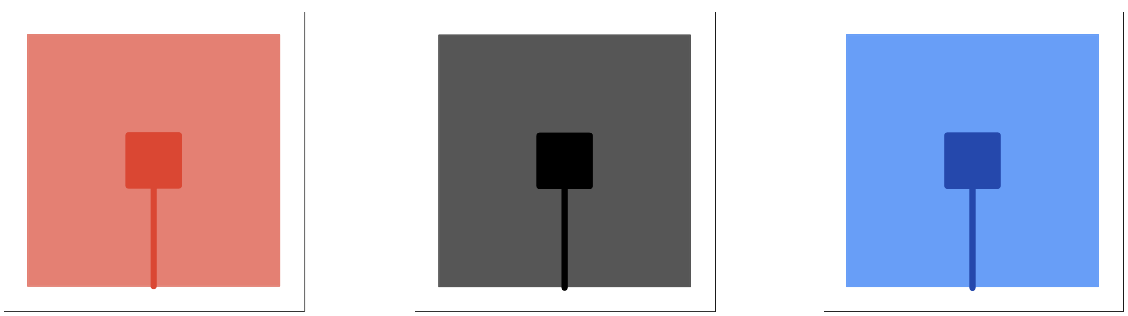
with 3-morphisms for join and unit



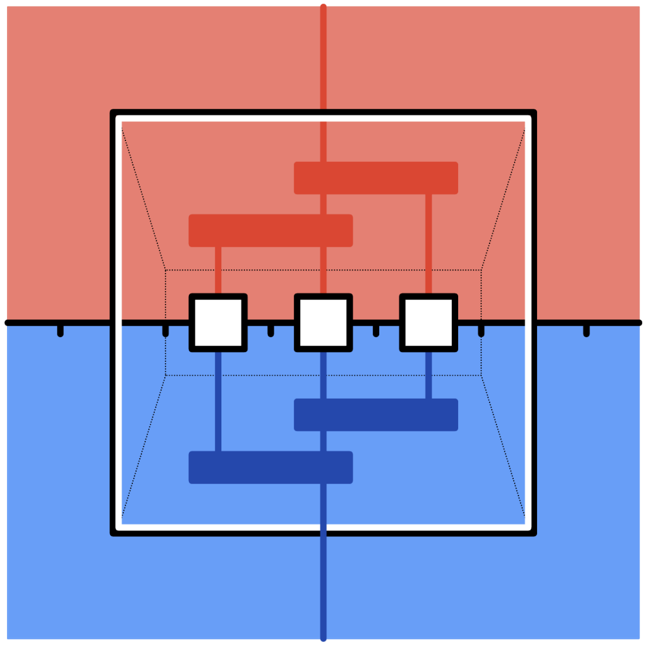
join
$\mathtt{comp}\colon \mathtt{t} * \mathtt{t} \Rightarrow \mathtt{t}$

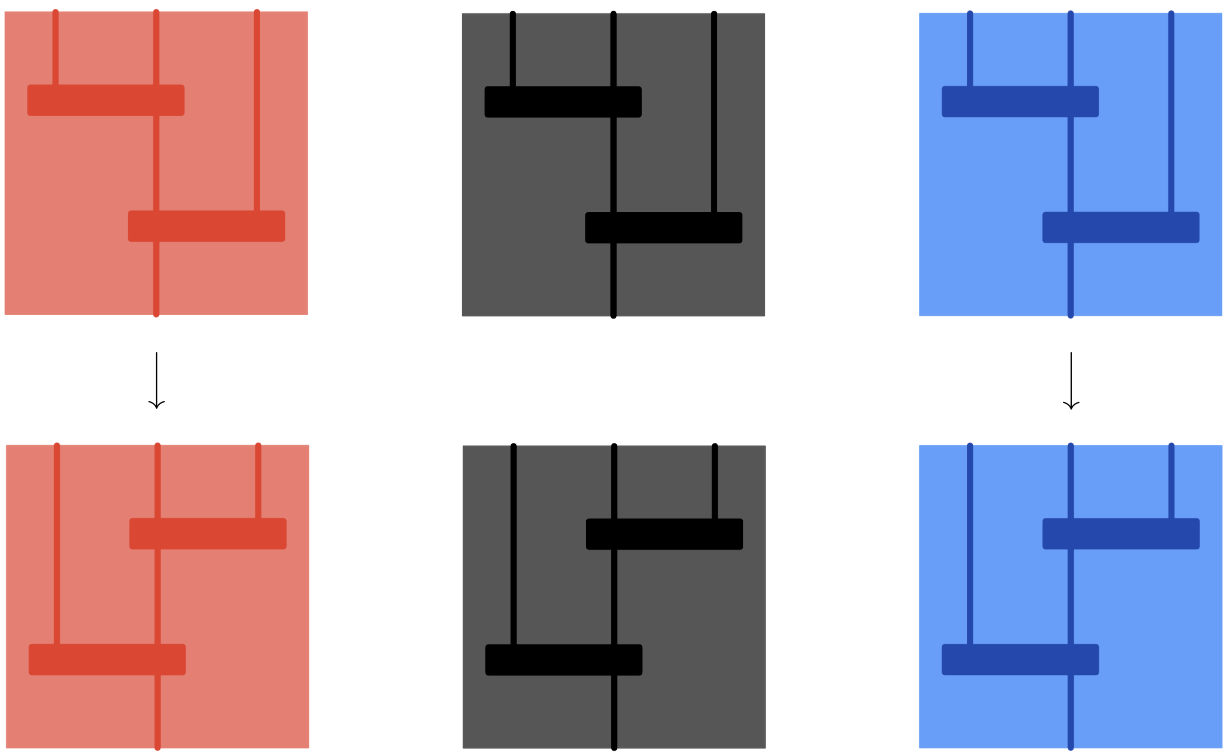


unit
$\mathtt{id} : \underline{\mathtt{t}} \Rightarrow \mathtt{t}$

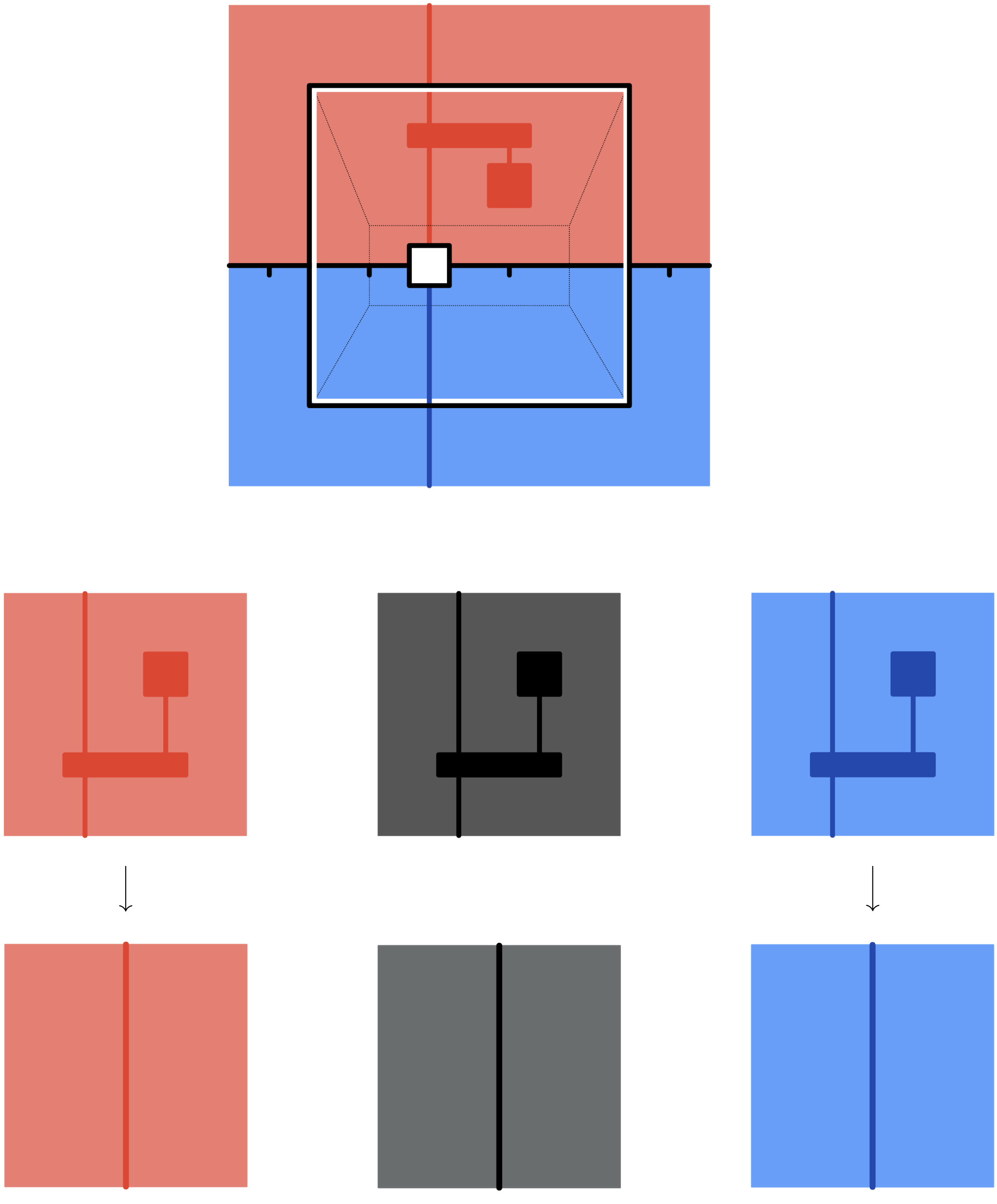


satisfying coherence with the associators and unitors of $\mathbb{A}$ and $\mathbb{B}$.

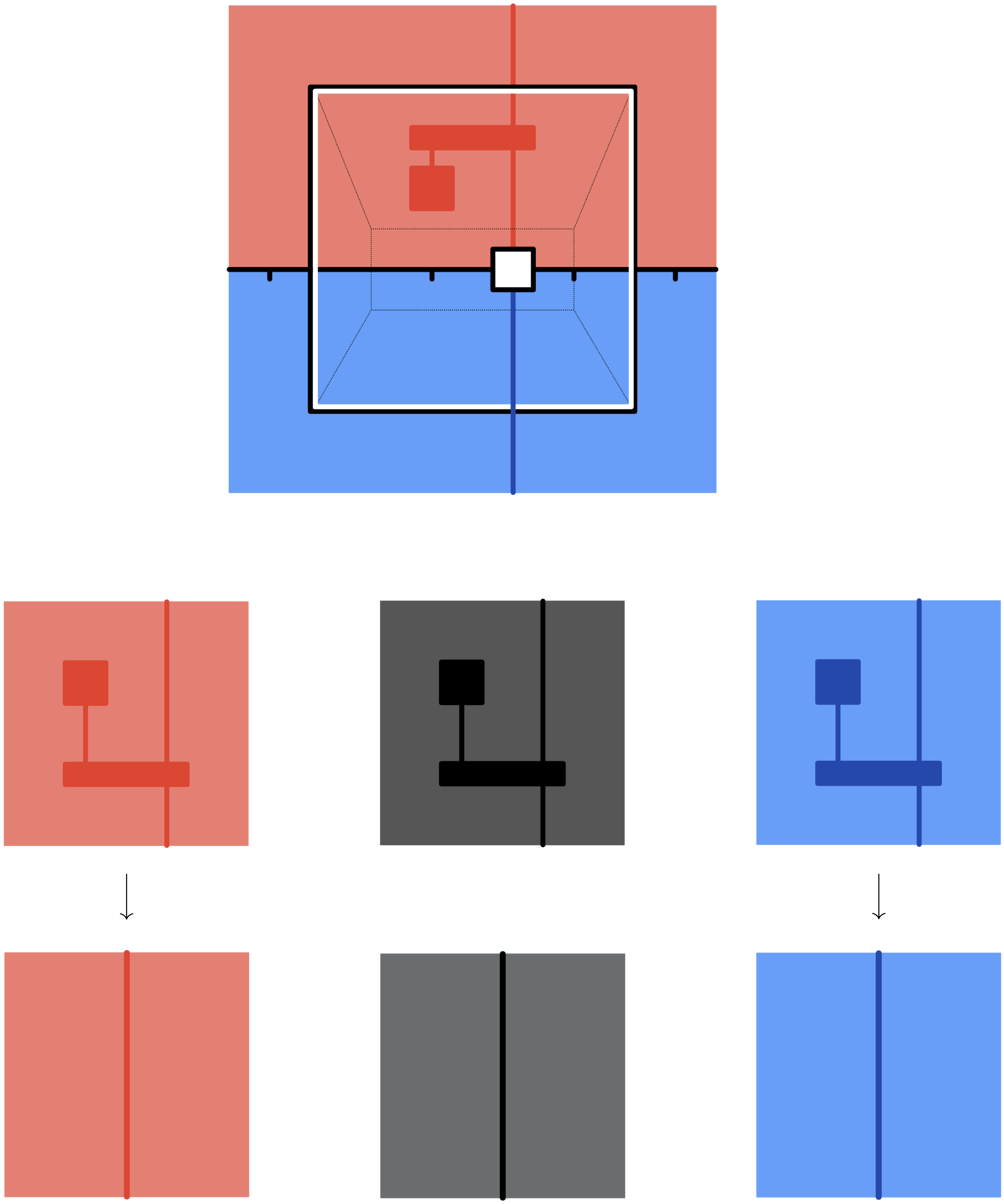




[M.V4A] VRel: Associator



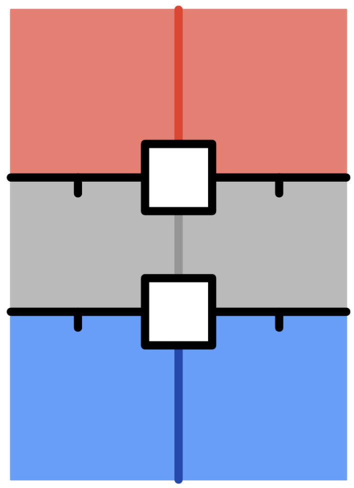
[M.V4R] VRel: Unitor-R



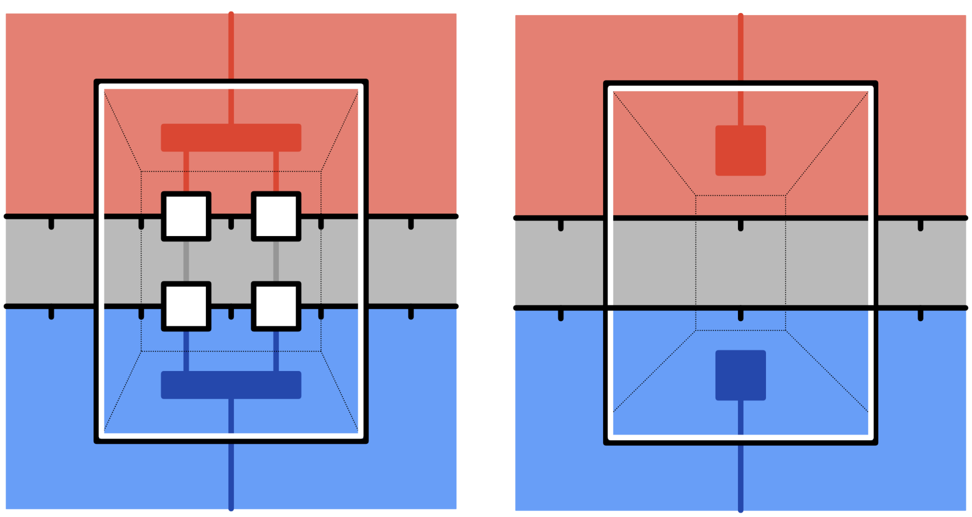
[M.V4L] VRel: Unitor-L

In Logos, a V-relation is a **metaterm**. The V-morphism is a profunctor, which for each pair of types gives a set of terms; and the HV-morphism is a fibered profunctor, which for each pair terms gives a profunctor of inferences. Join is parallel composition of these inferences, and the unit is the rule to introduce terms.

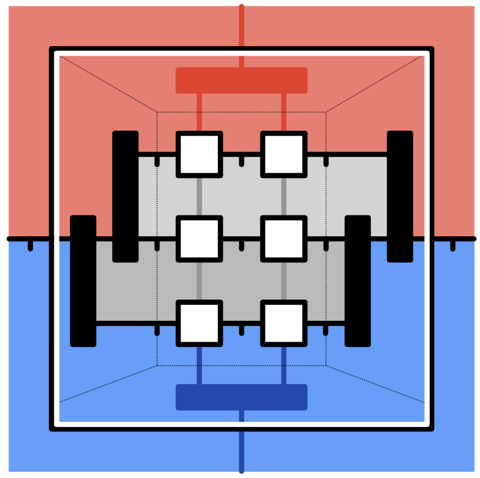
Definition 3. Let $\mathtt{s} : [\mathbb{A} \mid \mathbb{X}]$ and $\mathtt{t} : [\mathbb{X} \mid \mathbb{B}]$ be vertical relations. The **sequential composite** $\mathtt{s} \cdot \mathtt{t} : [\mathbb{A} \mid \mathbb{B}]$ is defined as follows. The V-morphism and HV-morphism are the sequential composites in $\mathcal{E}$.

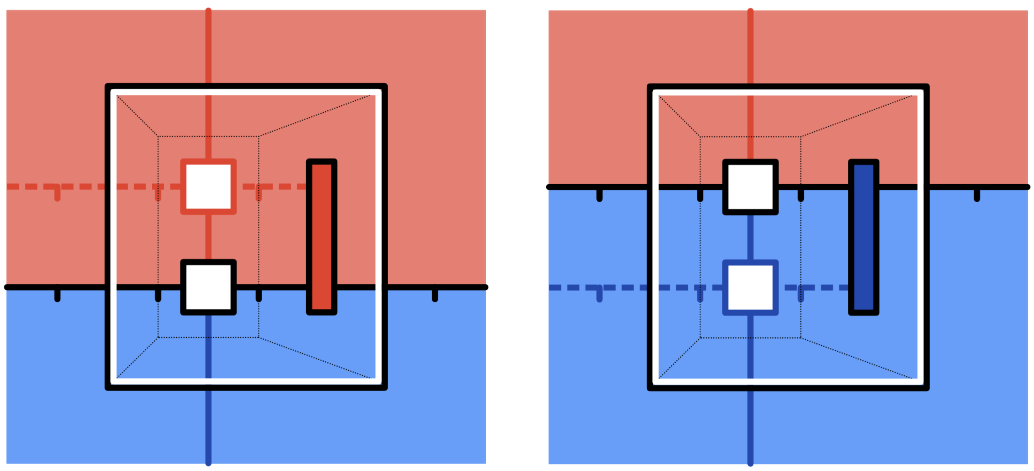


The join and unit are the sequential composites of the joins and units of each.



The associator and unitors of this composition is that of sequential composition in E.

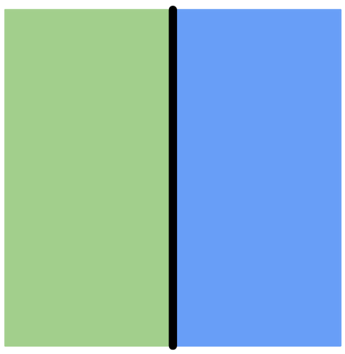




As one can see, the concept of "metaterm" between logics or "vertical relation" between pseudomonads is precisely the concept of *monad on a module*. Our construction creates a triple category with *two dimensions* of polarization: this is the first, and the second is "metajudgement" or "horizontal relation", which we now define.

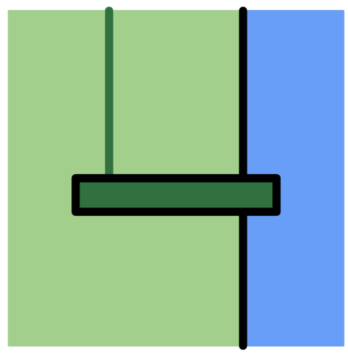
Definition 4. Let $\mathbb{A}, \mathbb{B}$ be pseudomonads in $\mathcal{E}$. A **horizontal relation** $\mathcal{M} : [\mathbb{A} \,||\, \mathbb{B}]$ is an H-morphism with HT-morphisms

h. relation



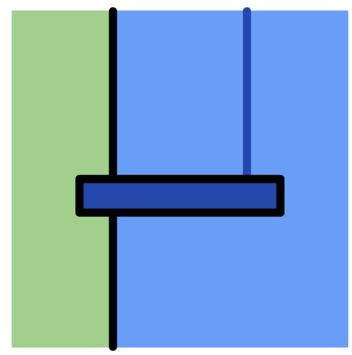
$\mathcal{R} : [\mathbb{A}_0 \,||\, \mathbb{B}_0]$

source action



$\mathtt{s} : \mathbb{A} \circ \mathcal{R} \Rightarrow \mathcal{R}$

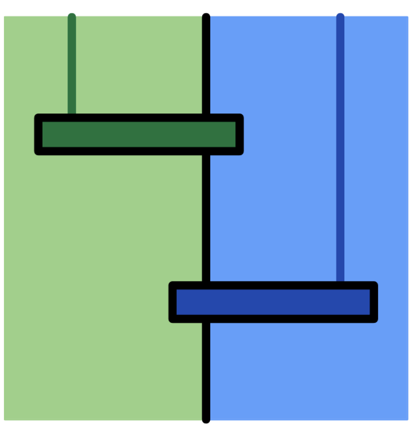
target action



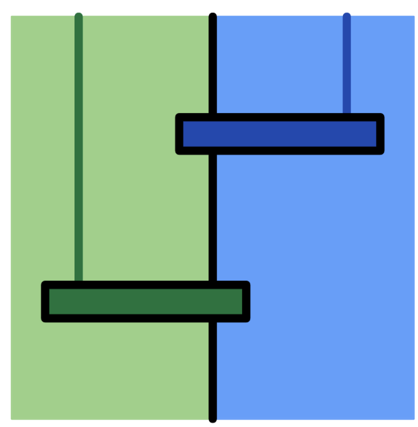
$\mathtt{t} : \mathcal{R} \circ \mathbb{B} \Rightarrow \mathcal{R}$

and invertible 3-morphisms

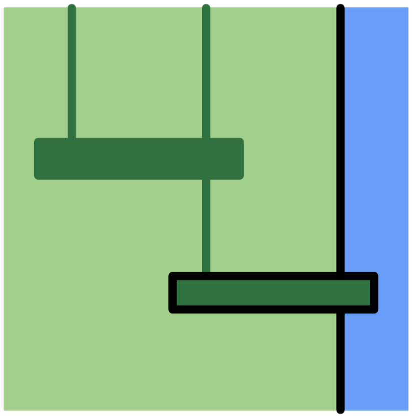
associator

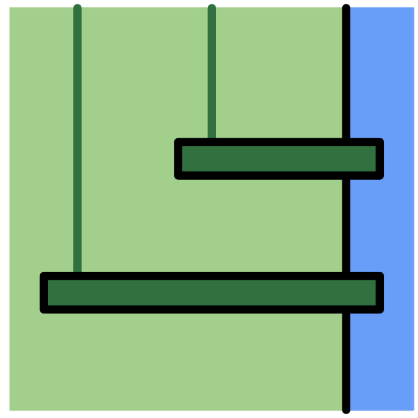


$(\mathtt{s}\mathbb{B})\mathtt{t} \xrightarrow{\mathcal{R}.\mathtt{assoc}} (\mathbb{A}\mathtt{t})\mathtt{s}$



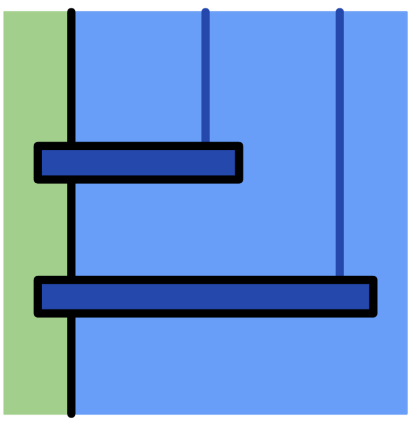
source associator

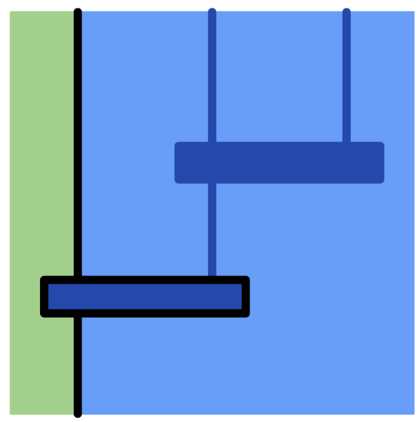




$$(\mu_{\mathbb{A}}\mathcal{R})\mathtt{s} \xrightarrow{\mathtt{s.assoc}} (\mathbb{A}\mathtt{s})\mathtt{s}$$

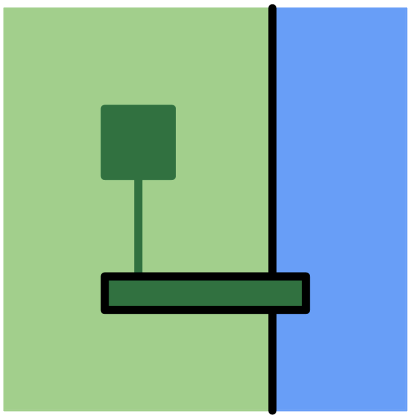
target associator

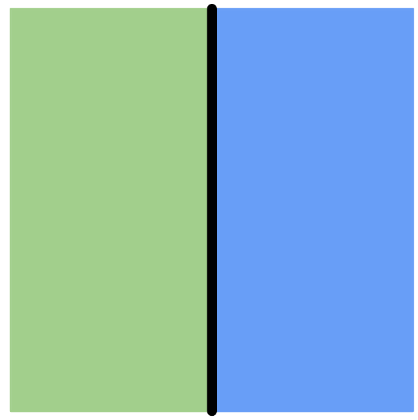




$$(\mathtt{t}\mathbb{B})\mathtt{t} \xrightarrow{\mathtt{t.assoc}} (\mathcal{R}\mu_{\mathbb{B}})\mathtt{t}$$

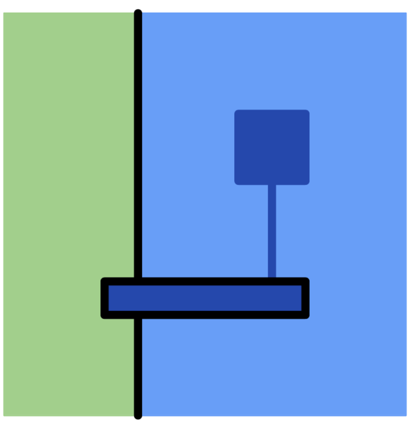
source unitor

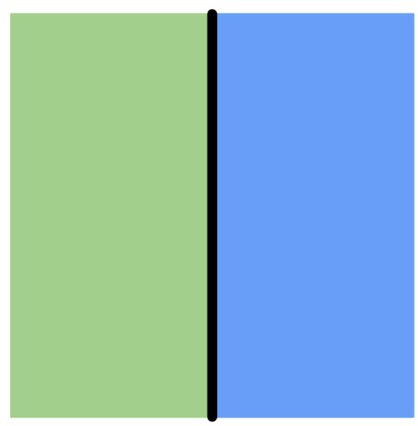




$$(\mu_{\mathbb{A}}\mathcal{R})\mathtt{s} \xrightarrow{\mathtt{s.unit}} (\mathbb{A}\mathtt{s})\mathtt{s}$$

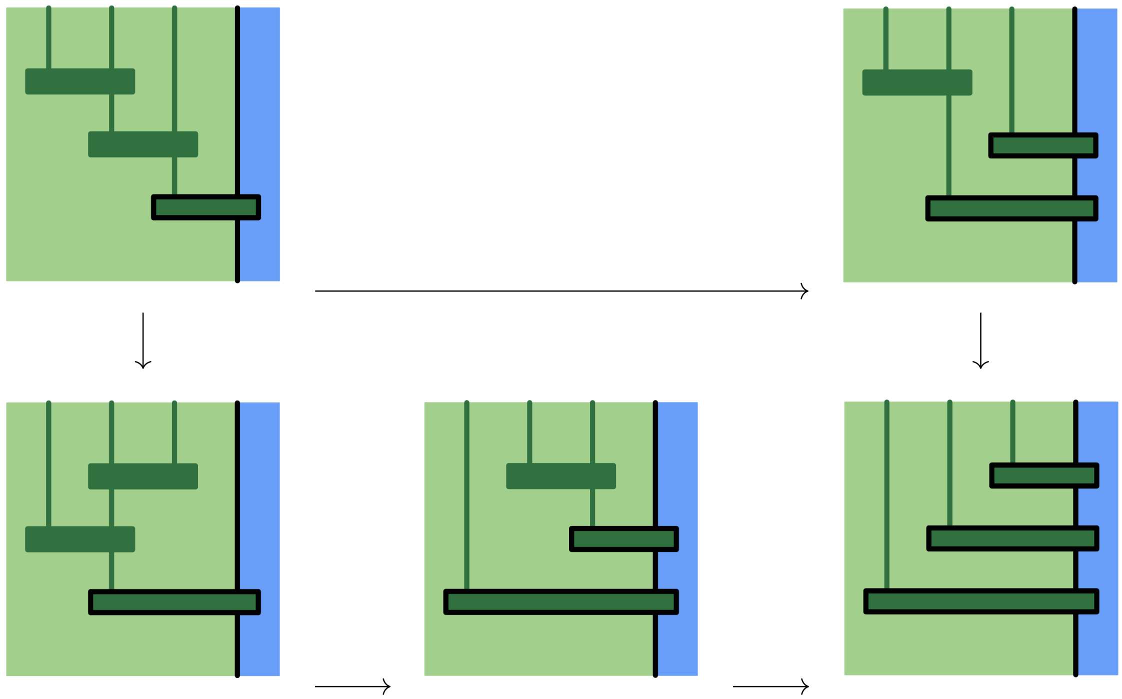
target unitor





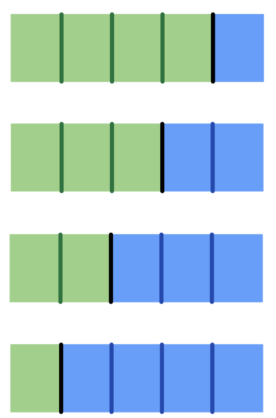
$$(\mu_{\mathbb{A}}\mathcal{R})\mathtt{s} \xrightarrow{\mathtt{s.unit}} (\mathbb{A}\mathtt{s})\mathtt{s}$$

satisfying the following coherence: a “pentagon equation” (ref) for each of the four possible 4-composites of $\mathbb{A}, \mathcal{R}, \mathbb{B}$



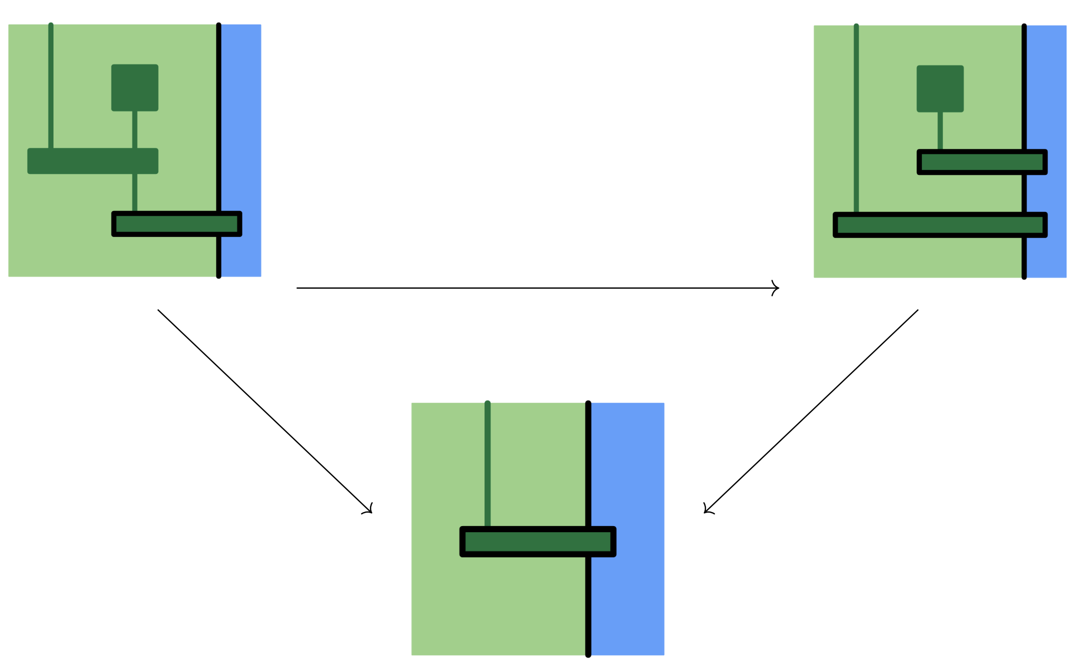
[M.H4S] Source Associator Equation

for each of

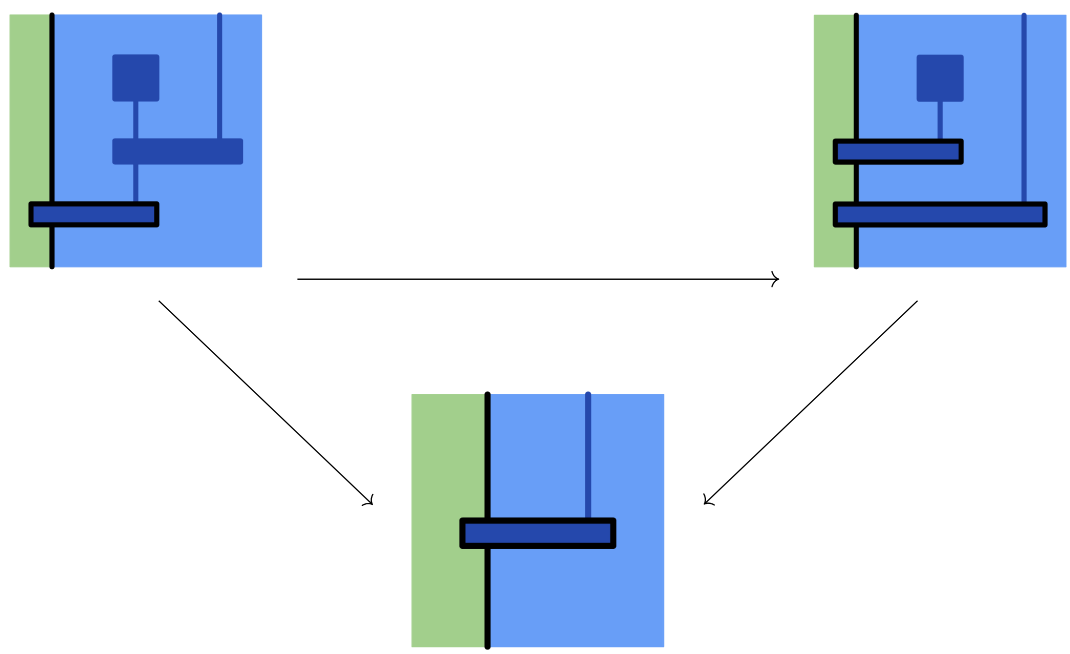


as well as “triangle equations” (ref) for each unitor:

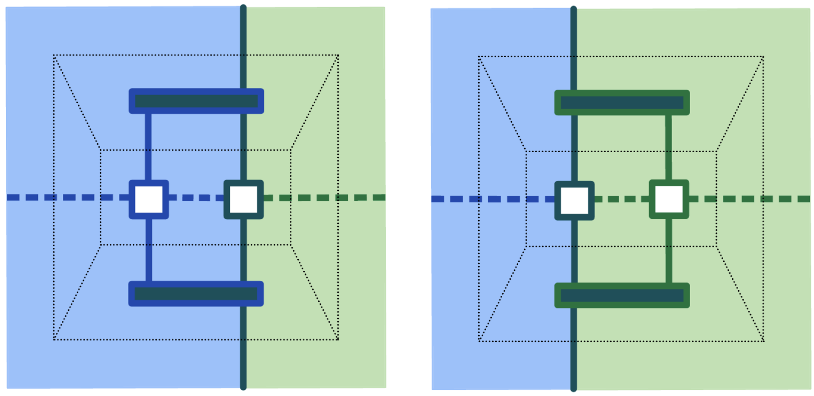
In $\mathrm{Logos} = \mathrm{Mnd}(\mathrm{MatCat})$, an H-module between logics is a **metajudgement**. In the same way that a profunctor encodes heteromorphisms between categories, a metajudgement encodes hetero-judgements between logics. Each action is parallel composition of inferences by each logic.



[M.H4S] Source Unitor Equation



[M.H4S] Target Unitor Equation



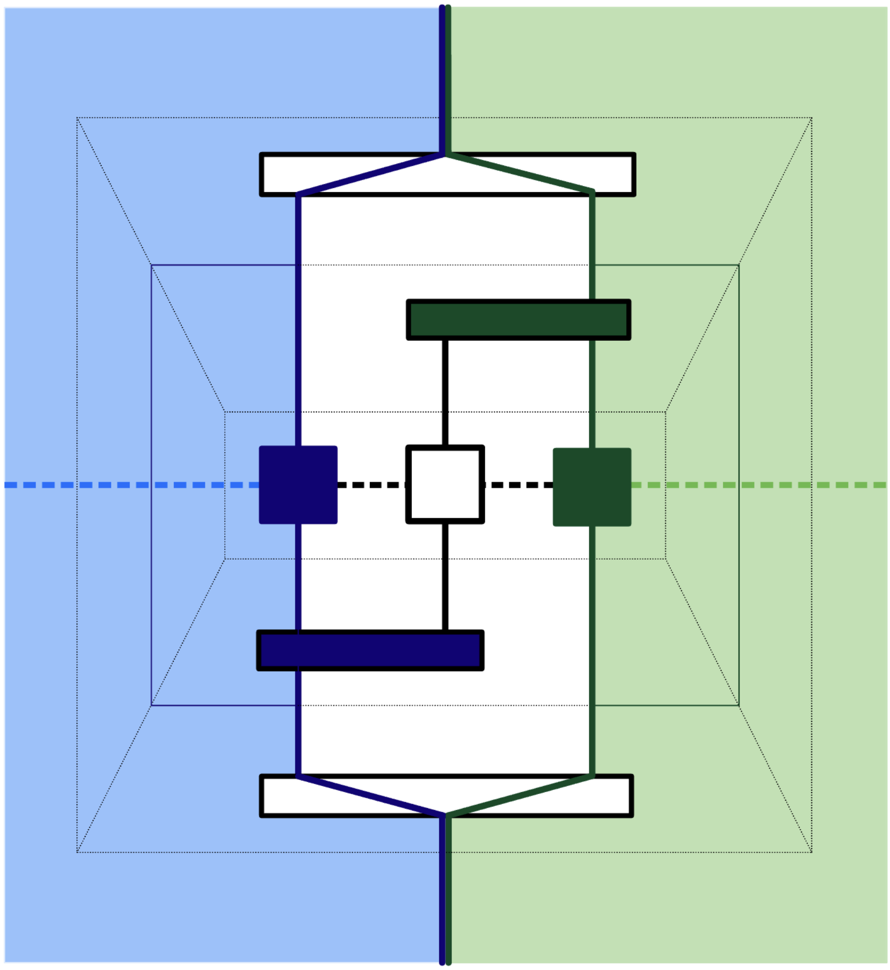
Composition of H-modules is the most complex aspect of the monad construction.

Parallel composition (H)

A metajudgement is a matrix of categories, acted on by logics on either side. Hence its composition is the most complex of the three dimensions; it requires forming a two-dimensional colimit in the hom-bicategory $[\mathbb{A} \mid \mathbb{B}]$ of metajudgements, judgement systems, and inference systems.

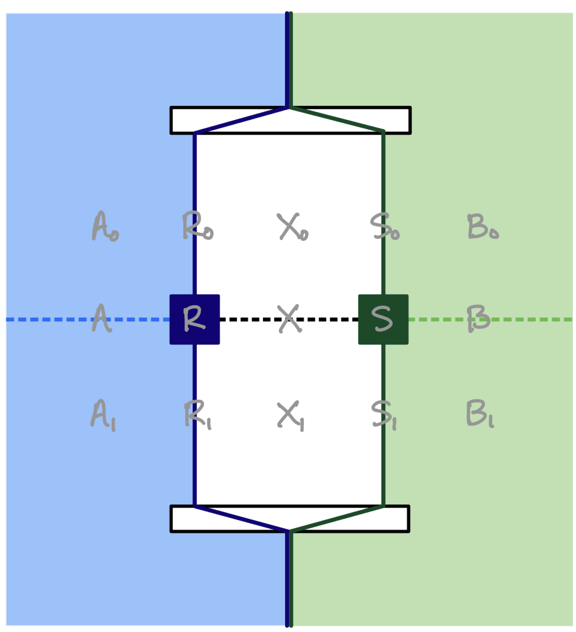
Definition 5. Let $\mathbb{A}, \mathbb{B}, \mathbb{X}$ be logics, and let $\mathcal{R} : [\mathbb{A} \mid \mathbb{X}]$ and $\mathcal{S} : [\mathbb{X} \mid \mathbb{B}]$ be metajudgements. The **composite metajudgement** $\mathcal{R} \circ \mathcal{S}$ is defined as follows.

First, form the isocoinserter of $\mathcal{R}(\mathbb{X}.\mathcal{S})$ and $(\mathcal{R}.\mathbb{X})\mathcal{S}$, the inner actions of $\mathbb{X}$.

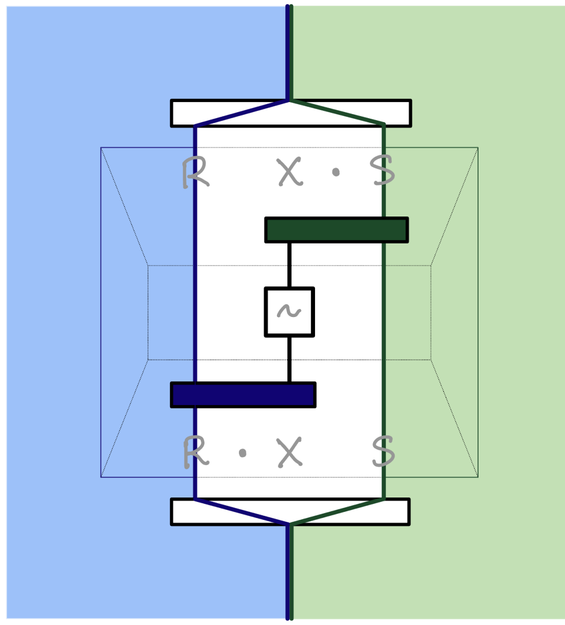


$$\mathcal{R} * \mathcal{S} \equiv \text{isocoinserter}[\ \mathcal{R}(\mathbb{X}.\mathcal{S}), (\mathcal{R}.\mathbb{X})\mathcal{S}\ ]$$

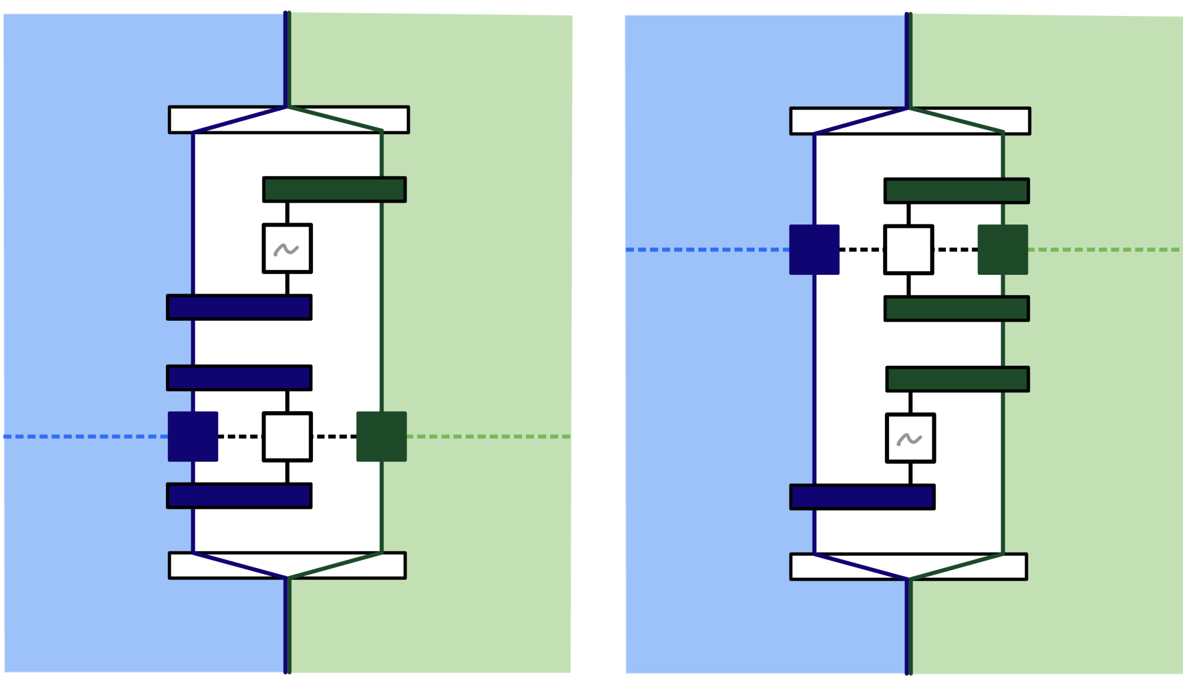
Hence $A(\mathcal{R} * \mathcal{S})B$ consists of objects and morphisms



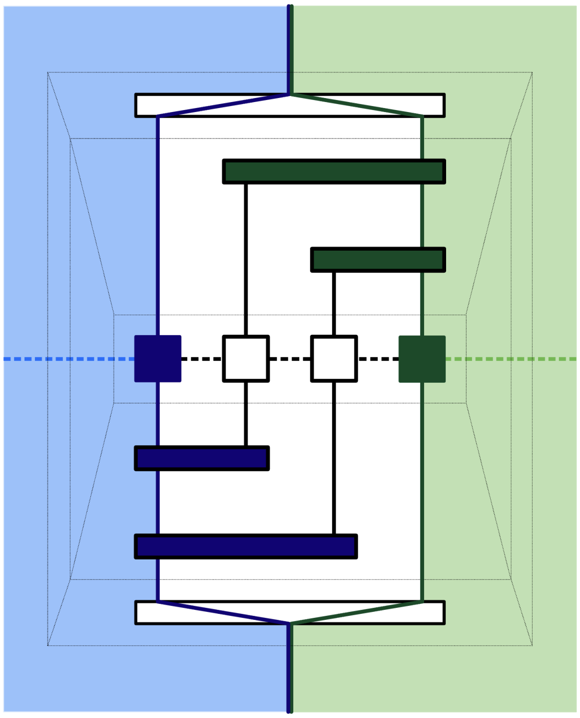
plus the family of isomorphisms



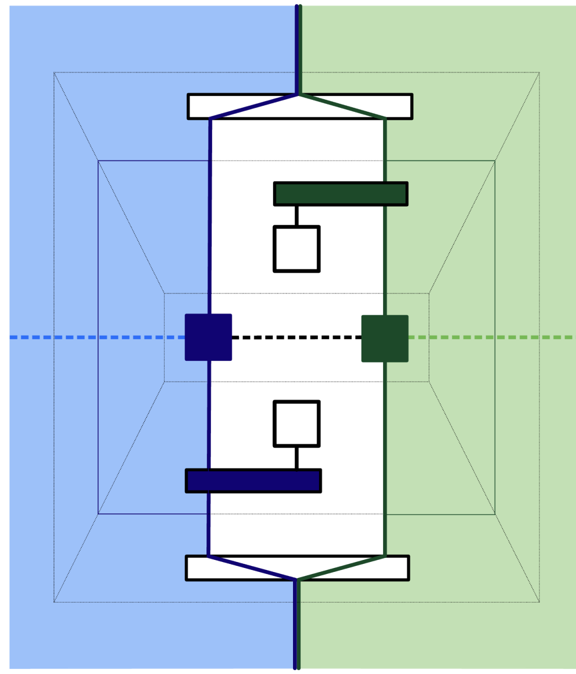
which are natural with respect to other morphisms, i.e. the following are equal.



Second, form the following coequifiers to cohere the family with the associators and unitors of the modules.



coequ[μ]



coequ[η]

The result is the **composite metajudgement** $\mathcal{R}\circ\mathcal{S}$. It is defined by the **codescent formula**:

$$\mathcal{R}\circ\mathcal{S}(a,b) \equiv \mathsf{coequ}[\mu,\eta]\big(\mathsf{isocoins}[\mathbb{X}]\big(\Sigma x : \mathbb{X}.\ \mathcal{R}(a,x)\times\mathcal{S}(x,b)\big)\big)$$

which we abbreviate as follows.

$$\mathcal{R}\circ\mathcal{S} = \overset{\circ}{\Sigma} x : \mathbb{X}.\ \mathcal{R}(a,x)\times\mathcal{S}(x,b)$$

This defines horizontal composition in Logos. We now characterize its coherence.

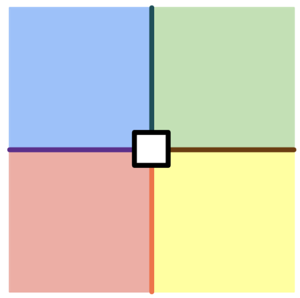
Definition 6. The **associator** of metajudgement composition is defined as follows.

Definition 7. unitor (coYoneda!!)

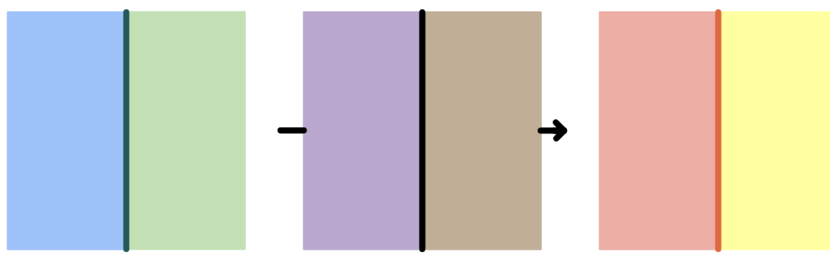
As existential $\exists$ composes in binary logic (ref), indexed sum Σ composes in matrix logic (ref), and coend composes in active logic (ref): this is the connecting "sum" of metalogic. It is more complex because a metajudgement is *two-dimensional* and often large, each carrying a whole way of thinking. Yet for all its richness, metalogic is just as well-behaved as other logics: there is still a perfect duality of "some" and "all", between composition and inference.

Horizontal and vertical modules "fit together" to form two-dimensional modules.

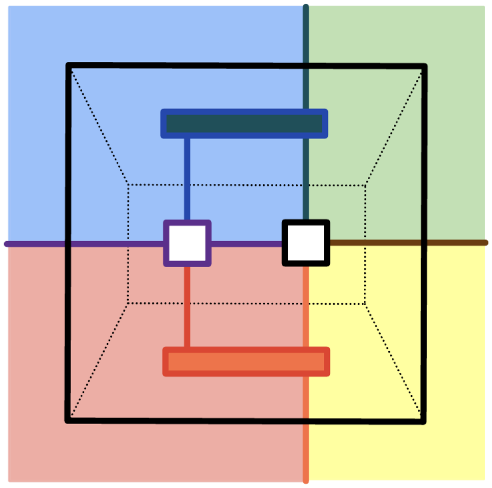
Definition 8. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be pseudomonads, $M : \mathbb{X} \nrightarrow \mathbb{Y}$, $N : \mathbb{A} \nrightarrow \mathbb{B}$ be H-modules, and $S : \mathbb{X} \to \mathbb{A}$, $T : \mathbb{Y} \to \mathbb{B}$ be V-modules. A **double module** $I : M \Rightarrow N(S,T)$ is an HV-morphism



$$I : M \xrightarrow[T_0]{S_0} N$$

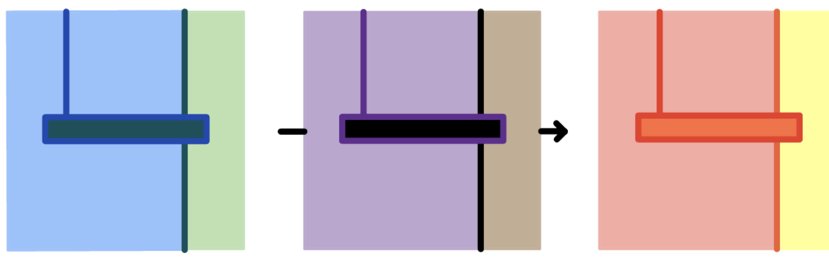


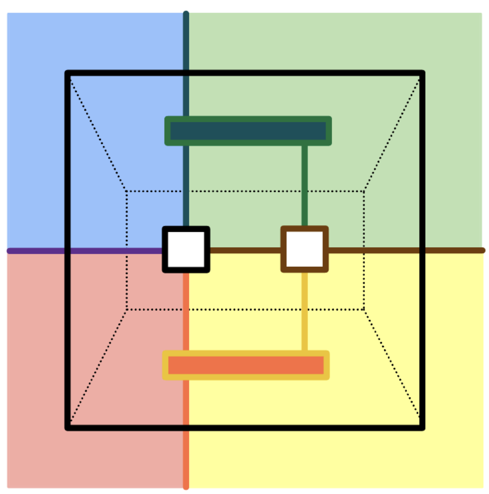
with left and right actions



left action

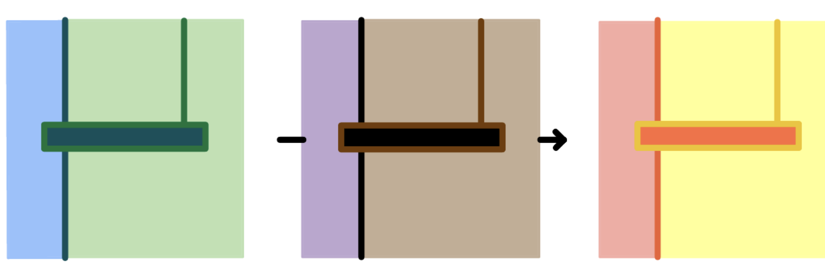
$\lambda : S * I \Rightarrow I$





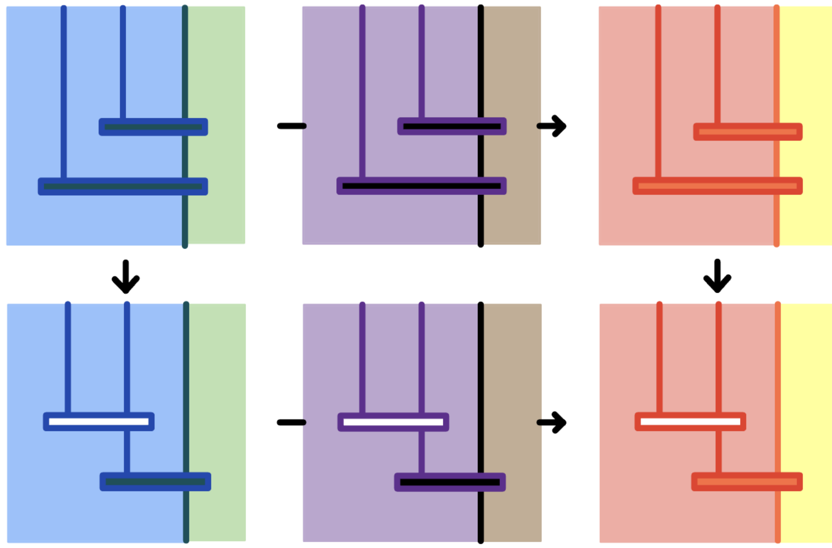
right action

$\rho : I * T \Rightarrow I$

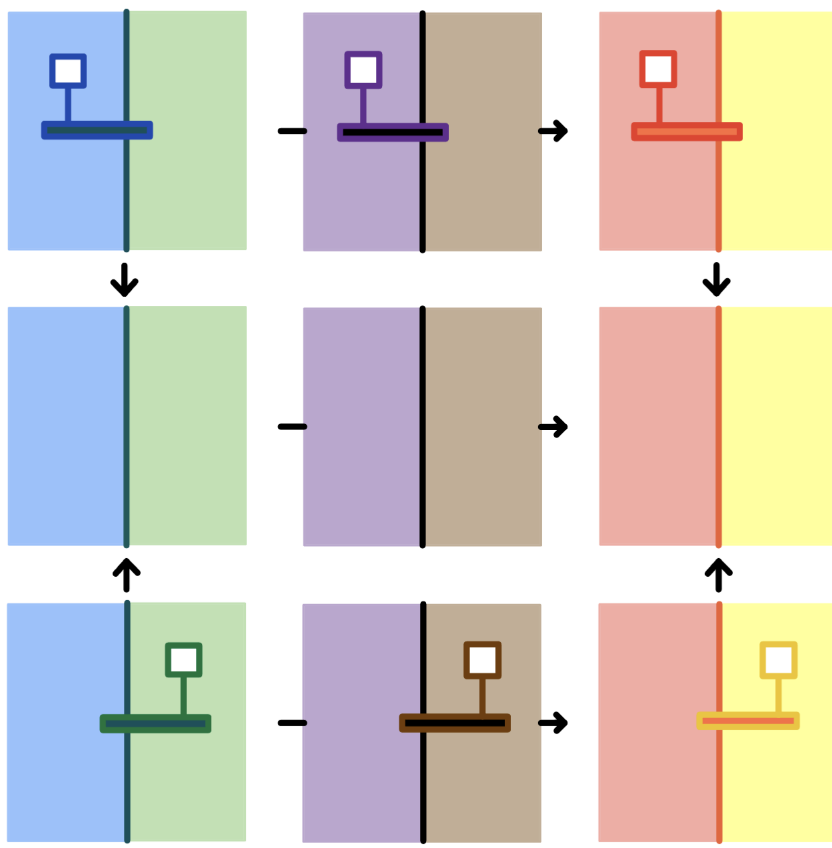


satisfying coherence with the associators and unitors of the H-modules.

left pentagon



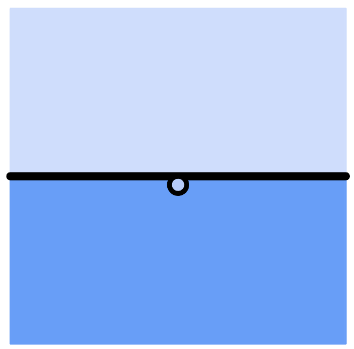
left and right units

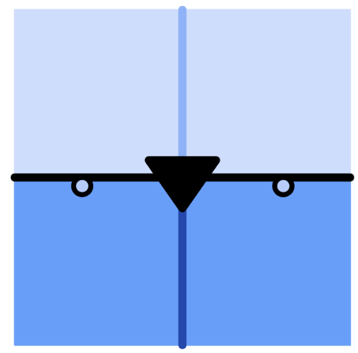


In Logos, a double module is a **metainference**. The fibered profunctor gives for each pair of hetero-terms in S, T a profunctor, which for each pair of judgements in M, N gives a set of “hetero-inferences” lying over those hetero-terms.

Definition 9. Let $\mathbb{A}, \mathbb{B}$ be pseudomonads in $\mathcal{E}$. A **pseudofunctor** $f : \mathbb{A} \to \mathbb{B}$ is a T-morphism and an HT-morphism

functor **morphism**

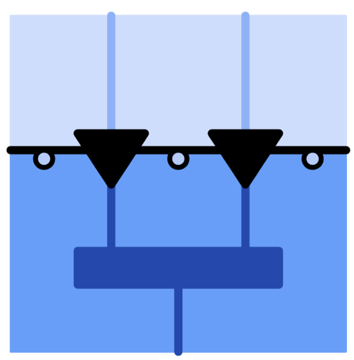


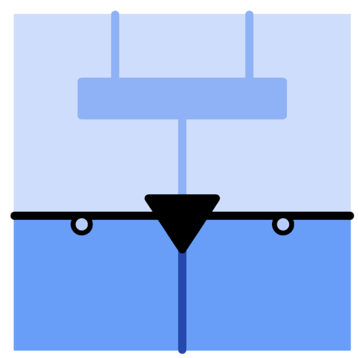


$f_0 : \mathbb{A}_0 \to \mathbb{B}_0$ $f : \mathbb{A} \to \mathbb{B}(f_0, f_0)$

with invertible 3-morphisms for join and unit

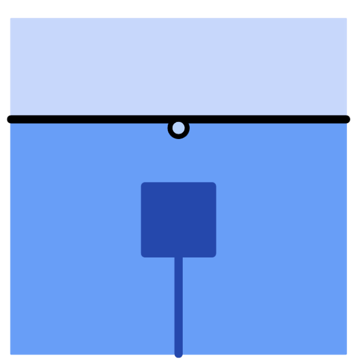
joinor

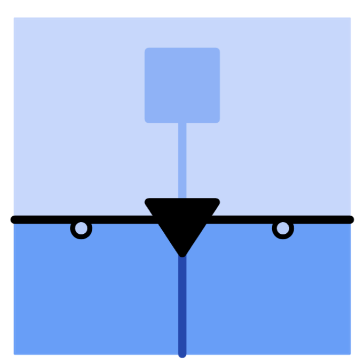




$$(ff)\mu_{\mathbb{B}} \xrightarrow[\sim]{\mu} \mu_{\mathbb{A}} f$$

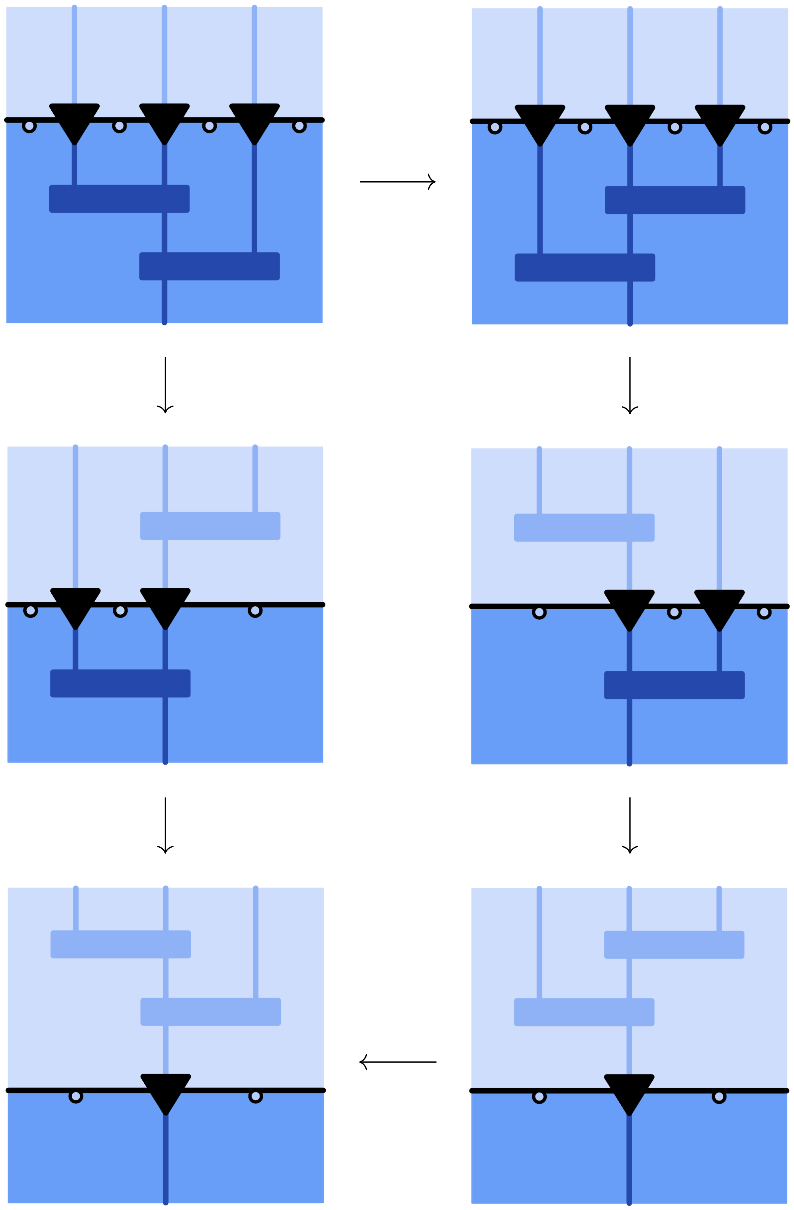
unor



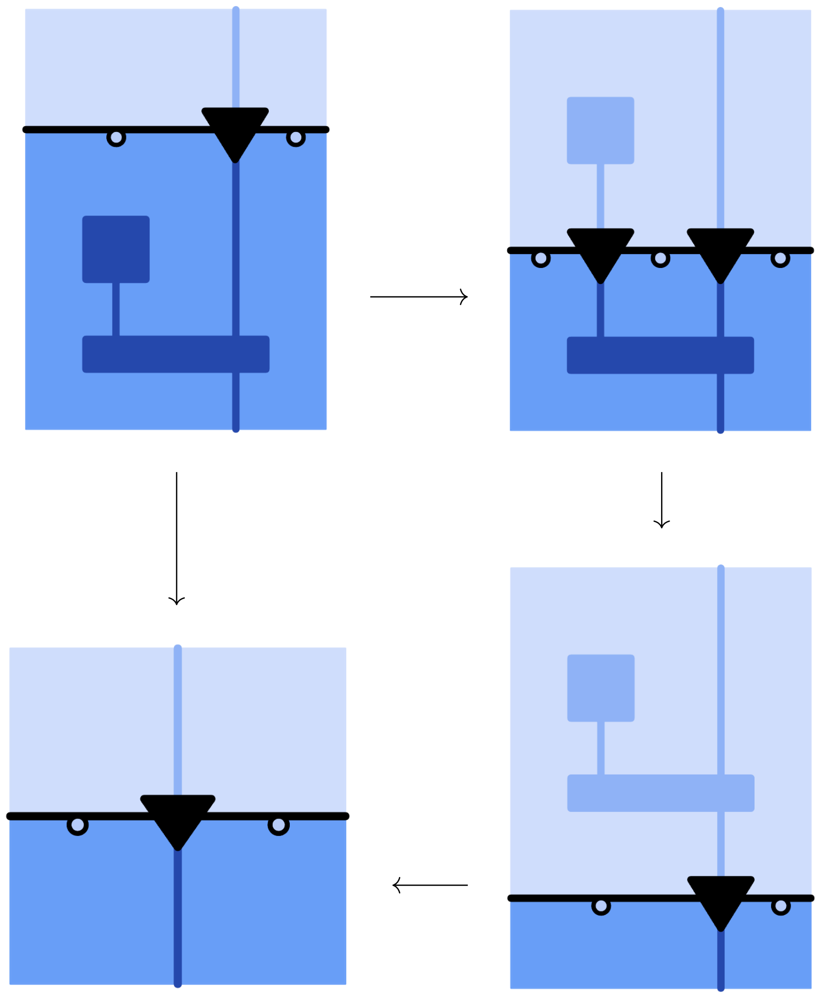


$$f_0\eta_{\mathbb{B}} \xrightarrow[\sim]{\eta} \eta_{\mathbb{A}} f$$

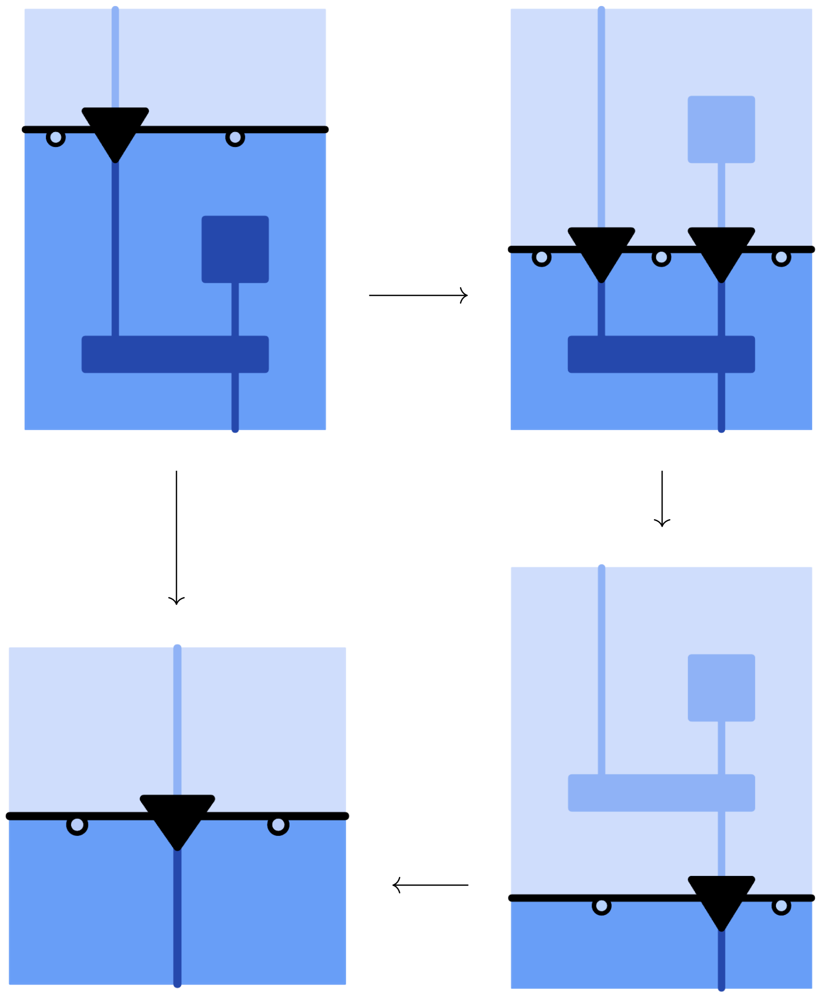
which satisfy the following coherence.



[M.F4J] Joinor Equation

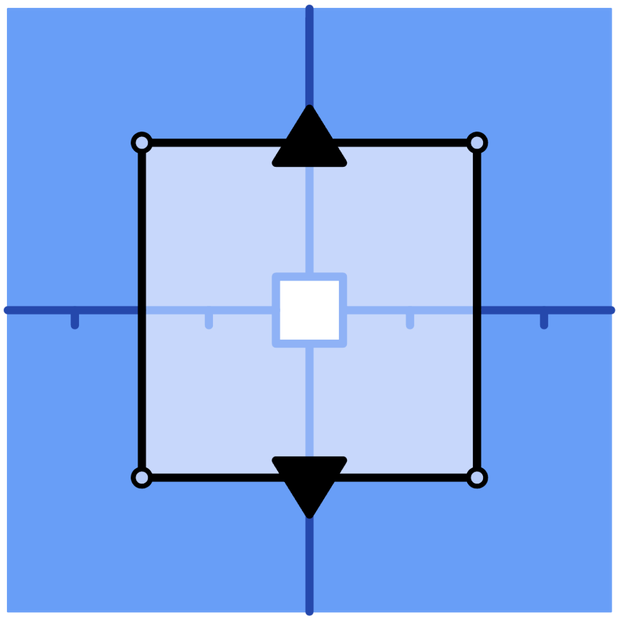


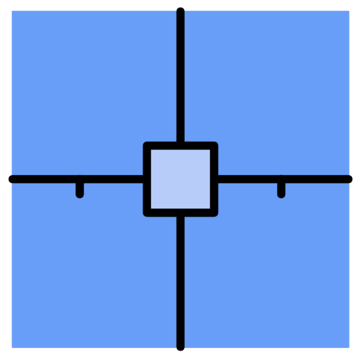
[M.F4L] Left Unor Equation



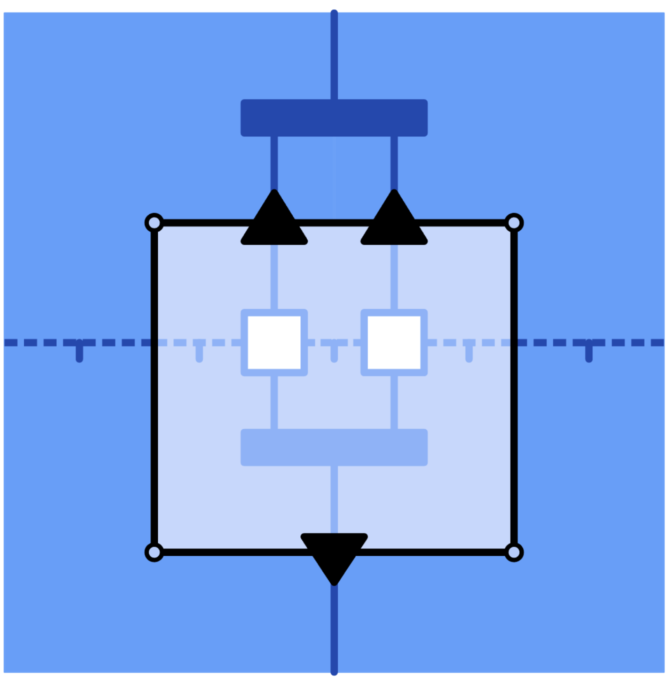
[M.F4R] Right Unor Equation

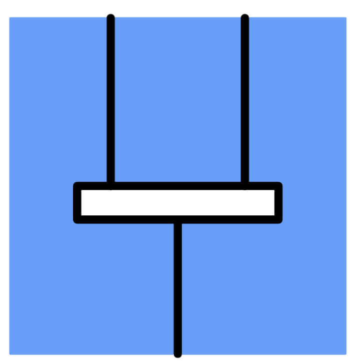
In three dimensions, a (pseudo)functor appears as the following cube. Adjacent, a "simplified view" of the cube represents the *image* of the functor in the target $\mathbb{B}$.

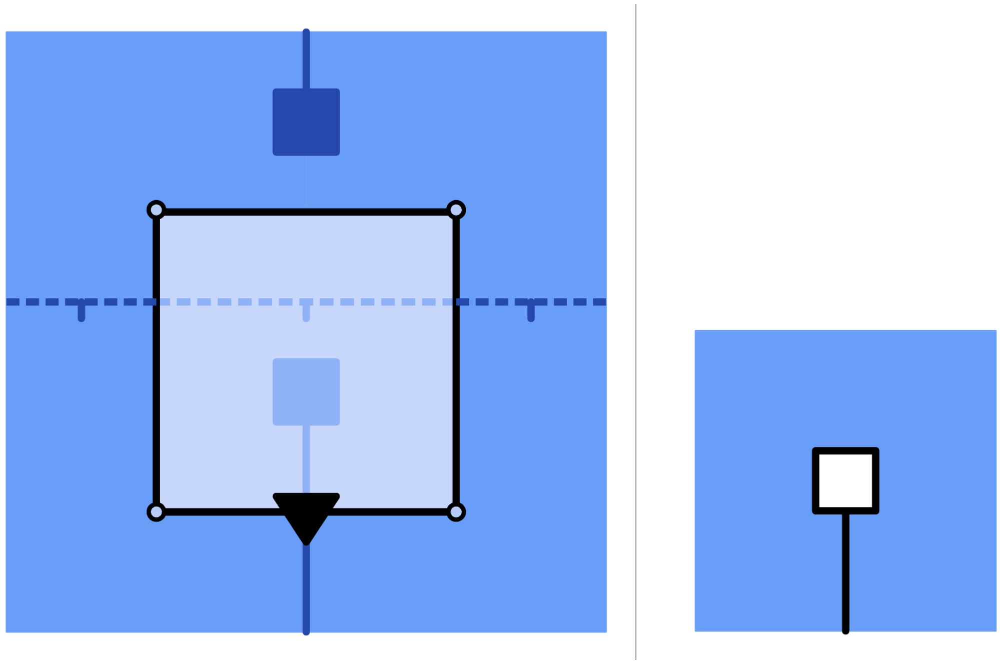




The 3-morphisms generate a coherent family of 2-morphisms in $\mathbb{B}$.



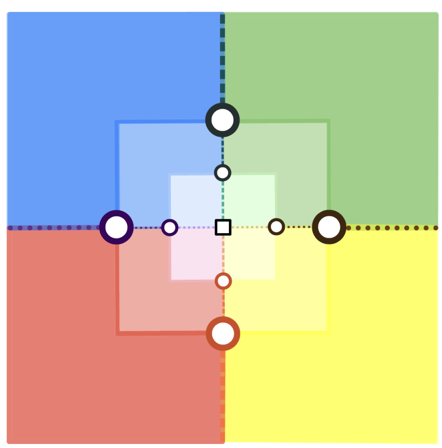




Note These look just like the join and unit of a monad! If we do not define μ and η to be invertible, then the above concept is known as a **lax functor** of pseudomonads (ref). It can be understood as a "polymorphic monad": the image of the functor consists of *many* types and a judgement between each pair, rather than a single type and judgement. This vastly expands the concept of monad, which as we saw already subsumes all notions of "category" in mathematics (ref).

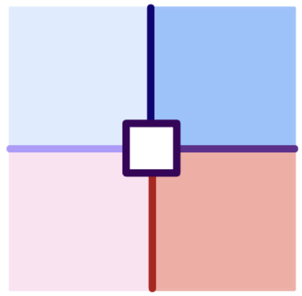
To better understand the structure, first note that

Substitution (T)



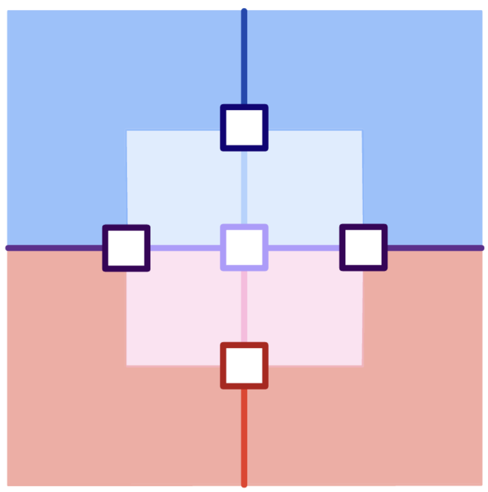
Definition 10. Let $f : \mathbb{X}_1 \to \mathbb{X}_2$, $g : \mathbb{X}_2 \to \mathbb{X}_3$ be pseudofunctors. The **substitution** $f; g : \mathbb{X}_1 \to \mathbb{X}_3$ is defined as follows. The T-morphism is $(f; g)_0 = f_0; g_0$, and the HT-morphism is the composite $f; g$. The coherence cells are composed as follows.

Definition 11. Let $\mathbb{A}_1, \mathbb{B}_1, \mathbb{A}_2, \mathbb{B}_2$ be pseudomonads, $f : \mathbb{A}_1 \to \mathbb{A}_2$, $g : \mathbb{B}_1 \to \mathbb{B}_2$ be pseudofunctors, and $S : \mathbb{A}_1 \to \mathbb{B}_1$, $T : \mathbb{A}_2 \to \mathbb{B}_2$ be V-modules. A **V-transformation** $\theta : S \Rightarrow T(f, g)$ is a VT-morphism



$\theta_0 : S_0 \Rightarrow T_0(f_0, g_0)$

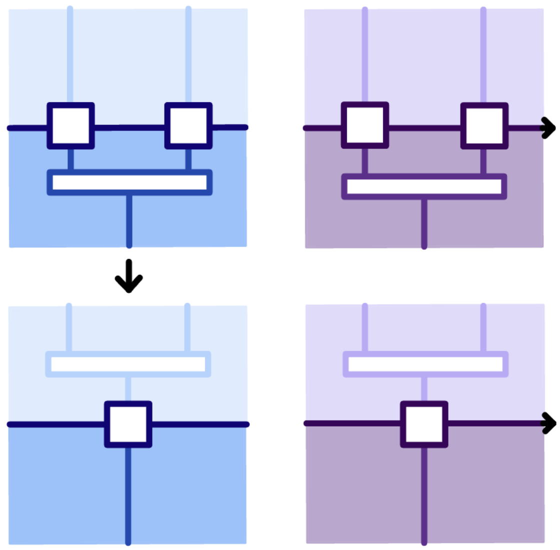
and 3-morphism



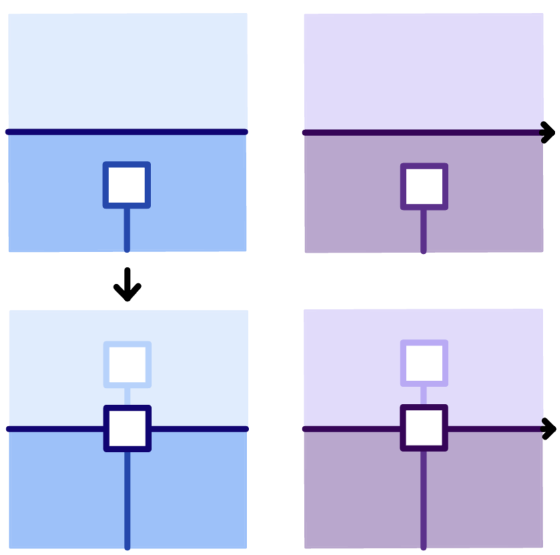
$\theta : S \Rightarrow T(\theta_0, \theta_0)$

with coherence for composition and unit.

composition identity

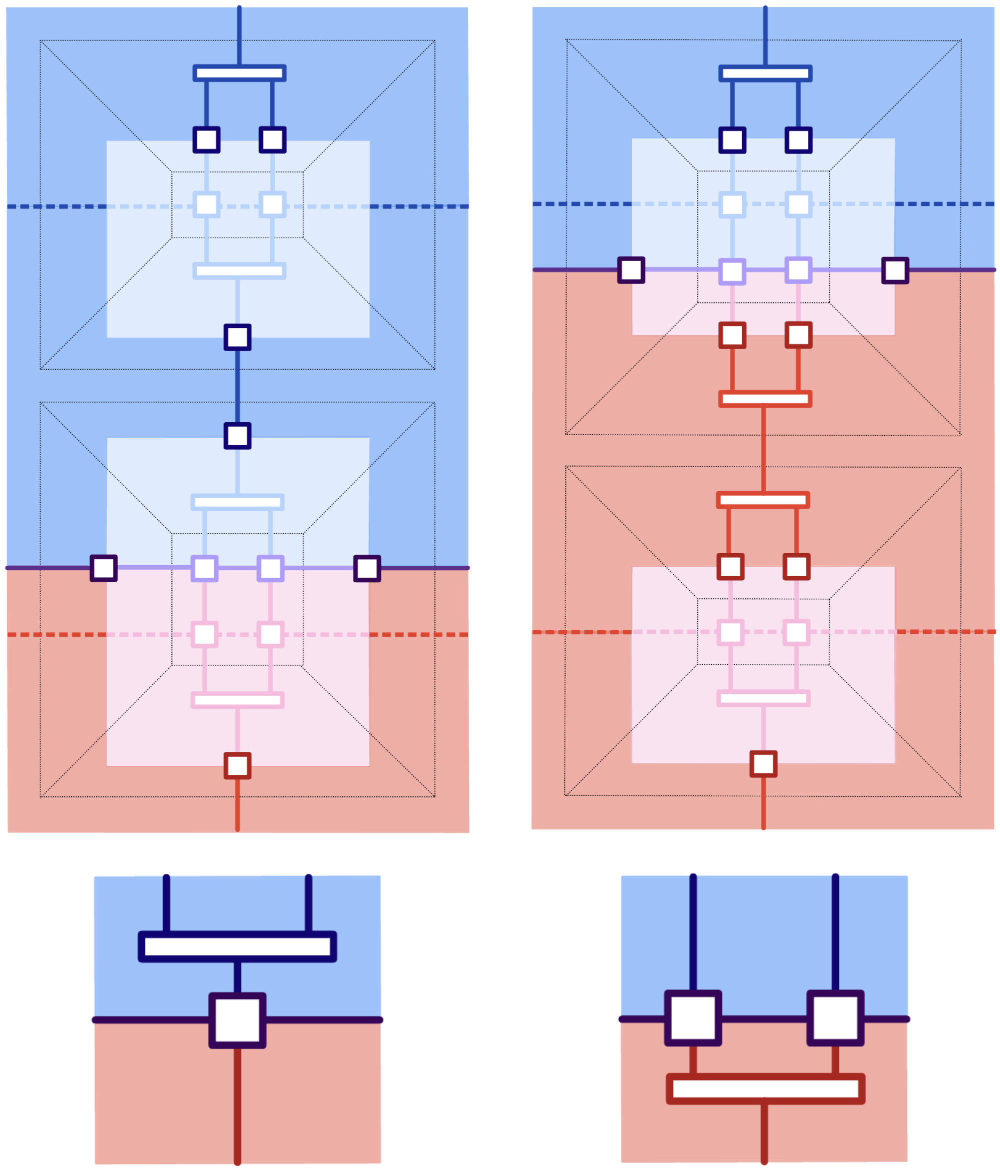


unit identity

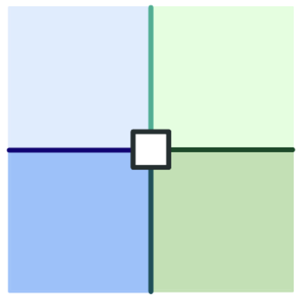


In Logos, a V-transformation is a **term system**: a map from inner metaterm S to outer metaterm T, framed by systems f and g. The VT-morphism is a natural transformation of profunctors $\theta_0 : S_0 \to T_0(f_0, g_0)$, which maps hetero-terms; and the 3-morphism is a fibered transformation of fibered profunctors $\theta : S \Rightarrow T(\theta_0, \theta_0)$ which maps hetero-inferences.

In three dimensions, one can see that the coherence equations form a "many-object functor" between "many-object monads".



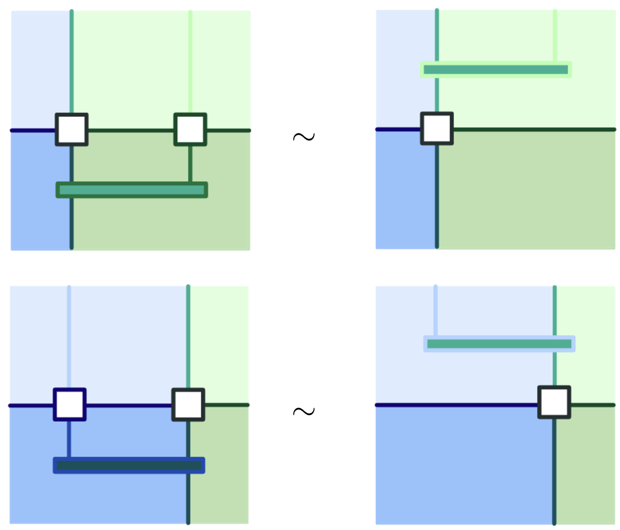
Definition 12. Let $f : \mathbb{X} \to \mathbb{A}$, $g : \mathbb{Y} \to \mathbb{B}$ be pseudofunctors, and $M : \mathbb{X} \to \mathbb{Y}$, $N : \mathbb{A} \to \mathbb{B}$ be H-modules. An **H-transformation** $\gamma : M \to N(f, g)$ is an HT-morphism $\gamma : M \to N(f, g)$



with invertible globular 3-morphisms to preserve left and right actions

left naturalizer
$\gamma_r : (\gamma g) \circ N_r \to M_r \circ \gamma$

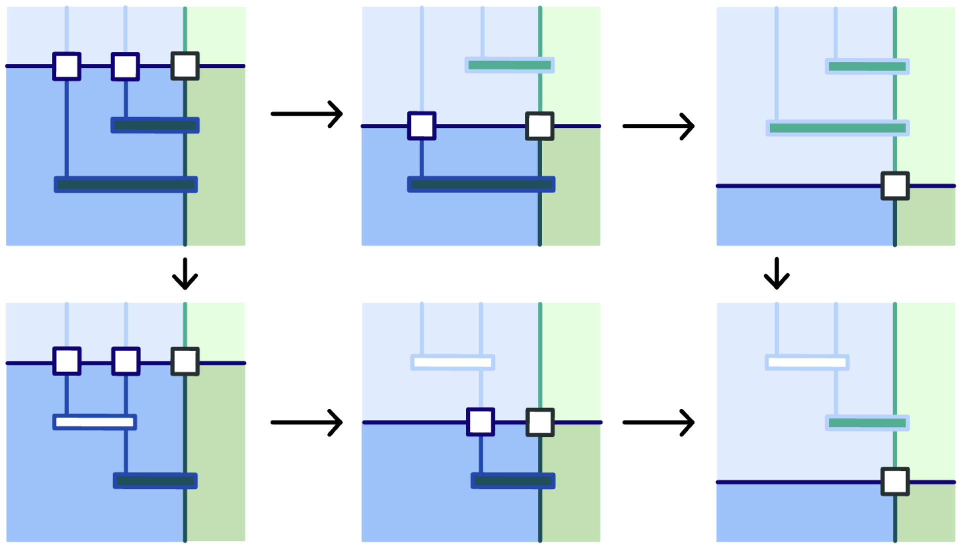
right naturalizer
$\gamma_r : (f\gamma) \circ N_l \to M_l \circ \gamma$

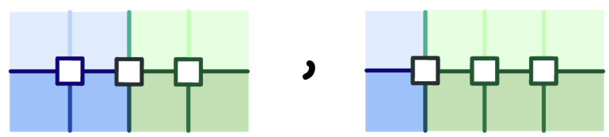


satisfying coherence for associativity and unitality.

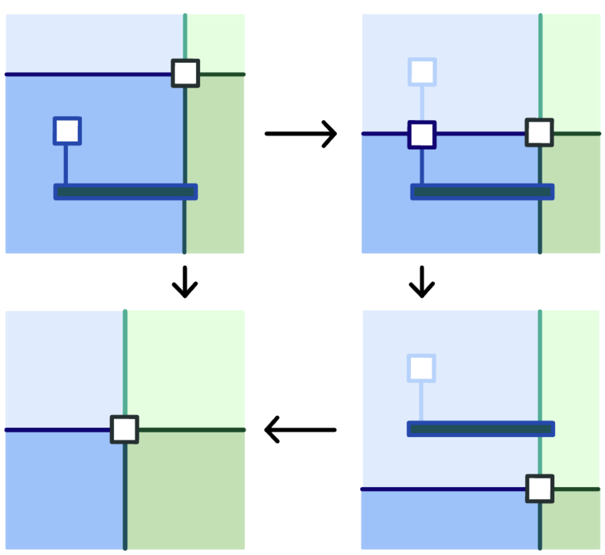
v-center the text, and fit left/right units

left pentagon identity

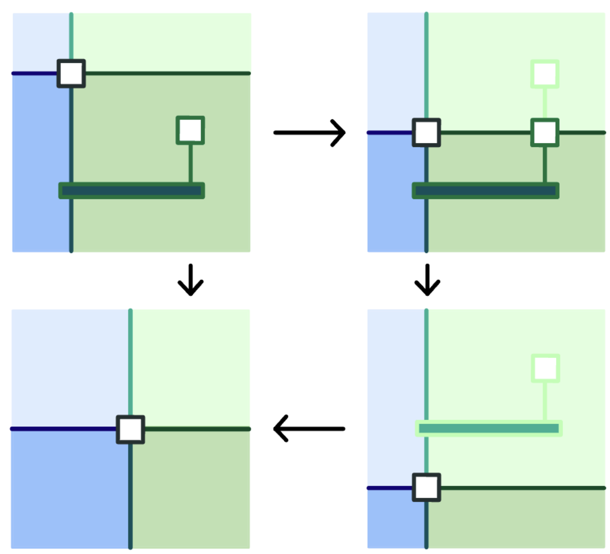


and same for 

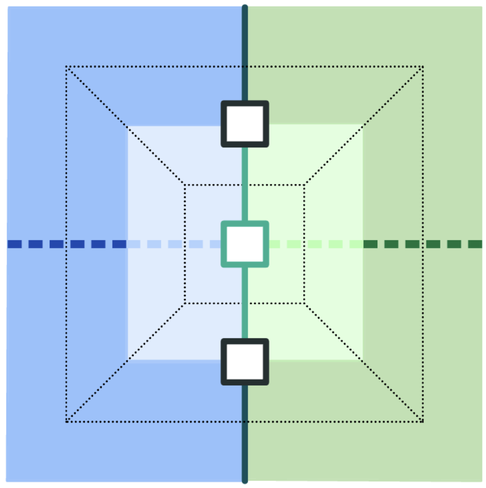
left unit identity



right unit identity



In Logos, an H-transformation is a **judgement system**: a map from inner hetero-judgements, the metajudgement $M : \mathbb{X} \to \mathbb{Y}$, to outer hetero-judgements, the metajudgement $N : \mathbb{A} \to \mathbb{B}$, framed by the systems $f : \mathbb{X} \to \mathbb{A}$ and $g : \mathbb{Y} \to \mathbb{B}$.

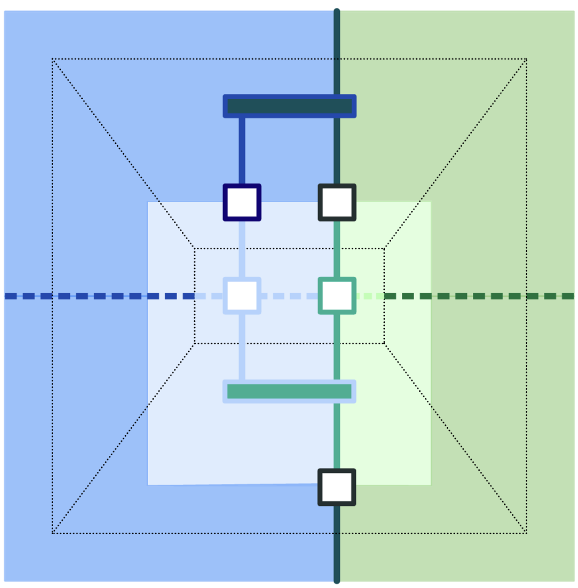


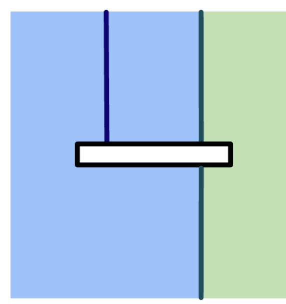
In three dimensions, one can see that each "naturalizer" 3-morphism above is manifested in the target metajudgement N as a family of hetero-inferences: compare the beads on the top and bottom faces of the cube with the string diagram of N.

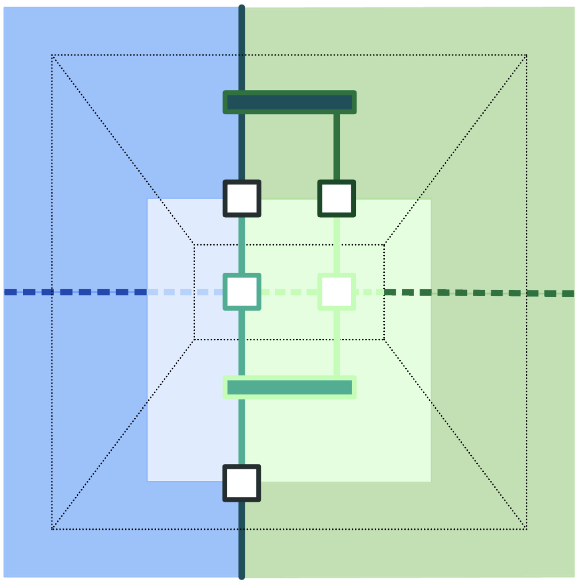
3d =¿ metalogic

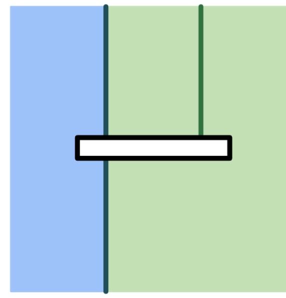
can have terms

string thickness









In the same way that a lax functor can be understood as a “polyad” or many-object monad, a lax H-module can be understood as a many-object module between lax functors. This was the key to the inverse image of a natural transformation of profunctors (ref). It is further explored in (ref).

Definition 13. Let $\mathbb{A}$ and $\mathbb{B}$ be logics. Metajudgements from $\mathbb{A}$ to $\mathbb{B}$ define a 2-category $[\mathbb{A} \mid \mathbb{B}]_0$, with judgement systems and (termless) inference systems forming hom-categories. This is given by the following construction.

Let $\mathcal{P}, \mathcal{Q}$ be metajudgements from $\mathbb{A}$ to $\mathbb{B}$. In MatCat, these are fibered relations with structure. Their hom in MatCat is the category of fibered functors and transformations:

$$[\mathcal{P} \vdash \mathcal{Q}] = \Pi a : \mathbb{A}\ \Pi b : \mathbb{B}\ \mathcal{P}(a,b) \vdash \mathcal{Q}(a,b)$$

This category is equipped with two functors, pre-action on $\mathcal{P}$ and post-action on $\mathcal{Q}$:

$$\Pi ab\ [\mathcal{P}(a,b) \vdash \mathcal{Q}(a,b)] \quad \underset{(-)(\mathbb{A}.\mathcal{Q}.\mathbb{B})}{\overset{(\mathbb{A}.\mathcal{P}.\mathbb{B})(-)}{\rightrightarrows}} \quad \Pi a_{01}, b_{01}\ [\mathcal{P}(a_0, b_0) \vdash \mathcal{Q}(a_1, b_1)]$$

We form the *inserter* of these two functors: this is the category whose objects are fibered functors $\mathcal{P} \vdash \mathcal{Q}$ equipped with an invertible fibered transformation (-), and whose morphisms are fibered transformations (draw cube) such that...

Finally, we define transformations of double modules. These are known in category theory as *modifications*; though now they are fully "heterogenized".

Definition 14. Let - be pseudomonads; let - be double modules; let - be horizontal transformations; and let - be vertical transformations. A **double transformation** is...

In Logos, a double transformation is an **inference system**, a map of an inner metainference into an outer metainference. It is a "system" of inferences, parameterized by...