The Yoneda Lemma

sarahzrf

March 31, 2020

1 Presheaves and Copresheaves

1.1 A Phenomenon

Let **Field** be the category of fields. Fix some ring R. If we have an object $F \in$ **Field**, then it's pretty obvious what "a morphism $R \to F$" should mean, even if R is not a field, and hence not a priori an object in the category that F is from. In this case, it's because **Field** is a subcategory of a larger category **Ring** to which R belongs; but note that we could define the notion of "a homomorphism from R to a field" even if we didn't know about **Ring**'s existence.

In a similar spirit, fix a set S. If we have an object $X \in$ **Top**, the category of topological spaces, then we can talk about "a function from S to X". While it is true that this corresponds to a morphism of **Top** from the discrete space on S into X, this is not necessarily a priori obvious; the most obvious actual definition of such a thing is not a bona fide morphism into X—which would be a morphism of **Top**—but rather a morphism of **Set**, into X's underlying set.

Hmm, maybe some of this language is hinting more specifically at adjunctions than just presheaves here... Could be distracting if the reader already knows a bit about that.

In both of these cases, we have the phenomenon of something which functions as the domain of "morphisms" of some kind into objects of some category, despite not necessarily being itself an object of that category.

Let's look at an example with something functioning as a codomain instead, and one which is a bit more abstract. Let $\mathcal{C}$ be any category, and let A and B be objects in it. $\mathcal{C}$ may or may not have a product of A and B, but regardless of whether it does, we can still say what "a morphism $X \to A \times B$" should be (or at least correspond to) for any $X \in \mathcal{C}$: a pair of morphisms $X \to A$ and $X \to B$.

One thing shared by all of these examples is that they have special cases or equivalent formulations where the "morphisms" involved correspond to actual morphisms of the category, by corresponding the domain or codomain to a real object of the category: for example 1, we have the case where R really is a field; for example 2, we can use continuous functions from a discrete space; and for example 3, we have the case where a product does exist. Let's finish with an example of our running phenomenon which cannot arise "concretely" like this.

Let **Diff** be the category of smooth manifolds and smooth maps between them. The intuitive idea of a smooth function is that, "at an infinitesimal scale", it becomes linear. Classically speaking, there is no such thing as an actual infinitesimal scale in a manifold as typically defined; instead, there is the formal stand-in of a tangent space. But if we imagine an interval $D = (-\epsilon, \epsilon) \subseteq \mathbb{R}$ of infinitesimal size, we can reason our way to a sensible answer of what the smooth maps $D \to M$ should be for $M \in \mathbf{Diff}$, even though there is not actually such an object D! In particular: if a smooth map is linear at an infinitesimal scale, then the smooth maps $D \to M$ should be exactly the linear maps $D \to M$. That is, a smooth map $D \to M$ should be characterized fully by the value and velocity it takes at 0—so we can identify smooth maps $D \to M$ with points of the tangent bundle of M!

More examples?

1.2 A Definition

Given a category $\mathcal{C}$, a *presheaf on* $\mathcal{C}$ is something that can be a codomain for morphisms from objects of $\mathcal{C}$, even if it is not itself an object of $\mathcal{C}$. More precisely, a presheaf is *what is left of* any such thing once you forget everything else about it except how it behaves in that role. Dually, a *copresheaf on* $\mathcal{C}$ is something that can be a *domain* for morphisms into objects of $\mathcal{C}$, even if it is not itself an object of $\mathcal{C}$—or rather, a copresheaf is what is left of any such thing once you forget everything else about it except how it behaves in that role.

Thus, example 1 was of a class of copresheaves on **Field**; example 2 was of a class of copresheaves on **Top**; example 3 was of a class of presheaves on any category; and example 4 was of a particular copresheaf on **Diff**.

Here's a formal definition, albeit in a non-standard presentation.

Definition 1. For a category $\mathcal{C}$, the data of a *presheaf* X *on* $\mathcal{C}$ consists of:

1. For each object $A \in \mathcal{C}$, a set $\mathrm{Hom}(A, X)$, whose elements are to be thought of as "the morphisms from A to X". (Note that this is not actually a hom-set, just part of the data of X!) If $f \in \mathrm{Hom}(A, X)$, then I will write $f : A \rightsquigarrow X$, use quotation marks when referring to f as a "morphism", or call it a "squiggly arrow". (None of this is standard notation or terminology, to my knowledge.)

2. For each actual $\mathcal{C}$ morphism $f : A \to B$ and claimed-by-X "morphism" $g : B \rightsquigarrow X$, a choice of composite $g \cdot f : A \rightsquigarrow X$. This "composition" operation must satisfy the following axioms:

   (a) For all $f : A \rightsquigarrow X$, we must have $f \cdot \mathrm{id}_A = f$.

   (b) For all $f : A \to B, g : B \to C, h : C \rightsquigarrow X$, we must have $h \cdot (g \circ f) = (h \cdot g) \cdot f$.

   In light of these identities, it is actually largely unambiguous to write $\cdot$ as $\circ$, except insofar as it conflates "morphisms" with morphisms, and so I will.

hmm...

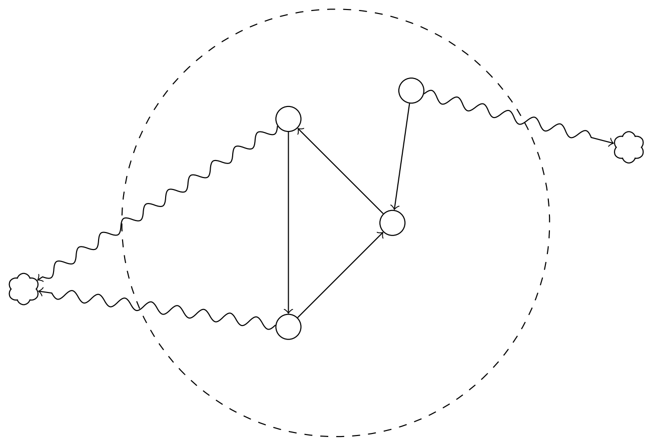

Figure 1: $\mathcal{C}$ and presheaves on it.

The *standard* definition of "presheaf on $\mathcal{C}$", which is equivalent, is "a contravariant functor $\mathcal{C} \to \mathbf{Set}$". This is more concise, but in my opinion, it obfuscates the significance of the data involved. To make the equivalence explicit, a presheaf X corresponds to the functor F which is defined on objects by $F(A) = \mathrm{Hom}(A, X)$ and on morphisms by $F(f) = g \mapsto g \cdot f$—note that for $f : A \to B$, $F(f) : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$.

A *co*presheaf on $\mathcal{C}$ can be defined in components as above, or as a *co*variant functor $\mathcal{C} \to \mathbf{Set}$, or as a presheaf on $\mathcal{C}^{\mathrm{op}}$; exercise to the reader ;)

Carry out the initial examples in the formalism.

I'm now going to introduce some imagery for depicting the concepts discussed so far, shown in Figure 1. The inside of the dashed line represents $\mathcal{C}$; all of $\mathcal{C}$'s objects and morphisms—depicted as circles and arrows—reside in this space. Outside of the dashed line lies the world of things that $\mathcal{C}$ may perceive by mapping into—presheaves on it, depicted as somewhat blobbier shapes, to suggest a less restricted class of objects. Finally, "morphisms" from objects of C to presheaves are depicted as squiggly arrows that cross over the boundary.