Categorical Databases

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$\Sigma \dashv \Delta \dashv \Pi$

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SEMF

Introduction

- This talk describes a new algebraic (purely equational) way to formalize databases and migrate data based on category theory.
- Category theory was designed to migrate **theorems** from one **area of mathematics** to another, so it is a very natural language with which to describe migrating **data** from one **schema** to another.
- Research has culminated in an open-source ETL and data migration tool, CQL, available at categoricaldata.net.
- Outline:
  - Review of basic category theory.
  - Introduction to CQL.
  - CQL demo.
  - Optional: additional CQL constructions.
  - Extra slides: How CQL instances model the simply-typed λ-calculus.

Motivation / Background

- CQL is a ‘category-theoretic’ SQL, used as an ETL tool.
  - Users define schemas and mappings, which induce data transformations.
- CQL schema mappings must preserve data integrity constraints, requiring the use of an automated theorem prover at compile time.
  - CQL catches mistakes at compile time that existing ETL / data migration tools catch at runtime – if at all.
- Some projects using CQL:
  - NIST - several projects.
  - DARPA BRASS project.
  - Empower Retirement.
  - Stanford Chemistry Department.
  - Uber/Tinkerpop
  - Fortune 50 energy and finance companies
  - and more

Category Theory

- A category $\mathcal{C}$ consists of (where "set" is understood naively):
  - a set of *objects*, $\mathsf{Ob}(\mathcal{C})$
  - forall $X, Y \in \mathsf{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$ of *morphisms* a.k.a *arrows*
  - forall $X \in \mathsf{Ob}(\mathcal{C})$, a morphism $\mathsf{id}_X \in \mathcal{C}(X, X)$
  - forall $X, Y, Z \in \mathsf{Ob}(\mathcal{C})$, a function $\circ \colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ s.t.

$$f \circ \mathsf{id} = f \qquad \mathsf{id} \circ f = f \qquad\qquad (f \circ g) \circ h = f \circ (g \circ h)$$

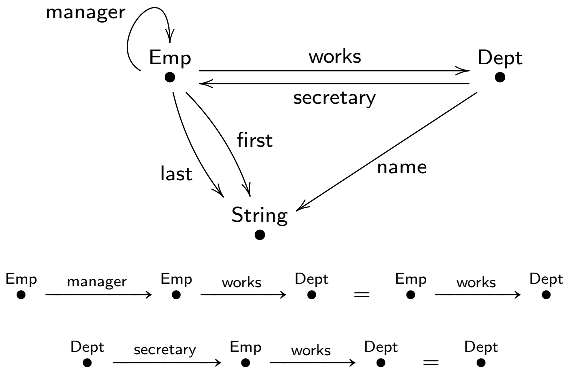
- The category $\mathbf{Set}$ has sets as objects and functions as arrows, and the "category" $\mathbf{Haskell}$ has types as objects and programs as arrows.

- A functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}, \mathcal{D}$ consists of
  - a function $\mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D})$
  - forall $X, Y \in \mathsf{Ob}(\mathcal{C})$, a function $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ s.t.

$$F(\mathsf{id}_X) = \mathsf{id}_{F(X)} \qquad F(f \circ g) = F(f) \circ F(g)$$

- The functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ takes each set to its power set, and the functor $\mathsf{List} : \mathbf{Haskell} \to \mathbf{Haskell}$ takes each type t to the type $\mathsf{List}\ t$.

Example Categorical Schema and Database



Emp				
ID	**mgr**	**works**	**first**	**last**
101	103	q10	Al	Akin
102	102	x02	Bob	Bo
103	103	q10	Carl	Cork

Dept		
ID	**sec**	**name**
q10	101	CS
x02	102	Math

String
ID
Al
Bob
...

A CQL Schema: Code

```
entities
    Emp
    Dept

foreign keys
    manager : Emp -> Emp
    works : Emp -> Dept
    secretary : Dept -> Emp

attributes
    first last : Emp -> string
    name : Dept -> string

path equations
    manager.works = works
    secretary.works = Department
```

Categorical Semantics of Schemas and Instances

- The meaning of a schema S is a category $[\![S]\!]$.
  - Ob($[\![S]\!]$) is the nodes of S.
  - Forall nodes X, Y, $[\![S]\!](X, Y)$ is the set of finite paths $X \to Y$, modulo the path equivalences in S.
  - Path equivalence in S may not be decidable! ("the word problem")
- A morphism of schemas (a "**schema mapping**") $S \to T$ is a functor $[\![S]\!] \to [\![T]\!]$.
  - It can be defined as an equation-preserving function:

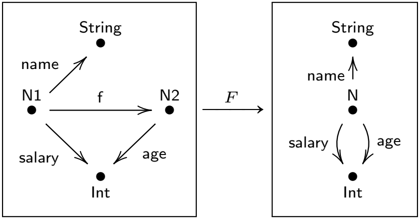
$$nodes(S) \to nodes(T) \qquad edges(S) \to paths(T).$$

- An S-instance is a functor $[\![S]\!] \to \mathbf{Set}$.
  - It can be defined as a set of tables, one per node in S and one column per edge in S, satisfying the path equivalences in S.
- A morphism of S-instances $I \to J$ (a "**data mapping**") is a natural transformation $I \to J$.
  - Instances on S and their mappings form a category, written S-inst.

Schema Mappings

A **schema mapping** $F : S \to T$ is an equation-preserving function:

$$nodes(S) \to nodes(T) \qquad edges(S) \to paths(T)$$



$$F(\text{Int}) = \text{Int} \qquad F(\text{String}) = \text{String}$$
$$F(\text{N1}) = \text{N} \qquad F(\text{N2}) = \text{N}$$
$$F(\text{name}) = [\text{name}] \qquad F(\text{age}) = [\text{age}] \qquad F(\text{salary}) = [\text{salary}]$$
$$F(\text{f}) = []$$

Functorial Data Migration

A schema mapping $F\colon S \to T$ induces three data migration functors:

- $\Delta_F\colon T\text{-inst} \to S\text{-inst}$ (like project)

$$S \xrightarrow{F} T \xrightarrow{I} \mathbf{Set}, \qquad \Delta_F(I) := I \circ F$$

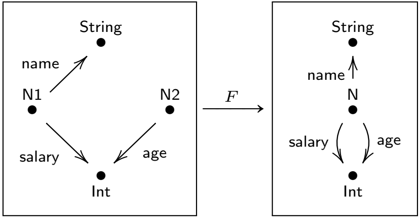
- $\Pi_F\colon S\text{-inst} \to T\text{-inst}$ (right adjoint to Δ_F; like join)

$$\forall I, J. \quad S\text{-inst}(\Delta_F(I), J) \cong T\text{-inst}(I, \Pi_F(J))$$

- $\Sigma_F\colon S\text{-inst} \to T\text{-inst}$ (left adjoint to Δ_F; like outer union then merge)

$$\forall I, J. \quad S\text{-inst}(J, \Delta_F(I)) \cong T\text{-inst}(\Sigma_F(J), I)$$

Δ (Project)



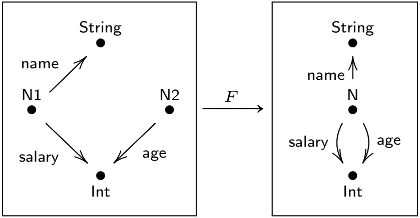
N1		
ID	**name**	**salary**
1	Alice	\$100
2	Bob	\$250
3	Sue	\$300

N2	
ID	**age**
4	20
5	20
6	30

$\xleftarrow{\Delta_F}$

N			
ID	**name**	**salary**	**age**
a	Alice	\$100	20
b	Bob	\$250	20
c	Sue	\$300	30

Π (Product)



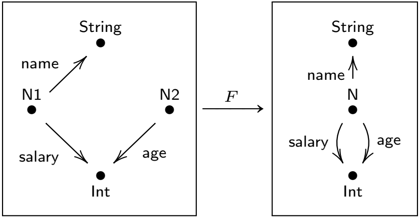
N1		
ID	**name**	**salary**
1	Alice	$100
2	Bob	$250
3	Sue	$300

N2	
ID	**age**
4	20
5	20
6	30

$\xrightarrow{\Pi_F}$

N			
ID	**name**	**salary**	**age**
a	Alice	$100	20
b	Alice	$100	20
c	Alice	$100	30
d	Bob	$250	20
e	Bob	$250	20
f	Bob	$250	30
g	Sue	$300	20
h	Sue	$300	20
i	Sue	$300	30

Σ (Outer Union)



N1		
ID	**Name**	**Salary**
1	Alice	$100
2	Bob	$250
3	Sue	$300

N2	
ID	**Age**
4	20
5	20
6	30

$\xrightarrow{\Sigma_F}$

N			
ID	**Name**	**Salary**	**Age**
a	Alice	$100	$null_1$
b	Bob	$250	$null_2$
c	Sue	$300	$null_3$
d	$null_4$	$null_5$	20
e	$null_6$	$null_7$	20
f	$null_8$	$null_9$	30

Unit of $\Sigma_F \dashv \Delta_F$

N1		
ID	**Name**	**Salary**
1	Alice	\$100
2	Bob	\$250
3	Sue	\$300

N2	
ID	**Age**
4	20
5	20
6	30

$\xrightarrow{\Sigma_F}$

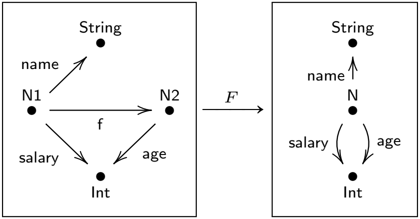
N			
ID	**Name**	**Salary**	**Age**
a	Alice	\$100	$null_1$
b	Bob	\$250	$null_2$
c	Sue	\$300	$null_3$
d	$null_4$	$null_5$	20
e	$null_6$	$null_7$	20
f	$null_8$	$null_9$	30

$\downarrow \eta$ $\quad \swarrow \Delta_F$

N1		
ID	**Name**	**Salary**
a	Alice	\$100
b	Bob	\$250
c	Sue	\$300
d	$null_4$	$null_5$
e	$null_6$	$null_7$
f	$null_8$	$null_9$

N2	
ID	**Age**
a	$null_1$
b	$null_2$
c	$null_3$
d	20
e	20
f	30

A Foreign Key



N1			
ID	**name**	**salary**	**f**
1	Alice	$100	4
2	Bob	$250	5
3	Sue	$300	6

N2	
ID	**age**
4	20
5	20
6	30

$\xleftarrow{\Delta_F}$ $\xrightarrow{\Pi_F, \Sigma_F}$

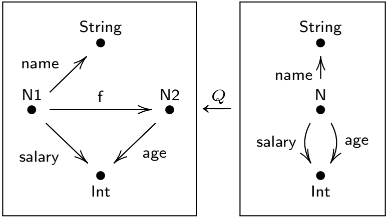
N			
ID	**name**	**salary**	**age**
a	Alice	$100	20
b	Bob	$250	20
c	Sue	$300	30

Queries

A **query** $Q : S \to T$ is a schema X and mappings $F : S \to X$ and $G : T \to X$.

$$eval_Q \cong \Delta_G \circ \Pi_F \qquad coeval_Q \cong \Delta_F \circ \Sigma_G$$

These can be specified using comprehension notation similar to SQL.

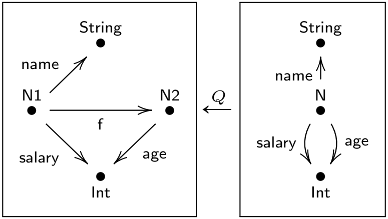



```
N1 -> select n1.name as name, n1.salary as salary
  from N as n1

N2 -> select n2.age as age
  from N as n2

f -> {n2 -> n1}
```

A Foreign Key



N1			
ID	**name**	**salary**	**f**
1	Alice	$100	4
2	Bob	$250	5
3	Sue	$300	6

N2	
ID	**age**
4	20
5	20
6	30

$\xleftarrow{eval_Q}$

$\xrightarrow{coeval_Q}$

N			
ID	**name**	**salary**	**age**
a	Alice	$100	20
b	Bob	$250	20
c	Sue	$300	30

CQL Demo

- CQL implements Δ, Σ, Π, and more in software.
  - Commercial support / services: conexus.com

Interlude - Additional Constructions

- What is "algebraic" here?
- CQL vs SQL.
- Pivot.
- Non-equational data integrity constraints.
- Data integration via pushouts.
- CQL vs comprehension calculi.

Why "Algebraic"?

- A schema can be identified with an algebraic (equational) theory.

$$\text{Emp Dept String} : \text{Type} \qquad \text{first last} : \text{Emp} \to \text{String} \qquad \text{name} : \text{Dept} \to \text{String}$$

$$\text{works} : \text{Emp} \to \text{Dept} \qquad \text{mgr} : \text{Emp} \to \text{Emp} \qquad \text{secr} : \text{Dept} \to \text{Emp}$$

$$\forall e : \text{Emp}.\ \text{works}(\text{manager}(e)) = \text{works}(e) \qquad \forall d : \text{Dept}.\ \text{works}(\text{secretary}(d)) = d$$

- This perspective makes it easy to add functions such as $+ : \text{Int}, \text{Int} \to \text{Int}$ to a schema. See *Algebraic Databases*.

- An S-instance can be identified with the initial algebra of an algebraic theory extending S.

$$\text{101 102 103} : \text{Emp} \qquad \text{q10 x02} : \text{Dept}$$

$$\text{mgr}(101) = 103 \qquad \text{works}(101) = \text{q10} \qquad \ldots$$

- Treating instances as theories allows instances that are infinite or inconsistent (e.g., Alice=Bob).

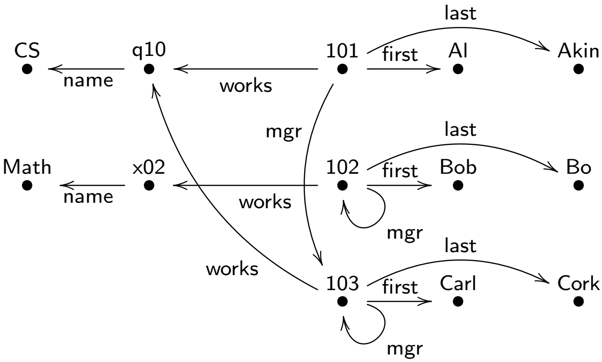
CQL vs SQL

- Data migration triplets of the form

$$\Sigma_F \circ \Pi_G \circ \Delta_H$$

  can be expressed using (difference-free) relational algebra and keygen, provided:
  - F is a discrete op-fibration (ensures union compatibility).
  - G is surjective on attributes (ensures domain independence).
  - All categories are finite (ensures computability).
- The difference-free fragment of relational algebra can be expressed using such triplets. See *Relational Foundations*.
- Such triplets can be written in "foreign-key aware" SQL-ish syntax.
- For arbitrary F, Σ_F can be implemented using canonical/deterministic chase (fire all active triggers across all rules at once.)

Pivot (Instance ⇔ Schema)

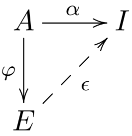


Emp				
ID	**mgr**	**works**	**first**	**last**
101	103	q10	Al	Akin
102	102	x02	Bob	Bo
103	103	q10	Carl	Cork

Dept	
ID	**name**
q10	CS
x02	Math

Richer Constraints

- Not all data integrity constraints are equational (e.g., keys).
- A data mapping $\varphi : A \to E$ defines a constraint: instance I satisfies φ if for every $\alpha : A \to I$ there exists an $\epsilon : E \to I$ s.t $\alpha = \epsilon \circ \varphi$.



- Most constraints used in practice can be captured the above way. E.g.,

$$\forall d_1, d_2 : \text{Dept. name}(d_1) = \text{name}(d_2) \to d_1 = d_2$$

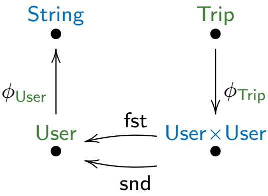
  is captured as

$$A(\text{Dept}) = \{d_1, d_2\} \qquad A(\text{name})(d_1) = A(\text{name})(d_2)$$

$$E(\text{Dept}) = \{d\} \qquad \varphi(d_1) = \varphi(d_2) = d$$

- See *Database Queries and Constraints via Lifting Problems* and *Algebraic Model Management*.

Algebraic Property Graphs with Product Schemas



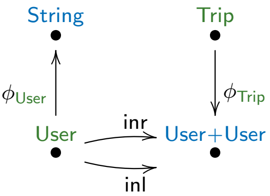
User × User		
ID	fst	snd
(u_1, u_1)	u_1	u_1
(u_1, u_2)	u_1	u_2
(u_1, u_3)	u_1	u_3
(u_2, u_1)	u_2	u_1
(u_2, u_2)	u_2	u_2
(u_2, u_3)	u_2	u_3
(u_3, u_1)	u_3	u_1
(u_3, u_2)	u_3	u_2
(u_3, u_3)	u_3	u_3

User	
ID	ϕ_{User}
u_1	Alice
u_2	Bob
u_3	Chaz

Trip	
ID	ϕ_{Trip}
t_1	(u_1, u_2)
t_2	(u_1, u_3)

String
ID
Alice
Bob
Chaz

Algebraic Property Graphs with Sum Schemas



User + User
ID
inl(u_1)
inl(u_2)
inl(u_3)
inr(u_1)
inr(u_2)
inr(u_3)

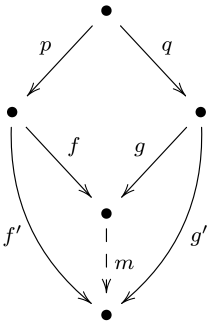
User			
ID	ϕ_{User}	inl	inr
u_1	Alice	inl(u_1)	inr(u_1)
u_2	Bob	inl(u_2)	inr(u_2)
u_3	Chaz	inl(u_3)	inr(u_3)

Trip	
ID	ϕ_{Trip}
t_1	inl(u_1)
t_2	inr(u_2)

String
ID
Alice
Bob
Chaz

Pushouts

- A pushout of p, q is f, g s.t. for every f', g' there is a unique m s.t.:



- The category of schemas has all pushouts.
- For every schema S, the category S-inst has all pushouts.
- Pushouts of schemas, instances, and Σ are used together to integrate data - see *Algebraic Data Integration*.

Using Pushouts for Data Integration

- Step 1: integrate schemas. Given input schemas S_1, S_2, an overlap schema S, and mappings F_1, F_2:

$$S_1 \stackrel{F_1}{\leftarrow} S \stackrel{F_2}{\rightarrow} S_2$$

we propose to use their pushout T as the integrated schema:

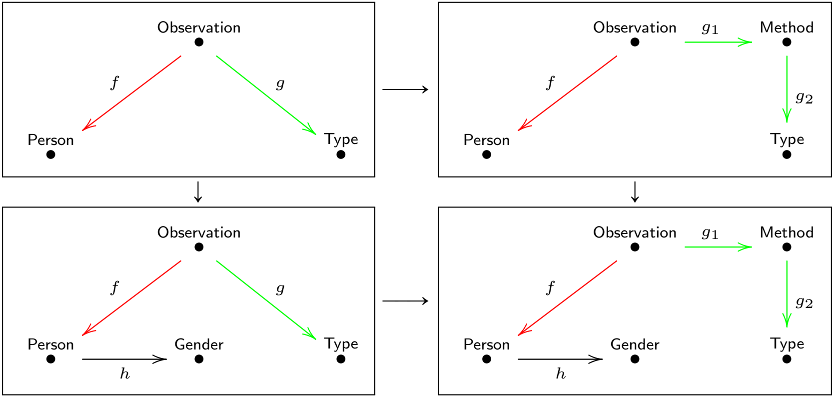
$$S_1 \stackrel{G_1}{\rightarrow} T \stackrel{G_2}{\leftarrow} S_2$$

- Step 2: integrate data. Given input S_1-instance I_1, S_2-instance I_2, overlap S-instance I and data mappings $h_1 : \Sigma_{F_1}(I) \rightarrow I_1$ and $h_2 : \Sigma_{F_2}(I) \rightarrow I_2$, we propose to use the pushout of:

$$\Sigma_{G_1}(I_1) \stackrel{\Sigma_{G_1}(h_1)}{\leftarrow} \big(\Sigma_{G_1 \circ F_1}(I) = \Sigma_{G_2 \circ F_2}(I)\big) \stackrel{\Sigma_{G_2}(h_2)}{\rightarrow} \Sigma_{G_2}(I_2)$$

as the integrated T-instance.

Schema Integration



Data Integration

Observation

ID	f	g

Person

ID
p

Type

ID
BP
Wt

→

Method

ID	g2
m_1	BP
m_2	BP
m_3	Wt
m_4	Wt

Type

ID
BP
Wt

Observation

ID	f	g1
o_1	Pete	m_1
o_2	Pete	m_2
o_3	Jane	m_3
o_4	Jane	m_1

Person

ID
Jane
Pete

↓

Gender

ID
F
M

Type

ID
BP
Wt
HR

Observation

ID	f	g
o_5	Peter	BP
o_6	Paul	HR
o_7	Peter	Wt

Person

ID	h
Paul	M
Peter	M

→

Method

ID	g2
$null_1$	BP
$null_2$	Wt
$null_3$	HR
m_1	BP
m_2	BP
m_3	Wt
m_4	Wt

Observation

ID	f	g1
o_1	Peter	m_1
o_2	Peter	m_2
o_3	Jane	m_3
o_4	Jane	m_1
o_5	Peter	$null_1$
o_6	Paul	$null_2$
o_7	Peter	$null_3$

Gender

ID
F
M
$null_4$

Type

ID
BP
Wt
HR

Person

ID	h
Jane	$null_4$
Paul	M
Peter	M

Quotients for Integration

- In practice, rather than providing entire schema mappings and instance transforms to define pushouts, it is easier to provide equivalence relations and use quotients. In CQL:

```
schema T = S1 + S2 /
  S1_Observation = S2.Observation
  S1_Person = S2_Patient
  S1_ObsType = S2_Type
  S1_f = S2_f
  S1_g = S2_g1.S2_g2

instance J = sigma F1 I1 + sigma F2 I2 /
  Peter = Pete
  BloodPressure = BP
  Wt = BodyWeight
```

Conclusion

- We described a new algebraic (equational) approach to databases based on category theory.
  - Schemas are categories, instances are set-valued functors.
  - Three adjoint data migration functors, Σ, Δ, Π manipulate data.
  - Instances on a schema model the simply-typed λ-calculus.
- Our approach is implemented in CQL, an open-source project, available at categoricaldata.net. Collaborators welcome!

Partial Bibliography

- *Patrick Schultz, Ryan Wisnesky.* **Algebraic Data Integration**. (JFP-PlanBig 2017)
- *Patrick Schultz, David I. Spivak, Christina Vasilakopoulou,, Ryan Wisnesky.* **Algebraic Databases**. (TAC 2017)
- *Patrick Schultz, David I. Spivak, Ryan Wisnesky.* **Algebraic Model Management: A Survey**. (WADT 2016)
- *David I. Spivak, Ryan Wisnesky.* **Relational Foundations for Functorial Data Migration**. (DBPL 2015).

Extra Slides

CQL is "one level up" from LINQ

- LINQ
  - Schemas are collection types over a base type theory

$$\mathsf{Set}\ (\mathsf{Int} \times \mathsf{String})$$

  - Instances are terms

$$\{(1, \mathsf{CS})\} \cup \{(2, \mathsf{Math})\}$$

  - Data migrations are functions

$$\pi_1 \colon \mathsf{Set}\ (\mathsf{Int} \times \mathsf{String}) \to \mathsf{Set\ Int}$$

- CQL
  - Schemas are type theories over a base type theory

$$\mathsf{Dept},\ \ \mathsf{name} \colon \mathsf{Dept} \to \mathsf{String}$$

  - Instances are term models (initial algebras) of theories

$$\mathsf{d}_1, \mathsf{d}_2 \colon \mathsf{Dept},\ \ \mathsf{name}(\mathsf{d}_1) = \mathsf{CS},\ \ \mathsf{name}(\mathsf{d}_2) = \mathsf{Math}$$

  - Data migrations are functors

$$\Delta_{\mathsf{Dept}} \colon (\mathsf{Dept}, \mathsf{name} \colon \mathsf{Dept} \to \mathsf{String})\text{-}\mathsf{inst} \ \to\ (\mathsf{Dept})\text{-}\mathsf{inst}$$

Part 2

- For every schema S, S-inst models simply-typed λ-calculus (STLC).
- The STLC is the core of typed functional languages ML, Haskell, etc.
- We will use the internal language of a cartesian closed category, which is equivalent to the STLC.
- Lots of "point-free" functional programming ahead.
- The category of schemas and mappings is also cartesian closed - see talk at Boston Haskell.

Categorical Abstract Machine Language (CAML)

- Types t:

$$t ::= 1 \mid t \times t \mid t^t$$

- Terms f, g:

$$id_t : t \to t \qquad ()_t : t \to 1 \qquad \pi^1_{s,t} : s \times t \to s \qquad \pi^2_{s,t} : s \times t \to t$$

$$eval_{s,t} : t^s \times s \to t \qquad \frac{f : s \to u \qquad g : u \to t}{g \circ f : s \to t} \qquad \frac{f : s \to t \qquad g : s \to u}{(f, g) : s \to t \times u}$$

$$\frac{f : s \times u \to t}{\lambda f : s \to t^u}$$

- Equations:

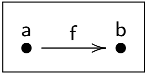
$$id \circ f = f \qquad f \circ id = f \qquad f \circ (g \circ h) = (f \circ g) \circ h \qquad () \circ f = ()$$

$$\pi^1 \circ (f, g) = f \qquad \pi^2 \circ (f, g) = g \qquad (\pi^1 \circ f, \pi^2 \circ f) = f$$

$$eval \circ (\lambda f \circ \pi^1, \pi^2) = f \qquad \lambda(eval \circ (f \circ \pi^1, \pi^2)) = f$$

Programming CQL in CAML

- For every schema S, the category S-inst is cartesian closed.
  - Given a type t, you get an S-instance $[t]$.
  - Given a term $f : t \to t'$, you get a data mapping $[f] : [t] \to [t']$.
  - All equations obeyed.
- S-inst is further a topos (model of higher-order logic / set theory).
- We consider the following schema in the examples that follow:



Programming CQL in CAML: Unit

- The unit instance 1 has one row per table:

a	
ID	**f**
x	x

b
ID
x

- The data mapping $()_t : t \to 1$ sends every row in t to the only row in 1. For example,

$t =$

a	
ID	**f**
p	q
r	t

b
ID
q
t

$\xrightarrow{()_t}$

a	
ID	**f**
x	x

b
ID
x

$= 1$

$$\text{p, q, r, t} \xrightarrow{()_t} \text{x}$$

Programming CQL in CAML: Products

- Products $s \times t$ are computed row-by-row, with evident projections $\pi^1 : s \times t \to s$ and $\pi^2 : s \times t \to t$. For example:

a	
ID	f
1	3
2	3

b
ID
3
4

$\times$

a	
ID	f
a	c
b	c

b
ID
c
d

$=$

a	
ID	f
(1,a)	(3,c)
(1,b)	(3,c)
(2,a)	(3,c)
(2,b)	(3,c)

b
ID
(3,c)
(3,d)
(4,c)
(4,d)

- Given data mappings $f : s \to t$ and $g : s \to u$, how to define $(f, g) : s \to t \times u$ is left to the reader.
  - hint: try it on π^1 and π^2 and verify that $(\pi^1, \pi^2) = id$.

Programming CQL in CAML: Exponentials

- Exponentials t^s are given by finding all data mappings $s \to t$:

a	
ID	**f**
1	3
2	3

b
ID
3
4

$\to$

a	
ID	**f**
a	c
b	c

b
ID
c
d

$=$

a	
ID	**f**
$1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto b, 2 \mapsto a, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto a, 2 \mapsto a, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto b, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$	$3 \mapsto c, 4 \mapsto d$
$1 \mapsto a, 2 \mapsto b, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto b, 2 \mapsto a, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto a, 2 \mapsto a, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$
$1 \mapsto b, 2 \mapsto b, 3 \mapsto d, 4 \mapsto c$	$3 \mapsto d, 4 \mapsto c$

b
ID
$3 \mapsto c, 4 \mapsto c$
$3 \mapsto c, 4 \mapsto d$
$3 \mapsto d, 4 \mapsto c$
$3 \mapsto d, 4 \mapsto d$

- Defining $eval$ and λ are left to the reader.