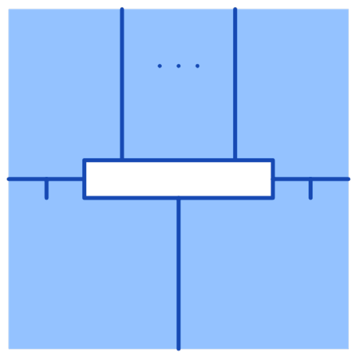
Let $\mathcal{L}$ be a (virtual) triple category. The dimensions are transversal (T), vertical (V), and horizontal (H). We denote 0-cells as $\mathbb{A}$, T-cells as $f : \mathbb{A} \to \mathbb{B}$, V-cells as $P : \mathbb{A} \,|\, \mathbb{B}$, and H-cells as $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$. We denote horizontal composition by $\mathcal{R} \otimes \mathcal{S}$, vertical composition by $P \circ Q$, and transversal composition by $f \cdot g$.

Our motivating example is $\mathrm{Span}\mathbb{C}\mathrm{at}$: dimension 0 is categories, T is functors, V is profunctors, and H is spans of categories.

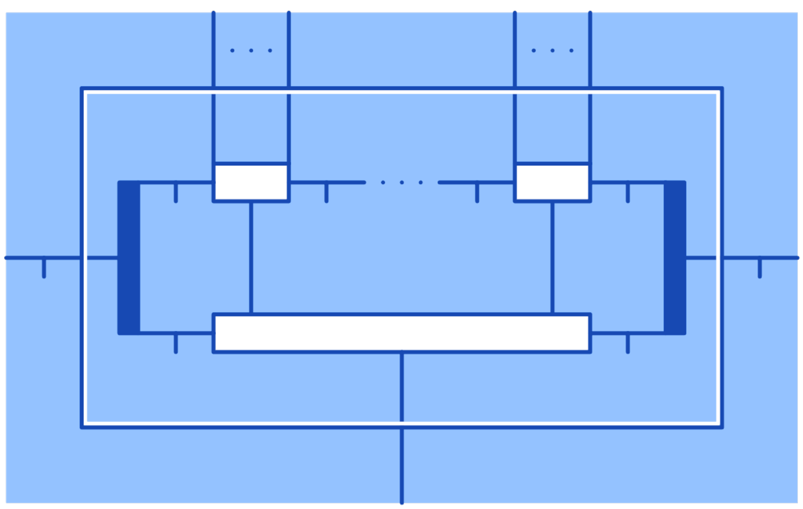
Definition 1. A **multimonad** in $\mathcal{L}$ is
- a 0-cell $\underline{\mathbb{A}}$, called the **base**,
- an H-cell $\mathbb{A} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{A}}$, called the **unary hom**,
- for each $n : \mathbb{N}$ an HV-cell called the **n-ary hom**

$$\alpha[n] : (\mathbb{A} \otimes \cdots \otimes \mathbb{A}) \,|\, \mathbb{A}$$



- for each $i_1, \ldots, i_n : \mathbb{N}$ a 3-cell called **multicomposition**

$$(\alpha[i_1] \otimes \cdots \otimes \alpha[i_n]) \circ \alpha[n] \Rightarrow \alpha[\Sigma i_j]$$

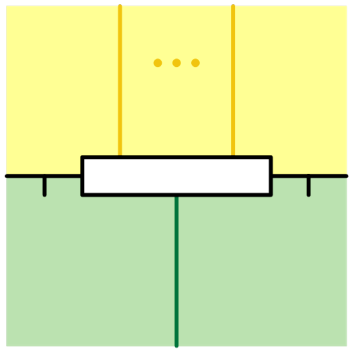


- so that multicomposition is associative and unital.

A multimonad in $\mathrm{Span}\mathbb{C}\mathrm{at}$ is a **virtual double category**.

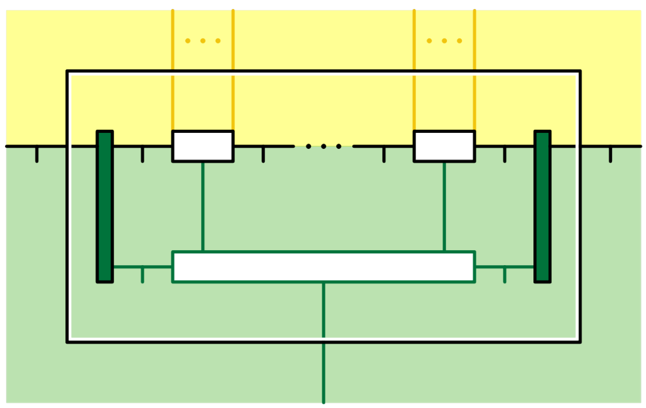
Definition 2. Let $\mathbb{X}, \mathbb{A}$ be multimonads in $\mathcal{L}$. A **vertical multimodule** $P : \mathbb{X} \,|\, \mathbb{A}$ is

- a V-cell $\underline{P} : \underline{\mathbb{X}} \,|\, \underline{\mathbb{A}}$
- for each $n : \mathbb{N}$ an HV-cell $P[n] : \otimes_n \mathbb{X} \,|\, \mathbb{A}$



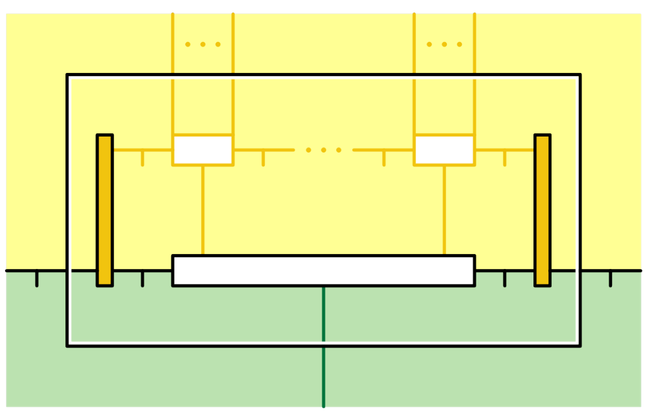
- for each $i_1, \ldots, i_n : \mathbb{N}$ a 3-cell

$$(\chi[i_1] \otimes \cdots \otimes \chi[i_n]) \circ P[n] \Rightarrow P(\Sigma i_j)$$



- for each $n_1, \ldots, n_k : \mathbb{N}$, a 3-cell

$$(P[n_1] \otimes \cdots \otimes P[n_k]) \circ \alpha[k] \Rightarrow P[\Sigma n_j]$$

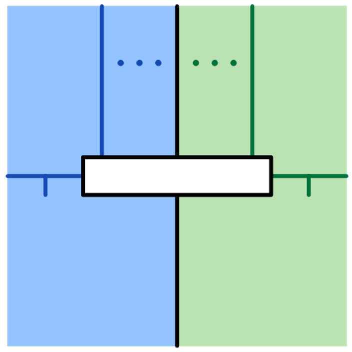


- so these actions are associative and unital.

A vertical multimodule (or "V-module") in $\mathrm{Span}\mathbb{C}\mathrm{at}$ is a **vertical profunctor** of virtual double categories.

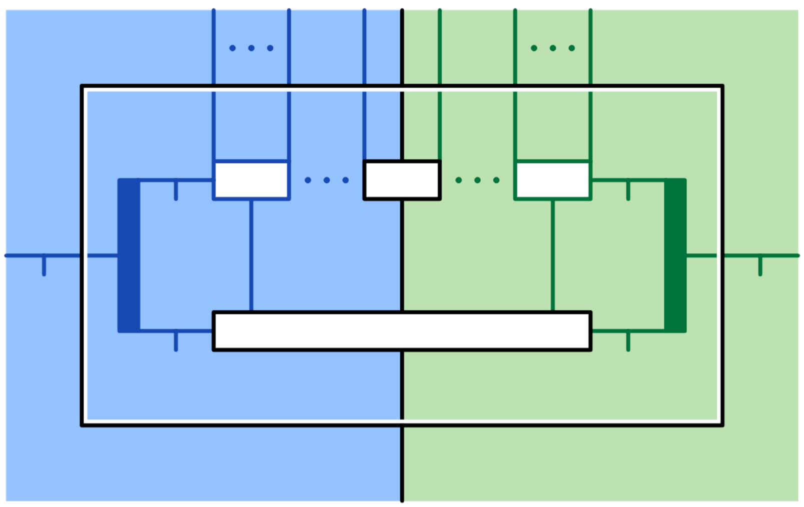
Definition 3. Let $\mathbb{A}, \mathbb{B}$ be multimonads in $\mathcal{L}$. A **horizontal multimodule** $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is

- an H-cell $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$
- for each $m, n : \mathbb{N}$ an HV-cell $\mathcal{R}[m,n] : (\mathbb{A}^m \otimes \mathcal{R} \otimes \mathbb{B}^n) \,|\, \mathcal{R}$



- for each $i_1, \ldots, i_m : \mathbb{N}$ and $j_1, \ldots, j_n : \mathbb{N}$ and $k, \ell : \mathbb{N}$ a 3-cell

$$((\alpha[i_1] \otimes \cdots \otimes \alpha[i_m]) \otimes \mathcal{R}[k,\ell] \otimes (\beta[j_1] \otimes \cdots \otimes \beta[j_n])) \circ \mathcal{R}[m,n] \Rrightarrow \mathcal{R}[\Sigma i + k, \ell + \Sigma j]$$



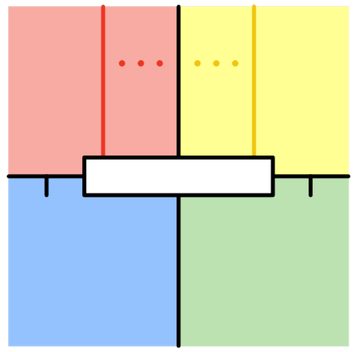
- so that this is associative and unital.

A horizontal multimodule (or “H-module”) in $\mathbb{S}\mathrm{pan}\mathbb{C}\mathrm{at}$ is a **horizontal profunctor** of virtual double categories.

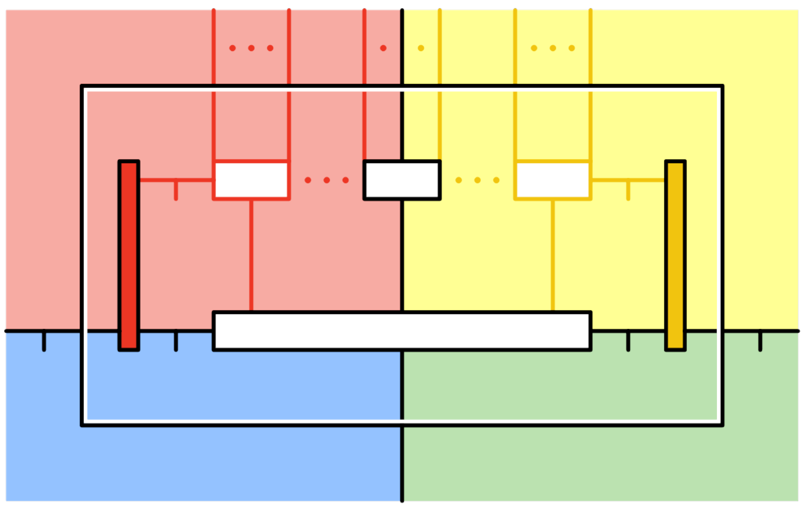
Definition 4. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be multimonads in $\mathcal{L}$. Let $P : \mathbb{X} \,|\, \mathbb{A}$ and $Q : \mathbb{Y} \,|\, \mathbb{B}$ be V-modules. Let $\mathcal{R} : \mathbb{X} \,\|\, \mathbb{Y}$ and $\mathcal{S} : \mathbb{A} \,\|\, \mathbb{B}$ be H-modules.

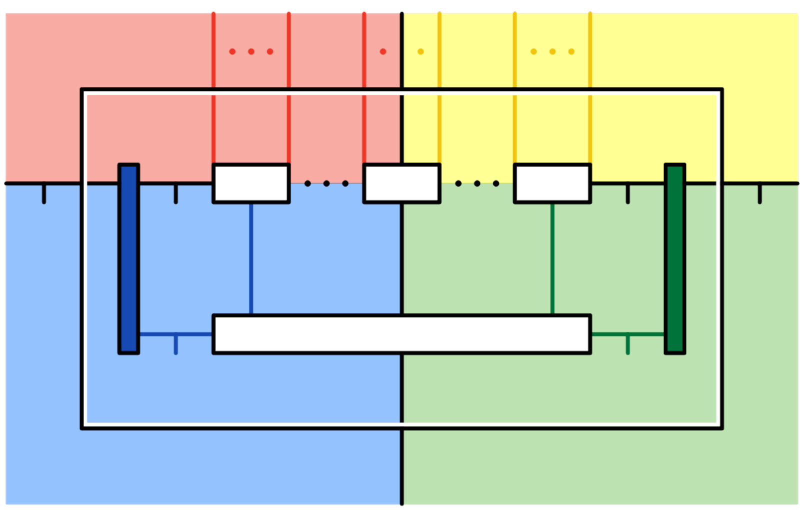
A **double multimodule** $i : \mathcal{R} \,|\, \mathcal{S}$ over P, Q is

- for each $m, n : \mathbb{N}$ an HV-cell $i[m, n] : (\mathbb{X}^m \otimes \mathcal{R} \otimes \mathbb{Y}^n) \,|\, \mathcal{S}$ over $\underline{P}, \underline{Q}$



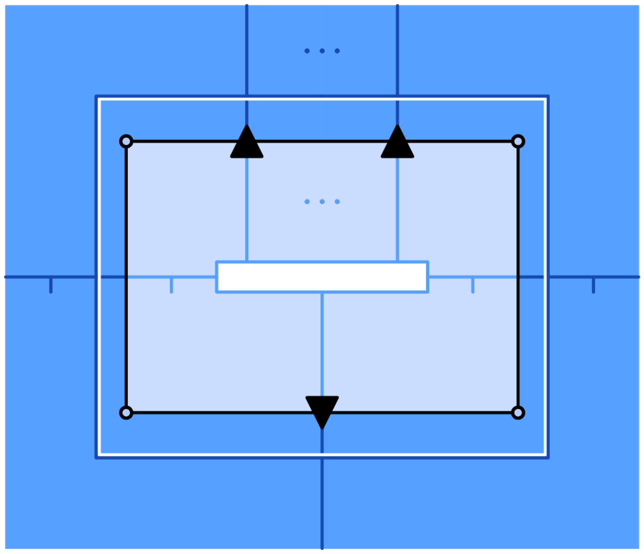
- with multicomposition that is associative and unital.



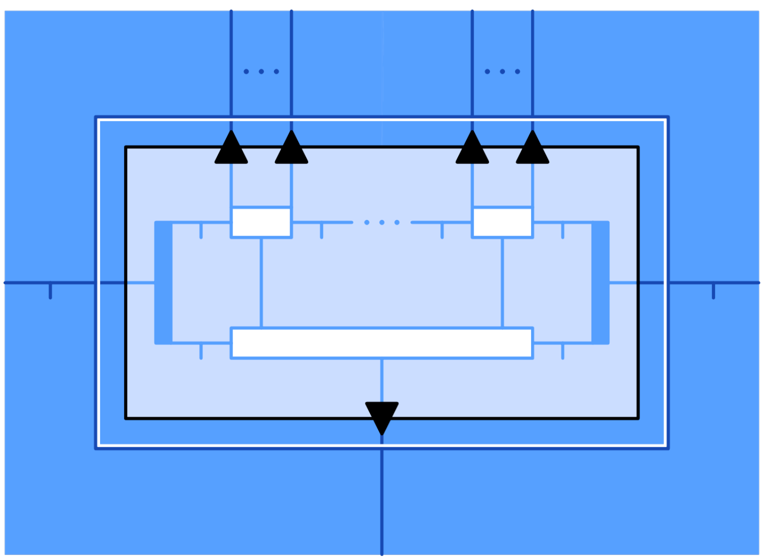


A double multimodule in $\mathbb{S}\mathrm{pan}\mathbb{C}\mathrm{at}$ is a **double profunctor** of virtual double categories.

Definition 5. Multifunctor

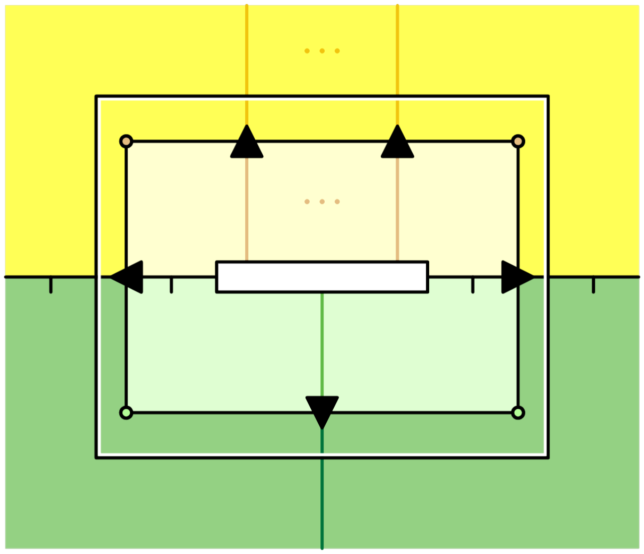


such that



A multifunctor in $\mathrm{Span}\mathbb{C}\mathrm{at}$ is a **virtual double functor**.

Definition 6. V-multitransform



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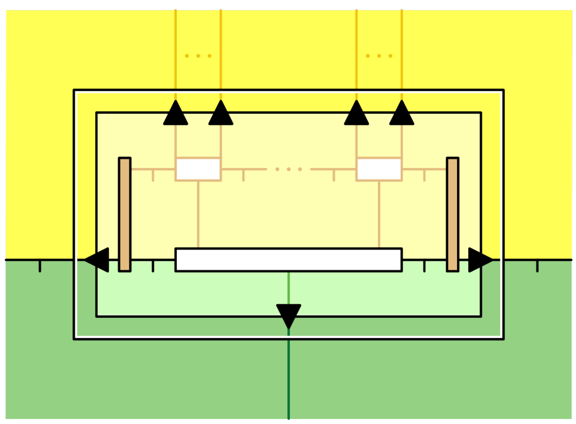
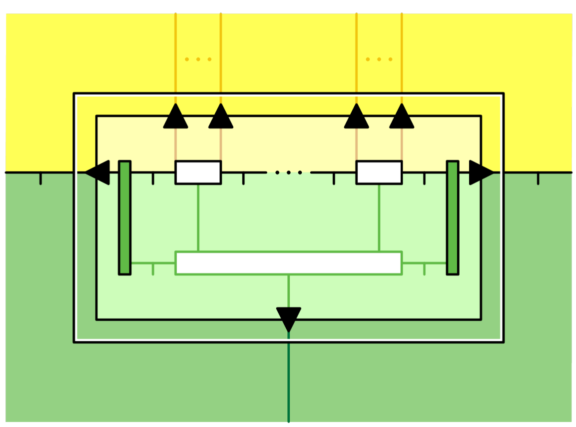
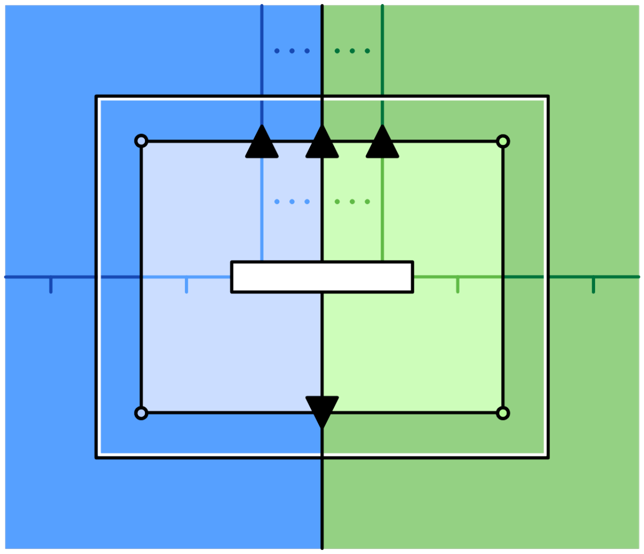


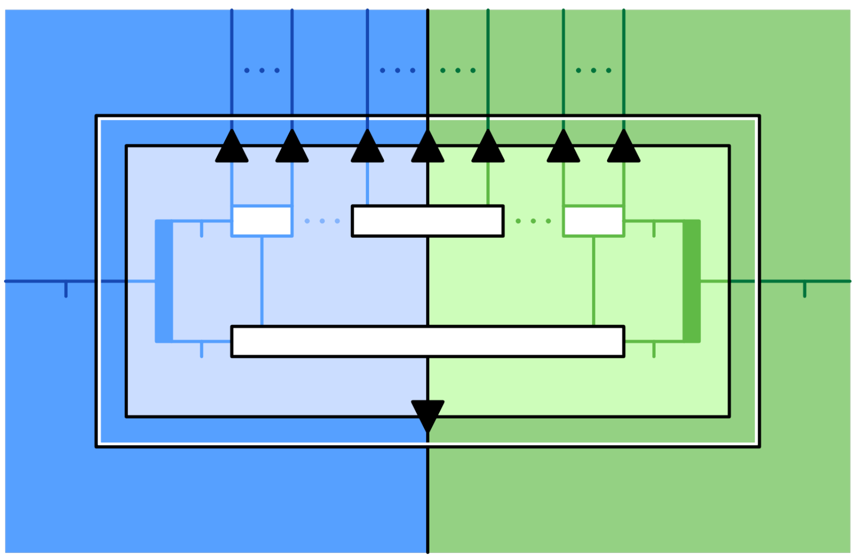


A V-multitransform in $\mathbb{S}\mathrm{pan}\mathbb{C}\mathrm{at}$ is a **vertical transformation** of virtual double functors.

Definition 7. H-multitransform

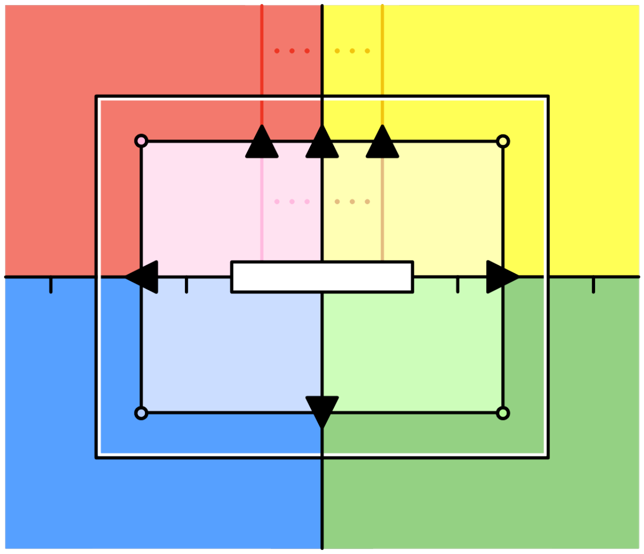


such that

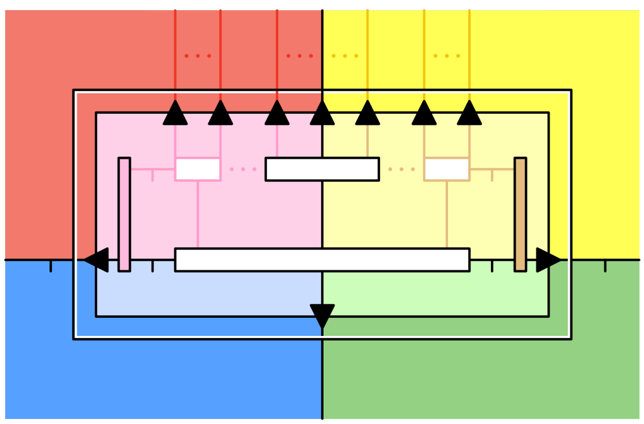


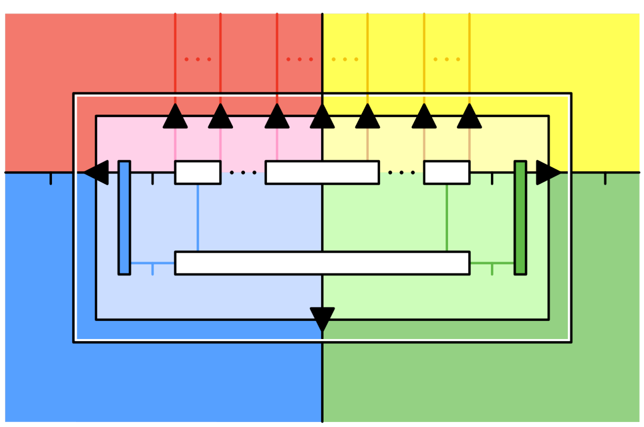
An H-multitransform in $\mathrm{Span}\mathbb{C}\mathrm{at}$ is a **horizontal transformation** of virtual double functors.

Definition 8. Double multitransform



such that





A double multitransform in SpanCat is a **double transformation** of virt-dbl-profs.