Parametricity of Extensionally Collapsed Term Models of Polymorphism and Their Categorical Properties

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Abstract

In the preceding paper, the author proved that parametric natural models have many categorical data types: finite products, finite coproducts, initial and terminal fixed points. In this paper, we show the second order minimum model is parametric, and thus enjoys the property. In addition to that, we give representation of internal right and left Kan extensions. We also show that extensionally collapsed models of closed types/terms collection are partially parametric, and that they have a part of the categorical data types above.

1 Introduction

When we speak of *term models* of lambda calculi, there are two possibilities: one is the collection of all terms including open terms, and the other the collection of closed terms. The former is used to prove completeness for simply typed lambda calculus [8] and for second order lambda calculus [5]. In contrast, it is even misleading to call the collection of closed terms a model. It does not satisfy extensionality in case of second order lambda calculus [16] (even in case of system $\mathbf{T}$ [2]). It is, however, still worth while considering the collection of closed terms (of closed types), in particular, for second order lambda calculus.

One reason is that it is natural for second order lambda calculus to have empty types. It is showed in [16] that the completeness by the collection of all terms is applied only for the axioms and inference rules including the rule called *nonempty*, which is not sound for models having empty types. On the other hand, although the collection of closed terms is not a model, it gives an interpretation such that some types may be interpreted as an empty set.

Another reason we are concerned with in this paper is that the collection of closed terms is a *parametric* interpretation [22, 13]. Second order lambda calculus is often called *polymorphic* lambda calculus. It is because a universal type $\forall X.\sigma$ behaves as a type of polymorphic data. A *polymorphic data* is a family of terms $\{M_X \mid X \text{ ranges over types}\}$. A term M of type $\forall X.\sigma$ is regarded to be a polymorphic data whose X-indexed member is extracted by applying X to the term M.

If M is a closed term of type $\forall X.\sigma$, its normal form is $M = \Lambda X.N[X]$. It means that the members of M as a polymorphic data have uniform definitions $N[X]$ with parameter X. Strachey called polymorphism with uniform definitions *parametric* polymorphism in contrast to *ad hoc* polymorphism. Polymorphism of second order lambda calculus is parametric in nature.

Reynolds invented to use binary relations to reflect parametricity in semantics [20]. In the preceding paper [13], the author adopt the same idea in the framework of BMM models, and obtained a result that if an interpretation is a parametric natural model (see the following sections) then it has many canonical categorical data types, as finite products, finite coproducts, cartesian closedness, initial and terminal fixed points of some endofunctors. In fact, the universal conditions these categorical constructions should satisfy just corresponds to parametricity.

The result is, however, valid only for parametric *models*. Although the collection of closed terms is parametric, it is not a model. Breazu-Tannen and Coquand developped *polymorphic extensional collapse* which can be used to obtain models from closed types/terms interpretation. Then we face a problem whether the extensionally collapsed models are parametric. We show that all syntactically definable universal types are parametric. Then the models have a part of categorical data types mentioned above.

Another model treated in this paper is the second order minimum model by Moggi and Statman [18]. It is obtained from a special theory, called the maximum consistent theory, which satisfies the ω rule. We show the second order minimum model is fully parametric and the categorical data types above all exist in the model. Furthermore we give representation of internal right and left Kan extensions, including left and right adjoints as a special case.

2 Syntax of polymorphism

Suppose given a set $\mathcal{K}$ of *type constants*. A *type judgement* has the form $\Gamma \vdash \sigma$ where Γ is a finite sequence of mutually distinct type variables. There are four inference rules for generating type judgements.

(Ty proj)	$\Gamma \vdash X$	(X appears in Γ)
(Ty const)	$\Gamma \vdash A$	(A in $\mathcal{K}$)

(Ty ⇒)
$$\frac{\Gamma \vdash \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma \Rightarrow \tau}$$

(Ty ∀)
$$\frac{\Gamma, X \vdash \sigma}{\Gamma \vdash \forall X.\sigma}$$

As usual, we do not distinguish types which differ only in bound type variables.

Suppose given a set $\mathcal{C}$ of *individual constants*, to each of which a *closed* type is assigned. We denote the type by Type(c) for $c \in \mathcal{C}$. A *term judgement* is written $\Gamma\,;\,\Theta \vdash M : \sigma$ where Θ is a type assignment for individual variables of the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$. It is assumed here that $\Gamma \vdash \sigma_i$ $(i = 1, \ldots, n)$ and $\Gamma \vdash \sigma$ hold. Term judgement has six inference rules.

(te proj) $\qquad \Gamma; \Theta \vdash x : \sigma \qquad (x : \sigma \in \Theta)$

(te const) $\qquad \Gamma; \Theta \vdash c : \text{Type}(c) \qquad (c \in \mathcal{C})$

where Γ is arbitrary, since Type(c) is assumed to be closed.

(te ⇒I)
$$\frac{\Gamma; \Theta, x : \sigma \vdash M : \tau}{\Gamma; \Theta \vdash \lambda x^{\sigma}.M : \sigma \Rightarrow \tau}$$

(te ⇒E)
$$\frac{\Gamma; \Theta \vdash M : \sigma \Rightarrow \tau \qquad \Gamma; \Theta \vdash N : \sigma}{\Gamma; \Theta \vdash MN : \tau}$$

(te ∀I)
$$\frac{\Gamma, X; \Theta \vdash M : \tau}{\Gamma; \Theta \vdash \Lambda X.M : \forall X.\tau}$$

where X does not appear as free type variables in Θ.

(te ∀E)
$$\frac{\Gamma; \Theta \vdash M : \forall X.\tau}{\Gamma; \Theta \vdash M\{\sigma\} : \tau[X{:=}\sigma]}$$

where σ is such that $\Gamma \vdash \sigma$ holds.

Four conversion rules (β), (η), (Type β) and (Type η) are defined as usual. We call $\Sigma = (\mathcal{K}, \mathcal{C})$ a *signature*. Let $\lambda^{\forall}(\Sigma)$ denote the second order lambda calculus generated by the inference rules above with the four conversion rules.

3 Parametric semantics

Polymorphism of second order lambda calculus is parametric polymorphism. $\forall X.\sigma$ is regarded to be a type of polymorphic data and the closed terms of the type have uniform definitions with parameter X. Then a problem is how to reflect uniformity (= parametricity) in semantics. Reynolds proposed in [20] an idea to use binary relations. In usual semantics [5], a variable type is interpreted as a function from the type domain (a set of sets) to itself. In Reynolds idea, a variable type has another interpretation as an endofunction on the set of binary relations between members of the type domain. Let F be such an interpretation of a variable type, and $\{a_X \in F(X) \mid X$ ranges over the type domain$\}$ a polymophic data. Reynolds asserted that a *parametric* polymorphic data should be subject to the condition $(a_A, a_B) \in F(r)$ for all r a subset of $A \times B$ (i.e., a binary relation from A to B). He introduced binary relations in trying to construct the full set-theoretic model [20] (but failed [21]). In the author's preceding paper, Reynolds parametricity is translated into BMM models [4, 5] to yield *relational models*. Adjoining binary relations makes difference. For example, Girard's coherence space model [9, 10] is not a BMM model (due to Moggi, see [6]), while it is a relational model [13]. We discuss below briefly the framework of relational model. A similar approach is developped by Wadler [22]. It also closely relates to the parametric HEO model by Bainbridge, Freyd, Scedrov and Scott [1, 13].

We are concerned with binary relations, for which identity relations are important but composition not. So we define an *r-frame* as the one obtained from category by removing the part involved in composition.

Definition 3.1 An *r-frame* is a pair of classes $(\mathcal{U}, \mathcal{R})$ with three maps $d_0, d_1 : \mathcal{R} \rightrightarrows \mathcal{U}$ (the domain and codomain maps) and $id : \mathcal{U} \to \mathcal{R}$ (identity map). $\mathcal{U}$ is the class of *objects*, and $\mathcal{R}$ is the class of *(binary) relations*.

R-frames are denoted by bold capital letters, as $\mathbf{T} = (\mathcal{U}, \mathcal{R})$. The notation $\mathrm{Obj}(\mathbf{T})$ and $\mathrm{Rel}(\mathbf{T})$ are used to indicate $\mathcal{U}$ and $\mathcal{R}$ respectively. For $A, B \in \mathcal{U}$, a binary relation r from A to B (i.e., $d_0(r) = A$, $d_1(r) = B$) is denoted as $r : A \multimap B$.

Definition 3.2 An *r-frame morphism* $F : \mathbf{T} \to \mathbf{T}'$ is a pair of functions $F : \mathrm{Obj}(\mathbf{T}) \to \mathrm{Obj}(\mathbf{T}')$ and $F : \mathrm{Rel}(\mathbf{T}) \to \mathrm{Rel}(\mathbf{T}')$ (both denoted by F) such that if $r : A \multimap B$ then $F(r) : F(A) \multimap F(B)$ (in other words, F preserves d_0 and d_1).

Note that it is not required for F to preserve id. An n-ary r-frame morphism is similarly defined. The case of $n = 0$ is worth being mentioned. Such an r-frame morphism is $F : \mathbf{1} \to \mathbf{T}$ where $\mathbf{1}$ is the r-frame consisting of one object 1 and one relation $id_1 : 1 \multimap 1$. Namely F is a pair of $A \in \mathrm{Obj}(\mathbf{T})$ and a relation $r : A \multimap A$ in $\mathrm{Rel}(\mathbf{T})$.

Definition 3.3 A *relation of r-frame morphisms* $q : F \multimap G$ for $F, G : \mathbf{T} \rightrightarrows \mathbf{T}'$ is a function $q : \mathrm{Rel}(\mathbf{T}) \to \mathrm{Rel}(\mathbf{T}')$ such that if $r : A \multimap B$ then $q(r) : F(A) \multimap G(B)$.

Note that the definition of q depends only on the object function part of F and G. An *identity relation of r-frame morphisms* $id_F : F \multimap F$ is defined by $id_F = F$.

Now we turn to defining relational model.

A type domain is an r-frame $\mathbf{T} = (\mathcal{U}, \mathcal{R})$. For each $n \leq 0$, $[\mathbf{T}^n \to \mathbf{T}]$ is an r-frame such that $\mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ is a class of r-frame morphisms $F : \mathbf{T}^n \to \mathbf{T}$ ($\mathbf{T}^0 = \mathbf{1}$) and $\mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ is a class of relations of r-frame morphisms $q : F \multimap G$ where

$F, G \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$. In order to interpret implication and universal types, we should be given two r-frame morphisms

$$\Rightarrow: \mathbf{T}^2 \to \mathbf{T}$$
$$\forall : [\mathbf{T} \to \mathbf{T}] \to \mathbf{T}\,,$$

and one function $I : \mathcal{K} \to \mathcal{U}$ (but for simplicity, we write just A for $I(A)$) to interpret constant types. From the $\forall$ above, we obtain $\tilde{\forall} : [\mathbf{T} \to \mathbf{T}] \to [\mathbf{T}^0 \to \mathbf{T}]$ as $\tilde{\forall} F = (\forall F, \forall id_F : \forall F \twoheadrightarrow \forall F)$ for $F \in \mathrm{Obj}\,[\mathbf{T} \to \mathbf{T}]$, and $\tilde{\forall} q = \forall q$ for $q \in \mathrm{Rel}\,[\mathbf{T} \to \mathbf{T}]$.

A type judgement is interpreted as an object of $[\mathbf{T} \to \mathbf{T}]$ ($n = |\Gamma|$).

(Ty proj) $[\![\Gamma \vdash X_i]\!]$ is the i-th projection.

(Ty const) $[\![\Gamma \vdash A]\!]$ is a constant r-frame morphism returning A for objects and id_A for relations

(Ty $\Rightarrow$) $[\![\Gamma \vdash \sigma \Rightarrow \tau]\!] = \Rightarrow \circ([\![\Gamma \vdash \sigma]\!], [\![\Gamma \vdash \tau]\!])$.

(Ty $\forall$) $[\![\Gamma \vdash \forall X.\sigma]\!] = \tilde{\forall}[\![\Gamma, X \vdash \sigma]\!]$.

For all types to be interpreted well, it is necessary that $[\mathbf{T}^n \to \mathbf{T}]$ ($n \leq 0$) is closed under projections, weakenings (i.e., adjunction of dummy variables), $\Rightarrow$, $\forall$, and composition. For the detail, see [13].

To each $A \in \mathcal{U}$, a set (possibly empty) D_A is associated, and to each $r : A \twoheadrightarrow B$ is associated a subset $D_r \subseteq D_A \times D_B$. For the present purpose, however, it suffices to regard A and D_A, r and D_r to be identical. Accordingly each $F \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ and $q \in \mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ have natural meanings as functions sending sets (of the form D_A) to sets and set-theoretic binary relations (of the form D_r) to set-theoretical binary relations.

The following term interpretation is the same as the counterpart of BMM interpretation [5, 4]. We associate *expansion functions*

$$\Phi^{\Rightarrow}_{A,B} : A \Rightarrow B \to (A \to B) \qquad \text{to each } A, B \in \mathcal{U}$$
$$\Phi^{\forall}_F : \forall F \to \Pi(X \in \mathcal{U})F(X) \quad \text{to each } F \in \mathrm{Obj}\,[\mathbf{T} \to \mathbf{T}]$$

where $(A \to B)$ is the set of functions from A to B and $\Pi(X \in \mathcal{U})F(X)$ is the collection of $\mathcal{U}$-indexed families such that the component of index X belongs to $F(X)$. We also assume a function I which assigns to each $c \in \mathcal{C}$ an element of $[\![\vdash \mathrm{Type}(c)]\!]$. For simplicity, $I(c)$ is written c in most cases.

A term judgement $\Gamma\,;\Theta \vdash M : \sigma$ is interpreted as a member of $\Pi(\underline{X} \in \mathcal{U}^{|\Gamma|})([\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{X}) \to [\![\Gamma \vdash \sigma]\!](\underline{X}))$ (we use underline to denote a sequence). $[\![\Gamma \vdash \sigma]\!]$ is recursively defined but $[\![\Gamma\,; \Theta \vdash M : \sigma]\!]$ is not. Instead the latter should be subject to a condition. See for the detail [4, p.93][13] (a little different with that in [5]).

Definition 3.4 A *second order relational interpretation* of $\lambda^{\forall}(\Sigma)$ (or *interpretation* in short) is a tuple

$$\xi = (\mathbf{T}, [\mathbf{T}^n \to \mathbf{T}](n \leq 0), \Rightarrow, \forall, D, \Phi^{\Rightarrow}, \Phi^{\forall}, I, [\![\,]\!]) .$$

In an interpretation, all types and terms are given meanings by $[\![\,]\!]$. However $[\![\,]\!]$ does not always respect conversions. Namely even if $M \rhd N$, it does not necessarily hold that $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!] = [\![\Gamma\,;\,\Theta \vdash N : \sigma]\!]$. It is easy to overcome this defect.

Definition 3.5 A *second order relational model* of $\lambda^{\forall}(\Sigma)$ (or in short *model*) is an interpretation in which all components of $\Phi^{\Rightarrow}$ and of $\Phi^{\forall}$ are one-to-one.

In a model, $[\![\,]\!]$ is uniquely determined by the other data, and it respects conversions. But the interpretation is still important notion since the collection of closed types and terms is an interpretation but not a model.

An ordinary BMM interpretation is a special case of relational interpretation. We have only to ignore the binary relation part. Formally let $\mathcal{R}$ consist of only identity relations id_A for $A \in \mathcal{U}$, and all $F \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$, $q \in \mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ send identity relations to identity relations.

If $\Rightarrow : \mathbf{T}^2 \to \mathbf{T}$ and $\forall : [\mathbf{T} \to \mathbf{T}] \to \mathbf{T}$ have canonical meanings as follows (logical, in other terms, an analogue of second order logical relation [17]), then we say ξ is a natural interpretation.

Definition 3.6 (i) $\Phi^{\Rightarrow}$ is *natural* if, for $r : A \rightarrow\!\!\!\!\Rightarrow A'$ and $s : B \rightarrow\!\!\!\!\Rightarrow B'$, the relation $r \Rightarrow s$ is defined as $r \Rightarrow s : f \mapsto\!\!\!\!\Rightarrow f'$ iff $s : \Phi^{\Rightarrow}_{A,B}(f)(a) \mapsto\!\!\!\!\Rightarrow \Phi^{\Rightarrow}_{A',B'}(f')(a')$ for any $r : a \mapsto\!\!\!\!\Rightarrow a'$. (N.B. we write $r : a \mapsto\!\!\!\!\Rightarrow a'$ for $(a, b) \in r$.)

(ii) $\Phi^{\forall}$ is *natural* if, for $q : F \rightarrow\!\!\!\!\Rightarrow F'$ in $\mathrm{Rel}\,[\mathbf{T} \to \mathbf{T}]$, the relation $\forall q : \forall F \rightarrow\!\!\!\!\Rightarrow \forall F'$ is defined as $\forall q : a \mapsto\!\!\!\!\Rightarrow a'$ iff $q(r) : \Phi^{\forall}_F(a)(A) \mapsto\!\!\!\!\Rightarrow \Phi^{\forall}_{F'}(a')(A')$ for any $r : A \rightarrow\!\!\!\!\Rightarrow A'$.

(iii) An interpretation ξ is *natural* if both $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are natural.

The next theorem by Reynolds is the principal theorem [20, 22, 13].

Theorem 3.7 (Abstraction Theorem) *Let ξ be a natural interpretation and suppose that for every $c \in C$ there holds $[\![\vdash \mathrm{Type}(c)]\!] : c \mapsto\!\!\!\!\Rightarrow c$. Then, for any $\Gamma\,;\,\Theta \vdash M : \sigma$ and n relations $\underline{r} : \underline{A} \rightarrow\!\!\!\!\Rightarrow \underline{B}$ ($n = |\Gamma|$), if $[\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{r}) : \underline{a} \mapsto\!\!\!\!\Rightarrow \underline{a'}$ then there holds*

$$[\![\Gamma \vdash \sigma]\!](\underline{r}) : b \mapsto\!\!\!\!\Rightarrow b'$$

where $b = [\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{A})(\underline{a})$ and $b' = [\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{A'})(\underline{a'})$.

Finally we define semantical parametricity.

Definition 3.8 (i) $a \in \forall F$ is *parametric* if, for any $r : A \twoheadrightarrow B$, there holds

$$F(r) : \Phi^{\forall}_{F}(a)(A) \mapsto \Phi^{\forall}_{F}(a)(B)\,.$$

(Using $[\![\,]\!]$, we can write it $F(r) : [\![a\{A\}]\!] \mapsto [\![a\{B\}]\!]$.)

(ii) $\forall F$ is *parametric* if all $a \in \forall F$ is pararmetric.

(iii) An interpretation ξ is parametric if all $\forall F$ is parametric.

In a natural model ξ such that all $F \in \mathrm{Obj}\,[\mathbf{T} \to \mathbf{T}]$ preserves identities, we can characterize parametricity by the assetion that ξ is parametric iff $\forall : [\mathbf{T} \to \mathbf{T}] \to \mathbf{T}$ preserves identities (i.e., $\forall id_F = id_{\forall F}$). The following theorem [20] that is valid only for parametric natural models is used below.

Theorem 3.9 (Identity Extension Lemma) *Let ξ be a parametric natural model. Then every syntactically defined r-frame morphism $[\![\Gamma \vdash \sigma]\!]$ preserves identity relations.*

4 Polymorphic extensional collapse of pretheories

As is well-known, the collection of closed types/terms of second order lambda calculus is a BMM interpretation but not a BMM model [4]. Polymorphic extensional collapse by Breazu-Tannen and Coquand [4] gives a method to obtain a BMM model from the closed types/terms collection. In this section, we show every syntactically defined universal type is parametric for extensionally collapsed models of the closed types/terms collection.

First we construct a BMM interpretation $\xi_{\Sigma,E}$ from a pretheory E of closed terms of $\lambda^{\forall}(\Sigma)$. Next we give an extensionally collapsed model $Coll(\xi, \mathcal{P})$ for a logical per collection $\mathcal{P}$ as in [4], but this time we adjoin binary relations to $Coll(\xi, \mathcal{P})$ so that it gives a natural relational model. Then it is proved that every syntactically defined universal type is parametric. The result is, in fact, an immediate consequence of Abstraction Theorem.

A *pretheory* E is an equivalence relation of closed terms of $\lambda^{\forall}(\Sigma)$ such that $M\,E\,N$ implies that M and N have the same closed type, and is subject to the condition

(1) E respects (β) and (Type-β) conversions

(2) E is congruent w.r.t. application and Type application.

(1) means that if M' is obtained from M by reducing some β (or Type-β) redex in M then $M\,E\,M'$, and (2) means that if $M\,E\,M'$ of type $\sigma \Rightarrow \tau$ and $N\,E\,N'$ of type σ then $(MN)\,E\,(M'N')$, and that if $M\,E\,M'$ of type $\forall X.\sigma$ then $(M\{\tau\})\,E\,(M'\{\tau\})$ for any closed type τ. Of course α-conversion is dealt implicitly. Let $[M]$ denote the equivalence class of M modulo E.

This definition is different with that in [18] in respect that our pretheory does not necessarily respect η and Type-η conversions.

A *theory* is a pretheory E such that there is a model ξ and $M\,E\,N$ is defined by $[\![M]\!] = [\![N]\!]$.

We have a straightforward construction of a BMM interpretation $\xi_{\Sigma,E}$ by collecting all the closed types and closed terms and dividing them by E. It is formalized as follows.

- $\mathcal{U}$ is the set of closed types (modulo the names of bound type variables). For each $\sigma \in \mathcal{U}$, D_σ is the set of $[M]$ where M is such that $; \vdash M : \sigma$.
- $\mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ (i.e., $\mathcal{U}^n \to \mathcal{U}$) is the set of type judgements $\Gamma \vdash \sigma$ ($|\Gamma| = n$) modulo the names of bound and/or free type variables. For $\sigma_1, \ldots, \sigma_n \in \mathcal{U}^n$, $(\Gamma \vdash \tau)(\underline{\sigma})$ is defined to be $\tau[\Gamma := \underline{\sigma}]$.
- $\Rightarrow (\sigma, \tau) = \sigma \Rightarrow \tau$, and $\forall(X \vdash \sigma) = \forall X.\sigma$.
- $\Phi^{\Rightarrow}_{\sigma,\tau}([M])$ is a function sending $[N]$ to $[MN]$. $\Phi^{\forall}_{X\vdash\sigma}([M])$ is a function sending τ to $[M\{\tau\}]$.
- $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{\tau})(\underline{[N]})$ is defined to be $[M[\Gamma := \underline{\tau}][\Theta := \underline{N}]]$.

Well-definedness of $\Phi^{\Rightarrow}$ (and $\Phi^{\forall}$) follows from the assumption that E is congruent w.r.t. application (Type application).

Lemma 4.1 $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]$ *is well-defined.*

(*Proof*) We must check that if $\underline{N}\,E\,\underline{N'}$ then

$$[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{\tau})(\underline{N}) = [\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{\tau})(\underline{N'})\,.$$

Since E is reflexive, $(\Lambda \underline{X} \lambda \underline{x}^\sigma M)\,E\,(\Lambda \underline{X} \lambda \underline{x}^\sigma M)$. where $\Gamma = \underline{X}$ and $\Theta = \underline{x : \sigma}$. Then use (Type) application and (Type-) β conversions to show

$$(M[\Gamma := \underline{\tau}][\Theta := \underline{N}])\,E\,(M[\Gamma := \underline{\tau}][\Theta := \underline{N'}])\,. \qquad \square$$

We must also check that $[\![\Gamma \vdash \sigma]\!]$ and $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]$ fulfill their required conditions. We omit the detail.

Theorem 4.2 $\xi_{\Sigma,E}$ *is a BMM interpretation.*

Remark 4.3 If E satisfies the ω rule

- if $(MN)\,E\,(M'N')$ for all $N\,E\,N'$ then $M\,E\,M'$, and
- if $(M\{\tau\})\,E\,(M\{\tau'\})$ for all τ then $M\,E\,M'$,

then $\xi_{\Sigma,E}$ is a model (i.e., all components of $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are all one-to-one).

Next we construct a relational model $Coll(\xi, \mathcal{P})$ from any BMM interpretation ξ and logical per collection $\mathcal{P}$. We use the notation $(\text{-})^o$ to distinguish the constructions of the 'o'riginal interpretation from those of a model $Coll(\xi, \mathcal{P})$ to be defined. For example, $\mathcal{U}^o$, $\Rightarrow^o$, etc.

Definition 4.4 ([4]) Suppose given a signature Σ and a BMM interpretation ξ. A *logical per collection* $\mathcal{P}$ is a pair $(\mathcal{P}^U, \mathcal{P}^K)$ where

- $\mathcal{P}^U$ is a $\mathcal{U}$-indexed family $\{\mathcal{P}_A \mid A \in \mathcal{U}^o\}$ where $\mathcal{P}_A$ is a set of pers on D^o_A such that
  - if $R \in \mathcal{P}_A$ and $S \in \mathcal{P}_B$ then $R \Rightarrow^p S \in \mathcal{P}_{A \Rightarrow^o B}$ where $R \Rightarrow^p S$ is a per defined as $f\,(R \Rightarrow^p S)\,f'$ iff $\Phi^{\Rightarrow^o}_{A,B}(f)(a)\;S\;\Phi^{\Rightarrow^o}_{A',B'}(f')(a')$ for any $a\,R\,a'$.
  - suppose Q is a function sending $R \in \mathcal{P}_A$ to $Q(R) \in \mathcal{P}_{FA}$ where F is an object of $[\mathbf{T} \to \mathbf{T}]^o$. Then $\forall^p Q \in \mathcal{P}_{\forall^o F}$ where $\forall^p Q$ is a per defined as $f\,(\forall^p Q)\,f'$ iff $\Phi^{\forall^o}_F(f)(A)\;(Q(R))\;\Phi^{\forall^o}_F(f')(A)$ for any $R \in \mathcal{P}_A$.
- $\mathcal{P}^K$ is a function assigning a per in $\mathcal{P}_A$ to each $A \in \mathcal{K}$.

Note there exists at least one logical per collecion, namely the collection of all pers with arbitrary $\mathcal{P}^K$.

Our construction of an extensionally collapsed model $Coll(\xi, \mathcal{P})$ follows [4], but with adjoined binary relations. Note that in the construction below binary relations have no effect in the object part (i.e., the part involved in BMM models). Now we begin with the definition of $Coll(\xi, \mathcal{P})$.

- $\langle A, R\rangle$ belongs to $\mathcal{U}$ iff $A \in \mathcal{U}^o$ and $R \in \mathcal{P}_A$. $D_{\langle A,R\rangle}$ is the subquotient set A/R. For $a \in Dom(R)$ (i.e., $a\,R\,a$), let $[a]_R$ denote an equivalence class of a.
- $r : \langle A, R\rangle \twoheadrightarrow \langle B, S\rangle$ belongs to $\mathcal{R}$ iff it is a subset of $A/R \times B/S$.
- $\langle F, Q\rangle$ is an object of $[\mathbf{T}^n \to \mathbf{T}]$ iff $F \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]^o$ and Q is a pair of functions (Q_p, Q_r) where Q_p sends $R_i \in \mathcal{P}_{A_i}$ $(i = 1, \ldots, n)$ to $Q_p(\underline{R}) \in \mathcal{P}_{F(\underline{A})}$, and Q_r sends $r_i : \langle A_i, R_i\rangle \twoheadrightarrow \langle B_i, S_i\rangle$ $(i = 1, \ldots, n)$ to $Q_r(\underline{r}) : \langle F(\underline{A}), Q_p(\underline{R})\rangle \twoheadrightarrow \langle F(\underline{B}), Q_p(\underline{S})\rangle$. The behaviour of $\langle F, Q\rangle$ is determined by

$$\langle F, Q\rangle(\underline{\langle A, R\rangle}) \stackrel{def}{=} \langle F(\underline{A}), Q_p(\underline{R})\rangle$$
$$\langle F, Q\rangle(\underline{r}) \stackrel{def}{=} Q_r(\underline{r})\,.$$

- $p : \langle F, Q\rangle \twoheadrightarrow \langle F', Q'\rangle$ belongs to $\mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ iff p sends $r_i : \langle A_i, R_i\rangle \twoheadrightarrow \langle A'_i, R'_i\rangle$ $(i = 1, \ldots n)$ to $p(\underline{r}) : \langle F(\underline{A}), Q_p(\underline{R})\rangle \twoheadrightarrow \langle F'(\underline{A'}), Q'_p(\underline{R'})\rangle$ (note that p is not involved in Q_r and Q'_r).

- $\Rightarrow: \mathbf{T}^2 \to \mathbf{T}$ is determined by; for objects, $\langle A, R\rangle \Rightarrow \langle B, S\rangle \stackrel{def}{=} \langle A \Rightarrow^o B, R \Rightarrow^p S\rangle$; for relations $r : \langle A, R\rangle \twoheadrightarrow \langle A', R'\rangle$ and $s : \langle B, S\rangle \twoheadrightarrow \langle B', S'\rangle$, the relation $r \Rightarrow s$ is defined as $r \Rightarrow s : [f]_{R \Rightarrow^p S} \mapsto [f']_{R' \Rightarrow^p S'}$ iff $s : [\Phi^{\Rightarrow^o}_{A,B}(f)(a)]_S \mapsto [\Phi^{\Rightarrow^o}_{A',B'}(f')(a')]_{S'}$ for any $r : [a]_R \mapsto [a']_{R'}$. It is well-defined by the definition of $\Rightarrow^p$.

- $\forall : [\mathbf{T} \to \mathbf{T}] \to \mathbf{T}$ is determined by; for an object, $\forall\langle F, Q\rangle \stackrel{def}{=} \langle \forall^o F, \forall^p Q_p\rangle$; and for a relation $p : \langle F, Q\rangle \twoheadrightarrow \langle F', Q'\rangle$, the relation $\forall p$ is defined as $\forall p : [f]_{\forall^p Q_p} \mapsto [f']_{\forall^p Q'_p}$ iff $p(r) : [\Phi^{\forall^o}_F(f)(A)]_{Q_p(R)} \mapsto [\Phi^{\forall^o}_{F'}(f')(A')]_{Q'_p(R')}$ for any $r : \langle A, R\rangle \twoheadrightarrow \langle A', R'\rangle$. Well-definedness follows from the definition of $\forall^p(\text{-})$.

- $\Phi^{\Rightarrow}_{\langle A,R\rangle\langle B,S\rangle}([f]_{R\Rightarrow^p S})$ is a function sending $[a]_R$ to $[\Phi^{\Rightarrow^o}_{A,B}(f)(a)]_S$.

- $\Phi^{\forall}_{\langle F,Q\rangle}([f]_{\forall^p(Q_p)})$ is a function sending $\langle A, R\rangle$ to $[\Phi^{\forall^o}_F(f)(A)]_{Q_p(R)}$.

- $I(A) \stackrel{def}{=} \langle A, \mathcal{P}^K(A)\rangle$ and $I(c) = [c]_R$ where R is the per part of $[\![\vdash \mathrm{Type}\,\Theta]\!]$. Remark that $I(c)$ makes sense only in case there holds $c\,R\,c$. In this case we say c is *self-related* in $\mathcal{P}$.

- $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!](\underline{\langle A, R\rangle})(\underline{[a]}_{[\![\Gamma\vdash\mathrm{Type}(a)]\!](\underline{R})})$ is defined to be $[[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]^o(\underline{A})(\underline{a})\,]_{[\![\Gamma\vdash\sigma]\!](\underline{R})}$ where $[\![\Gamma \vdash \sigma]\!](\underline{R})$ is the per part of $[\![\Gamma \vdash \sigma]\!](\underline{\langle A, R\rangle})$.

Lemma 4.5 *(i)* $\Phi^{\Rightarrow}_{\langle A,R\rangle\langle B,S\rangle}$ *and* $\Phi^{\forall}_{\langle F,Q\rangle}$ *are all one-to-one.*

(ii) $\Phi^{\Rightarrow}$ *and* $\Phi^{\forall}$ *are natural.*

(*Proof*) Immediate from definitions. □

Well-definedness of $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]$ (in case all individual constant is self-related in $\mathcal{P}$) is proved as in [4], or by Abstraction Theorem remarking that $\Rightarrow^p$ and $\forall^p$ is defined so that $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are natural w.r.t. pers in $\mathcal{P}$.

What remains to check is the conditions for $[\![\Gamma \vdash \sigma]\!]$ and $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]$. They are satisfied, in fact, inheriting the conditions $[\![\Gamma \vdash \sigma]\!]^o$ and $[\![\Gamma\,;\,\Theta \vdash M : \sigma]\!]^o$ fulfill.

Theorem 4.6 *If all individual constant is self-related in* $\mathcal{P}$, *then* $Coll(\xi, \mathcal{P})$ *is a natural model.*

Now let us focus on polymorphic extensional collapse of closed types/terms interpretation $Coll(\xi_{\Sigma,E}, \mathcal{P})$. We say $c \in C$ is *self-related in* $\mathcal{R}$ iff $[\![\vdash \mathrm{Type}(c)]\!] : c \mapsto c$.

Theorem 4.7 *Consider* $Coll(\xi_{\Sigma,E}, \mathcal{P})$ *and suppose all individual constant is self-related in* $\mathcal{R}$. *All syntactically defined universal type* $[\![\vdash \forall X.\sigma]\!]$ *is parametric.*

(*Proof*) Since $Coll(\xi_{\Sigma,E}, \mathcal{P})$ is a natural model, Abstraction Theorem asserts that, for any $\vdash M : \forall X.\sigma$, there holds $[\![\vdash \forall X.\sigma]\!] : [\![M]\!] \mapsto [\![M]\!]$, that is, $[\![M]\!]$ is parametric. But any member of $[\![\vdash \forall X.\sigma]\!]$ have the form $[\![M]\!]$ for some closed term $M : \forall X.\sigma$. Therefore $[\![\vdash \forall X.\sigma]\!]$ is parametric. □

There are many syntactically undefinable universal types $\forall\langle F, Q\rangle$. A question is whether all $\forall\langle F, Q\rangle$ is parametric or there is any counterexample.

Remark 4.8 If the signature Σ is empty, the theorem can be extended a little. For example, $[\![\forall X(((A,R\rangle \Rightarrow \langle B,S\rangle \Rightarrow X) \Rightarrow X)]\!]$ is parametric for any $\langle A,R\rangle$ and $\langle B,S\rangle$. Every element of the type has the form $\Lambda X \lambda y^{\langle A,R\rangle \Rightarrow \langle B,S\rangle \Rightarrow X} y[M]_R[N]_S$. Hence $[M]_R$ and $[N]_S$ can be treated as individual constants of constant types $\langle A,R\rangle$ and $\langle B,S\rangle$. Similar arguments show that $[\![\forall X((\langle A,R\rangle \Rightarrow X) \Rightarrow ((\langle B,S\rangle \Rightarrow X) \Rightarrow X)]\!]$ is parametric.

5 Second order minimum model

Moggi and Statman discovered a construction of maximum consistent theory [18]. We denote it by E_m. The theory is interesting because of its satisfying the ω rule. Hence the corresponding BMM interpretation ξ_{ϕ,E_m} (ϕ is the empty signature) turns into a BMM model. Indeed ξ_{ϕ,E_m} is easily lifted into a parametric natural model. In this section we assume the signature is always empty

A pretheory is *consistent* iff it is *not* the case that $\texttt{true}\, E\, \texttt{false}$ where $\texttt{true} = \Lambda X \lambda x^X \lambda y^X x$ and $\texttt{false} = \Lambda X \lambda x^X \lambda y^X y$ of type $\texttt{Bool} = \forall X(X \Rightarrow X \Rightarrow X)$. All pretheory but one is consistent. Indeed if $\texttt{true}\, E\, \texttt{false}$ then any two closed terms of the same type are equivalent modulo E.

The set of all consistent pretheories has the structure of a lattice w.r.t. inclusion [18]. There is a maximal element of the lattice, called the *maximum consistent theory*. It is characterized as follows.

Definition 5.1 E_m is a pretheory defined as $M\, E_m\, N$ (M, N of type σ) iff, for all closed term K of type $\sigma \Rightarrow \texttt{Bool}$, there holds $KM =_{\beta\eta} KN$.

It is easily checked that E_m is actually a pretheory. And if we note that $\texttt{true}$ and $\texttt{false}$ are the only $\beta\eta$-normal closed terms of type $\texttt{Bool}$, and at the same time the only β-normal closed terms, then the following proposition is easily proved.

Proposition 5.2 *E_m is maximal in the lattice of consistent pretheories.*

We can infer, from the following proposition [18], that E_m is a theory.

Proposition 5.3 *E_m satisfies the ω rule:*

(i) for $M, M' : \sigma \Rightarrow \tau$, if $MN \, E_m \, M'N$ for all $N : \sigma$ then $M \, E_m \, M'$.

(ii) for $M, M' : \forall X.\sigma$, if $M\{\tau\} \, E_m \, M'\{\tau\}$ for all τ, then $M \, E_m \, M'$.

(*Proof*) Let E' be the theory of $Coll(\xi_{\phi,E_m}, All)$ where All is the collection of all pers. By the construction of polymorphic extensional collapse, $E_m \subseteq E'$. Since E_m is maximal, $E_m = E'$. Then (i) follows by replacing E_m by E'. As for (ii), if $M\{\tau\} \, E_m \, M'\{\tau\}$ for all τ, then $M \, E' \, M'$ by the construction of $Coll(\xi_{\phi,E_m}, All)$. (A direct proof is also possible.) □

By the ω rule, ξ_{ϕ,E_m} is a BMM model, called the *second order minimum model.* A quesion is whether there are any other consistent theories satisfying the ω rule. For those theories, if any, the following argument is valid.

Here we adjoin binary relations to ξ_{ϕ,E_m} so that it is to be a relational model. The additional structure is given as follows.

- $r : \sigma \twoheadrightarrow \tau$ iff $r \subseteq D_\sigma \times D_\tau$. $\mathcal{R}$ is the collection of such r's.
- $p : (\Gamma \vdash \tau) \twoheadrightarrow (\Gamma \vdash \tau')$ belongs to $\mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ iff p sends $\underline{r} : \underline{\sigma} \twoheadrightarrow \underline{\sigma}'$ to $p(r) : \tau[\Gamma := \underline{\sigma}] \twoheadrightarrow \tau'[\Gamma := \underline{\sigma}']$.
- For $r : \sigma \twoheadrightarrow \sigma'$ and $s : \tau \twoheadrightarrow \tau'$, the relation $r \Rightarrow s$ is defined as $r \Rightarrow s : [M] \mapsto [M']$ $(M : \sigma \Rightarrow \tau,\ M' : \sigma' \Rightarrow \tau')$ iff $s : [MN] \mapsto [M'N']$ for any $r : [N] \mapsto [N']$.
- For $p : (X \vdash \sigma) \twoheadrightarrow (X \vdash \sigma')$, the relation $\forall p$ is defined as $\forall p : [M] \mapsto [M']$ $(M : \forall X.\sigma,\ M' : \forall X.\sigma')$ iff $p(r) : [M\{\tau\}] \mapsto [M'\{\tau'\}]$ for any $r : \tau \twoheadrightarrow \tau'$.

Since all $F \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ is syntactically definable, the behaviour of F for relation arguments is completely determined by $r \Rightarrow s$ and $\forall p$. Note that $r \Rightarrow s$ and $\forall p$ are defined so that $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are natural. Then the next theorem is immediate.

Theorem 5.4 *ξ_{ϕ,E_m} (with the additional structure above) is a parametric natural model.*

(*Proof*) For parametricity, all universal type is syntactically defined and its elements are all interpretations of some closed terms of the type. Then parametricity follows from Abstraction Theorem. □

6 Categorical properties

An interesting feature of parametric model is about categorical data types. Prawitz showed [19] that in intuitionistic second order logic implication and universal quantification represent other logical connectives, as

$$
\begin{aligned}
\bot &= \forall X.X \\
A \wedge B &= \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X) \\
A \vee B &= \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) \\
\exists X.F(X) &= \forall Y(\forall X(F(X) \Rightarrow Y) \Rightarrow Y).
\end{aligned}
$$

It is showed in [13] that if a model ξ has enough binary relations and is a parametric natural model then in the categorya $\mathbf{C}_\xi$ generated by ξ (see below) the representations of Prawitz have natural meanings. For example, $\bot$ is an initial object, $A \wedge B$ is a product of A and B, and $A \vee B$ is a coproduct of A and B.

Definition 6.1 Let ξ be a model. A category $\mathbf{C}_\xi$ is defined as $\mathrm{Obj}(\mathbf{C}_\xi) = \mathcal{U}$, $\mathbf{C}_\xi = A \Rightarrow B$, $1_A = [\![\lambda x^A x]\!]$ and $g \circ f = [\![\lambda x^A g(fx) : A \Rightarrow C]\!]$ ($f : A \to B$ and $g : B \to C$).

To each arrow $f \in A \Rightarrow B$, we associate a binary relation $|f| : A \multimap B$, a *graph* of f, defined as $|f| : a \mapsto b$ iff $b = \Phi^{\Rightarrow}_{A,B}(f)(a)$. A model ξ *has enough relations* iff all graphs $|f|$ belong to $\mathcal{R}$. The following theorem is a general property of a natural model which has enough relations.

Theorem 6.2 *Suppose ξ is a natural model which has enough binary relations. Then, in the category $\mathbf{C}_\xi$,*

(i) if $\bot = \forall X.X$ is parametric, $\bot$ is an initial object.

(ii) if $\top = \forall X(X \Rightarrow X)$ is parametric, then $\top$ is a terminal object.

(iii) if $A + B = \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$ is parametric, then $A + B$ is a coproduct of A and B.

(iv) if $A \times B = \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X)$ is parametric, then $A \times B$ is a product of A and B. Moreover if $A \times B$ is parametric for all A then adjunction $(\text{-}) \times B \dashv B \Rightarrow (\text{-})$ holds.

(N.B. we write, for example, simply $\forall X.X$ for $[\![\forall X.X]\!]$.)

Since ξ_{ϕ,E_m} has enough relations (in fact, it has all binary relations) and parametric (Theorem 5.4), the above theorem is applied for ξ_{ϕ,E_m}. As for $Coll(\xi_{\Sigma,E}, \mathcal{P})$, if the signature Σ is empty, by the remark following Theorem 4.7, the above theorem is applied, too. In these cases, however, the theorem is proved also by directly analyzing the closed terms of the types.

See [13] for the proof of the theorem. In an appropriate setting, even the converse is proved. For example, if $\forall X.X$ is an initial object then it is parametric w.r.t. the collection of all graphs. The essence of the proof of the theorem is to use parametricity for graphs in place of commutativity of diagrams.

Other syntactically definable data types are initial and terminal fixed points of *universally strong* endofunctors.

Definition 6.3 $F \in \mathrm{Obj}\,[\mathbf{T} \to \mathbf{T}]$ is a *universally strong functor* iff both

(i) there is a *universal strength* $\overline{F} \in \forall X \forall Y((X \Rightarrow Y) \Rightarrow (FX \Rightarrow FY))$ so that $\overline{F}$ induces a functor $\mathbf{C}_\xi \to \mathbf{C}_\xi$, and

(ii) for $f \in A \Rightarrow B$, there holds $F(|f|) = |\overline{F}(f)|$ where $\overline{F}(f)$ means $[\![\overline{F}\{A\}\{B\}(f)]\!]$.

For an endofunctor $F : \mathbf{C} \to \mathbf{C}$, an initial object of a comma category $(F \downarrow Id)$ is called, if any, an *initial fixed point* of F, and a terminal object of $(Id \downarrow F)$ a *terminal fixed point* of F. Initial fixed points are used to encode algebraic types and terminal fixed points to encode some lazy stream types [3, 11, 23].

The following theorem is a general property of parametric natural models [13].

Theorem 6.4 *Suppose ξ is a natural model which has enough relations, and $F : \mathbf{C}_\xi \to \mathbf{C}_\xi$ is a universally strong functor. Then*

(i) if $\mu F = \forall X((FX \Rightarrow X) \Rightarrow X)$ is parametric, then μF is an initial fixed point of F.

(ii) if $\nu F = \exists X((X \Rightarrow FX) \times X)$ and $\forall X(((X \Rightarrow FX) \times X) \Rightarrow \nu F)$ are parametric, then νF is a terminal fixed point of F.

To apply the theorem to the minimum model and extensionally collapsed model, we should know what objects of $[\mathbf{T} \to \mathbf{T}]$ are universally strong functors.

For $\Gamma \vdash \sigma$ and $X \in \Gamma$, we say X is $+$ ($-$, 0) variant if X occurs at most positively (at most negatively, possibly both positively and negatively) in σ. Moreover we say, for example, $X, Y, Z \vdash \forall W((Y \Rightarrow X) \Rightarrow Z) \Rightarrow Y$ is $+0-$ variant where X, Y, Z are $+$, 0, $-$ variant respectively.

$X \vdash \sigma$ of $+$ variant is a candidate of a universally strong functor. In fact, is has a universal strength, constructed by induction on σ. But do not look over the second condition of Definition 6.3. For that, we need the next lemma and parametricity.

Lemma 6.5 *Let ξ be a natural model, $f \in A \Rightarrow A'$, and $r : B \rightarrow\!\!\!\!\Rightarrow B'$.*

(i) for $F = [\![X, Y \vdash \sigma]\!]$ where $X, Y \vdash \sigma$ is $+0$ variant,

$$F(id_A, r); |\overline{F}(f, 1_{B'})| \subseteq F(|f|, r) \subseteq |\overline{F}(f, 1_B)|; F(id_{A'}, r)\,.$$

(ii) for $F = [\![X, Y \vdash \sigma]\!]$ where $X, Y \vdash \sigma$ is -0 variant,

$$|\overline{F}(f, 1_B)|^{op}; F(id_{A'}, r) \subseteq F(|f|, r) \subseteq F(id_A, r); |\overline{F}(f, 1_{B'})|^{op}\,.$$

(N.B. $r^{op} : a \mapsto\!\!\!\!\Rightarrow b$ iff $r : b \mapsto\!\!\!\!\Rightarrow a$. And $r; s : a \mapsto\!\!\!\!\Rightarrow c$ iff there is b such that $r : a \mapsto\!\!\!\!\Rightarrow b$ and $s : b \mapsto\!\!\!\!\Rightarrow c$.)

Proposition 6.6 *If ξ is a parametric natural model, every $[\![X \vdash \sigma]\!]$ of $+$ variant is a universally strong functor.*

(*Proof*) Use the lemma above together with Identity Extension Lemma. □

Since the minimum model ξ_{ϕ,E_m} is a parametric natural model, for all $X \vdash \sigma$ of $+$ variant, $\forall X((\sigma \Rightarrow X) \Rightarrow X)$ and $\exists X((X \Rightarrow \sigma) \times X)$ are initial and terminal fixed points of $(X \vdash \sigma)$.

For $Coll(\xi_{\phi,E}, \mathcal{P})$, the situation is more subtle, because the model is only partially parametric (we shall assume the signature is empty ϕ). We must investigate Identity Extension Lemma for partially parametric models. A sufficient condition for $[\![\Gamma \vdash \sigma]\!]$ of $+$ variant to give a universally strong functor is that X is not captured in the scope of universal quanitifier. This covers, for example, the 'power set of power sets' functor $F = [\![X \vdash (X \Rightarrow \mathtt{Bool}) \Rightarrow \mathtt{Bool}]\!]$. Functors for defining algebraic types are also parametric. For example, the natural numbers object is an initial fixed point of $F = [\![X \vdash \top + X]\!]$ which is universally strong by the remark following Theorem 4.7.

Example 6.7 For $\xi = \xi_{\phi,E_m}$ and $= Coll(\xi_{\phi,E}, \mathcal{P})$, the following data types are initial (terminal) fixed points in the category $\mathbf{C}_\xi$.

(i) the type of ordinals [11]:

$$\mu X(\top + (\mathtt{Nat} \Rightarrow X))$$

where $\mathtt{Nat}$ is $\mu X(\top + X)$ the type of natural numbers.

(ii) the type of infinite lists of A [23]:

$$\nu X(A \times X).$$

Second order types also provide representation of internal right and left Kan extensions. It seems to be new, as far as the author knows, but is a straightforward analogue of the representation by ends [15]. What follows is applicable for the minimum model ξ_{ϕ,E_m}, but not clear for $Coll(\xi_{\phi,E}, \mathcal{P})$, since the represetations include parameters.

Definition 6.8 Let ξ be a parametric natural model. $\mathbf{Us}(\mathbf{C}_\xi)$ is a category defined as; $\mathrm{Obj}(\mathbf{Us}(\mathbf{C}_\xi))$ is the collection of all universally strong functors; $\mathbf{Us}(\mathbf{C}_\xi)(F, G)$ is defined to be $\forall X(FX \Rightarrow GX)$ together with

$$1_F = [\![\Lambda X \lambda x^{FX} x]\!]$$
$$f' \circ f = [\![\Lambda X \lambda x^{FX} f'\{X\}(f\{X\}x)]\!]$$

for $f : F \to F'$ and $f' : F' \to F''$.

$\mathbf{Us}(\mathbf{C}_\xi)$ is a $\mathbf{C}_\xi$-enriched category, It is also a subcategory of the functor category $\mathbf{C}_\xi^{\mathbf{C}_\xi}$, since all $f \in \mathbf{Us}(\mathbf{C}_\xi)(F, G)$ is a natural transformation by the following lemma.

Lemma 6.9 *Let F and G be universally strong functors. If $\forall X(FX \Rightarrow GX)$ is parametric, then every $f \in \forall X(FX \Rightarrow GX)$ is a natural transformation.*

Recall the definition of right Kan extension. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{S}$ be categories, and $K : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{C} \to \mathbf{S}$. A right Kan extension of T along K is a functor $\mathrm{Ran}_K T : \mathbf{D} \to \mathbf{S}$, such that adjunction at T

$$\mathbf{S}^{\mathbf{C}}((\text{-}) \circ K, T) \cong \mathbf{S}^{\mathbf{D}}((\text{-}), \mathrm{Ran}_K T)$$

holds where $\mathbf{S}^{\mathbf{C}}$ and $\mathbf{S}^{\mathbf{D}}$ are the functor categories. We say a universally strong functor $\mathrm{Ran}_K T : \mathbf{C}_\xi \to \mathbf{C}_\xi$ is an *internal right Kan extension* of T along K (K and T are both universally strong endofunctors on $\mathbf{C}_\xi$) if

$$\mathbf{Us}(\mathbf{C}_\xi)((\text{-}) \circ K, T) \cong \mathbf{Us}(\mathbf{C}_\xi)((\text{-}), \mathrm{Ran}_K T)$$

is a natural equivalence in (-) where the equivalence is an isomorphism in $\mathbf{C}_\xi$ (recall $\mathbf{Us}(\mathbf{C}_\xi)$ is a $\mathbf{C}_\xi$-enriched category). An internal left Kan extension $\mathrm{Lan}_K T$ is defined dually.

Theorem 6.10 *In $\mathbf{C}_\xi$ for ξ a parametric natural model (thus in particular for $\xi = \xi_{\phi, E_m}$),*

(i) $\mathrm{Ran}_K T = \forall X(((\text{-}) \Rightarrow KX) \Rightarrow TX)$ *gives an internal right Kan extension of T along K.*

(ii) $\mathrm{Lan}_K T = \exists X((KX \Rightarrow (\text{-})) \times TX)$ *gives an internal left Kan extension of T along K.*

Corollary 6.11 *Let ξ be a parametric natural model and $K : \mathbf{C}_\xi \to \mathbf{C}_\xi$ a universally strong functor.*

(i) If K preserves $\mathrm{Ran}_K 1_{\mathbf{C}_\xi}$, then K has an internal left adjoint $\forall X(((\text{-}) \Rightarrow KX) \Rightarrow X)$.

(ii) If K preserves $\mathrm{Lan}_K 1_{\mathbf{C}_\xi}$, then K has an internal right adjoint $\exists X((KX \Rightarrow (\text{-})) \times X)$.

$A \times B = \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X)$ is a special case of (i) by $K = B \Rightarrow (\text{-})$. There is no other case, however, as pointed out in [7], namely,

Corollary 6.12 *Let ξ be a parametric natural model.*

(i) Any universally strong functor K preserving $\mathrm{Ran}_K 1_{\mathbf{C}_\xi}$ is representable as

$$K \cong (\forall X((\top \Rightarrow KX) \Rightarrow X)) \Rightarrow (\text{-}).$$

(ii) Any universally strong functor K perserving $\mathrm{Lan}_K 1_{\mathbf{C}_\xi}$ is corepresentable as

$$K \cong (\exists X((KX \Rightarrow \top) \times X)) \times (\text{-}).$$

I conjecture that if we consider a full subcategory of $Coll(\xi_{\Sigma,E}, \mathcal{P})$ consisting of all syntactically defined objects, then all the theorems above are applied. But the detail has not yet been checked.

Remark 6.13 Huwig and Poigné showed [14] that some kinds of looping combinators cannot coexist with cartesian closedness. For example, if $\mathbf{2} = \mathbf{1} + \mathbf{1}$ exists in a non-degenerating cartesian closed category, no arrow $Y : (\mathbf{2} \Rightarrow \mathbf{2}) \to \mathbf{2}$ can be a looping combinator (i.e., an arrow such that for any $f : \mathbf{2} \to \mathbf{2}$, there holds $Y \circ \lceil f \rceil = f \circ Y \circ \lceil f \rceil : \mathbf{1} \to \mathbf{2}$). Hence in $\mathbf{C}_\xi$ where $\xi = \xi_{\phi,E_m}$ or $Coll(\xi_{\phi,E}, \mathcal{P})$, there is no looping combinator of $\mathbf{2}$. More generally we can show that, in any natural interpretation with full binary relations, if $Y \in (\mathtt{Bool} \Rightarrow \mathtt{Bool}) \Rightarrow \mathtt{Bool}$ is a looping combinator and $[\![\vdash (\mathtt{Bool} \Rightarrow \mathtt{Bool}) \Rightarrow \mathtt{Bool}]\!] : Y \mapsto Y$ then $[\![\mathtt{true}]\!] = [\![\mathtt{false}]\!]$ is derived.

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