DIRECTED EQUALITY WITH DINATURALITY

ANDREA LARETTO, FOSCO LOREGIAN, AND NICCOLÒ VELTRI

Abstract. We show how dinaturality plays a central role in the interpretation of directed type theory where types are interpreted as (1-)categories and directed equality is represented by hom-functors. We present a general elimination principle based on dinaturality for directed equality which very closely resembles the J-rule used in Martin-Löf type theory, and we highlight which syntactical restrictions are needed to interpret this rule in the context of directed equality. We then use these rules to characterize directed equality as a left relative adjoint to a functor between (para)categories of dinatural transformations which contracts together two variables appearing naturally with a single dinatural one, with the relative functor imposing the syntactic restrictions needed. We then argue that the quantifiers of such a directed type theory should be interpreted as ends and coends, which dinaturality allows us to present in adjoint-like correspondences to a weakening functor. Using these rules we give a formal interpretation to Yoneda reductions and (co)end calculus, and we use logical derivations to prove the Fubini rule for quantifier exchange, the adjointness property of Kan extensions via (co)ends, exponential objects of presheaves, and the (co)Yoneda lemma. We show transitivity (composition), congruence (functoriality), and transport (coYoneda) for directed equality by closely following the same approach of Martin-Löf type theory, with the notable exception of symmetry. We formalize our main theorems in Agda.

Contents

1. Introduction

Propositional equality is one of the most interesting aspects of Martin-Löf type theory. Its precise analysis gave birth to entire fields of research, starting from the seminal work of Hoffmann and Streicher [HS98] to the development of homotopy type theory [Uni13] and cubical type theory [CCHM15]. In these contexts, the inherently symmetric nature of equality is what enables types to be interpreted as (∞-)groupoids, where equality is precisely interpreted by morphisms which are always invertible. A natural question follows: can there be a variant of Martin-Löf type theory which enables types to be interpreted as *categories*, where morphisms need not be invertible? Such a system should take the name of *directed type theory*, where the directed aspect comes from a non-symmetric interpretation of "equality", which now naturally possesses both a source and a target in the same way that morphisms do in a category.

Directed type theory has been a sought-after goal of recent type theoretical research, with several attempts aimed at pinpointing precisely both its syntactic and semantic aspects. One can speculate how the traditional aspects of type theory should be transported in a system where propositional equality is directed: in the same way that terms in standard type theory respect equality, terms of directed type theory now should resemble functors between categories, which transport morphisms from one type to another whilst respecting the structure of the types. Since directed equality types model morphisms of a category (and often take the name of hom-types), composing directed equalities together can exhibit highly non-trivial behaviour; a similar situation arises in homotopy type theory, where the points in a generic type can have non-trivial equalities between them. Using a directed generalization of quotient inductive types [ACD+18], one should moreover be able to postulate the structure of a category by giving both the points (objects) along with the non-trivial equalities among them (morphisms), and then specify how these directed equalities interact with each other possibly using higher paths. When eliminating out of a quotient inductive type one must prove an obligation stating that the equalities are respected by the function; in directed type theory, this should correspond with the fact that when eliminating out of a category one must provide not only an action on objects, but also a functorial action on morphisms.

Polarity. Another fundamental aspect of the category interpretation of type theory is the fact that with each type (category) $\mathbb{C}$ there is a naturally associated type $\mathbb{C}^{\mathsf{op}}$, where the objects are the same but all directed equalities are reversed, which corresponds with the usual notion of opposite of a category. This idea allows us to make sense of the directed equality type $\hom_{\mathbb{C}}(a, b) : \mathsf{Set}$ as receiving a "contravariant" argument $a : \mathbb{C}^{\mathsf{op}}$ and a "covariant" one $b : \mathbb{C}$, mirroring the interpretation of the functor $\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$. A directed type theory should therefore have some notion of "*polarity*" or "*variance*" which allows variables to be distinguished and appear only in the appropriate position, as similarly treated in [LH11, NL23, Nor19].

Directed equality introduction. In standard type theory, each type former is typically characterized with an introduction and a corresponding elimination rule. The introduction rule for equality is typically given by the term constructor $\mathsf{refl}_a : a = a$, which expresses the reflexivity of symmetric equality. In a directed type theory where types are categories, this introduction rule should be motivated semantically with the fact that, for any point a in a type $\mathbb{C}$ (i.e., for any object in a category), the set $\hom_{\mathbb{C}}(a, a)$ is pointed with the identity id_a. However, naïvely stating this typing rule involves both a contravariant and a covariant occurrence of the same variable $a : \mathbb{C}$, and is thus not defined functorially with respect to the variance of $\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$:

$$\frac{a : \mathbb{C}}{\mathsf{refl}_a : \hom_{\mathbb{C}}(a, a)} \ (\textsf{hom-intro?})$$

One possible solution considered by North [Nor19] is to replace the category $\mathbb{C}$ with its maximal subgroupoid $\mathbb{C}^{\mathsf{core}}$ in order to collapse the two variances, since $(\mathbb{C}^{\mathsf{core}})^{\mathsf{op}} \cong \mathbb{C}^{\mathsf{core}}$. The previous rule for identity introduction can then be expressed by applying the embeddings $\mathsf{i} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}$ and $\mathsf{i}^{\mathsf{op}} : \mathbb{C}^{\mathsf{core}} \to \mathbb{C}^{\mathsf{op}}$ to $a : \mathbb{C}^{\mathsf{core}}$ on both sides of the equality, obtaining $\mathsf{refl}_a : \hom(\mathsf{i}^{\mathsf{op}}(a), \mathsf{i}(a))$.

Directed equality elimination. The second fundamental component needed to effectively work with equalities is a rule to *eliminate* them, dually to the introduction rule refl_a. In standard Martin-Löf type theory, symmetric equalities can be eliminated (or "contracted", following the homotopical interpretation) with a typing rule called the J-principle [Hof97], which can be stated as follows,

$$\frac{C : \mathsf{Type}, \quad P : \prod_{a,b:C}(a =_C b \to \mathsf{Type}), \quad t : \prod_{x:C} P(x, x, \mathsf{refl}_x)}{J : \prod_{a,b:C} \prod_{e:a=_C b} P(a, b, e)} \ (J\textsf{-rule})$$

where the computation rule $J(x, x, \mathsf{refl}_x) \equiv t(x)$ holds definitionally.

The intuition behind this principle is that whenever we want to prove a proposition P which assumes an equality $e : a = b$ between two (possibly different) terms $a, b : A$, it is sufficient to consider the case where a and b are exactly the same term, and the equality e is refl_a.

This statement "it is enough to consider the case when the equality is refl" bears a striking similarity to the same fundamental principle underlying the Yoneda lemma, one of the most central and praised results in category theory [ML98]. The Yoneda lemma states that, for any object $a : \mathbb{C}$ and functor $P : \mathbb{C} \to \mathsf{Set}$, there is an isomorphism (natural in a and P) between the set $P(a)$ and the set of natural transformations of the form $\alpha_x : \hom_{\mathbb{C}}(a, x) \to P(x)$ from a representable functor $\hom_{\mathbb{C}}(a, -)$, which plays here the role of directed equality. The idea behind the proof is that *natural* transformations α_x are uniquely determined by their component α_a in a and their action on the identity morphism id_a. The connection between the Yoneda lemma as a sort of (based) directed J-principle is investigated in HoTT in [Esc15], and is practically used in [RS17] in the context of simplicial type theory.

Quantifiers and coends. A central yet unexplored question is how *quantifiers* should be interpreted in the types-as-categories interpretation of directed type theory. A well-known rephrasing of the Yoneda lemma (called "ninja" Yoneda lemma [Lei10]) provides inspiration for a possible answer, which we present in detail in Section 5 and introduce here at an intuitive level.

The set of natural transformations appearing in the Yoneda lemma can be characterized in terms of a universal object called the *end* of a functor of a specific type [ML98, IX.5], [Lor21]. Given a functor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, the *end* of P, denoted as $\int_{x:\mathbb{C}} P(\overline{x}, x)$, is an object of $\mathbb{D}$ with a certain terminal universal property. Notationally, the integral sign of ends binds positive and negative variables until the end of expressions, and for convenience we will explicitly indicate with $\overline{x} : \mathbb{C}^{\mathsf{op}}$ the contravariant occurrences of variables $x : \mathbb{C}$. As it turns out, ends can be computed as certain limits indexed over a category that depends only on $\mathbb{C}$, and therefore exist whenever $\mathbb{D}$ is complete.

Ends of certain functors into Set characterize natural transformations, in the sense that for any two functors $F, G : \mathbb{C} \to \mathbb{D}$ there is an isomorphism (natural in F, G) as follows:

$$\mathsf{Nat}(F, G) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{D}}(F(\overline{x}), G(x)).$$

Note the resemblance between the end of the above functor and the universal quantification expressed in the elementary definition of natural transformation:

$$\mathsf{Nat}(F, G) := \{\forall x : \mathbb{C}, \alpha_x \in \hom_{\mathbb{D}}(F(x), G(x)) \mid \textit{naturality condition}\}$$

With this characterization, one can rephrase the original statement of the Yoneda lemma as the following isomorphism (natural in $a : \mathbb{C}$ and $P : \mathbb{C} \to \mathsf{Set}$),

$$P(a) \cong \int_{x:\mathbb{C}} \hom_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x).$$

By turning directed equality back into symmetric equality and viewing ends as a sort of universal quantifier, this celebrated result of category theory essentially corresponds to a simple equivalence of formulas in first order logic, where for any constant $a : C$ and predicate P (which can be seen as a degenerate notion of presheaf) we have that

$$P(a) \Leftrightarrow \forall (x : C).\, a =_C x \Rightarrow P(x).$$

A similar logical correspondence also holds, using existential rather than universal quantifiers. Semantically, the existential quantifier is now interpreted by the dual (in the formal, categorical sense) construction of ends, *coends* [Lor21], which are denoted as $\int^{x:\mathbb{C}} P(\overline{x}, x)$ for any functor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$. In particular, the following isomorphism and equivalence hold[1],

$$\begin{array}{ccccccc} P(a) & \cong & \displaystyle\int^{x:\mathbb{C}} & \hom_{\mathbb{C}}(x,a) & \times & P(x) \\ \hline\hline P(a) & \Leftrightarrow & \exists(x:C). & x =_C a & \wedge & P(x) \end{array}$$

and since coends are colimits, this result also takes the well-known category-theoretical slogan of "presheaves are colimits of representables" [Lei14] or "coYoneda lemma" [Lor21].

In first order logic, one can validate the equivalences presented so far through a formal system, such as sequent calculus or type theory; however, there is currently no formal system in which one can obtain the corresponding isomorphisms in the directed case. In the logical tradition, such a system should allow a modular treatment of each logical connective and quantifier, e.g., with appropriate introduction and elimination rules specific to directed equality and coends.

Categorical logic. There is a well-known characterization for (pointwise) Kan extensions in terms of certain (co)ends [ML98]; in the particular case where the functor to be extended is a (co)presheaf $P : \mathbb{C} \to \mathsf{Set}$, its extension along a functor $F : \mathbb{C} \to \mathbb{D}$ can be computed with the following formulas:

$$\begin{aligned} (\mathsf{Lan}_F(P))(x) &:= \int^{y:\mathbb{C}} \hom_{\mathbb{C}}(F^{\mathsf{op}}(\overline{y}), x) \times P(y) \\ (\mathsf{Ran}_F(P))(x) &:= \int_{y:\mathbb{C}} \hom_{\mathbb{C}}(x, F^{\mathsf{op}}(\overline{y})) \Rightarrow P(y) \end{aligned}$$

This situation is particularly reminiscent of the following scenario in categorical logic: given a general hyperdoctrine $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Cat}$ with logical operators $\forall, \exists, \wedge, \Rightarrow, =$ [Jac99], one can explicitly compute both the left and right adjoints to the functor $f^* := \mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ reindexing along *any* morphism $f : A \to B$ of the base category $\mathbb{C}$, and these adjoints take the logical name of "generalized quantifiers" [Pit95, 5.6.6]:

$$\begin{aligned} \forall_f(P) &:= \forall_{X,Y}((\mathsf{id}_Y \times f)^*(\mathsf{Eq}_Y(\top_Y)) \Rightarrow \pi^*_{X,Y} P) \\ \exists_f(P) &:= \exists_{X,Y}((\mathsf{id}_Y \times f)^*(\mathsf{Eq}_Y(\top_Y)) \wedge \pi^*_{X,Y} P) \end{aligned}$$

In the (initial) syntactic model, these correspond to the following formulas:

$$\begin{aligned} (\forall_f(P))(y) &:= \forall x.(y = f(x) \Rightarrow P(x)) \\ (\exists_f(P))(y) &:= \exists x.(f(x) = y \wedge P(x)) \end{aligned}$$

We stress the similarity between the above and the formulas for Kan extensions, where again quantifiers and logical operators are substituted by (co)ends, the cartesian closed structure of Set, and directed equality: in particular, $\mathsf{Lan}_F, \mathsf{Ran}_F$ become the functors adjoint to reindexing for the presheaf hyperdoctrine [Law73, MZ16]. Moreover, the above syntactic formulas can be *proven* to be adjoint to precomposition functors using precisely the logical rules of the syntactic model [Pit95, 5.6.6]; however, there is not yet a formal sense in which the (co)ends used in the formulas for Kan extensions can be similarly considered as quantifiers in a doctrine and, similarly, a sense in which equality can be generalized to the directed case.

Dinaturality. The notion of (co)end was introduced by Yoneda [Yon60] in the context of homological algebra. However, it was only later recognized that there is a close connection between (co)ends and the generalized notion of naturality that we now call *dinaturality*: *dinatural transformations* were first introduced by Dubuc and Street [DS70] in order to enlarge the class of transformations of functors to which we recognize a *parametricity* property [HRR14], further generalizing the fundamental idea of *extranaturality* introduced by Eilenberg and Kelly [EK66]. Either notion finds

[1] We use double lines in this case to suggest a correspondence between the connectives of both formulas, without giving it here a formal meaning.

its necessity in the need to understand the precise role of variance in category theory, especially in contexts where a *parametric adjunction* [ML98, IV.7] is given; a typical example of such a situation is elucidated in [Str03], where the counit ("evaluation") map $\varepsilon_{AB} : (A \Rightarrow B) \otimes A \to B$ of a monoidal closed category $(\mathbb{C}, \otimes, I, \Rightarrow)$ is *extranatural* when its domain is considered as the "diagonalized component" of a functor

$$(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times \mathbb{C} \to \mathbb{C} : (A, A', B) \mapsto (A \Rightarrow B) \otimes A'.$$

Dinatural transformations generalize at once natural and extranatural transformations, by considering families of morphisms between functors with different variances $F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$. Famously, however, such generalized natural transformations *do not always compose.* Since their very introduction the problem of establishing conditions under which composition is possible has been at the heart of the subject. A sufficient condition for the composability of extranaturals is the absence of loops in a suitably associated graph [EK66], and this idea constitutes the core of a recently revived line of research [Pet03, MS21].

Despite the apparent lack of composition in general, there are numerous examples of settings in which all dinaturals compose: the fundamental idea is to single out a class of "definable" dinaturals, all of which are composable because their formation rules are precisely defined and are given syntactically. This point of view finds its most natural application in the logical setting, where composition corresponds with cut elimination [BS96, BS98, Blu93, GSS92], and in certain parametric models for System F [BFSS90], for which the connection between the notion of parametricity in programming languages and dinaturality is well-studied [FRR92, Pis19, Voi20]. An in-depth review on the issue of dinaturality and its importance for both computer science and category theory can be found in the introductory chapter of [San19], and in [Sco00, Sec. 3].

Our interest in the issue of composability of dinaturals is outlined towards the end of Section 5 (most notably, in Theorem 5.10) and in Section 6.

1.1. Contribution

In this paper we describe how dinaturality allows us to semantically validate an introduction and an elimination rule for directed equality in the style of Martin-Löf type theory. Moreover, we show how dinaturality justifies (co)ends as the "directed quantifiers" of the (1-)categorical interpretation of directed type theory, which we present in a correspondence reminiscent of the quantifiers-as-adjoint paradigm of Lawvere [Law69] (the choice of the word 'reminiscent' should suggest that this correspondence is not perfect: Remark 5.7 will clarify the matter). This also suggests how (a restricted form of) dinatural transformations might play a central role towards a satisfactory account of directed type theory, both from the semantic and the syntactic point of view.

The intuition behind dinatural transformations is that they allow the same variable to appear both covariantly and contravariantly: this is exactly what allows us to resolve the variance problems previously mentioned in the directed refl rule, which is precisely validated using identities in hom-sets. Crucially, we identify a directed equality *elimination* rule which is syntactically extremely reminiscent of the J-rule as used in standard Martin-Löf type theory, and dinaturality is again what permits the same variable x to appear with both variances in the expression $P(x, x, \mathsf{refl}_x)$ in (J-rule). The elimination rule is semantically motivated by the connection between dinaturality and naturality, and sheds a light on the syntactic restrictions imposed in a full type theory where equality is now directed rather than symmetric: in short, the syntactic requirement for directed equality to be contracted is that, given a directed equality $\hom_{\mathbb{C}}(x, y)$ in context for $x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C}$, both x and y must appear only positively (i.e., with the same variance) in the conclusion and only negatively (i.e., with the opposite variance) in the context.

Symmetric equality has a well-known characterization as a left adjoint to contraction functors [Jac99, 3.4.1], first noticed by Lawvere in [Law70]. In Theorem 4.3 we present a similar characterization for directed equality in terms of a *relative adjunction* [Ulm68], where hom-functors are characterized as *relative left adjoints* to certain "contraction-like" functors between (para)categories of dinatural transformations which, intuitively, join two *natural* variables into a single *dinatural* one. The relative adjunction is semantically justified by the rules for directed equality elimination and, intuitively, the *relativeness* of the adjunction is precisely needed to capture the syntactic restriction prescribed by the rule for directed equality elimination. This suggests a tentative answer to a problem first posed by Lawvere on the precise role played by $\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ for the presheaf hyperdoctrine [Law70, p.11], [MZ16].

The setting in which we validate our rules semantically is by considering a categorification (both proof-relevant and directed) of (non-dependent) first-order logic: types are (small) categories (possibly with $-^{\mathsf{op}}$), contexts are lists of categories, terms are functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, propositions are functors of type $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, and entailments are dinatural transformations. We do not provide a semantic account of these rules using categorical semantics or type theoretical methods, precisely because dinaturals do not compose in general. Despite this lack of general composition, the rules for directed equality and coends-as-quantifiers can be used to give concise proofs of several central theorems in category theory using a distinctly *logical* flavour: e.g., the Yoneda and coYoneda lemma, Kan extensions computed via coends are adjoint to precomposition, presheaves form a closed category; each of these theorems easily follows by modularly using the logical properties of each connective used.

The rules for directed equality allow us to recover the same type theoretical definitions about symmetric equality that one expects in standard Martin-Löf type theory, except for symmetry: e.g., transitivity of directed equality (composition in a category), congruences of terms along directed equalities (the action of a functor on morphisms), transport along directed equalities (i.e., the Yoneda lemma). We highlight how the syntactic restrictions imposed by the rule for directed equality elimination do not allow us to obtain that directed equalities are symmetric.

Our treatment of coends as quantifiers is a concrete step towards formally understanding the so-called "coend calculus" [Lor21], a set of elementary results that elegantly express the universal property of several central concepts in category theory, such as natural transformations, the Yoneda lemma, Kan extensions and the calculus of bimodules (or *profunctors*, encoded in the study of the bicategory Prof [B0́0, AH10]). There is a formal aspect to the manipulation of ends and coends which is common knowledge among category theorists, outlined in [Lor21], and which allows non-trivial theorems to be proven using simple formal rules reminiscent of a deductive system. Apart from formal category theory [Lor21, Str81], this has proven to be useful in a diverse array of disciplines using category theory, such as algebraic topology [MMSS00], universal algebra [Cur12, Hyl14], as well as theoretical computer science, for example in the context of profunctor optics [CEG⁺22, BG18] and their string diagrams [Rom20, Boi20], strong monads and functional programming [Asa10, AH10, Hin12], logic [Pis18, PT21], and quantum circuits [HC23]. Leveraging on this variety of applications, [Lor21] hints at the existence of such deductive system, but falls short of the expectation to precisely pinpoint its structural rules. This application to coend calculus is in particular what motivates our focus on a first-order and non-dependent presentation of directed type theory.

1.2. Related work

Directed type theory with groupoids. North [Nor19] describes a dependent directed type theory with semantics in the categories of (small) categories Cat, but uses the groupoid structure to deal with the problem of variance in both the introduction and elimination rules for directed equality elimination. This line of research has been recently expanded in [CMN24] by extending the judgements of Martin-Löf type theory with variance annotations.

We focus on non-dependent semantics, and tackle the variance issue precisely with the notion of dinatural transformation; this allows us to characterize directed equality intrinsically, without using any of the groupoidal structure of categories.

Directed type theory, judgemental models. Another approach to modeling directed equality is at the judgemental level. This line of research started with Licata and Harper [LH11] who introduced a directed type theory with a model in Cat. Since directed equality is treated judgementally, there are no rules governing its behaviour in terms of elimination and introduction principles, although variances are present in the context as we similarly do in our approach. Ahrens et al. [ANvdW23] similarly identify a type theory with judgmental directed equalities with sound semantics in comprehension bicategories, and extensively compare previous work on both judgemental and propositional directed type theory.

Synthetic logics for category theory. New and Licata [NL23] give a sound and complete presentation for the internal language of (hyperdoctrines of) certain virtual equipments. These models capture enriched, internal, and fibred categories and have an intrinsically directed flavour. In these contexts, the type theory can give synthetic proofs of Fubini, Yoneda, and Kan extensions as adjoints. This generality however comes at the cost of a non-standard syntactic structure of the logic, for example when compared to standard Martin-Löf type theory, along with some non-trivial syntactic judgements prescribed by the structure of the models. Directed equality elimination here takes the shape of the identity laws axiomatized in virtual equipments [CS10], which in Prof is essentially the coYoneda-lemma. Their quantifiers are given by the universal properties of tensor and (left/right) internal homs, which in the Prof model are given by certain restricted coends which always come combined with the tensors and internal homs of Set. Our work is similar in spirit in that we provide a formal setting for proving category theoretical theorems using logical methods, but we only focus on the elementary 1-categorical model of categories and do not yet capture enriched and internal settings. However, we treat ends and coends as quantifiers *directly* as adjoints to contextual functors which only act on the variables of the context, without the need for quantifiers to include (restricted forms of) conjunction and implication. Our rules for directed equality are more direct and reminiscent of standard Martin-Löf type theory, and have a different semantic justification based on dinaturality. Since we consider less general models, our contexts do not have any linear nor ordered restriction, and the same variable can appear multiple times both in equalities and contexts. This allows us to consider profunctors of many variables and different variances as typically needed in coend calculus.

Directed type theory with variances. Preliminary work on a directed type theory with variances is explored by Nuyts [Nuy15]. Notably, the notion of positive and negative variance in a context is introduced, along with the general idea of what a directed notion of univalence might look like; however, no formal set of rules is provided and there is not a precise semantic interpretation in a model.

Geometric models of directed type theory. Riehl and Shulman [RS17] introduce a simplicial type theory based on a synthetic description of $(\infty, 1)$-categories. A directed interval type is axiomatized in a style reminiscent of cubical type theory [CCHM15], which allows a form of (dependent) Yoneda lemma to be proven using the structure of the identity type. This type theory has been implemented in practice in the Rzk proof assistant [KRW24]. On this line of research, Weaver and Licata [WL20] present a *bicubical* type theory with a directed interval and investigate a directed analogue of the univalence axiom; the same objective was recently explored in Gratzer et al. [GWB24] with triangulated type theory and modalities.

In comparison with the above works, we do not explore the geometrical interpretation of directedness and focus on elementary 1-categorical semantics; moreover, our treatment of directed equality is done intrinsically with elimination rules as in Martin-Löf type theory rather than with synthetic intervals, with semantics directly provided by hom-functors.

Coend calculus, formally. Caccamo and Winskel [CW01] give a formal system in which one can work with coends and establish non-trivial theorems with a few syntactical rules. The flavour is explicitly that of an axiomatic system, and does not take inspiration from type-theoretic rules: for instance, presheaves are *postulated* to be equivalent under the swapping of quantifiers (Fubini), so this principle is not derived from structural rules as typically done in a logical presentation.

1.3. **Synopsis**

We start in Section 2 by giving a general overview of the semantic setting in which we work and by recalling basic notions about dinatural transformations.

In Section 3 we present rules for introduction and elimination of directed equality, which we validate using dinatural transformations. Finally, we provide examples for the rules of directed equality and how they can be used in exactly the same style of Martin-Löf type theory.

In Section 4 we characterize directed equality and hom-functors in terms of a relative (para)adjunction with the operation which contracts two natural variables into a dinatural one.

In Section 5 we give rules and semantics to ends and coends in terms of adjoint-like situations with weakening functors, and analyze their relation to the compositionality of dinatural transformations. We then combine all the logical rules previously introduced for directed equality and quantifiers to give concise logical proofs of classical theorems in category theory.

We provide a formalization of the theorems in this paper using the Agda proof assistant and the `agda-categories` library. Whenever present, the symbol (🐓) next to theorems links to the formal proof, for which we report here just the core idea. The full formalization can be accessed at the following link:

`https://github.com/iwilare/dinaturality`

2. Semantics

Given our motivation of investigating the semantics of directed type theory with 1-categories, we will consider the following interpretation:

- **types** will be considered as *(small) categories* (possibly with $\mathbb{C}^{\mathsf{op}}$),
- **contexts** as *finite products of categories*,
- **terms** as *difunctors*, i.e., functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$,
- **predicates** as *dipresheaves*, i.e., functors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$,
- **propositional contexts** as *pointwise products of dipresheaves*,
- **entailments** as *dinatural transformations* (without requiring composition),
- **quantifiers** $\forall, \exists$ as *ends and coends*, logically representing universal and existential quantifiers respectively,
- **propositional directed equality** as hom-*functors* $\hom_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$.

Warning 2.1. Because of the lack of composition of dinatural transformations, we do not consider a fully-fledged type theoretical account of this system; we will only identify suitable rules and semantically validate them using dinatural transformations, e.g., the introduction and elimination rules for hom-types. In particular, type theoretical notations are to be understood here merely as suggestive shorthands for semantic judgements, and not as formal syntactic objects.

The reader fluent in categorical logic can imagine our presentation to essentially revolve around the analysis of a specific doctrine $\mathsf{Dinat} : \mathsf{coKleisli}(\Delta) \to \mathsf{CAT}$, where $\Delta : \mathsf{Cat} \to \mathsf{Cat}$ is the (strict)

2-comonad on Cat sending $\mathbb{C}$ to $\mathbb{C}^{\mathsf{op}} \times \mathbb{C}$: Dinat is defined by sending a small category $\mathbb{C}$ to the (non-)category where objects are endoprofunctors $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ and morphisms are *dinatural transformations* between them. We do not give a precise description of our analysis in terms of the language of doctrines precisely because of the lack of composition.

Despite this lack of compositionality, we can put these semantic rules into practice by showing (in Section 5.3) how we can prove theorems about category theory using a distinctly logical flavour, as well as showcasing (in Section 3.1) how our rules about directed equality can be used in precisely the same way as it is done in Martin-Löf type theory for symmetric equality.

2.1. Dinaturality

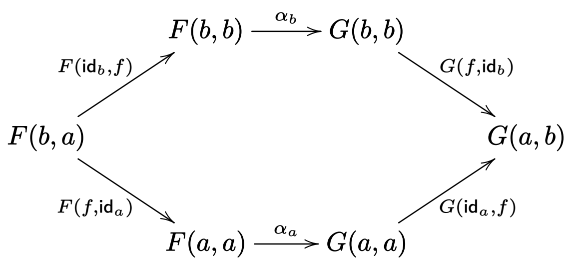
We recall the definition of dinatural transformation along with some elementary properties. For convenience, we give an explicit name to the specific shape of functors used in our logical interpretation of dinaturality.

Definition 2.2 (Difunctors and dipresheaves)**.** A *difunctor* from $\mathbb{C}$ to $\mathbb{D}$ is simply a functor $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$; similarly, a *dipresheaf* in $\mathbb{C}$ is a functor $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$.

Definition 2.3 (Dinatural transformation [DS70])**.** Given two difunctors $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, a *dinatural transformation* $\alpha : F \xrightarrow{\bullet\bullet} G$ is a family of arrows indexed by objects $x : \mathbb{C}$,

$$\alpha_x : F(x, x) \longrightarrow G(x, x)$$

such that for any $a, b : \mathbb{C}$ and $f : a \to b$ the following hexagon commutes:



Using $u\,;v$ to denote diagrammatic composition of morphisms $x \xrightarrow{u} y \xrightarrow{v} z$, this means that the following equation holds,

$$F(\mathsf{id}_b, f)\,;\alpha_b\,;G(f, \mathsf{id}_b) = F(f, \mathsf{id}_a)\,;\alpha_a\,;G(\mathsf{id}_b, f).$$

Lemma 2.4 (Dinaturality subsumes naturality [DS70])**.** Given $F, G : \mathbb{C} \to \mathbb{D}$, a dinatural transformation from $\alpha : (\pi_2\,;F) \xrightarrow{\bullet\bullet} (\pi_2\,;G) : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ is simply a natural transformation $F \longrightarrow G$.

Proof. Two sides of the hexagon collapse, obtaining a naturality square. □

Lemma 2.5 (Naturality to dinaturality)**.** Naturality in two variables with different variance can be "collapsed" to dinaturality in a single variable: given $F, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ and a natural transformation $\alpha : F \longrightarrow G$ one obtains a dinatural transformation $\Delta(\alpha) : F \xrightarrow{\bullet\bullet} G$.

Proof. The map is $\Delta(\alpha)_x := \alpha_{xx}$, and dinaturality follows using naturality twice. □

The idea behind Lemma 2.5 will be crucial in Section 4 to present directed equality as a relative left adjoint.

We introduce specific notation to emphasize the logical nature of dinatural transformations and to deal with contravariance explicitly.

Remark 2.6 (Notation for variance of variables)**.** We will explicitly indicate with "$\overline{x}$" the *negative* (or "contravariant") occurrences of $x : \mathbb{C}$, and simply "x" for the *positive* (or "covariant") ones. Note that we shall still use the same terminology even in the case in which $\mathbb{C} := (\mathbb{C}')^{\mathsf{op}}$ is the opposite of some category. Using Lemma 2.4, we shall consider a natural transformation to simply be a dinatural transformation where $\overline{x}$ does not appear syntactically.

Remark 2.7 (Notation for dinaturals)**.** The type-theoretic notation

$$[x : \mathbb{C}, y : \mathbb{D}]\ F(\overline{x}, \overline{y}, x, y) \vdash \alpha : G(\overline{x}, \overline{y}, x, y)$$

denotes a dinatural α between functors $F, G : (\mathbb{C} \times \mathbb{D})^{\mathsf{op}} \times (\mathbb{C} \times \mathbb{D}) \to \mathsf{Set}$, where we use $[x : \mathbb{C}, y : \mathbb{D}]$ to indicate the indices of the dinatural: this is to be thought of as a (term) context, reminiscent of the situation where in a fibration one has a fibre of entailments over an object of the base category [Jac99]. Since we give names to variables in context, we will reorder the indices of dipresheaves whenever convenient.

Remark 2.8 (Negative variables or negative context)**.** Given $F : \mathbb{C}^{\mathsf{op}} \to \mathbb{D}, G : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ and a dinatural transformation $\alpha : \pi_1 ; F \xrightarrow{\bullet\bullet} G$, the following two ways of indicating the family α are clearly equivalent:

$$[x : \mathbb{C}]\ F(\overline{x}) \vdash \alpha\ : G(\overline{x}, x)$$
$$[x : \mathbb{C}^{\mathsf{op}}]\ F(x) \vdash \alpha' : G(x, \overline{x})$$

For simplicity, we shall always prefer to pick the presentation with the most *positive* variables x as possible (α' in this case), even when variables comes from categories with an explicit $-^{\mathsf{op}}$ (rather than choosing the type to be $\mathbb{C}$ and then using contravariant variables $\overline{x} : \mathbb{C}^{\mathsf{op}}$). Note that identifying these two families together is only possible because the functor $-^{\mathsf{op}} : \mathsf{Cat} \to \mathsf{Cat}$ is a strict involution in the sense of [Shu18].

Remark 2.9 (Notation for inference rules of dinaturals)**.** Following the interpretation of dinaturals as entailments, we will use (trees of) inference rules to indicate that, given certain dinatural(s) as in the rule premise, one obtains a dinatural as in the rule conclusion, for instance:

$$\frac{[x : \mathbb{C}_1]\ \Gamma_1(\overline{x}, x) \vdash \alpha_1 : P_1(\overline{x}, x) \qquad \cdots \qquad [x : \mathbb{C}_n]\ \Gamma_n(\overline{x}, x) \vdash \alpha_2 : P_n(\overline{x}, x)}{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash \alpha(\alpha_1, ..., \alpha_2) : P(\overline{x}, x)}\ (\mathsf{rule\ name})$$

A "one-directional rule" is simply a (parametric) function between sets of dinatural transformations, and we will use double lines to indicate an isomorphism of sets of dinaturals.

Theorem 2.10 (Identity dinatural transformation)**.** ( ) For any difunctor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ the family of identity morphisms is a dinatural transformation:

$$\frac{}{[x : \mathbb{C}]\ P(\overline{x}, x) \vdash \mathsf{id} : P(\overline{x}, x)}\ (\mathsf{id})$$

Definition 2.11 (Paracategories and parafunctors [Fre96, HM03, HM04])**.** We shall use the terms "paracategory" and "parafunctors" to refer to a structure which satisfies the axioms of a category, besides the fact that not all composable pairs of arrows in its underlying quiver admit a composition; a notion of parafunctor is similarly obtained by disregarding laws involving composition, which we will simply refer to as "functors" when it is clear we talking about paracategories. We use these to hint at the lack of a general notion of composition between dinaturals, and we will simply talk about *sets of dinaturals* rather than hom-sets.

Definition 2.12 (Paracategory of difunctors)**.** For any $\mathbb{C}, \mathbb{D}$ we indicate with $[\mathbb{C}^{\diamond}, \mathbb{D}]$ the paracategory of *difunctors from* $\mathbb{C}$ *to* $\mathbb{D}$ and *dinatural transformations* between them.

We now show how dinaturals and dipresheaves, viewed as entailments with generalized predicates, support the interpretation of the usual propositional connectives of conjunction and implication: despite the lack of composition, we show how $[\mathbb{C}^{\diamond}, \mathbb{D}]$ is cartesian closed via the presentation of adjunction as a (natural) isomorphism of sets. We assume $\mathbb{D}$ to have a terminal object, products, and exponentials for the rest of the section.

Theorem 2.13 (Product of difunctors)**.** (🗸) The product of difunctors is computed pointwise, and there is a natural isomorphism of sets of dinaturals as follows, given naturally for any $\Gamma, P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$:

$$\frac{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash p = \langle a, b \rangle : P(\overline{x}, x) \times Q(\overline{x}, x)}{[x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash a = \pi_1(p) : P(\overline{x}, x), \qquad [x : \mathbb{C}]\ \Gamma(\overline{x}, x) \vdash b = \pi_2(p) : Q(\overline{x}, x)} \ (\mathsf{prod})$$

where the bottom side of the isomorphism indicates the product of sets of dinatural transformations.

Similarly, $\top : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D} := (c, c') \mapsto \top_{\mathbb{D}}$ satisfies the universal property of the terminal object in $[\mathbb{C}^{\diamond}, \mathbb{D}]$ (note that defining 'terminal objects' does not require a category structure: in a quiver, a vertex $\top$ is 'terminal' if there exists exactly one edge into $\top$, for every other edge X).

Dinatural transformations always compose with projections both on the left and on the right; composition on the right with π_1, π_2 is given by Theorem 2.13, and composition on the left is given by the following theorem.

Theorem 2.14 (Composition with projections on the left)**.** (🗸) Given $\alpha : \Gamma \xrightarrow{\bullet\bullet} P$ for $\Gamma, P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, there is a dinatural $\mathsf{wk}_2(\alpha)$ as follows:

$$\frac{[x : \mathbb{C}] \qquad \Gamma(\overline{x}, x) \vdash \alpha : P(\overline{x}, x)}{[x : \mathbb{C}]\ Q(\overline{x}, x) \times \Gamma(\overline{x}, x) \vdash \mathsf{wk}_2(\alpha) : P(\overline{x}, x)} \ (\mathsf{weakening})$$

defined by $(\mathsf{wk}_2(\alpha))_x := \pi_2 \mathbin{;} \alpha_x$, and similarly for the other projection π_1.

Remark 2.15 (Notation for contravariance of functors)**.** We will pedantically indicate contravariant uses of functors as follows: given $F : \mathbb{C} \to \mathbb{D}$ we indicate the opposite functor explicitly with $F^{\mathsf{op}} : \mathbb{C}^{\mathsf{op}} \to \mathbb{D}^{\mathsf{op}}$. In particular for dipresheaves and difunctors we establish that whenever $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, then $P^{\mathsf{op}} : \mathbb{C} \times \mathbb{C}^{\mathsf{op}} \to \mathbb{D}^{\mathsf{op}}$, and no swapping of variables is involved. Similarly, we will always take the internal hom-functor of Set as a functor $- \Rightarrow - : \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set} \to \mathsf{Set}$.

Contrary to the situation with natural transformations, the exponential object in the paracategory of dipresheaves is computed *pointwise*. This idea is similarly introduced in [GSS92, BFSS90], where this construction takes the name of *twisted exponential*.

Theorem 2.16 (Exponential of dipresheaves)**.** (🗸) There is an isomorphism of sets of dinaturals as follows, for any $F, G, H : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$:

$$\frac{[x : \mathbb{C}]\ F(\overline{x}, x) \times G(\overline{x}, x) \vdash H(\overline{x}, x)}{[x : \mathbb{C}] \qquad G(\overline{x}, x) \vdash F^{\mathsf{op}}(x, \overline{x}) \Rightarrow H(\overline{x}, x)} \ (\mathsf{exp})$$

Proof. Obvious by currying the families of morphisms of the underlying category. □

Intuitively, dipresheaves "switch" between the two sides of the turnstile by inverting the variance of all their variables. Theorem 2.16 elucidates why the exponential object in the category of presheaves and *natural* transformations is non-trivial, and is not the pointwise hom in Set: directly applying the isomorphism would result in the following situation,

$$\frac{[x : \mathbb{C}]\ F(x) \times G(x) \vdash H(x)}{[x : \mathbb{C}] \qquad G(x) \vdash F^{\mathsf{op}}(\overline{x}) \Rightarrow H(x)} \ (\mathsf{exp})$$

but the second family of morphisms is *dinatural* in x, since it appears both covariantly and contravariantly. We show in Example 5.18 how Theorem 2.16 and the rules for directed equality can be used to give a logical proof that the usual definition of exponential object for presheaves is indeed the correct one.

Following the hyperdoctrinal presentation of logic (see [Jac99, Pit95] for standard accounts), dinatural transformations can be "reindexed" by difunctors, i.e., variables in entailments can be substituted with concrete difunctors, which are viewed as terms.

Theorem 2.17 (Reindexing with difunctors)**.** Given a difunctor $F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ and a dinatural transformation $\alpha : P \xrightarrow{\bullet\bullet} Q$ for $P, Q : \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathbb{E}$, there is a dinatural transformation $F^*(\alpha)$ as follows,

$$\frac{[x : \mathbb{D}] \qquad P(\overline{x}, x) \vdash \alpha : Q(\overline{x}, x)}{[x : \mathbb{C}]\ P(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x)) \vdash F^*(\alpha) : Q(F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x))} \ (\text{reindex})$$

defined by $F^*(\alpha)_x := \alpha_{F(x,x)}$.

3. Directed equality with dinaturality

We start our semantic analysis of directed equality by giving introduction and elimination rules for hom-types and showing how dinatural transformations validate them.

The introduction rule for hom-types logically corresponds to reflexivity of directed equality, and is precisely motivated by the fact that hom-sets are pointed with identity morphisms.

Theorem 3.1 (Directed refl)**.** The following is a dinatural transformation,

$$\frac{}{[x : \mathbb{C}]\ \top \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)} \ (\text{hom-intro})$$

where $\top$ denotes the terminal dipresheaf.

Proof. For a concrete object $x : \mathbb{C}$, the map is given by $\alpha_x(*) := \mathsf{id}_x$. Dinaturality prescribes that for any $f : a \to b$, $f \mathbin{;} \mathsf{id}_b = \mathsf{id}_a \mathbin{;} f$. □

Before introducing the general rule for directed equality elimination, we describe the fundamental idea behind it using the following result, relating dinaturality back to ordinary naturality.

Theorem 3.2 (Characterization of dinaturals via naturality)**.** For any $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, there is an isomorphism between the set of dinatural transformations $P \xrightarrow{\bullet\bullet} Q$ and certain natural transformations, as follows:

$$\frac{[x : \mathbb{C}]\ P(\overline{x}, x) \xrightarrow{\bullet\bullet} Q(\overline{x}, x)}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ \hom(a, b) \longrightarrow P^{\mathsf{op}}(b, a) \Rightarrow Q(a, b)} \ (\text{dinat-homnat})$$

Proof. We describe the maps in both directions:

- ($\Downarrow$) Given a dinatural $\alpha : P \xrightarrow{\bullet\bullet} Q$ and a morphism $f : \hom(a, b)$, the map $P(b, a) \to Q(a, b)$ corresponds precisely with the sides of the hexagon given in Definition 2.3 for dinaturality, which is obtained by applying the functorial action of P and Q.
- ($\Uparrow$) Take $a = b$ and precompose with $\mathsf{id}_a \in \hom(a, a)$.

The fact that this is an isomorphism notably follows from the (di)naturality of both sets of maps. Note the similarity between the above argument and the proof of the Yoneda lemma, where the two central ideas are precisely applying the functorial action and instantiating at id, with the isomorphism following from (di)naturality. □

Theorem 3.2 is the central idea behind directed equality elimination with dinatural transformations; most of the rules we identify later come as equivalent generalizations of this principle.

Remark 3.3 (Notation for assumptions in context)**.** We will henceforth use the standard notation

$$[x : \mathbb{C}]\ p : P(\overline{x}, x), q : Q(\overline{x}, x) \vdash h[p, q] : R(\overline{x}, x)$$

to give names p, q to assumptions in the "propositional context". We use commas to separate assumptions, which is to be interpreted semantically by taking the (pointwise) product of dipresheaves. We will use square brackets $h[p, q]$ both to refer to these assumptions as "free variables" and to denote application of functions in Set. For instance, $h_c[a, b] \in R(c, c)$ whenever we are given a concrete object $c \in \mathbb{C}$ and $a \in P(c, c), b \in Q(c, c)$.

Theorem 3.4 (Directed equality elimination)**.** ( ) For any $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathsf{Set}$, given $h : \Gamma \xrightarrow{\bullet\bullet} P$ one obtains a dinatural $J(h)$ as follows:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad\qquad k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e, k] : P(a, b, \overline{x}, x)}\ \text{(hom-elim)}$$

The dinatural $J(h)$ satisfies the following "*computation rule*",

$$J(h)_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k] = h_{zx}[k]$$

for any object $z : \mathbb{A}, x : \mathbb{C}$ and $k \in \Gamma(z, z, x, x)$.

Proof. The dinatural $J(h)$ is simply obtained using the ($\Downarrow$) map in Theorem 3.2 and uncurrying the twisted exponential using ($\Uparrow$) of Theorem 2.16. Explicitly, the map is given by

$$J(h)_{abx}[e, k] := (\Gamma(\mathsf{id}_b, e, \mathsf{id}_x, \mathsf{id}_x) \mathbin{;} h_{bx} \mathbin{;} P(e, \mathsf{id}_b, \mathsf{id}_x, \mathsf{id}_x))[k].$$

The computation rule is clearly satisfied when $a = b = z$ and $e = \mathsf{id}_z$, without the need to use dinaturality. □

The operational meaning behind this rule is the following: having identified a position $a : \mathbb{A}^{\mathsf{op}}$ and a position $b : \mathbb{A}$ in the proposition P, if there is a directed equality $\hom_{\mathbb{A}}(a, b)$ from a to b then it is enough to prove that the proposition holds "on the diagonal", where the two positions have been identified with the same variable $z : \mathbb{A}$. Moreover, the variables can be identified in the context as long as they appear *contravariantly* (i.e., using only the variables $\overline{a}$ and $\overline{b}$).

Following Remark 2.8, one can equivalently state (hom-elim) by picking $a : \mathbb{A}$ (instead of $a : \mathbb{A}^{\mathsf{op}}$) and inverting its variance correspondingly whenever it appears. We choose the present formulation in which $a : \mathbb{A}^{\mathsf{op}}$ and $b : \mathbb{A}$ have different types to emphasize the fact that the two variables play two asymmetric roles.

Remark 3.5 (Failure of symmetry for directed equality)**.** The syntactic constraints given in Theorem 3.4 show why it is *not* in general possible to obtain that directed equality is symmetric:

$$[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}]\ e : \hom(a, b) \vdash \mathsf{sym} : \hom(\overline{b}, \overline{a})$$

The equality $e : \hom(a, b)$ cannot be contracted because $\overline{a}$ appears in the conclusion negatively (similarly with $\overline{b}$), whereas the directed J rule (hom-elim) requires that the conclusion only has *positive* occurrences of the variables being contracted. (Strictly speaking, it is always possible to contract a directed equality by simply asking that the conclusion is dummy in the relevant variables: it is not possible in this case to contract the equality *and* simplify the conclusion at the same time.)

The equality in (hom-elim) always composes on the left with *any* dinatural and can be given "in context Γ", in the following sense:

Theorem 3.6 (J composes with any equality)**.** Given two dinatural transformations h and e, there is a dinatural transformation $J(h,e)$ as follows, for any $\mathit{\Gamma}, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathsf{Set}$:

$$\frac{\begin{array}{c}[z : \mathbb{A}, x : \mathbb{C}]\ \mathit{\Gamma}(\overline{z}, z, \overline{x}, x) \vdash h : P(\overline{z}, z, \overline{x}, x) \\ [a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ \mathit{\Gamma}(\overline{a}, \overline{b}, \overline{x}, x) \vdash e : \hom(a,b)\end{array}}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ \mathit{\Gamma}(\overline{a}, \overline{b}, \overline{x}, x) \vdash J(h,e) : P(a, b, \overline{x}, x)}$$

Proof. By currying the context of $J(h)$ to the right, the variables a, b appear naturally as in Theorem 3.2, and therefore the desired map is obtained by composing the dinatural $\langle e, \mathsf{id}_{\mathit{\Gamma}(\overline{a},\overline{b})}\rangle$ with the *natural* $\mathsf{curry}(J(h))$ (which *can* be composed together [DS70]). □

Theorem 3.7 (Directed J as isomorphism)**.** Rule (hom-elim) in Theorem 3.4 is an isomorphism, i.e., the following is a (natural) isomorphism of set of dinaturals:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \qquad k : \mathit{\Gamma}(\overline{z}, z, \overline{x}, x) \vdash h[k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a,b), k : \mathit{\Gamma}(\overline{b}, \overline{a}, \overline{x}, x) \vdash J(h)[e,k] : P(a, b, \overline{x}, x)} \text{(hom)}$$

Proof. Obvious since the rule is obtained with Theorem 3.2 and Theorem 2.16. Given a dinatural α as in the bottom sequent, the inverse map $J^{-1}(\alpha)$ is explicitly defined by

$$J^{-1}(\alpha)_{zx}[k] := \alpha_{zzx}[\mathsf{refl}_{\mathbb{A}z}, k].$$

The computation rule of hom-elimination corresponds precisely with the fact that $J \mathbin{;} J^{-1} = \mathsf{id}$. On the other hand, $J^{-1} \mathbin{;} J = \mathsf{id}$ follows from (di)naturality. □

Remark 3.8 (Notation for contraction and composition with refl)**.** The dinaturality of $J^{-1}(\alpha)_{zx}$ for some α_{abc} ensures that contracting the indices $a = b = z$ of α and composing with refl_z is again dinatural; thus we will directly write '$\alpha_{zzc}[\mathsf{refl}_z]$' (or even '$\alpha[\mathsf{refl}_z]$' by omitting the indices) to indicate $J^{-1}(\alpha)$ whenever appropriate.

We have seen how Theorem 3.2 (and its equivalent formulation with Theorem 3.7) justifies the rule for directed equality *elimination*; the (hom-intro) rule can similarly either be given directly, as done in Theorem 3.1, or using the other direction of the J principle:

Theorem 3.9 (refl from J-isomorphism)**.** The rules (hom-intro) and (hom-elim) are logically equivalent to (hom) (which states that (hom-elim) is an isomorphism).

Proof. Clearly (hom-elim) is the top-to-bottom direction. We verify that (hom-intro) logically follows from the J^{-1} direction of Theorem 3.7. Take $\mathbb{C} := \mathbb{1}$ the terminal category, $P := \top$ the terminal presheaf and $Q := \hom$. A map with the desired type can be obtained by picking the bottom side of Theorem 3.7 to be $J(h) := \mathsf{id}$. Equationally, the computation rule states that $(J(h)_{zz*})[\mathsf{refl}_{Az}] = h_{z*}$ where h_{z*} is the map above the sequent, but since we picked $J(h)$ to be the identity we have that $\mathsf{refl}_{Az} = h_{z*}$ as desired. □

Since we work with proof-relevant dipresheaves, we need to introduce a directed J-principle for the "judgement" stating that two entailments are equal: whenever we want to prove the equality between two dinaturals which have a directed equality in context (for which J can be applied), it is enough to prove that they are equal in the case where the directed equality is effectively contracted. This principle embodies a *dependent* form of directed J (specialized on the equality predicate of Set), since it depends on the specific directed equality given in the propositional context.

We first define an equality judgement between dinaturals along with its semantics.

Definition 3.10 (Judgement for equality of entailments)**.** Given two dinaturals α, β with signature given from a list Γ of categories and a list Φ of dipresheaves in Γ, the judgement

$$[a : \mathbb{C}, b : \mathbb{C}, ...\Gamma]\ h : P(\overline{a}, a, \overline{b}, b, ...\Gamma), h' : Q(\overline{a}, a, \overline{b}, b, ...\Gamma), ...\Phi \\ \vdash \alpha[h, h', ...\Phi] = \beta[h, h', ...\Phi] : R(\overline{a}, a, \overline{b}, b, ...\Gamma)$$

is interpreted as $\alpha = \beta$, i.e., the following (extensional) equality holds:

$$\forall (a : \mathbb{C}), (b : \mathbb{D}), ...\Gamma,\ \forall (h : P(\overline{a}, a, \overline{b}, b, ...\Gamma)), (h' : Q(\overline{a}, a, \overline{b}, b, ...\Gamma)), ...\Phi, \\ \alpha_{ab...\Gamma}[h, h', ...\Phi] = \beta_{ab...\Gamma}[h, h', ...\Phi].$$

Example 3.11 (J computation as judgemental equality)**.** We can express the computation rule given in Theorem 3.7 in terms of the judgement for equality of entailments; the following judgement holds for any dinatural $h : \Gamma \xrightarrow{\bullet\bullet} P$:

$$\frac{}{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash J(h)[\mathsf{refl}_z, k] = h : P(\overline{z}, z, \overline{x}, x)}\ (J\text{-comp})$$

i.e., the computation rule simply states that $J \,;\, J^{-1} = \mathsf{id}$.

Theorem 3.12 (Directed J for judgemental equality)**.** (✐) For any $\Gamma, P : (\mathbb{A}^{\mathsf{op}}) \times (\mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathsf{Set}$, given two dinaturals α, β as follows,

$$[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : \Gamma(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k], \beta[e, k] : P(a, b, \overline{x}, x),$$

then the above judgement implies the one below:

$$\frac{[z : \mathbb{A}, x : \mathbb{C}]\ k : \Gamma(\overline{z}, z, \overline{x}, x) \vdash \alpha[\mathsf{refl}_z, k] = \beta[\mathsf{refl}_z, k] : P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ e : \hom(a, b), k : P(\overline{a}, \overline{b}, \overline{x}, x) \vdash \alpha[e, k] = \beta[e, k] : P(a, b, \overline{x}, x)}\ (J\text{-eq})$$

where by Remark 3.8 the top hypothesis simply indicates $J^{-1}(\alpha) = J^{-1}(\beta)$.

More explicitly, to prove that $\alpha = \beta$, i.e., that

$$\forall a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}.\ \forall e : \hom(a, b), k : P(a, b, x, x).\ \alpha_{abx}[e, k] = \beta_{abx}[e, k],$$

it is enough to prove that

$$\forall z : \mathbb{A}, x : \mathbb{C}.\ \forall k : P(z, z, x, x).\ \alpha_{zzx}[\mathsf{refl}_z, k] = \beta_{zzx}[\mathsf{refl}_z, k].$$

Proof. Since $\alpha_{zzx}[\mathsf{refl}_z, k] = J^{-1}(\alpha)_{zx}[k]$, the assumption simply states that $J^{-1}(\alpha) = J^{-1}(\beta)$; using Theorem 3.7 one obtains $\alpha = \beta$ simply by applying J to both sides. □

3.1. Examples for directed equality

We show how the rules we gave for directed equality can be used to obtain the same properties one can define in Martin-Löf type theory about symmetric equality. Most importantly, defining the necessary maps and proving equational properties about them follow precisely the same steps of the standard proofs.

Remark 3.13 (On dinaturality and composition)**.** Note that all dinaturals considered in this section have the specific form which allows every directed equality in context to be contracted with J, and thus the equality can be composed on the left with another dinatural, following Theorem 3.6. This compositionality is needed to make sure that, e.g., we can express the composition of comp with itself in $\mathsf{comp}[\mathsf{comp}[f, g], h]$ and still obtain a dinatural family; dinaturality is required in the proof of Theorem 3.7 to have that $J^{-1} \,;\, J = \mathsf{id}$, which is then used in Theorem 3.12.

We start by considering transitivity of directed equality, which corresponds to the fact that there is a composition map for any category.

Example 3.14 (Composition in a category)**.** The following derivation constructs the *composition map for* $\mathbb{C}$, which is natural in $a : \mathbb{C}, c : \mathbb{C}^{\mathsf{op}}$ and dinatural in $b : \mathbb{C}$:

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \quad \hom(\overline{z}, c) \vdash \mathsf{id} : \hom(\overline{z}, c)}\ (\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, c : \mathbb{C}] \ \hom(a, b), \hom(\overline{b}, c) \vdash J(\mathsf{id}) : \hom(a, c)}\ (\text{hom-elim})$$

We choose here to eliminate the first equality $\hom(a, b)$. The (hom-elim) rule can be applied since a, b appear only negatively in context (a does not appear) and positively in the conclusion ($\overline{b}$ does not appear).

We want to prove that $\mathsf{comp}[f, g] := J(\mathsf{id})$ is unital with respect to identities (i.e., $\mathsf{refl}_{\mathbb{C}}$) and associative. Since we chose to contract the first equality, the computation rule ensures that it is unital on the left:

$$\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \ g : \hom(\overline{z}, c) \vdash \mathsf{comp}[\mathsf{refl}_z, g] = g : \hom(\overline{z}, c)}\ (J\text{-comp})$$

essentially because $J^{-1}(J(\mathsf{id})) = \mathsf{id}$. On the other hand, to show that composition is unital on the right we must use directed J for the equality judgement (Theorem 3.12), which states that it is enough to prove the case where $a = z = w$ and $f = \mathsf{refl}_w$:

$$\dfrac{[w : \mathbb{C}] \ \top \vdash \mathsf{comp}[\mathsf{refl}_w, \mathsf{refl}_w] = \mathsf{refl}_w : \hom(\overline{w}, w)}{[a : \mathbb{C}, z : \mathbb{C}] \ f : \hom(\overline{a}, z) \vdash \mathsf{comp}[f, \mathsf{refl}_z] = f : \hom(\overline{a}, z)}\ (J\text{-eq})$$

which follows by the computation rule for comp since refl_w appears on the left.

Similarly, to prove associativity in the following derivation it is enough to consider the case in which $a = b = z$ and $f = \mathsf{refl}_z$,

$$\dfrac{\dfrac{}{\begin{array}{l}[z : \mathbb{C}, c : \mathbb{C}, d : \mathbb{C}] \ \ g : \hom(\overline{z}, c), h : \hom(\overline{c}, d) \\ \qquad \vdash \mathsf{comp}[\mathsf{refl}_z, \mathsf{comp}[g, h]] = \mathsf{comp}[\mathsf{comp}[\mathsf{refl}_z, g], h] : \hom(\overline{z}, d)\end{array}}}{\begin{array}{l}[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}, d : \mathbb{C}] \ \ f : \hom(\overline{a}, b), g : \hom(\overline{b}, c), h : \hom(\overline{c}, d) \\ \qquad \vdash \mathsf{comp}[f, \mathsf{comp}[g, h]] = \mathsf{comp}[\mathsf{comp}[f, g], h] : \hom(\overline{a}, d)\end{array}}$$

where the top sequent follows since

$$\mathsf{comp}[\mathsf{refl}_z, \mathsf{comp}[g, h]] = \mathsf{comp}[g, h] = \mathsf{comp}[\mathsf{comp}[\mathsf{refl}_z, g], h]$$

by the computation rules for $\mathsf{comp} := J(\mathsf{id})$.

Example 3.15 (Functorial action on morphisms)**.** For any functor $F : \mathbb{C} \to \mathbb{D}$, the functorial action on morphisms of F corresponds with the fact that any term/functor F respects directed equality, i.e., directed equality is a congruence:

$$\dfrac{\dfrac{}{[z : \mathbb{C}] \ \top \vdash F^*(\mathsf{refl}_{\mathbb{C}}) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{z}), F(z))}\ (\text{reindex})+(\text{hom-intro})}{[x : \mathbb{C}, y : \mathbb{C}] \ \hom_{\mathbb{C}}(\overline{x}, y) \vdash J(F^*(\mathsf{refl}_{\mathbb{C}})) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(y))}\ (\text{hom-elim})$$

and thus we define $\mathsf{map}_F[f] := J(F^*(\mathsf{refl}_{\mathbb{C}}))$. More precisely, we need to use reindexing (Theorem 2.17) through the difunctor $\pi_2 \,;\, F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ in the top sequent. The computation rule gives that F maps identities in identities,

$$\dfrac{}{[z : \mathbb{C}] \ \top \vdash \mathsf{map}_F[\mathsf{refl}_z] = F^*(\mathsf{refl}_z) : \hom_{\mathbb{D}}(F^{\mathsf{op}}(\overline{x}), F(x))}$$

and to prove functoriality, it is enough to prove the case where $a = b = z$ and $f = \mathsf{refl}_z$, as follows,

$$\dfrac{\dfrac{}{[z : \mathbb{C}, c : \mathbb{C}] \;\; g : \hom(\overline{z}, c) \vdash \mathsf{map}_F[\mathsf{comp}[\mathsf{refl}_z, g]] = \mathsf{comp}[\mathsf{comp}[\mathsf{refl}_z, g], h] : \hom(\overline{z}, d)}\,(J\text{-comp})}{[a : \mathbb{C}, b : \mathbb{C}, c : \mathbb{C}] \;\; f : \hom(\overline{a}, b), g : \hom(\overline{b}, c) \vdash \mathsf{map}_F[\mathsf{comp}[f, g]] = \mathsf{comp}[\mathsf{comp}[f, g], h] : \hom(\overline{a}, d)}\,(J\text{-eq})$$

for which both sides compute down to

$$\mathsf{map}_F[\mathsf{comp}[\mathsf{refl}_z, g]] = \mathsf{map}_F[g] = \mathsf{comp}[\mathsf{comp}[\mathsf{refl}_z, g], h]$$

using two computation rules on the left and one of the equational properties of comp previously shown by directed equality induction.

Example 3.16 (Transport)**.** In the directed case, the $\mathsf{transport}$ map along equality corresponds with the fact that any copresheaf $P : \mathbb{C} \to \mathsf{Set}$ has a functorial action on morphisms, where (co)presheaves are considered as (proof-relevant) predicates on directed types.

$$\dfrac{\dfrac{}{[z : \mathbb{C}] \;\; P(z) \vdash \mathsf{id} : P(z)}\,(\mathsf{id})}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}] \;\; \hom(a, b), P(\overline{a}) \vdash J(\mathsf{id}) : P(b)}\,(\text{hom-elim})$$

The computation rule simply states that transporting a point of $P(a)$ along the identity morphism with $\mathsf{trans}[f, k] := J(\mathsf{id})$ is the same as giving the point itself,

$$\dfrac{}{[z : \mathbb{C}] \; k : P(z) \vdash \mathsf{trans}[\mathsf{refl}_z, k] = k : P(z)}\,(J\text{-comp})$$

4. Directed equality as relative adjoint

Symmetric equality has a well-known characterization as a left adjoint to contraction functors [Jac99, 3.4.1]. The following shows how the usual rules of symmetric equality can be obtained as special cases of the directed ones.

Remark 4.1 (J for groupoids is symmetric equality)**.** In the case where both $\mathbb{A}$ and $\mathbb{C}$ are groupoids, all variances become irrelevant and Theorem 3.7 expresses the well-known *Lawvere adjunction for equality (with Frobenius)* in doctrines [Jac99, 3.2.4] where symmetric equality is presented as a left adjoint to reindexing with a diagonal functor which identifies together a, b with z in $P(z, z, x)$ using the following rule:

$$\dfrac{[z : \mathbb{A}^{\mathsf{core}}, x : \mathbb{C}^{\mathsf{core}}] \qquad\qquad k : \Gamma(z, z, x) \vdash h[k] : P(z, z, x)}{[a : \mathbb{A}^{\mathsf{core}}, b : \mathbb{A}^{\mathsf{core}}, x : \mathbb{C}^{\mathsf{core}}] \; e : \hom_{\mathbb{A}^{\mathsf{core}}}(a, b), k : \Gamma(b, a, x) \vdash J(h)[e, k] : P(a, b, x)}\,(\mathsf{eq})$$

It is not obvious how to use Theorem 3.7 to similarly present directed equality as a left adjoint to a contraction-like functor, as exemplified for the case of symmetric equality in Remark 4.1: postponing the compositionality problem for paracategories of dinaturals and parafunctors, the main difficulty comes from the syntactic restrictions on variances imposed on both sides of the dinatural, which in practice forces functors to have an action on objects but not on morphisms.

Our solution to characterize directed equality is to consider instead a *relative adjunction* [Ulm68, AM24] between a functor introducing directed equality in context (which is the relative *left* adjoint, as in the case of equality) and a contraction-like functor (which is a *right* adjoint like the classical case) which contracts two *natural* variables $x : \mathbb{A}^{\mathsf{op}}, y : \mathbb{A}$ with opposite variances into a single *dinatural* one $x : \mathbb{A}$, using the same idea behind Lemma 2.5. The relativeness is taken with respect to a projection-like functor, which is precisely what allows us to impose the syntactic restriction on the propositional context in such a way that it does not depend on positive occurrences of the variables that we collapse together.

We recall the definition of relative adjunction that we will employ.

Definition 4.2 (Relative adjunction [AM24, 5.1]). Consider an arrangement of categories and functors as follows:

$$\begin{array}{ccc} & \mathbb{D} & \\ {\scriptstyle L}\nearrow & \dashv & \searrow{\scriptstyle R} \\ \mathbb{C} & \xrightarrow[J]{} & \mathbb{X} \end{array}$$

In this situation, we say that L *is the J-relative left adjoint to* R, written $L \dashv_J R$ and indicated in the above diagram with a central '$\dashv$', if there is a bijection

$$\mathbb{D}(L(x), y) \cong \mathbb{X}(J(x), R(y))$$

natural in both arguments $x : \mathbb{C}, y : \mathbb{D}$.

One obtains the standard definition of adjunction when $\mathbb{X} = \mathbb{C}$, $J = \mathsf{id}_{\mathbb{C}}$. The most notable difference with plain adjunctions is that this relation between functors is *not* symmetric, i.e. if $L \dashv_J R$, it is not true in general that $R \vdash_J L$ (for a suitably defined notion of relative adjunction for right adjoints), and R is not uniquely determined by the pair (L, J). However, L is uniquely determined by the pair (R, J), and it arises as the *absolute left lifting* of J along R [AM24, 5.9].

Theorem 4.3 (Directed equality as relative adjunction). Take $[\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}]$ to be the paracategory where morphisms are dinatural transformations *natural* in $\mathbb{A}^{\mathsf{op}}, \mathbb{A}$ and *dinatural* in $\mathbb{C}$. Similarly, $[\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathsf{Set}] := [(\mathbb{A} \times \mathbb{C})^{\diamond}, \mathsf{Set}]$ up to reordering of variables. Let the functor

$$\pi^*_{\mathbb{A}} : [\mathbb{C}^{\diamond}, \mathsf{Set}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathsf{Set}]$$

be defined in the intuitive way by precomposing with the projection.

There is a dipresheaf $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \mathsf{Set}]$ such that the functor

$$\begin{aligned} \hom_{\mathbb{A}} \times - &: [\mathbb{C}^{\diamond}, \mathsf{Set}] \to [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}] \\ \hom_{\mathbb{A}} \times \Gamma &:= \hom_{\mathbb{A}} \times (\pi^*_{\mathbb{A}^{\mathsf{op}} \times \mathbb{A}})(\Gamma) = (a', a, x', x) \mapsto \hom(a', a) \times \Gamma(x', x) \\ (\hom_{\mathbb{A}} \times \alpha_x)_{abc} &:= \lambda(e \in \hom(a, b), k \in \Gamma(c, c)).(e, \alpha_c(k)) \end{aligned}$$

determines a $\pi^*_{\mathbb{A}}$-relative left adjoint to the functor

$$\begin{aligned} \Delta_{\mathbb{A}} \times - &: [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}] \to [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathsf{Set}] \\ \Delta_{\mathbb{A}} \times P &:= P \\ (\Delta_{\mathbb{A}} \times \alpha_{abc})_{zx} &:= \alpha_{zzx} \end{aligned}$$

which is defined on morphisms by following the same idea of Lemma 2.5. Thus the relative adjointness situation $\hom_{\mathbb{A}} \times - \dashv_{\pi^*_{\mathbb{A}}} \Delta_{\mathbb{A}} \times -$ is as follows:

$$\begin{array}{ccc} & [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}] & \\ {\scriptstyle \hom_{\mathbb{A}} \times -}\nearrow & \dashv & \searrow{\scriptstyle \Delta_{\mathbb{A}} \times -} \\ [\mathbb{C}^{\diamond}, \mathsf{Set}] & \xrightarrow[\pi^*_{\mathbb{A}}]{} & [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathsf{Set}] \end{array}$$

Moreover, a Frobenius-like condition for directed equality is automatic in this model thanks to the presence of exponentials [Jac99, 1.9.12].

Proof. The required isomorphism is given by the following,

$$\frac{[z : \mathbb{A}, x : \mathbb{C}] \quad (\pi^*_{\mathbb{A}}(\Gamma)) = \Gamma(\overline{x}, x) \vdash P(\overline{z}, z, \overline{x}, x)}{[a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}] \ \hom_{\mathbb{A}}(a, b) \times \Gamma(\overline{x}, x) \vdash P(a, b, \overline{x}, x)} \ \text{(hom-rel-adj)}$$

which simply holds by Theorem 3.7 and picking Γ to not depend on $\overline{a} : \mathbb{A}^{\mathsf{op}}, \overline{b} : \mathbb{A}$. Note that (hom-rel-adj) is actually equivalent to Theorem 3.7 simply by picking $P := \Gamma^{\mathsf{op}}(b, a, x, \overline{x}) \Rightarrow P(a, b, \overline{x}, x)$ and applying twisted exponentials. □

Compare the above characterization of directed equality as a left adjoint with the usual presentation of equality in (elementary) doctrines [MR15, Sec. 2], in particular with the existence of a point $\delta_A \in P(A \times A)$ for a doctrine $P : \mathbb{C}^{\mathsf{op}} \to \mathsf{InfSL}$ for which this role is played in our case by $\hom_{\mathbb{A}} \in [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \mathsf{Set}]$. One can view the above situation as a concretization of a remark by Lawvere [Law70, p.11] on the "vitality" of the presheaf hyperdoctrine and the role played by hom-functors. Lawvere comments that for this model the usual definition of equality $\mathsf{Eq}_Y(\top_Y)$ does not provide the "right" definition of equality, which really should be semantically represented as hom; in his words (*ibi*),

> [. . .] This should not be taken as indicative of a lack of vitality of [the presheaf hyperdoctrine] as a hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. Equality should be the "graph" of the identity term. But present categorical conceptions indicate that [. . .] the graph of a functor $f : \mathbb{B} \to \mathbb{C}$ should be [. . .] a binary attribute of mixed variance in $P(\mathbb{B}^{\mathsf{op}} \times \mathbb{C})$ [*the presheaf category* $[\mathbb{B}^{\mathsf{op}} \times \mathbb{C}, \mathsf{Set}]$]. Thus in particular "equality" should be the functor $\hom_{\mathbb{B}}$ [. . .]. The term which would take the place of δ in such a more enlightened theory of equality would then be the forgetful functor $\mathsf{Tw}(\mathbb{B}) \to \mathbb{B}^{\mathsf{op}} \times \mathbb{B}$ from the "twisted morphism category" [. . .]. Of course to abstract from this example would require at least the addition of a functor $T \xrightarrow{\mathsf{op}} T$ to the structure of a [doctrine]. [Law70, p. 11]

There is an equivalent formulation of relative adjunctions in terms of suitable universal arrows playing the role of *unit* and *counit* of the adjunction, satisfying certain conditions, which we instantiate in the case of Theorem 4.3.

Definition 4.4 (Unit and counit for a relative adjunction [AM24, 5.5(3)])**.** Given a situation as in Definition 4.2, an equivalent condition for $L \dashv_J R$ is given by the existence of the following: a "unit" 2-cell $J \Rightarrow L \,;\, R$ and a "counit" 2-cell $\mathbb{D}(1, L), \mathbb{X}(J, R) \Rightarrow \mathbb{D}(1, 1)$ satisfying suitable equations.

We explicitly unfold the unit and counit maps given by the relative adjunction in Theorem 4.3 and see how they similarly justify the presentation of directed equality via introduction and elimination rules.

Theorem 4.5 (Unit and counit for Theorem 4.3)**.** The signature of the unit is given as the following 2-cell, for which we progressively unfold the definition:

$$\eta : \pi_{\mathbb{A}}^* \Rightarrow (\hom_{\mathbb{A}} \times -) \,;\, (\Delta_{\mathbb{A}} \times -)$$
$$\eta(P : [\mathbb{C}^{\diamond}, \mathsf{Set}]) : [a : \mathbb{A}, x : \mathbb{C}]\ P(\overline{x}, x) \vdash \hom_{\mathbb{A}}(\overline{a}, a) \times P(\overline{x}, x)$$

i.e., it essentially provides a semantic justification for the (hom-intro) rule stating that directed equality is pointed with identities. (Note that η is a collection of dinatural transformations η_P indexed by functors $P : [\mathbb{C}^{\diamond}, \mathsf{Set}]$, hence η is a natural transformation [HM03, HM04].)

Following Definition 4.4, the counit is given as the following 2-cell:

$$\varepsilon : \mathbb{D}(1, \hom_{\mathbb{A}} \times -), \mathbb{X}(\pi_{\mathbb{A}}^*, \Delta_{\mathbb{A}} \times -) \Rightarrow \mathbb{D}(1, 1)$$

Explicitly, composing the profunctors in the domain:

$$\begin{array}{rl} \varepsilon(P, Q : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}]) & : \displaystyle\int^{X:[\mathbb{C}^{\diamond},\mathsf{Set}]} [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}](P, \hom_{\mathbb{A}} \times X) \\ & \qquad \times [\mathbb{A}^{\diamond}, \mathsf{Set}](\pi^*_{\mathbb{A}}(X), \Delta_{\mathbb{A}} \times Q) \\ & \Rightarrow [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}](P, Q) \end{array}$$

The individual maps out of the coend are defined as follows (for simplicity we henceforth omit the quotient imposed on X),

$$\begin{array}{rcl} \varepsilon(P, Q : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}], X : [\mathbb{C}^{\diamond}, \mathsf{Set}]) & : & [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}](P, \hom_{\mathbb{A}} \times X) \\ & \times & [\mathbb{A}^{\diamond} \times \mathbb{C}^{\diamond}, \mathsf{Set}](\pi^*_{\mathbb{A}}(X), \Delta_{\mathbb{A}} \times Q) \\ & \Rightarrow & [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}](P, Q) \end{array}$$

which, using our logical notation for sets of dinaturals, becomes the following rule (parametric in P, Q, X):

$$\begin{array}{rl} & \varepsilon(P, Q : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}^{\diamond}, \mathsf{Set}], X : [\mathbb{C}^{\diamond}, \mathsf{Set}]) \\ : & ([a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ P(a, b, \overline{x}, x) \vdash \hom(a, b, \overline{x}, x) \times X(\overline{x}, x)) \\ \wedge & ([z : \mathbb{A}, x : \mathbb{C}]\ X(\overline{x}, x) \vdash Q(\overline{z}, z, \overline{x}, x)) \\ \Rightarrow & ([a : \mathbb{A}^{\mathsf{op}}, b : \mathbb{A}, x : \mathbb{C}]\ P(a, b, \overline{x}, x) \vdash Q(a, b, \overline{x}, x)) \end{array}$$

which is essentially a generalized form of the J principle with an equality in context given in Theorem 3.6. In particular, it is equivalent to Theorem 3.6 by picking $\mathbb{C} := \mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}', X := P$ and then suitably selecting the projections.

5. (Co)ends as quantifiers

In this section we describe how dinaturality allows us to give an interpretation of ends and coends as quantifiers à-la-Lawvere [Law69] for the (1-)category interpretation of directed type theory; in particular, we give logical rules for dinaturals which identify ends and coends as, respectively, the right and left adjoints to a common structural "weakening-like" functor which only operates on the context [Jac99, 1.9.1]. This is semantically motivated by the well-known connection between (co)ends and dinatural transformations.

5.1. Background

We start by recalling some background material on (co)end calculus and their relation to dinaturality in order to keep this paper self-contained; the reader can find more details on (co)end calculus and its applications in [ML98, IX.5-6] and [Lor21, Ch. 1]).

Definition 5.1 ((Co)wedges for P [Lor21, 1.1.4])**.** Fix a functor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$. A *wedge for P* is a pair object/dinatural $(X : \mathbb{D}, \alpha : \mathsf{K}_X \xrightarrow{\bullet\bullet} P)$, where K_X is the constant functor in X. A *wedge morphism* from (X, α) to (X, α') is a morphism $f : X \to Y$ such that $\alpha_c = u \mathbin{;} \alpha'_c$ for every $c : C$. Similarly, a *cowedge* is a wedge in $\mathbb{D}^{\mathsf{op}}$. The category of wedges for P is denoted by $\mathsf{Wedge}(P)$, similarly with $\mathsf{Cowedge}(P)$.

Definition 5.2 ((Co)ends [Lor21, 1.1.6])**.** Given a functor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$, the *end* of P is defined to be the terminal object of $\mathsf{Wedge}(P)$, i.e., a pair comprising of an object, denoted as $\int_{x:\mathbb{C}} P(\overline{x}, x) : \mathbb{D}$, and a dinatural

$$[c : \mathbb{C}] \int_{x:\mathbb{C}} P(\overline{x}, x) \vdash \pi : P(\overline{c}, c).$$

Similarly, the *coend* of P is the initial object of $\mathsf{Cowedge}(P)$, i.e., a pair with an object $\int_{x:\mathbb{C}} P(\overline{x}, x) : \mathbb{D}$ and a dinatural

$$[c : \mathbb{C}]\ P(\overline{c}, c) \vdash \iota : \int^{x:\mathbb{C}} P(\overline{x}, x).$$

Remark 5.3 (On the notation for (co)ends)**.** The integral symbol acts as a binder which makes the expression '$P(\overline{x}, x)$' independent from the parameter x, in the sense that the expressions '$\int_{c:\mathbb{C}} P(\overline{c}, c)$' and '$\int_{c:\mathbb{C}} P(\overline{c}, c)$' are α-equivalent.

Note that there might be other parameters $b : \mathbb{C}, c : \mathbb{D}, \ldots$ on which P depends, i.e. it is possible (and encoded from the very beginning in our syntax) that P has type $(\mathbb{A}^{\mathsf{op}} \times \mathbb{A}) \times (\mathbb{B}^{\mathsf{op}} \times \mathbb{B}) \to \mathbb{D}$ and thus $\int_{b:\mathbb{B}} P(\overline{a}, a, \overline{b}, b)$ has type $\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \to \mathbb{D}$.

A natural question is to ask when either of these universal objects exists for a given P; it turns out that existence of (co)ends is ensured by the existence of certain (co)limits in $\mathbb{D}$, since a (co)end can be computed as follows:

Proposition 5.4 ((Co)ends as equalizers [Lor21, 1.2.4])**.** Given a functor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ with $\mathbb{D}$ complete, the **end** of P, denoted as '$\int_{x:\mathbb{C}} P(\overline{x}, x)$' is the object of $\mathbb{D}$ obtained as an equalizer of the conjoint actions of P on morphisms:

$$\int_{x:\mathbb{C}} P(\overline{x}, x) := \mathsf{eq}\left(\prod_{x:\mathbb{C}} P(x, x) \overset{P_\ell}{\underset{P_r}{\rightrightarrows}} \prod_{a:\mathbb{C}, b:\mathbb{C}, f:a\to b} P(a, b) \right)$$

where P_ℓ, P_r are the induced morphisms by the universal property of the product,

$$P_\ell := \langle \pi_b \mathbin{;} P(f, \mathsf{id}_b) \rangle_{a,b,f} \qquad P_r := \langle \pi_a \mathbin{;} P(\mathsf{id}_a, f) \rangle_{a,b,f}$$

and we denote with $\langle ... \rangle_{a,b,f}$ the universal map of the second product and a, b, f are the indices over which the product is taken.

Dually, the **coend** of P is denoted '$\int^{x:\mathbb{C}} P(\overline{x}, x)$' and is described by the following coequalizer:

$$\int^{x:\mathbb{C}} P(\overline{x}, x) := \mathsf{coeq}\left(\sum_{a:\mathbb{C}, b:\mathbb{C}, f:a\to b} P(b, a) \overset{P^\ell}{\underset{P^r}{\rightrightarrows}} \sum_{c:\mathbb{C}} P(c, c) \right)$$

and P^ℓ, P^r are the induced morphisms by the universal property of the coproduct:

$$P^\ell := [P(f, \mathsf{id}_a) \mathbin{;} \iota_a]_{a,b,f} \qquad P^r := [P(\mathsf{id}_b, f) \mathbin{;} \iota_b]_{a,b,f}$$

Moreover, sending a functor to its (co)end is a functorial operation for natural transformations:

Lemma 5.5 ((Co)ends as functors for naturals)**.** Given functors $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, a *natural* transformation $P \longrightarrow Q$ induces a morphism

$$\int_{x:\mathbb{C}} P(\overline{x}, x) \to \int_{x:\mathbb{C}} Q(\overline{x}, x)$$

in Set between the (co)ends. One can show that sending a functor to its (co)end is a functorial operation with respect to naturals, i.e., there are functors as follows:

$$\textstyle\int_{\mathbb{C}}, \int^{\mathbb{C}} : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \mathsf{Set}] \to \mathsf{Set}$$

and more generally the following "parameterized ends" are functorial,

$$\textstyle\int_{\mathbb{C}[\mathbb{A}]}, \int^{\mathbb{C}[\mathbb{A}]} : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A} \times \mathbb{C}, \mathsf{Set}] \to [\mathbb{C}, \mathsf{Set}]$$

where the morphisms of all categories involved are natural transformations.

5.2. (Co)ends as quantifiers

Our goal for the rest of the section is to make precise sense of (co)ends as quantifiers; to do this we follow the approach of categorical logic which characterizes quantifiers as adjoints to a mutual structural operation [Jac99, 1.9.1] only operating on the context. This presentation has the advantage that several properties of quantifiers, e.g., that they can be exchanged and permuted whenever possible, follow automatically from certain structural properties of contexts. For example, in first order logic the formulas $\forall x.\forall y.Q(x,y) \iff \forall y.\forall x.Q(x,y) \iff \forall (x,y).Q(x,y)$ are logically equivalent for any predicate P: this property is indeed also verified in the case of ends and takes the name of "Fubini rule" [ML98, First proposition of IX.8], [Lor21, 1.3.1] (similarly with coends and existential quantifiers):

$$\begin{array}{c}
\displaystyle\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} Q(\overline{x},x,\overline{y},y) \;\cong\; \int_{y:\mathbb{D}} \int_{x:\mathbb{C}} Q(\overline{x},x,\overline{y},y) \;\cong\; \int_{(x,y):\mathbb{C}\times\mathbb{D}} Q(\overline{x},x,\overline{y},y) \\
\hline\hline
\forall x.\ \forall y.\ Q(x,y) \;\Leftrightarrow\; \forall y.\ \forall x.\ Q(x,y) \;\Leftrightarrow\; \forall (x,y).\ Q(x,y)
\end{array}$$

The universal property of ends states that morphisms into the end of a functor P are essentially the same as wedges for P; viewing dinatural transformations as entailments of a logical system, this defining property of ends is precisely what enables them to be considered as "universal quantifiers" in a correspondence reminiscent of right adjointness with weakening functors. Coends have a similar universal property for morphisms out of them, which characterizes them as left adjoint to the same weakening functor. For simplicity we assume for the rest of the section to only take the (co)ends of dipresheaves.

Theorem 5.6 (Ends and coends as quantifiers)**.** Given a dipresheaf $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, we denote with $\pi^*_{\mathbb{A}}(P) : (\mathbb{A}^{\mathsf{op}} \times \mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathsf{Set}$ the functor defined by $(\overline{a}, a, \overline{x}, x) \mapsto P(\overline{x}, x)$ which precomposes with the projection.

There are isomorphisms of sets of dinatural transformations (natural in $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, $Q : (\mathbb{A}^{\mathsf{op}} \times \mathbb{A}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathsf{Set}$):

$$\begin{array}{c}
[a:\mathbb{A}, x:\mathbb{C}]\ \pi^*_{\mathbb{A}}(P)(\overline{a},a,\overline{x},x) \vdash Q(\overline{a},a,\overline{x},x) \\
\hline\hline
[x:\mathbb{C}] \qquad P(\overline{x},x) \vdash \displaystyle\int_{a:\mathbb{A}} Q(\overline{a},a,\overline{x},x)
\end{array}\ (\text{end})$$

$$\begin{array}{c}
[x:\mathbb{C}]\ \displaystyle\int^{a:\mathbb{A}} Q(\overline{a},a,\overline{x},x) \vdash P(\overline{x},x) \\
\hline\hline
[a:\mathbb{A}, x:\mathbb{C}] \qquad Q(\overline{a},a,\overline{x},x) \vdash \pi^*_{\mathbb{A}}(P)(\overline{a},a,\overline{x},x)
\end{array}\ (\text{coend})$$

Proof. The result follows since the set of morphisms into(/out of) the (co)end of P are isomorphic to (co)wedges for P, up to an extra variable in the term context. ☐

As customary in logic, we shall leave the weakening functors implicit for the rest of the paper whenever we use the above rules.

Remark 5.7. In order to make the analogy of (co)ends as quantifiers precise and view Theorem 5.6 as an adjunction between functors, we need to consider the functoriality of sending a difunctor to its (co)end. As shown in Lemma 5.5, taking (co)ends is indeed a functorial operation for natural transformations. However, it is *not* true in general that (co)ends are also functorial operators with respect to dinatural transformations; intuitively because dinaturals do not have enough maps to induce a morphism using the universal property of (co)ends.

In the following we give a characterization for *ends* to be indeed functorial: this is the case precisely when ends are defined as functors from categories of families of dinaturals which all compose. For simplicitly, we will not consider the case of parameterized ends, but the more general case can be similarly proved with the same idea. Coends are also functorial when all dinaturals compose, but we do not yet know if the converse is true.

Lemma 5.8 (Dinaturals as ends [DS70, Thm. 1]). Given $P, Q : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, the set of dinaturals $\mathsf{Dinat}(P, Q) := \{P \xrightarrow{\bullet\bullet} Q\}$ can be characterized in terms of the following end:

$$\mathsf{Dinat}(P,Q) \cong \int_{x:\mathbb{C}} P^{\mathsf{op}}(x,\overline{x}) \Rightarrow Q(\overline{x},x)$$

Proof. We give a simple proof by characterizing all points of the end in terms of the logical rule introduced previously for ends:

$$\mathsf{Dinat}(P,Q) := \cfrac{\cfrac{[x:\mathbb{C}]\ P(\overline{x},x) \vdash Q(\overline{x},x)}{[x:\mathbb{C}]\ \top \vdash P^{\mathsf{op}}(x,\overline{x}) \Rightarrow Q(\overline{x},x)}\ (\mathsf{exp})}{[\,]\ \top \vdash \int_{x:\mathbb{C}} P^{\mathsf{op}}(x,\overline{x}) \Rightarrow Q(\overline{x},x)}\ (\mathsf{end})$$

(both inference lines are double) □

Lemma 5.9 (Functoriality of hom-functors with dinaturals). Given $P, P', Q, Q' : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ where $\mathbb{D}$ is cartesian closed, the internal hom-functor of $\mathbb{D}$ is functorial with respect to dinatural transformations:

$$\frac{[x:\mathbb{C}]\ P'(\overline{x},x) \vdash p : P(\overline{x},x), \qquad [x:\mathbb{C}]\ Q(\overline{x},x) \vdash q : Q'(\overline{x},x)}{[x:\mathbb{C}]\ (P^{\mathsf{op}}(x,\overline{x}) \Rightarrow Q(\overline{x},x)) \vdash (p^{\mathsf{op}} \Rightarrow q) : (P'^{\mathsf{op}}(x,\overline{x}) \Rightarrow Q'(\overline{x},x))}\ (\Rightarrow\text{-}\mathsf{func})$$

Theorem 5.10 (Ends are functorial ⇔ all dinaturals compose). Sending a dipresheaf to its end is functorial with respect to dinaturals (i.e., there are functors $\int_{\mathbb{C}}, \int^{\mathbb{C}} : [\mathbb{A}^{\diamond}, \mathsf{Set}] \to \mathsf{Set}$) *if and only if* all the families of dinaturals considered in the domain $[\mathbb{A}^{\diamond}, \mathsf{Set}]$ compose, i.e., it is indeed a category.

Proof. We describe the proof in detail:

(⇒) Assume to have two dinaturals $\alpha \in \mathsf{Dinat}(P,Q)$ and $\beta \in \mathsf{Dinat}(Q,R)$; we need to construct a dinatural $\alpha;\beta : \mathsf{Dinat}(P,R)$. Using Lemma 5.9, the following is a dinatural transformation:

$$\frac{\overline{[x:\mathbb{C}]\ P(\overline{x},x) \vdash \alpha : Q(\overline{x},x),} \qquad \overline{[x:\mathbb{C}]\ R(\overline{x},x) \vdash \mathsf{id} : R(\overline{x},x)}}{[x:\mathbb{C}]\ (Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)) \vdash (\alpha^{\mathsf{op}} \Rightarrow \mathsf{id}) : (P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x))}\ (\Rightarrow\text{-}\mathsf{func})$$

By applying the assumed functoriality of ends on the dinatural $\alpha^{\mathsf{op}} \Rightarrow \mathsf{id}$, we obtain a morphism in Set as follows,

$$\textstyle\int_{\mathbb{C}}(\alpha^{\mathsf{op}} \Rightarrow \mathsf{id}) : \left(\displaystyle\int_{x:\mathbb{C}} Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)\right) \to \left(\displaystyle\int_{x:\mathbb{C}} P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)\right).$$

By Lemma 5.8, we are given a point in the end $\beta \in \mathsf{Dinat}(Q,R) \cong \int_{x:\mathbb{C}} Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)$, and by applying the previous morphism to it we obtain a point $(\int_{\mathbb{C}} \alpha^{\mathsf{op}} \Rightarrow \mathsf{id})(\beta) \in \mathsf{Dinat}(P,R)$ as desired.

(⇐) Assume all dinaturals compose, and take a dinatural $\alpha : P \xrightarrow{\bullet\bullet} Q$ for $P, Q : \mathbb{A}^{\mathsf{op}} \times \mathbb{A} \to \mathsf{Set}$ to apply the functorial action of ends to. By instantiating (end) with $\mathbb{C} := \mathbb{1}$, $P := \int_{x:\mathbb{A}} P(\overline{x},x)$ and picking the bottom side to be the identity dinatural, we obtain the following "counit-like" dinatural ε:

$$\frac{\overline{[\,]\ \int_{x:\mathbb{A}} P(\overline{x},x) \vdash \mathsf{id} : \int_{x:\mathbb{A}} P(\overline{x},x)}\ (\mathsf{id})}{[x:\mathbb{A}]\ \pi_{\mathbb{A}}^{*}\left(\int_{x:\mathbb{A}} P(\overline{x},x)\right) \vdash \varepsilon : P(\overline{x},x)}\ (\mathsf{end}).(\Uparrow)$$

Composing α with ε and then applying the other direction of (end) gives us the desired map:

$$\dfrac{\dfrac{[x:\mathbb{A}]\ \pi^*_{\mathbb{A}}\left(\int_{x:\mathbb{A}} P(\overline{x},x)\right) \vdash \varepsilon : P(\overline{x},x), \qquad [x:\mathbb{A}]\ P(\overline{x},x) \vdash \alpha : Q(\overline{x},x)}{[x:\mathbb{A}]\ \pi^*_{\mathbb{A}}\left(\int_{x:\mathbb{A}} P(\overline{x},x)\right) \vdash \varepsilon \,;\, \alpha : Q(\overline{x},x)}\ \text{(composition)}}{[]\ \int_{x:\mathbb{A}} P(\overline{x},x) \vdash \mathsf{end}(\varepsilon \,;\, \alpha) : \int_{x:\mathbb{A}} Q(\overline{x},x)}\ \text{(end)}.(\Downarrow)$$

□

In the previous theorem we only used composition on the left by the map ε; this map is universal, in the sense that always being able to compose on the left with this specific map entails that all dinaturals compose:

Theorem 5.11. Consider a family of dinaturals Λ closed under ($\Rightarrow$-func) for which every map has a composite on the left with the map $\varepsilon : \mathsf{K}_{\int_{x:\mathbb{A}} P(\overline{x},x)} \xrightarrow{\bullet\bullet} P$. Then the composition between a map $\alpha \in \Lambda$ and any dinatural β (not necessarily in Λ) is again dinatural.

Proof. Assume $\alpha \in \mathsf{Dinat}(P,Q)$ for which $\alpha \in \Lambda$. We can construct a map $\gamma_\alpha \in \Lambda$:

$$\dfrac{\overline{[x:\mathbb{C}]\ P(\overline{x},x) \vdash \alpha : Q(\overline{x},x)}, \qquad \overline{[x:\mathbb{C}]\ R(\overline{x},x) \vdash \mathsf{id} : R(\overline{x},x)}}{[x:\mathbb{C}]\ (Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)) \vdash \gamma_\alpha := (\alpha^{\mathsf{op}} \Rightarrow \mathsf{id}) : (P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x))}\ (\Rightarrow\text{-func})$$

By composing ε instantiated on $Q \Rightarrow R$ with γ_α and then reintroducing the end,

$$\dfrac{\dfrac{\begin{array}{c}[x:\mathbb{A}]\ \pi^*_{\mathbb{A}}\left(\int_{x:\mathbb{A}} Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)\right) \vdash \varepsilon : Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x),\\ [x:\mathbb{A}]\ Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x) \vdash \gamma_\alpha : P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)\end{array}}{[x:\mathbb{A}]\ \pi^*_{\mathbb{A}}\left(\int_{x:\mathbb{A}} Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)\right) \vdash \varepsilon \,;\, \gamma_\alpha : P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)}\ \text{(composition)}}{[]\ \int_{x:\mathbb{A}} Q^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x) \vdash \mathsf{end}(\varepsilon \,;\, \gamma_\alpha) : \int_{x:\mathbb{A}} P^{\mathsf{op}}(x,\overline{x}) \Rightarrow R(\overline{x},x)}\ \text{(end)}.(\Downarrow)$$

we obtain a map in Set which, using Lemma 5.8, expresses that any $\beta \in \mathsf{Dinat}(Q,R)$ composes with α to obtain a dinatural $(\mathsf{end}(\varepsilon \,;\, \gamma_\alpha))(\beta) : \mathsf{Dinat}(P,R)$ (both β and the composition are not necessarily in Λ). □

Lemma 5.12 ((Co)ends as adjoints to weakening)**.** Assume that (co)ends are functorial for dinaturals for some family of dinaturals in $[(\mathbb{A} \times \mathbb{C})^\diamond, \mathsf{Set}]$ as in Theorem 5.10.

Consider the functors

$$\textstyle\int_{\mathbb{A}[\mathbb{C}]}, \int^{\mathbb{A}[\mathbb{C}]} : [(\mathbb{A} \times \mathbb{C})^\diamond, \mathsf{Set}] \to [\mathbb{C}^\diamond, \mathsf{Set}]$$

sending dipresheaves into their (co)end in the variable $a : \mathbb{A}$ and the functor precomposing with projections

$$\pi^*_{\mathbb{A}[\mathbb{C}]} : [\mathbb{C}^\diamond, \mathsf{Set}] \to [(\mathbb{A} \times \mathbb{C})^\diamond, \mathsf{Set}].$$

Then, Theorem 5.6 states that the following adjunction situation holds:

$$\textstyle\int^{\mathbb{A}[\mathbb{C}]} \dashv \pi^*_{\mathbb{A}[\mathbb{C}]} \dashv \int_{\mathbb{A}[\mathbb{C}]}$$

Recall that in categorical logic one has that quantifiers typically have to satisfy additional requirements in order to faithfully model logical operations. In particular, the Beck-Chevalley condition [Jac99, 1.9.4] logically expresses that "quantifiers commute with substitution"; moreover, the Frobenius condition [Jac99, 1.9.12] is a distributivity condition for colimit-like operations, and it logically corresponds with the fact that rules are expressed parametrically with an extra context parameter [Jac99, Sec. 3.4]. We verify in the following that in the case of (co)ends for dinaturals these are indeed satisfied.

Theorem 5.13 (Beck-Chevalley and Frobenius condition for (co)ends)**.** The functors given in Lemma 5.12 for (co)ends satisfy a *Beck-Chevalley condition* in the following sense: for any difunctor $F : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$ acting as term, the following functors of type $[\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \mathsf{Set}] \to [\mathbb{C}^\diamond, \mathsf{Set}]$ coincide strictly:

$$\textstyle\int_{\mathbb{A}[\mathbb{D}]} ; F^* = (\mathsf{id}_{\mathbb{C}^\diamond} \times F)^* ; \int_{\mathbb{A}[\mathbb{C}]}$$

where $F^* : [\mathbb{D}^\diamond, \mathsf{Set}] \to [\mathbb{C}^\diamond, \mathsf{Set}]$ indicates reindexing by F as in Theorem 2.17, and the above similarly holds for coends.

Moreover, a *Frobenius condition* for coends is satisfied, since for any presheaf $\Gamma : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \mathsf{Set}], P : [\mathbb{C}^\diamond, \mathsf{Set}]$ the following dipresheaves in $[\mathbb{C}^\diamond, \mathsf{Set}]$ coincide strictly:

$$\int^{\mathbb{A}[\mathbb{C}]} (\pi^*_{\mathbb{A}[\mathbb{C}]}(P) \wedge \Gamma) = P \wedge \int^{\mathbb{A}[\mathbb{C}]} (\Gamma),$$

where $- \wedge - : [\mathbb{C}^\diamond, \mathsf{Set}] \times [\mathbb{C}^\diamond, \mathsf{Set}] \to [\mathbb{C}^\diamond, \mathsf{Set}]$ is the parafunctor taking the pointwise product of dipresheaves. This Frobenius condition for coends is satisfied automatically since (propositional) exponentials exist and they are given in Theorem 2.16 [Jac99, 1.9.12(i)].

Proof. Explicitly, Beck-Chevalley states that the following two dipresheaves coincide strictly for $P : [\mathbb{A}^\diamond \times \mathbb{D}^\diamond, \mathsf{Set}]$:

$$\frac{[\overline{x} : \mathbb{C}^{\mathsf{op}}, x : \mathbb{C}]\ ((\mathsf{id}_{\mathbb{C}^\diamond} \times F)^* ; \textstyle\int_{\mathbb{A}[\mathbb{C}]})(P) = \displaystyle\int_{a:\mathbb{A}[x:\mathbb{C}]} P(\overline{a}, a, F^{\mathsf{op}}(x, \overline{x}), F(\overline{x}, x)) : \mathsf{Set}}{[\overline{x} : \mathbb{C}^{\mathsf{op}}, x : \mathbb{C}]\ (\textstyle\int_{\mathbb{A}[\mathbb{D}]} ; F^*)(P) = (F ; -)\left(\lambda \overline{y}, y.\ \displaystyle\int_{a:\mathbb{A}[y:\mathbb{C}]} P(\overline{a}, a, \overline{y}, y)\right) : \mathsf{Set}}$$

To prove Frobenius explicitly, we can use our logical rules to apply exactly the same proof technique given in [Jac99, 1.9.12(i)] using a Yoneda-like reasoning, where for any $\Gamma : [\mathbb{A}^\diamond \times \mathbb{C}^\diamond, \mathsf{Set}], P : [\mathbb{C}^\diamond, \mathsf{Set}]$ and a generic $K : [\mathbb{C}^\diamond, \mathsf{Set}]$, we have the following isomorphism:

$$\begin{array}{c}
[y : \mathbb{C}]\ \displaystyle\int^{x:\mathbb{A}[y:\mathbb{C}]} P(\overline{y}, y), \Gamma(\overline{x}, x, \overline{y}, y) \vdash K(\overline{y}, y) \\
\hline\hline
[x : \mathbb{A}, y : \mathbb{C}]\ P(\overline{y}, y), \Gamma(\overline{x}, x, \overline{y}, y) \vdash K(\overline{y}, y) \quad (\mathsf{coend}) \\
\hline\hline
[x : \mathbb{A}, y : \mathbb{C}]\ \Gamma(\overline{x}, x, \overline{y}, y) \vdash P^{\mathsf{op}}(y, \overline{y}) \Rightarrow K(\overline{y}, y) \quad (\mathsf{exp}) \\
\hline\hline
[y : \mathbb{C}]\ \displaystyle\int^{x:\mathbb{A}[y:\mathbb{C}]} \Gamma(\overline{x}, x, \overline{y}, y) \vdash P^{\mathsf{op}}(y, \overline{y}) \Rightarrow K(\overline{y}, y) \quad (\mathsf{coend}) \\
\hline\hline
[y : \mathbb{C}]\ P(\overline{y}, y), \displaystyle\int^{x:\mathbb{A}[y:\mathbb{C}]} \Gamma(\overline{x}, x, \overline{y}, y) \vdash K(\overline{y}, y) \quad (\mathsf{exp})
\end{array}$$

where the weakening functors are left implicit in the derivation. We give the previous derivation the name of (coend+frobenius), which we later use in the examples. □

In order to frame the importance of Lemma 5.12, note that for categories of *natural transformations* a similar situation happens where ends and coends are, respectively, left adjoints and right adjoints. However, in the case of naturals, the functors to which (co)ends are adjoint to are *not* the same, and they are not in a sense structural. We are able to give an intuitive explanation for this phenomenon using the logical rules previously introduced for (co)ends and directed equality.

Lemma 5.14 (Ends and coends as adjoints for naturals)**.** The adjointness situation for (co)ends as functors from categories of *natural* transformations does not place them as adjoint to a common functor: viewing them as functors $\int_{\mathbb{A}}, \int^{\mathbb{A}} : [\mathbb{A}^{\mathsf{op}} \times \mathbb{A}, \mathsf{Set}] \to \mathsf{Set}$ from presheaf categories, the following situation arises:

$$\int^{\mathbb{A}} \dashv (\hom_{\mathbb{A}} \Rightarrow -) \qquad \ncong \qquad (\hom_{\mathbb{A}} \times -) \dashv \int_{\mathbb{A}}$$

where the two functors are defined as follows:

$$\begin{aligned} (\hom_{\mathbb{A}} \times X)(x,y) &:= \hom_{\mathbb{A}}(x,y) \times X, \\ (\hom_{\mathbb{A}} \Rightarrow X)(x,y) &:= \hom^{\mathsf{op}}_{\mathbb{A}}(y,x) \Rightarrow X. \end{aligned}$$

Proof. We give a direct proof for the adjunction using the logical rules previously introduced. For any presheaf $P : \mathbb{A}^{\mathsf{op}} \times \mathbb{A} \to \mathsf{Set}$ and object Γ, the following are (natural) isomorphisms of sets:

$$\dfrac{\dfrac{[x : \mathbb{A}^{\mathsf{op}}, y : \mathbb{A}] \;\; \hom_{\mathbb{A}}(x,y), \Gamma \vdash P(x,y)}{[z : \mathbb{A}] \; \Gamma \vdash P(\overline{z}, z)} \,(\hom)}{[\,] \; \Gamma \vdash \displaystyle\int_{z:\mathbb{A}} P(\overline{z}, z)} \,(\mathsf{end})$$

and for coends similarly:

$$\dfrac{\dfrac{\dfrac{[\,] \; \displaystyle\int^{z:\mathbb{A}} P(\overline{z}, z) \vdash \Gamma}{[z : \mathbb{A}] \; P(\overline{z}, z) \vdash \Gamma} \,(\mathsf{coend})}{[x : \mathbb{A}^{\mathsf{op}}, y : \mathbb{A}] \;\; \hom_{\mathbb{A}}(\overline{y}, \overline{x}), P(x,y) \vdash \Gamma} \,(\hom)}{[x : \mathbb{A}^{\mathsf{op}}, y : \mathbb{A}] \; P(x,y) \vdash \hom^{\mathsf{op}}_{\mathbb{A}}(y,x) \Rightarrow \Gamma} \,(\mathsf{exp})$$

Clearly in both situations the rule (hom) can be applied. □

Remark 5.15 (Adjunction for ends and naturals from the relative adjunction)**.** If the functor $J : \mathbb{A} \to \mathbb{X}$ has a right adjoint, relative adjunctions as in Definition 4.2 give rise to an ordinary adjunction; this can be done for the relative adjunction for hom of Theorem 4.3 using the right adjoint of $\pi^*_{\mathbb{A}}$, given by ends $\int_{\mathbb{A}}$ as in Lemma 5.12, for which one obtains exactly the ordinary adjunction for ends and natural transformations $(\hom_{\mathbb{A}} \times -) \dashv \int_{\mathbb{A}}$ given in Lemma 5.14.

5.3. **Coend calculus, syntactically**

We show how the rules for directed equality and (co)ends can be used to give concise proofs with a distinctly logical flavour to several central theorems of category theory. Note that our proofs follow a very different approach to that taken in [Lor21] and [CW01], since we explicitly use the correspondence between (co)ends and weakening operations and explicitly view expressions involving $\hom(a,b)$ in terms of directed equality and its rules.

The tecnique we henceforth apply is a Yoneda-like one: in order to prove that two functors or objects coincide, we assume to have a general functor/object Γ and then apply a series of natural isomorphisms of sets to obtain the desired equivalence. The equivalence then follows by the faithfulness of the Yoneda embedding. Note that no proof ever involves a "dinatural isomorphism", since we cannot apply Yoneda for generic sets which are not in general hom-sets: we shall only apply the (natural) isomorphisms of sets provided by the logical rules as intermediate steps.

Example 5.16 (Yoneda lemma)**.** For any presheaf $P : \mathbb{C} \to \mathsf{Set}$, and a presheaf $\Gamma : \mathbb{C} \to \mathsf{Set}$ acting as generic context, the following derivation expresses a (natural) isomorphism between the presheaf P and the presheaf sending an object $a : \mathbb{C}$ to the set of natural transformations $\mathsf{Nat}(\hom(a, -), P(-))$ computed as an end [Lor21, Thm. 1]:

$$\dfrac{\dfrac{\dfrac{[a : \mathbb{C}]\ \Gamma(a) \vdash \int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)}{[a : \mathbb{C}, x : \mathbb{C}]\ \Gamma(a) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(a, \overline{x}) \Rightarrow P(x)}\ (\mathsf{end})}{[a : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{a}, x) \times \Gamma(a) \vdash P(x)}\ (\mathsf{exp})}{[z : \mathbb{C}]\ \Gamma(z) \vdash P(z)}\ (\mathrm{hom})$$

Example 5.17 (coYoneda lemma)**.** For any $P, \Gamma : \mathbb{C} \to \mathsf{Set}$, the following derivation expresses that any presheaf P is isomorphic to a colimit (coend) of points of P "weighted" by representable functors [ML98, III.7, Theorem 1]:

$$\dfrac{\dfrac{[a : \mathbb{C}]\ \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, a) \times P(x) \vdash \Gamma(a)}{[a : \mathbb{C}, x : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{a}, x) \times P(a) \vdash \Gamma(x)}\ (\mathsf{coend})}{[z : \mathbb{C}]\ P(z) \vdash \Gamma(z)}\ (\mathrm{hom})$$

Example 5.18 (Presheaves are cartesian closed)**.** For any $A, B, \Gamma : \mathbb{C} \to \mathsf{Set}$, the following derivation expresses that the internal hom in the category of presheaves and natural transformation, given by $(A \Rightarrow B)(x) := \mathsf{Nat}(\hom(x, -) \times A, B)$, is indeed the correct definition, which we prove by showing the (natural) isomorphism of sets of the tensor/hom adjunction:

$$\begin{aligned} [x : \mathbb{C}]\ \Gamma(x) \vdash\ & (A \Rightarrow B)(x) \\ :=\ & \mathsf{Nat}(\hom_{\mathbb{C}}(x, -) \times A, B) \\ \cong\ & \int_{y:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y) \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : \mathbb{C}]\ \Gamma(x) \vdash \int_{y:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y)}{[x : \mathbb{C}, y : \mathbb{C}]\ \Gamma(x) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(x, \overline{y}) \times A^{\mathsf{op}}(\overline{y}) \Rightarrow B(y)}\ (\mathsf{end})}{[x : \mathbb{C}, y : \mathbb{C}]\ A(y) \times \hom_{\mathbb{C}}(\overline{x}, y) \times \Gamma(x) \vdash B(y)}\ (\mathsf{exp})}{[y : \mathbb{C}]\ A(y) \times \left(\int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, y) \times \Gamma(x) \right) \vdash B(y)}\ (\mathsf{coend{+}frobenius})}{[y : \mathbb{C}]\ A(y) \times \Gamma(y) \vdash B(y)}\ (\mathsf{coYoneda})$$

Note that (hom) cannot be used in this derivation since y appears positively in context in $A(y)$, whereas it should be negative in context to unify it with x using J. Instead, we apply the rule (coYoneda) given in Example 5.17 "extensionally", in the sense that the copresheaf $\Gamma : C \to \mathsf{Set}$ *as a whole* is isomorphic to a certain functor $C \to \mathsf{Set}$ computed with a coend, independently of the point on which it is evaluated (in this case y). We shall apply this tecnique again in the forthcoming examples.

Example 5.19 (Pointwise fomula for right Kan extensions)**.** Using our rules, we now give a logical proof that the functors $\mathsf{Ran}_F, \mathsf{Lan}_F : [\mathbb{C}, \mathsf{Set}] \to [\mathbb{D}, \mathsf{Set}]$ sending (co)presheaves to their extensions along $F : \mathbb{C} \to \mathbb{D}$ are, respectively, left and right adjoint to precomposition functors $F^* : [\mathbb{D}, \mathsf{Set}] \to [\mathbb{C}, \mathsf{Set}]$. Note the similarity between this derivation and the argument given in [Pit95, 5.6.6] for a general hyperdoctrine, in light of the discussion given in Section 1.

For any presheaves $P, \Gamma : \mathbb{C} \to \mathsf{Set}$ and a functor/term $\mathbb{C} \to \mathbb{D}$:

$$
\begin{array}{cl}
[x : \mathbb{C}]\ \Gamma(x) \vdash (\mathsf{Ran}_F P)(x) := \displaystyle\int_{y:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(x, F^{\mathsf{op}}(\overline{y})) \Rightarrow P(y) & \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{C}]\ \Gamma(x) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(x, F^{\mathsf{op}}(\overline{y})) \Rightarrow P(y) & (\mathsf{end}) \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(\overline{x}, F(y)) \times \Gamma(x) \vdash P(y) & (\mathsf{exp}) \\
\hline\hline
[y : \mathbb{C}]\ \displaystyle\int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x}, F(y)) \times \Gamma(x) \vdash P(y) & (\mathsf{coend}) \\
\hline\hline
[y : \mathbb{C}]\ \Gamma(F(y)) \vdash P(y) & (\mathsf{coYoneda})
\end{array}
$$

Example 5.20 (Pointwise fomula for left Kan extensions)**.** For any presheaves $A, B, \Gamma : \mathbb{C} \to \mathsf{Set}$ and a functor $\mathbb{C} \to \mathbb{D}$:

$$
\begin{array}{cl}
[x : \mathbb{C}]\ (\mathsf{Lan}_F P)(x) := \displaystyle\int^{y:\mathbb{C}} \hom_{\mathbb{C}}(F^{\mathsf{op}}(\overline{y}), x) \times P(y) \vdash \Gamma(x) & \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{C}]\ \hom_{\mathbb{C}}(F^{\mathsf{op}}(\overline{y}), x) \times P(y) \vdash \Gamma(x) & (\mathsf{coend}) \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{C}]\ P(y) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(F(y), \overline{x}) \Rightarrow \Gamma(x) & (\mathsf{exp}) \\
\hline\hline
[y : \mathbb{C}]\ P(y) \vdash \displaystyle\int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(F(y), \overline{x}) \Rightarrow \Gamma(x) & (\mathsf{end}) \\
\hline\hline
[y : \mathbb{C}]\ P(y) \vdash \Gamma(F(y)) & (\mathsf{Yoneda})
\end{array}
$$

Example 5.21 (Synthetic Fubini rule for ends)**.** For convenience we only show the case for ends. For any dipresheaf $\Gamma : \mathbb{1}^{\mathsf{op}} \times \mathbb{1} \to \mathsf{Set}$ (a dipresheaf in the empty context, i.e., simply an object $\Gamma : \mathsf{Set}$) and dipresheaf $P : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \to \mathsf{Set}$ one has:

$$
\begin{array}{cl}
[\,]\ \Gamma \vdash \displaystyle\int_{x:\mathbb{C}} \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & \\
\hline\hline
[x : \mathbb{C}]\ \Gamma \vdash \displaystyle\int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & (\mathsf{end}) \\
\hline\hline
[x : \mathbb{C}, y : \mathbb{D}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y) & (\mathsf{end}) \\
\hline\hline
[y : \mathbb{D}, x : \mathbb{C}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y) & (\mathsf{structural\ property}) \\
\hline\hline
[y : \mathbb{D}]\ \Gamma \vdash \displaystyle\int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y) & (\mathsf{end}) \\
\hline\hline
[\,]\ \Gamma \vdash \displaystyle\int_{y:\mathbb{D}} \int_{x:\mathbb{C}} P(\overline{x}, x, \overline{y}, y) & (\mathsf{end})
\end{array}
$$

where we used the fact that certain structural properties are true for contexts by the cartesian structure of Cat. Similarly,

$$
\begin{array}{cl}
[\,]\ \Gamma \vdash \int_{x:\mathbb{C}} \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & \\
\hline\hline
 & (\mathsf{end}) \\
[x:\mathbb{C}]\ \Gamma \vdash \int_{y:\mathbb{D}} P(\overline{x}, x, \overline{y}, y) & \\
\hline\hline
 & (\mathsf{end}) \\
[x:\mathbb{C}, y:\mathbb{D}]\ \Gamma \vdash P(\overline{x}, x, \overline{y}, y) & \\
\hline\hline
 & (\text{structural property}) \\
[p:\mathbb{C}\times\mathbb{D}]\ \Gamma \vdash P(\overline{p}, p) & \\
\hline\hline
 & (\mathsf{end}) \\
[\,]\ \Gamma \vdash \int_{p:\mathbb{C}\times\mathbb{D}} P(\overline{x}, x, \overline{y}, y) &
\end{array}
$$

Example 5.22 (Right rifts in profunctors)**.** We give a logical proof that composition in Prof has a right adjoint (on both sides) [Lor21, 5.2.5 and Exercise 5.2]. This makes Prof a bicategory where *right extensions* and *right liftings* exist. For simplicity we only treat composition on the left, although composition on the right is completely analogous. For any composable profunctors $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{A} \to \mathsf{Set}$, $Q : \mathbb{A}^{\mathsf{op}} \times \mathbb{D} \to \mathsf{Set}$ and a generic $\Gamma : \mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathsf{Set}$:

$$
\begin{array}{cl}
[x:\mathbb{C}^{\mathsf{op}}, z:\mathbb{D}]\ (P\,;-)(Q)(x,z) := \int^{y:\mathbb{A}} P(x,y)\times Q(\overline{y},z) \vdash \Gamma(x,z) & \\
\hline\hline
 & (\mathsf{coend}) \\
[x:\mathbb{C}^{\mathsf{op}}, y:\mathbb{A}, z:\mathbb{D}]\ P(x,y)\times Q(\overline{y},z) \vdash \Gamma(x,z) & \\
\hline\hline
 & (\mathsf{exp}) \\
[x:\mathbb{C}^{\mathsf{op}}, y:\mathbb{A}, z:\mathbb{D}]\ Q(\overline{y},z) \vdash P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \Gamma(x,z) & \\
\hline\hline
 & (\mathsf{end}) \\
[y:\mathbb{A}, z:\mathbb{D}]\ Q(\overline{y},z) \vdash \int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},\overline{y}) \Rightarrow \Gamma(x,z) & \\
\hline\hline
 & (\text{Remark 2.8}) \\
[y:\mathbb{A}^{\mathsf{op}}, z:\mathbb{D}]\ Q(y,z) \vdash \mathsf{Rift}_P(\Gamma)(y,z) := \int_{x:\mathbb{C}} P^{\mathsf{op}}(\overline{x},y) \Rightarrow \Gamma(x,z) &
\end{array}
$$

where the last (end) can be applied since $x : \mathbb{C}$ does not appear on the left.

Example 5.23 (hom-functors preserve limits)**.** Recall that (co)ends generalize limits, in the sense that the (co)end of a presheaf $P : \mathbb{C} \to \mathsf{Set}$, viewed as a dipresheaf mute in its contravariant variable, coincides with the (co)limit of that functor [Lor21], i.e., $\lim_{x:\mathbb{C}} P(x) := \int^{x:\mathbb{C}} P(x) = \int^{\mathbb{C}}(\pi_{\mathbb{C}^{\mathsf{op}}}\,; P)$. We can prove logically that hom-functors $- \Rightarrow - : \mathsf{Set}^{\mathsf{op}} \times \mathsf{Set} \to \mathsf{Set}$ preserve ends viewed as limits (i.e., colimits in $\mathsf{Set}^{\mathsf{op}}$ in its contravariant variable and limits in Set in its covariant variable). For any object $\Gamma, Q : \mathsf{Set}$ and a dipresheaf $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$, the following derivation states preservation of limits (colimits in $\mathsf{Set}^{\mathsf{op}}$) on the left:

$$
\begin{array}{cl}
[\,]\ \Gamma \vdash \left(\int^{x:\mathbb{C}} P(\overline{x}, x)\right) \Rightarrow Q & \\
\hline\hline
 & (\mathsf{exp}) \\
[\,]\ \left(\int^{x:\mathbb{C}} P(\overline{x}, x)\right), \Gamma \vdash Q & \\
\hline\hline
 & (\mathsf{coend{+}frobenius}) \\
[x:\mathbb{C}]\ P(\overline{x}, x), \Gamma \vdash Q & \\
\hline\hline
 & (\mathsf{exp}) \\
[x:\mathbb{C}]\ \Gamma \vdash P^{\mathsf{op}}(x, \overline{x}) \Rightarrow Q & \\
\hline\hline
 & (\mathsf{end}) \\
[\,]\ \Gamma \vdash \int_{x:\mathbb{C}} P^{\mathsf{op}}(x, \overline{x}) \Rightarrow Q &
\end{array}
$$

Similarly, hom-functors preserve limits on the right:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\,]\ \Gamma \vdash Q \Rightarrow \int_{x:\mathbb{C}} P(\overline{x}, x)}{[\,]\ Q, \Gamma \vdash \int_{x:\mathbb{C}} P(\overline{x}, x)}\ (\mathsf{exp})}{[x : \mathbb{C}]\ Q, \Gamma \vdash P(\overline{x}, x)}\ (\mathsf{end})}{[x : \mathbb{C}]\ \Gamma \vdash Q \Rightarrow P(\overline{x}, x)}\ (\mathsf{exp})}{[\,]\ \Gamma \vdash \int_{x:\mathbb{C}} (Q \Rightarrow P(\overline{x}, x))}\ (\mathsf{end})
$$

6. Conclusions and future work

In this paper we semantically motivated how dinaturality plays a crucial role in the treatment of a directed type theory with variances where types are interpreted as categories and (propositional) directed equality is represented by hom-functors.

Dinaturality. The most important piece missing from our work is the compositionality of dinatural transformations. The existence of a sufficiently expressive criterion for the composability of dinaturals would provide a full account of a true directed type theory, using the semantic rules we provided in a compositional and fully syntactic way; strong dinatural transformations provide a hint in this direction but lack in expressivity, e.g., they do not form a cartesian closed category in general [Uus]. Nevertheless, we have seen how our rules can still be used as a sound semantics (although not compositional) to succintly prove several useful theorems in category theory.

Abstract models. Our characterization of directed equality in terms of a relative adjunction between (para)categories of dinatural transformations is a first step towards a formal understanding of the role played by variance and directed equality. In particular, we wish to expand our semantic analysis towards an abstract account in the style of the doctrinal approach [Jac99, MR15], possibly by introducing a notion of *directed hyperdoctrine* which suitably axiomatizes the roles played by variance and the $-^{\mathsf{op}}$ involution, (di)naturality, and the relative adjunction for directed equality. This would allow to fully interpret our semantic rules, which we already provided with a notation reminiscent of type theoretical judgements, truly as as syntactic objects, with a suitable initiality result in a category of models.

Enrichment. Our semantic analysis does not rely on properties that are specific to Set (viewed as the base of enrichment of Cat), other than cartesian closedness to have a notion of implication and conjunction and the existence of (co)limits to express (co)ends. We conjecture that our analysis of dinaturals can be developed in more generality by taking enriched categories (over a sufficiently structured base of enrichment) as types, rather than simply categories (enriched over Set).

In particular, all our results can be specialized to the category of posets Pos rather than Cat: in that case, dinatural transformations all compose trivially, directed equality is precisely presented as a relative left adjoint, and our work provides a characterization of the "logic of posets" via the (*directed*) hyperdoctrine sending a poset to its category of (posetal) dipresheaves and dinatural transformations. The main issue is that the structure of (co)ends is also trivialized, and merely correspond with (co)products. Another degenerate but interesting case is obtained by replacing Cat with Gpd the (2-)category of groupoids, given its correspondence with models of Martin-Löf type theory with symmetric equality, for which we similarly conjecture a general theorem of compositionality for dinaturals in the fully groupoidal case.

Pseudo(co)ends. There are suitable relaxations of ends not as (strict) limits, but as pseudolimits in the 2-category Cat with pseudonatural transformations as 2-cells; in this sense, there are important examples of coend calculus which are not captured by our definition. In particular, the category of elements of a functor, reminiscent of a sigma type, can be implemented using a suitable coend in Cat [Lor21, 4.2.2], and $\mathsf{El}(F) \cong \int^{c:\mathbb{C}} c/\mathbb{C} \times F(c)$ where $c/\mathbb{C}$ is the coslice category (note that it is a contravariant assignment $-/\mathbb{C} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Cat}$) and $F(c)$ is seen as a discrete category. This particular case can be implemented using Cat as base of enrichment, but it is not obvious how and if the equational aspects of our work become too strict and would need, for example, to prescribe equivalences up-to coherent isomorphism.

Higher order and universes. In a similar spirit, our analysis could be extended by introducing a notion of universe type former, as typically done in type theories. This would for example enable us to express that composition maps exist in general for any category $\mathbb{C} : \mathsf{Cat}$, where this quantification is expressed by a suitable pseudo(co)end, with Cat acting as type of types [Hof97].

Type dependency. We have concentrated on a first order treatment because our original motivation of (co)end calculus (usually) exhibits only the simple dependencies of first order logic; a promising step towards a real directed type theory would be the introduction of a suitable notion of dependent types, and therefore of dependent/indexed dinatural transformations. It is not clear if the composability issues of dinatural transformations can be dealt in a way that is essentially independent from type dependency.

∞-category theory. Our type theory captures the elementary theory of 1-categories, without focusing however on the 2-categorical structure of Cat; it would be interesting to explore a generalization of this analysis to encompass higher-categorical models in the spirit of [RS17, GWB24, WL20], possibly using the same framework of dinaturality to pinpoint synthetic aspects of, e.g., ω-categories.

References

[ACD+18] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg, *Quotient Inductive-Inductive Types*, Springer International Publishing, 2018, pp. 293–310.

[AH10] Kazuyuki Asada and Ichiro Hasuo, *Categorifying Computations into Components via Arrows as Profunctors*, Electronic Notes in Theoretical Computer Science **264** (2010), no. 2, 25–45.

[AM24] Nathanael Arkor and Dylan McDermott, *The formal theory of relative monads*, Journal of Pure and Applied Algebra **228** (2024), no. 9.

[ANvdW23] Benedikt Ahrens, Paige Randall North, and Niels van der Weide, *Bicategorical type theory: semantics and syntax*, Mathematical Structures in Computer Science (2023), 868 – 912.

[Asa10] Kazuyuki Asada, *Arrows are strong monads*, MSFP '10, Association for Computing Machinery, 2010, pp. 33–42.

[B0́0] Jean Bénabou, *Distributors at work*, Notes by Thomas Streicher from lectures given at TU Darmstadt, 2000.

[BFSS90] E. S. Bainbridge, Peter J. Freyd, Andre Scedrov, and Philip J. Scott, *Functorial polymorphism*, Theoretical Computer Science **70** (1990), no. 1, 35–64.

[BG18] Guillaume Boisseau and Jeremy Gibbons, *What you needa know about Yoneda: profunctor optics and the Yoneda lemma (functional pearl)*, Proceedings of the ACM on Programming Languages **2** (2018), no. ICFP, 84:1–84:27.

[Blu93] Richard F. Blute, *Linear logic, coherence and dinaturality*, Theoretical Computer Science **115** (1993), no. 1, 3–41.

[Boi20] Guillaume Boisseau, *String Diagrams for Optics*, vol. 167, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020, pp. 17:1–17:18.

[BS96] Richard F. Blute and Philip J. Scott, *Linear Läuchli semantics*, Annals of Pure and Applied Logic **77** (1996), no. 2, 101–142.

[BS98] ———, *The Shuffle Hopf Algebra and Noncommutative Full Completeness*, Journal of Symbolic Logic **63** (1998), no. 4, 1413–1436.

[CCHM15] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg, *Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom*, vol. 69, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2015, pp. 5:1–5:34.

[CEG+22] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore, and Mario Román, *Profunctor optics, a categorical update*, 2022, arxiv.2001.07488.

[CMN24] Fernando Chu, Éléonore Mangel, and Paige Randall North, *A directed type theory for 1-categories*, 2024, p. 205.

[CS10] Geoffrey S.H. Cruttwell and Michael Shulman, *A unified framework for generalized multicategories*, Theory and Applications of Categories **24** (2010), 580–655.

[Cur12] Pierre-Louis Curien, *Operads, clones, and distributive laws*, World Scientific, 2012, pp. 25–49.

[CW01] Mario Cáccamo and Glynn Winskel, *A Higher-Order Calculus for Categories*, Lecture Notes in Computer Science, Springer, 2001, pp. 136–153.

[DS70] Eduardo Dubuc and Ross Street, *Dinatural transformations*, Lecture Notes in Mathematics, Springer, 1970, pp. 126–137.

[EK66] Samuel Eilenberg and G. Max Kelly, *A generalization of the functorial calculus*, Journal of Algebra **3** (1966), no. 3, 366–375.

[Esc15] Martin Escardó, *Using Yoneda rather than J to present the identity type*, `https://www.cs.bham.ac.uk/~mhe/yoneda/yoneda.html`, 2015.

[Fre96] Peter J. Freyd, *Paramathematics*, Unpublished manuscript, Talk delivered at XII MFPS, Boulder, 1996.

[FRR92] Peter J. Freyd, Edmund Robinson, and Giuseppe. Rosolini, *Dinaturality for free*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1992, pp. 107–118.

[GSS92] Jean-Yves Girard, Andre Scedrov, and Philip J. Scott, *Normal Forms and Cut-Free Proofs as Natural Transformations*, Mathematical Sciences Research Institute Publications, Springer, 1992, pp. 217–241.

[GWB24] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz, *Directed univalence in simplicial homotopy type theory*, 2024.

[HC23] James Hefford and Cole Comfort, *Coend Optics for Quantum Combs*, Electronic Proceedings in Theoretical Computer Science **380** (2023), 63–76.

[Hin12] Ralf Hinze, *Kan Extensions for Program Optimisation Or: Art and Dan Explain an Old Trick*, Lecture Notes in Computer Science, Springer, 2012, pp. 324–362.

[HM03] Claudio Hermida and Paulo Mateus, *Paracategories i: internal paracategories and saturated partial algebras*, Theoretical Computer Science **309** (2003), no. 1–3, 125–156.

[HM04] ———, *Paracategories ii: adjunctions, fibrations and examples from probabilistic automata theory*, Theoretical Computer Science **311** (2004), no. 1–3, 71–103.

[Hof97] Martin Hofmann, *Syntax and Semantics of Dependent Types*, Publications of the Newton Institute, Cambridge University Press, Cambridge, 1997, pp. 79–130.

[HRR14] Claudio Hermida, Uday S. Reddy, and Edmund P. Robinson, *Logical relations and parametricity – a reynolds programme for category theory and programming languages*, Electronic Notes in Theoretical Computer Science **303** (2014), 149–180.

[HS98] Martin Hofmann and Thomas Streicher, *The groupoid interpretation of type theory*, Oxford Logic Guides, vol. 36, Oxford Univ. Press, New York, New York, 1998, pp. 83–111. MR 1686862

[Hyl14] J. M. E. Hyland, *Elements of a theory of algebraic theories*, Theoretical Computer Science **546** (2014), 132–144, Models of Interaction: Essays in Honour of Glynn Winskel.

[Jac99] Bart P. F. Jacobs, *Categorical logic and type theory*, Studies in Logic and the Foundations of Mathematics, vol. 141, North-Holland, 1999.

[KRW24] Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger, *Formalizing the ∞-categorical Yoneda lemma*, CPP 2024, Association for Computing Machinery, 2024, p. 274–290.

[Law69] Francis William Lawvere, *Adjointness in Foundations*, Dialectica **23** (1969), no. 3/4, 281–296.

[Law70] ———, *Equality in hyperdoctrines and the comprehension schema as an adjoint functor*, American Mathematical Society, Providence RI, 1970, pp. 1–14.

[Law73] ———, *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano **43** (1973), no. 1, 135–166.

[Lei10] Tom Leinster, *Coend computation*, MathOverflow, 2010, MO question (2010-04-06).

[Lei14] ———, *Basic Category Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2014.

[LH11] Daniel R. Licata and Robert Harper, *2-Dimensional Directed Type Theory*, Electronic Notes in Theoretical Computer Science **276** (2011), 263–289.

[Lor21] Fosco Loregian, *Coend Calculus*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2021.

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag New York, 1998.

[MMSS00] Michael Mandell, Peter May, Stefan Schwede, and Brooke Shipley, *Model categories of diagram spectra*, Proceedings of the London Mathematical Society **82** (2000), no. 2, 441–512.

[MR15] Maria Emilia Maietti and Giuseppe Rosolini, *Unifying Exact Completions*, Applied Categorical Structures **23** (2015), no. 1, 43–52.

[MS21] Guy McCusker and Alessio Santamaria, *Composing dinatural transformations: Towards a calculus of substitution*, Journal of Pure and Applied Algebra **225** (2021), no. 10, 106689.

[MZ16] Paul-André Melliès and Noam Zeilberger, *A bifibrational reconstruction of Lawvere's presheaf hyperdoctrine*, LICS '16, Association for Computing Machinery, 2016, pp. 555–564.

[NL23] Max S. New and Daniel R. Licata, *A Formal Logic for Formal Category Theory*, Lecture Notes in Computer Science, Springer Nature Switzerland, 2023, pp. 113–134.

[Nor19] Paige Randall North, *Towards a Directed Homotopy Type Theory*, Electronic Notes in Theoretical Computer Science **347** (2019), 223–239.

[Nuy15] Andreas Nuyts, *Towards a directed homotopy type theory based on 4 kinds of variance*, Master's thesis, KU Leuven, 2015.

[Pet03] Zoran Petrić, *G-dinaturality*, Annals of Pure and Applied Logic **122** (2003), no. 1, 131–173.

[Pis18] Paolo Pistone, *Proof nets, coends and the Yoneda isomorphism*, EPTCS, vol. 292, 2018, pp. 148–167.

[Pis19] Paolo Pistone, *On completeness and parametricity in the realizability semantics of System F*, Logical Methods in Computer Science **15** (2019), 6:1–6:54.

[Pit95] Andrew M. Pitts, *Categorical logic*, Handbook of Logic in Computer Science: Volume 5: Logic and Algebraic Methods, Oxford University Press, 1995, p. 39–123.

[PT21] Paolo Pistone and Luca Tranchini, *The Yoneda Reduction of Polymorphic Types*, vol. 183, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2021, pp. 35:1–35:22.

[Rom20] Mario Román, *Open Diagrams via Coend Calculus*, ACT, vol. 333, 2020, pp. 65–78.

[RS17] Emily Riehl and Michael Shulman, *A type theory for synthetic ∞-categories*, Higher structures **1** (2017), no. 1.

[San19] Alessio Santamaria, *Towards a godement calculus for dinatural transformations*, Ph.D. thesis, University of Bath, Somerset, UK, 2019.

[Sco00] Philip J. Scott, *Some aspects of categories in computer science*, p. 3–77, Elsevier, 2000.

[Shu18] Michael Shulman, *Contravariance through enrichment*, Theory and Applications of Categories **33** (2018), no. 5, 95–130.

[Str81] Ross Street, *Conspectus of variable categories*, Journal of Pure and Applied Algebra **21** (1981), no. 3, 307–338.

[Str03] ———, *Functorial calculus in monoidal bicategories*, Applied Categorical Structures **11** (2003), no. 3, 219–227.

[Ulm68] Friedrich Ulmer, *Properties of dense and relative adjoint functors*, Journal of Algebra **8** (1968), no. 1, 77–95.

[Uni13] The Univalent Foundations Program, *Homotopy type theory: Univalent foundations of mathematics*, `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[Uus] Tarmo Uustalu, *A note on strong dinaturality, initial algebras and uniform parameterized fixpoint operators*, FICS 2010, pp. 77–82.

[Voi20] Janis Voigtländer, *Free Theorems Simply, via Dinaturality*, Springer International Publishing, 2020, pp. 247–267.

[WL20] Matthew Z. Weaver and Daniel R. Licata, *A Constructive Model of Directed Univalence in Bicubical Sets*, LICS '20, Association for Computing Machinery, 2020, pp. 915–928.

[Yon60] Nobuo Yoneda, *On Ext and exact sequences*, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 507–576.

Tallinn University of Technology, Department of Software Science, Tallinn, Estonia
Email address: {`andrea.laretto,fosco.loregian,niccolo.veltri`}`@taltech.ee`